The representation theory of somewhere-to-below shuffles

Darij Grinberg

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Abstract. The *somewhere-to-below shuffles* are the elements

$$t_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n}$$

(for $\ell \in \{1,2,\ldots,n\}$) in the group algebra $\mathbf{k}\,[S_n]$ of the n-th symmetric group S_n. Their linear combinations are called the *one-sided cycle shuffles*. We determine the eigenvalues of the action of any one-sided cycle shuffle on any Specht module $\mathcal{S}^\lambda$ of S_n.

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Contents

1. Introduction

This paper is a continuation of [GriLaf22] with other means. Specifically, our goal here is to answer some natural representation-theoretical questions around the somewhere-to-below shuffles in the symmetric group algebra (including [GriLaf22, Question 16.12]).

At the present moment, this paper is just an outline; one day it will grow to contain complete proofs.

We recall that the *somewhere-to-below shuffles* are n special elements $t_1, t_2, \ldots, t_n$ of the group algebra $\mathbf{k}[S_n]$ of a symmetric group S_n; they are defined by

$$ t_\ell := \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell, \ell+1} + \operatorname{cyc}_{\ell, \ell+1, \ell+2} + \cdots + \operatorname{cyc}_{\ell, \ell+1, \ldots, n} \in \mathbf{k}[S_n], $$

where $\operatorname{cyc}_{\ell, \ell+1, \ldots, k}$ denotes the cycle that sends $\ell \mapsto \ell+1 \mapsto \ell+2 \mapsto \cdots \mapsto k \mapsto \ell$ (and leaves all remaining elements of $[n] = \{1, 2, \ldots, n\}$ unchanged). Together with their linear combinations (called the *one-sided cycle shuffles*), they have been introduced and studied in the paper [GriLaf22] (published with abridgements as [GL24][1]) by Lafrenière and the present author. One of the main results is [GriLaf22, Theorem 11.1], which constructs a basis $(a_w)_{w \in S_n}$ of $\mathbf{k}[S_n]$ on which each of the shuffles $t_1, t_2, \ldots, t_n$ acts (by right multiplication) triangularly – i.e., which satisfies

$$ a_w t_\ell \in \operatorname{span} \{a_v \mid v \leq w\} \qquad \text{for all } w \in S_n \text{ and } \ell \in \{1, 2, \ldots, n\} $$

(for an appropriate total order $<$ on S_n). This entails that the shuffles $t_1, t_2, \ldots, t_n$ and their linear combinations have integer eigenvalues; these eigenvalues have indeed been found ([GriLaf22, §12]) along with their multiplicities ([GriLaf22, §13]). As a further consequence, the $\mathbf{k}$-subalgebra of $\mathbf{k}[S_n]$ generated by $t_1, t_2, \ldots, t_n$ is isomorphic to an algebra of upper-triangular matrices, and the commutators $[t_i, t_j] := t_i t_j - t_j t_i$ are nilpotent; a followup work [Grinbe23] proves even stronger claims.

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[1]The numbering of results in [GriLaf22] and in [GL24] is identical except for Section 9, so the reader can consult either version.

However, like any elements of the group algebra $\mathbf{k}[S_n]$, the shuffles $t_1, t_2, \ldots, t_n$ act not just on the whole algebra $\mathbf{k}[S_n]$, but on any of its modules, i.e., on any representation of S_n. Thus, the question about eigenvalues can be asked for each representation of S_n, in particular for the *Specht modules* (which are the irreducible representations of S_n, at least in characteristic 0).

The main goal of this paper is to answer this latter question. Let us give a quick outline of the answer (which was announced in [GriLaf24, §11])[2]:

We shall use some basic notions from the representation theory of S_n and from symmetric functions; the reader can find all prerequisites in [Fulton97, Chapters 6 and 7]. For any partition λ of n, a Specht module $\mathcal{S}^\lambda$ is defined, which is a representation of S_n with a basis indexed by standard tableaux of shape λ. (In [Fulton97], it is called S^λ.) This S_n-module $\mathcal{S}^\lambda$ is irreducible when $\mathbf{k}$ has characteristic 0. Each $u \in \mathbf{k}[S_n]$ acts (on the left) on this Specht module $\mathcal{S}^\lambda$; we let $L_\lambda(u)$ denote this action (viewed as a $\mathbf{k}$-module endomorphism of $\mathcal{S}^\lambda$).

We let Λ denote the ring of symmetric functions over $\mathbb{Z}$ (defined in [Fulton97, §6.2]). We recall that it has a basis $(s_\lambda)_{\lambda \text{ is a partition}}$ of *Schur functions s_λ*.

For each $m \in \mathbb{N}$, we let $h_m \in \Lambda$ denote the m-th complete homogeneous symmetric function. For each $m > 1$, we let $z_m \in \Lambda$ denote the Schur function

$$z_m := s_{(m-1,1)} = h_{m-1}h_1 - h_m \in \Lambda.$$

A set of integers is called *lacunar* if it contains no two consecutive integers. For each lacunar subset I of $[n-1]$, we define a symmetric function

$$z_I := h_{i_1-1} \prod_{j=2}^{m} z_{i_j - i_{j-1}} \in \Lambda,$$

where $i_1, i_2, \ldots, i_m$ are the elements of $I \cup \{n+1\}$ in increasing order (so that $i_m = n+1$ and $I = \{i_1 < i_2 < \cdots < i_{m-1}\}$). When this symmetric function z_I is expanded in the basis $(s_\lambda)_{\lambda \text{ is a partition}}$ of Λ, the coefficient of a given Schur function s_λ shall be called c_I^λ. This coefficient c_I^λ is actually a Littlewood–Richardson coefficient (since z_I is a skew Schur function), hence a nonnegative integer.

We now claim the following:

Theorem 1.1 (part of Theorem 4.3). Let λ be a partition. Let $\omega_1, \omega_2, \ldots, \omega_n \in \mathbf{k}$. Then, the eigenvalues of the operator $L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ on the Specht module $\mathcal{S}^\lambda$ are the linear combinations

$$\omega_1 m_{I,1} + \omega_2 m_{I,2} + \cdots + \omega_n m_{I,n} \qquad \text{for } I \subseteq [n-1] \text{ lacunar satisfying } c_I^\lambda \neq 0,$$

where the $m_{I,k}$ are certain nonnegative integers defined combinatorially (namely, $m_{I,k}$ is the distance between k and the smallest element of $I \cup \{n+1\}$ that is $\geq k$).

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[2]The formulation in [GriLaf24, §11] uses the Frobenius characteristic map, but this has turned out to be a red herring.

The algebraic multiplicities of these eigenvalues are the c_I^λ in the generic case (i.e., if no two I's produce the same linear combination; otherwise the multiplicities of colliding eigenvalues should be added together). Moreover, if all these linear combinations are distinct, then $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ is diagonalizable.

The proof of this theorem will rely on the filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n]$ of $\mathbf{k}[S_n]$ introduced in [GriLaf22, §8.1]. We call this the *Fibonacci filtration* of $\mathbf{k}[S_n]$, as its length f_{n+1} is the $(n+1)$-st Fibonacci number. We note that this filtration is not completely canonical, as it depends on the choice of a listing $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ of all lacunar subsets of $[n-1]$ in the order of increasing sum of elements (the ties can be broken arbitrarily, whence the non-canonicity). Much about this filtration was already understood in [GriLaf22], but we will need some additional information about the action of S_n on its subquotients F_i/F_{i-1}:

Let $\mathcal{A}$ be the $\mathbf{k}$-algebra $\mathbf{k}[S_n]$, and let $\mathcal{T}$ be its $\mathbf{k}$-subalgebra generated by $t_1, t_2, \ldots, t_n$. Then, each F_i is a left ideal of $\mathcal{A}$ but is also fixed under right multiplication by each t_ℓ; therefore, each F_i is an $(\mathcal{A}, \mathcal{T})$-subbimodule of $\mathcal{A}$. Thus, each subquotient F_i/F_{i-1} of the Fibonacci filtration is an $(\mathcal{A}, \mathcal{T})$-bimodule. As a right $\mathcal{T}$-module, it is *scalar* (meaning that each t_ℓ acts on it by a scalar, which is in fact the integer $m_{Q_i,\ell}$ from [GriLaf22, Theorem 8.1 **(c)**]). As a left $\mathcal{A}$-module (i.e., as a representation of S_n), we describe it explicitly here:

Theorem 1.2 (part of Theorem 3.3). Let $i \in [f_{n+1}]$.

Consider the lacunar subset Q_i of $[n-1]$ (from the above listing $Q_1, Q_2, \ldots, Q_{f_{n+1}}$). Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Note that $j_1 \geq 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = i_m - i_0 = n$.

For each $p \in \mathbb{N}$, we let $\mathcal{H}_p$ denote the trivial 1-dimensional representation of S_p (that is, the $\mathbf{k}$-module $\mathbf{k}$ on which S_p acts trivially), and we let $\mathcal{Z}_p$ denote the reflection quotient representation of S_p (that is, the free $\mathbf{k}$-module $\mathbf{k}^p$ on which S_p acts by permuting the coordinates, divided by the submodule consisting of all vectors of the form $(a, a, \ldots, a) \in \mathbf{k}^p$). Then,

$$F_i/F_{i-1} \cong \mathrm{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \underbrace{\left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \right)}_{\substack{\text{the first tensorand is an } \mathcal{H}, \\ \text{while all others are } \mathcal{Z}\text{'s}}}$$

as S_n-representations. Here, we embed $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$ into S_n by the usual parabolic embedding (since $j_1 + j_2 + \cdots + j_m = n$).

This theorem will be a crucial stepping stone on our way to Theorem 1.1.

We note that neither of our two main results requires any assumption about the characteristic of $\mathbf{k}$. However, in positive characteristic, care must be taken to distinguish between the reflection quotient representation $\mathcal{Z}_p$ in Theorem 1.2 and the reflection subrepresentation $\mathcal{R}_p$ (which consists of the zero-sum vectors in $\mathbf{k}^p$);

the two representations have the same dimension $p - 1$ (for $p \geq 1$), but are not isomorphic unless char $\mathbf{k} \neq p$ or $p \leq 2$.

We suspect that our results can be generalized ("q-deformed") from the symmetric group algebra to the Hecke algebra $\mathcal{H}_q(S_n)$. Most results from [GriLaf22] can definitely be generalized this way, as will be detailed in forthcoming work.

2. Definitions and notations

2.1. Basics

We recall some notations from [GriLaf22].

Let $\mathbf{k}$ be any commutative ring. (We don't require that $\mathbf{k}$ is a field or a $\mathbb{Q}$-algebra, but the reader can think of $\mathbf{k} = \mathbb{Q}$ as a standing example.)

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers.

For any integers a and b, we set

$$[a, b] := \{k \in \mathbb{Z} \mid a \leq k \leq b\} = \{a, a + 1, \ldots, b\}.$$

This is an empty set if $a > b$. In general, $[a, b]$ is called an *integer interval*.

For each $n \in \mathbb{Z}$, let $[n] := [1, n] = \{1, 2, \ldots, n\}$.

Fix an integer $n \in \mathbb{N}$. Let S_n be the n-th symmetric group, i.e., the group of all permutations of $[n]$. We multiply permutations in the "continental" way: that is, $(\pi\sigma)(i) = \pi(\sigma(i))$ for all $\pi, \sigma \in S_n$ and $i \in [n]$.

For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\mathrm{cyc}_{i_1, i_2, \ldots, i_k}$ be the permutation in S_n that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$, respectively while leaving all remaining elements of $[n]$ unchanged. This permutation is known as a *cycle*. Note that $\mathrm{cyc}_i = \mathrm{id}$ for any single $i \in [n]$.

For any $i \in [n-1]$, we denote the cycle $\mathrm{cyc}_{i,i+1}$ by s_i and call it a *simple transposition*.

2.2. Somewhere-to-below shuffles, $\mathcal{A}$ and $\mathcal{T}$

Let $\mathcal{A}$ be the group algebra $\mathbf{k}[S_n]$. In this algebra, define n elements $t_1, t_2, \ldots, t_n$ by setting[3]

$$t_\ell := \mathrm{cyc}_\ell + \mathrm{cyc}_{\ell,\ell+1} + \mathrm{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \mathrm{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n]$$

for each $\ell \in [n]$. Thus, in particular, $t_n = \mathrm{cyc}_n = \mathrm{id} = 1$ (where 1 means the unity of $\mathbf{k}[S_n]$). The n elements $t_1, t_2, \ldots, t_n$ are known as the *somewhere-to-below shuffles*.

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[3]We view S_n as a subset of $\mathbf{k}[S_n]$ in the obvious way.

We let $\mathcal{T}$ be the **k**-subalgebra of $\mathcal{A}$ generated by these n somewhere-to-below shuffles $t_1, t_2, \ldots, t_n$. Clearly, $\mathcal{A}$ is an $(\mathcal{A}, \mathcal{T})$-bimodule (with $\mathcal{A}$ acting from the left by multiplication, and $\mathcal{T}$ acting from the right by multiplication).

2.3. Some S_n-representation theory

We recall that the representations of the symmetric group S_n (over **k**) are precisely the left $\mathbf{k}[S_n]$-modules, i.e., the left $\mathcal{A}$-modules. We will use the following four classes of S_n-representations in particular:

1. *The Specht modules $\mathcal{S}^\lambda$:* If λ is any partition of n, then the *Specht module $\mathcal{S}^\lambda$* is a representation of S_n constructed using the Young diagram of shape λ. For its definition, see [Gri25, Definition 5.4.1 **(b)**] (where it is called $\mathcal{S}^{Y(\lambda)}$) or [Fulton97, §7.2] (where it is called S^λ). If **k** is a field of characteristic 0, then the Specht module $\mathcal{S}^\lambda$ is irreducible.

2. *The trivial representation $\mathcal{H}_n$:* We let $\mathcal{H}_n$ denote the **k**-module **k**, equipped with a trivial S_n-action (that is, $\sigma \cdot v = v$ for all $\sigma \in S_n$ and $v \in \mathbf{k}$). This is called the *trivial representation* of S_n. It is isomorphic to the Specht module $\mathcal{S}^{(n)}$.

3. *The natural representation $\mathcal{N}_n$:* We let $\mathcal{N}_n$ denote the free **k**-module $\mathbf{k}^n = \{(v_1, v_2, \ldots, v_n) \mid \text{all } v_i \in \mathbf{k}\}$, on which S_n acts by permuting the coordinates:

$$\sigma \cdot (v_1, v_2, \ldots, v_n) = \left(v_{\sigma^{-1}(1)}, v_{\sigma^{-1}(2)}, \ldots, v_{\sigma^{-1}(n)} \right) \qquad \text{for all } \sigma \in S_n.$$

   This is called the *natural representation* of S_n.

4. *The reflection quotient representation $\mathcal{Z}_n$:* If $n > 0$, then the natural representation $\mathcal{N}_n$ has a 1-dimensional subrepresentation

$$\mathcal{D}_n := \{(v_1, v_2, \ldots, v_n) \mid \text{all } v_i \text{ are equal}\} = \{(a, a, \ldots, a) \mid a \in \mathbf{k}\}.$$

   The quotient

$$\mathcal{Z}_n := \mathcal{N}_n / \mathcal{D}_n$$

   is thus another representation of S_n. This $\mathcal{Z}_n$ is called the *reflection quotient representation* of S_n. As a **k**-module, it is free of rank $n - 1$ (with basis $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_{n-1}})$, where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbf{k}^n$). Here and in the following, the notation $\overline{v}$ denotes the residue class of a vector v modulo some submodule (the submodule is to be inferred from the context).

   If **k** is a field of characteristic 0 (or, more generally, if n is invertible in **k**), then this representation $\mathcal{Z}_n$ is isomorphic to the Specht module $\mathcal{S}^{(n-1,1)}$. Without any such assumptions, $\mathcal{Z}_n$ is isomorphic to the dual $\left(\mathcal{S}^{(n-1,1)} \right)^*$ of this Specht module. (See Proposition 2.1 below.)

If V is any **k**-module, then V^* shall denote its dual **k**-module $\mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$. If V is an S_n-representation, then its dual V^* becomes an S_n-representation as well (see [Gri25, §5.19.3]).

Proposition 2.1. Let $n > 1$ be an integer. Then:

(a) The reflection quotient representation $\mathcal{Z}_n$ is isomorphic (as an S_n-representation) to the dual $\left(\mathcal{S}^{(n-1,1)}\right)^*$ of the Specht module $\mathcal{S}^{(n-1,1)}$.

(b) If n is invertible in **k**, then $\mathcal{Z}_n$ is isomorphic (as an S_n-representation) to the Specht module $\mathcal{S}^{(n-1,1)}$.

This proposition is clearly part of the folklore, but we outline a proof in the Appendix (Section A) for the sake of completeness.

2.4. Tensor products, induction and induction products

We shall now discuss certain ways to produce new representations from old.

The symbol "$\otimes$" shall always mean a tensor product over **k**, unless a different base ring is provided as a subscript.

It is well-known that if A and B are two **k**-algebras, then the tensor product $U \otimes V$ of any left A-module U and any left B-module V is canonically a left $A \otimes B$-module. An analogous construction exists for tensor products of k left modules. Thus, if U is a representation of a group G, and if V is a representation of a group H, then $U \otimes V$ is a representation of $G \times H$, and a similar fact holds for tensor products of k representations.

We recall the notion of an induced representation: If G is a group, and if H is a subgroup of G, then any H-representation V gives rise to a G-representation $\mathrm{Ind}_H^G V$ defined by

$$\mathrm{Ind}_H^G V = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V, \tag{1}$$

where we view $\mathbf{k}[G]$ as a $(\mathbf{k}[G], \mathbf{k}[H])$-bimodule while viewing V as a left $\mathbf{k}[H]$-module (so that the tensor product over $\mathbf{k}[H]$ becomes a left $\mathbf{k}[G]$-module). This G-representation $\mathrm{Ind}_H^G V$ is called the *induced representation* of V to G.

We furthermore recall the notion of an induction product ([Fulton97, §7.3]):

Definition 2.2. Let n and m be two nonnegative integers. Then, the direct product $S_n \times S_m$ can be canonically embedded as a subgroup into S_{n+m}, by the group morphism that sends each pair $(\sigma, \tau) \in S_n \times S_m$ to the permutation $\sigma * \tau \in S_{n+m}$ that applies σ to the first n elements while applying τ (appropriately shifted) to the last m elements of $[n+m]$. (To be fully precise: $\sigma * \tau$ is the permutation of $[n+m]$ that sends $1, 2, \ldots, n$ to $\sigma(1), \sigma(2), \ldots, \sigma(n)$ while sending $n+1, n+2, \ldots, n+m$ to $n + \tau(1), n + \tau(2), \ldots, n + \tau(m)$.) This is called the *parabolic embedding* of $S_n \times S_m$ into S_{n+m}.

Now, if U is an S_n-representation and if V is an S_m-representation, then the tensor product $U \otimes V$ is an $S_n \times S_m$-representation, and thus (by the embedding

of $S_n \times S_m$ into S_{n+m} we just explained) we can construct the induced representation
$$U * V := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (U \otimes V)$$
of S_{n+m}. This induced representation $U * V$ is called the *induction product* of U and V.

More generally, if $n_1, n_2, \ldots, n_k$ are any k nonnegative integers, and if U_i is an S_{n_i}-representation for each $i \in [k]$, then the *induction product* $U_1 * U_2 * \cdots * U_k$ is defined to be the $S_{n_1+n_2+\cdots+n_k}$-representation

$$\mathrm{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}}^{S_{n_1+n_2+\cdots+n_k}} (U_1 \otimes U_2 \otimes \cdots \otimes U_k),$$

where we embed $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$ in the obvious way (having each S_{n_i} act on an appropriate interval[4]). The latter embedding is again called the *parabolic embedding* of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$.

These induction products satisfy associativity up to isomorphism: e.g., we have $(U * V) * W \cong U * V * W \cong U * (V * W)$ for all U, V, W. More generally:

Proposition 2.3. Let $n_1, n_2, \ldots, n_k$ be any k nonnegative integers, and let U_i be an S_{n_i}-representation for each $i \in [k]$. Let $i \in [0, k]$. Then,

$$U_1 * U_2 * \cdots * U_k \cong (U_1 * U_2 * \cdots * U_i) * (U_{i+1} * U_{i+2} * \cdots * U_k).$$

This is again a folklore result, but we sketch a proof in the Appendix (Section A) to fill a little gap in the literature.

2.5. Lacunar sets and the submodules $F(I)$

Next, we recall some more concepts from [GriLaf22].

If I is a finite set of integers, then we let sum I denote the sum of all elements of I. For instance, sum $\{3, 7\} = 3 + 7 = 10$.

Let $(f_0, f_1, f_2, \ldots)$ be the *Fibonacci sequence*. This is the sequence of integers defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_m = f_{m-1} + f_{m-2} \text{ for all } m \geq 2.$$

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[4]To make this precise: Let $m_i := n_1 + n_2 + \cdots + n_i$ for each $i \in [0, k]$. Then, the integer interval $[n_1 + n_2 + \cdots + n_k]$ is partitioned into the intervals $[m_{i-1} + 1, \ m_i]$ for all $i \in [k]$. The embedding of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ into $S_{n_1+n_2+\cdots+n_k}$ sends each k-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$ to the permutation $\sigma_1 * \sigma_2 * \cdots * \sigma_k \in S_{n_1+n_2+\cdots+n_k}$ defined by

$$(\sigma_1 * \sigma_2 * \cdots * \sigma_k)(m_{i-1} + x) := m_{i-1} + \sigma_i(x) \qquad \text{for each } i \in [k] \text{ and each } x \in [n_i].$$

We shall say that a set $I \subseteq \mathbb{Z}$ is *lacunar* if it contains no two consecutive integers (i.e., there exists no $i \in I$ such that $i + 1 \in I$). For instance, the set $\{1, 4, 6\}$ is lacunar, while the set $\{1, 4, 5\}$ is not.

The number of lacunar subsets of $[n-1]$ is the Fibonacci number f_{n+1}. Let $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ be all these f_{n+1} lacunar subsets of $[n-1]$, listed in an order that satisfies

$$\operatorname{sum}(Q_1) \le \operatorname{sum}(Q_2) \le \cdots \le \operatorname{sum}\left(Q_{f_{n+1}}\right). \tag{2}$$

We fix this order once and for all[5]. Many of our constructions will formally (though rather shallowly) depend on this order.

For any subset I of $[n]$, we define the following:

- We let $I - 1$ denote the set $\{i - 1 \mid i \in I\} = \{j \in \mathbb{Z} \mid j + 1 \in I\}$. For instance, $\{2, 4, 5\} - 1 = \{1, 3, 4\}$. Note that I is lacunar if and only if $I \cap (I - 1) = \varnothing$.

- We let I' be the set $[n-1] \setminus (I \cup (I - 1))$. This is the set of all $i \in [n-1]$ satisfying $i \notin I$ and $i + 1 \notin I$. We shall refer to I' as the *non-shadow* of I.

  For example, if $n = 5$, then $\{2, 3\}' = [4] \setminus \{1, 2, 3\} = \{4\}$.

- We let
  $$F(I) := \left\{ q \in \mathbf{k}[S_n] \mid q s_i = q \text{ for all } i \in I' \right\}.$$

  We can rewrite this equality as
  $$\begin{aligned} F(I) &= \left\{ t \in \mathbf{k}[S_n] \mid t s_j = t \text{ for all } j \in I' \right\} \\ &= \left\{ t \in \mathcal{A} \mid t s_j = t \text{ for all } j \in I' \right\} \end{aligned} \tag{3}$$

  (since $\mathbf{k}[S_n] = \mathcal{A}$).

3. The first main theorem: the Fibonacci filtration

3.1. The theorem

For each $i \in [0, f_{n+1}]$, we define a $\mathbf{k}$-submodule

$$F_i := F(Q_1) + F(Q_2) + \cdots + F(Q_i) \qquad \text{of } \mathbf{k}[S_n]$$

(so that $F_0 = 0$). In [GriLaf22, Theorem 8.1], the following is shown:

> **Theorem 3.1.**
>
> **(a)** We have
> $$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}[S_n].$$
>
> In other words, the $\mathbf{k}$-submodules $F_0, F_1, \ldots, F_{f_{n+1}}$ form a $\mathbf{k}$-module filtration of $\mathbf{k}[S_n]$.

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[5]For $n \le 3$, this order is uniquely defined. For $n > 3$, we need to make a choice.

(b) We have $F_i \cdot t_\ell \subseteq F_i$ for each $i \in [0, f_{n+1}]$ and $\ell \in [n]$.

(c) For each $i \in [f_{n+1}]$ and $\ell \in [n]$, we have

$$F_i \cdot \left(t_\ell - m_{Q_i, \ell} \right) \subseteq F_{i-1}.$$

Here, $m_{Q_i, \ell}$ is a certain integer whose definition we will give in Subsection 4.1 (as we will not use it until then).

The filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k}\,[S_n]$ will be called the *Fibonacci filtration* of $\mathcal{A}$. We can easily see that it is a filtration of $(\mathcal{A}, \mathcal{T})$-bimodules:

Proposition 3.2. Let $i \in [0, f_{n+1}]$. Then, F_i is an $(\mathcal{A}, \mathcal{T})$-subbimodule of $\mathcal{A}$.

Proof. For any $I \subseteq [n]$, the set $F(I)$ is closed under addition and left action of $\mathcal{A}$ (by its very definition), hence is a left $\mathcal{A}$-submodule of $\mathcal{A}$. Thus, F_i (being defined as a sum of such sets $F(I)$) is also a left $\mathcal{A}$-submodule of $\mathcal{A}$. Moreover, F_i is also closed under right multiplication by each t_ℓ (by Theorem 3.1 **(b)**), and hence under the right action of $\mathcal{T}$ (since $\mathcal{T}$ is the subalgebra generated by $t_1, t_2, \ldots, t_n$). Thus, F_i is also a right $\mathcal{T}$-submodule of $\mathcal{A}$. Altogether, we conclude that F_i is an $(\mathcal{A}, \mathcal{T})$-subbimodule of $\mathcal{A}$. $\qquad\square$

Proposition 3.2 shows that the subquotients F_i / F_{i-1} are $(\mathcal{A}, \mathcal{T})$-bimodules as well. In particular, they are therefore left $\mathcal{A}$-modules, i.e., representations of S_n. Our second main theorem characterizes these representations:

Theorem 3.3. Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. (Note that $j_1 \geq 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = n$; this follows from Lemma 3.4 below (applied to $I = Q_i$).) Then,

$$F_i / F_{i-1} \cong \underbrace{\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}}_{\substack{\text{the first factor is an } \mathcal{H}, \\ \text{while all others are } \mathcal{Z}\text{'s}}}$$

as S_n-representations.

We will spend the rest of this section proving this theorem, then restating it (in the characteristic-0 case) using Littlewood–Richardson coefficients.

3.2. Lemmas on $F(Q_i)$

First, let us show some lemmas about lacunar sets I and the corresponding $\mathbf{k}$-modules $F(I)$:

Lemma 3.4. Let I be a lacunar subset of $[n-1]$. Write the set $I \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Then, $j_1 \geq 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = n$.

Proof. By definition, we have $j_1 = i_1 - i_0 \geq 0$, since $i_1 \geq 1 = i_0$.

Next, we recall that the set I is lacunar. This lacunarity is preserved even when we insert the new element $n+1$ into this set, since all existing elements of I are $\leq n-1$ (since $I \subseteq [n-1]$) and thus cannot be consecutive with $n+1$. That is, the set $I \cup \{n+1\}$ is again lacunar. Since we have written this set as $\{i_1 < i_2 < \cdots < i_m\}$, this yields that any $k \in [2, m]$ satisfies $i_k - i_{k-1} > 1$. In other words, any $k \in [2, m]$ satisfies $j_k > 1$ (since $j_k = i_k - i_{k-1}$). In other words, $j_2, j_3, \ldots, j_m > 1$.

It remains to prove that $j_1 + j_2 + \cdots + j_m = n$. But recall that $j_k = i_k - i_{k-1}$ for each $k \in [m]$. Hence,

$$\sum_{k=1}^{m} j_k = \sum_{k=1}^{m} (i_k - i_{k-1}) = \underbrace{i_m}_{=n+1} - \underbrace{i_0}_{=1} \qquad \text{(by the telescope principle)}$$

$$= n + 1 - 1 = n.$$

In other words, $j_1 + j_2 + \cdots + j_m = n$. Thus, Lemma 3.4 is fully proved. $\qquad \square$

Lemma 3.5. Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. Then, the **k**-module F_i / F_{i-1} is free of rank

$$\frac{n!}{j_1! j_2! \cdots j_m!} \cdot \prod_{k=2}^{m} (j_k - 1).$$

Proof. We have $Q_i \subseteq [n-1]$. Hence, $n+1$ is the largest element of $Q_i \cup \{n+1\}$. Thus, from $Q_i \cup \{n+1\} = \{i_1 < i_2 < \cdots < i_m\}$, we obtain

$$i_m = n+1 \qquad \text{and} \qquad Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}.$$

Lemma 3.4 (applied to $I = Q_i$) shows that $j_1 + j_2 + \cdots + j_m = n$. Let $\binom{n}{j_1, j_2, \ldots, j_m}$ denote the multinomial coefficient $\dfrac{n!}{j_1! j_2! \cdots j_m!}$. We know from [GriLaf22, Theorem 13.1 **(a)** and **(c)**] (applied to $p = m-1$) that the **k**-module F_i / F_{i-1} is free of rank

$$\delta_i = \binom{n}{j_1, j_2, \ldots, j_m} \cdot \prod_{k=2}^{m} (j_k - 1).$$

In view of $\binom{n}{j_1, j_2, \ldots, j_m} = \dfrac{n!}{j_1! j_2! \cdots j_m!}$, this is precisely the claim of Lemma 3.5. $\qquad \square$

Lemma 3.6. Let I be a subset of $[n]$. Let $j \in I$. Then, there exists a lacunar subset J of $[n-1]$ such that $\operatorname{sum} J < \operatorname{sum} I$ and $J' \subseteq I' \cup \{j\}$.

Proof. Set $K := (I \setminus \{j\}) \cup \{j-1\}$ if $j > 1$, and otherwise set $K := I \setminus \{j\}$. Then, K is a subset of $[n]$ and satisfies $\operatorname{sum} K < \operatorname{sum} I$ (since K is obtained from I by removing the element j and possibly inserting the smaller element $j-1$). Furthermore, [GriLaf22, Proposition 8.6 **(a)**] says that $K' \subseteq I' \cup \{j\}$.

Now, [GriLaf22, Corollary 8.8] (applied to K instead of I) shows that there exists a lacunar subset J of $[n-1]$ such that $\operatorname{sum} J \leq \operatorname{sum} K$ and $J' \subseteq K'$. Consider this J.

The set J is a lacunar subset of $[n-1]$ and satisfies $\operatorname{sum} J < \operatorname{sum} I$ (since $\operatorname{sum} J \leq \operatorname{sum} K < \operatorname{sum} I$) and $J' \subseteq I' \cup \{j\}$ (since $J' \subseteq K' \subseteq I' \cup \{j\}$). Hence, such a J exists. This proves Lemma 3.6. $\square$

Lemma 3.7. Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Let $k \in [m-1]$. Then,

$$\{t \in F(Q_i) \mid ts_{i_k} = t\} \subseteq F_{i-1}.$$

Proof. As in the proof of Lemma 3.5, we find $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$. Thus, $i_k \in Q_i$ (since $k \in [m-1]$). Hence, Lemma 3.6 (applied to $I = Q_i$ and $j = i_k$) shows that there exists a lacunar subset J of $[n-1]$ such that $\operatorname{sum} J < \operatorname{sum}(Q_i)$ and $J' \subseteq Q_i' \cup \{i_k\}$. Consider this J. Since J is lacunar, we have $J = Q_s$ for some $s \in [f_{n+1}]$. Consider this s. Thus, $Q_s = J$, so that $\operatorname{sum}(Q_s) = \operatorname{sum} J < \operatorname{sum}(Q_i)$ and therefore $s < i$ (by (2)). Hence, $s \leq i-1$, so that $F(Q_s) \subseteq F_{i-1}$ (since the definition of F_{i-1} says that $F_{i-1} = F(Q_1) + F(Q_2) + \cdots + F(Q_{i-1})$). In view of $Q_s = J$, we can rewrite this as
$$F(J) \subseteq F_{i-1}.$$

Now, (3) (applied to Q_i instead of I) shows that

$$F(Q_i) = \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i'\},$$

so that

$$
\begin{aligned}
&\{t \in F(Q_i) \mid ts_{i_k} = t\} \\
&= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i', \text{ and also } ts_{i_k} = t\} \\
&= \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i' \cup \{i_k\}\} \\
&\subseteq \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in J'\} \qquad (\text{since } J' \subseteq Q_i' \cup \{i_k\}) \\
&= F(J) \qquad (\text{by (3), applied to } J \text{ instead of } I) \\
&\subseteq F_{i-1}.
\end{aligned}
$$

Thus, Lemma 3.7 follows. $\square$

3.3. The elements $\nabla_{\mathbf{p}}$

Lemma 3.8. Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$. Furthermore, set $i_0 := 1$.

For each $k \in [m]$, let J_k denote the integer interval $[i_{k-1}, \ i_k - 1]$. Note that the intervals $J_1, J_2, \ldots, J_m$ are disjoint and – except possibly for J_1 – nonempty (J_1 is empty if and only if $1 \in Q_i$), and their union is $[n]$. Thus, we can view the direct product $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ as a subgroup of S_n in the obvious way (each factor S_{J_k} acts on the elements of J_k while leaving all remaining elements of $[n]$ unchanged).

For each $(m-1)$-tuple $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$ (that is, with $p_k \in J_k$ for each $k \in [2, m]$), we define an element

$$\nabla_{\mathbf{p}} := \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k \in [2,m]}} \sigma \in \mathcal{A}.$$

Then:

(a) For any $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ and $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$, we have

$$\tau \nabla_{\mathbf{p}} = \nabla_{\tau \mathbf{p}},$$

where

$$\tau \mathbf{p} := (\tau_2 (p_2), \tau_3 (p_3), \ldots, \tau_m (p_m)) = (\tau (p_2), \tau (p_3), \ldots, \tau (p_m)).$$

(b) The left $\mathcal{A}$-module $F(Q_i)$ is generated (as a left $\mathcal{A}$-module) by any single element of the form $\nabla_{\mathbf{p}}$ (with $\mathbf{p} \in J_2 \times J_3 \times \cdots \times J_m$).

(c) Let $\ell \in [2, m]$. For each $k \in [2, m] \setminus \{\ell\}$, let $p_k \in J_k$ be an element. Then,

$$\sum_{p_\ell \in J_\ell} \nabla_{(p_2, p_3, \ldots, p_m)} \in F_{i-1}.$$

(Note that the elements $p_2, p_3, \ldots, p_{\ell-1}, p_{\ell+1}, p_{\ell+2}, \ldots, p_m$ in this sum are fixed, whereas p_ℓ runs through the set J_ℓ.)

Proof. As in the proof of Lemma 3.5, we find $i_m = n + 1$ and $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$.

(a) Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ and $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. Recall that

$$\nabla_{\mathbf{p}} = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k \in [2,m]}} \sigma.$$

Multiplying this equality by τ from the left, we obtain

$$\tau \nabla_{\mathbf{p}} = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1})=p_k \text{ for each } k \in [2,m]}} \tau\sigma = \sum_{\substack{\sigma \in S_n; \\ \tau(\sigma(J_k))=\tau(J_k) \text{ for each } k \in [m]; \\ \tau(\sigma(i_{k-1}))=\tau(p_k) \text{ for each } k \in [2,m]}} \tau\sigma$$

$$\left(\begin{array}{c} \text{here, we have replaced the conditions } ``\sigma\left(J_k\right) = J_k" \\ \text{and } ``\sigma\left(i_{k-1}\right) = p_k" \text{ under the summation sign} \\ \text{by the conditions } ``\tau\left(\sigma\left(J_k\right)\right) = \tau\left(J_k\right)" \\ \text{and } ``\tau\left(\sigma\left(i_{k-1}\right)\right) = \tau\left(p_k\right)" \text{ (which are equivalent to} \\ \text{the former two conditions because } \tau \text{ is injective)} \end{array} \right)$$

$$= \sum_{\substack{\sigma \in S_n; \\ (\tau\sigma)(J_k)=J_k \text{ for each } k \in [m]; \\ (\tau\sigma)(i_{k-1})=\tau(p_k) \text{ for each } k \in [2,m]}} \tau\sigma$$

$$\left(\begin{array}{c} \text{since each } k \in [m] \text{ satisfies } \tau\left(\sigma\left(J_k\right)\right) = (\tau\sigma)\left(J_k\right) \\ \text{and } \tau\left(\sigma\left(i_{k-1}\right)\right) = (\tau\sigma)\left(i_{k-1}\right) \text{ (if } k > 1) \\ \text{and } \tau\left(J_k\right) = J_k \text{ (since } \tau \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}) \end{array} \right)$$

$$= \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k)=J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1})=\tau(p_k) \text{ for each } k \in [2,m]}} \sigma$$

$$\left(\begin{array}{c} \text{here, we have substituted } \sigma \text{ for } \tau\sigma \text{ in the sum,} \\ \text{since the map } S_n \to S_n, \ \sigma \mapsto \tau\sigma \text{ is a bijection} \end{array} \right)$$

$$= \nabla_{\tau\mathbf{p}}$$

(by the definition of $\nabla_{\tau\mathbf{p}}$, since $\tau\mathbf{p} = (\tau\left(p_2\right), \tau\left(p_3\right), \ldots, \tau\left(p_m\right))$). This proves Lemma 3.8 **(a)**.

(b) The definition of the non-shadow Q_i' yields

$$Q_i' = [n-1] \setminus (Q_i \cup (Q_i - 1))$$
$$= [n-1] \setminus (\{i_1 < i_2 < \cdots < i_{m-1}\} \cup \{i_1 - 1 < i_2 - 1 < \cdots < i_{m-1} - 1\})$$

(since $Q_i = \{i_1 < i_2 < \cdots < i_{m-1}\}$). In other words, Q_i' consists of all elements of $[n-1]$ except for those of the forms $i_k - 1$ and i_k for $k \in [m-1]$.

Let Γ be the subgroup of S_n generated by the simple transpositions s_j with $j \in Q_i'$. Thus, Γ is generated by all simple transpositions $s_1, s_2, \ldots, s_{n-1}$ except for those of the forms s_{i_k-1} and s_{i_k} for $k \in [m-1]$ (by the description of Q_i' in the previous paragraph). Hence, every permutation $\omega \in \Gamma$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$.

Conversely, if some permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$, then ω must belong to Γ (because such a permutation ω must preserve the intervals $J_1 = [i_0, \ i_1 - 1]$ as well as $J_k \setminus \{i_{k-1}\} = [i_{k-1} + 1, \ i_k - 1]$ for all $k \in [2, m]$ (since it preserves both J_k and i_{k-1}) as well as

the length-1 intervals $\{i_{k-1}\}$ for all $k \in [2, m]$, and thus must be a composition of permutations of these intervals; but any such permutation belongs to Γ (since any permutation of an integer interval $[a, b]$ can be written as a product of simple transpositions s_j with $j \in [a, b-1]$)).

The subgroup Γ of S_n acts from the right on S_n (simply by right multiplication), and thus also acts **k**-linearly from the right on $\mathcal{A} = \mathbf{k}[S_n]$ (by linear extension), making $\mathcal{A}$ into a permutation module[6] of Γ. Applying (3) to $I = Q_i$, we see that

$$F(Q_i) = \{t \in \mathcal{A} \mid ts_j = t \text{ for all } j \in Q_i'\}$$
$$= \{t \in \mathcal{A} \mid t\omega = t \text{ for all } \omega \in \Gamma\}$$

(since Γ is the group generated by the s_j with $j \in Q_i'$, and therefore the condition "$ts_j = t$ for all $j \in Q_i'$" is equivalent to "$t\omega = t$ for all $\omega \in \Gamma$"). Thus, $F(Q_i)$ is the space of fixed points[7] of the right Γ-action on $\mathcal{A}$.

However, we know from the basic theory of group actions (see, e.g., [Lor18, §3.3.1, "Invariants of Permutation Representations"]) that when a finite group G acts on a set X, the space of fixed points of the corresponding permutation module is spanned by the orbit sums[8]. Hence, the **k**-module $F(Q_i)$ is spanned by the orbit sums of the right Γ-action on S_n (since $F(Q_i)$ is the set of fixed points of the right Γ-action on $\mathcal{A}$, which is the permutation module corresponding to the right Γ-action on S_n). In other words, $F(Q_i)$ is spanned by the orbit sums $\sum_{\sigma \in \tau\Gamma} \sigma$ for $\tau \in S_n$ (since each orbit of the right Γ-action on S_n has the form $\tau\Gamma$ for some $\tau \in S_n$). As a left $\mathcal{A}$-module, $F(Q_i)$ is therefore generated by any **one** of these orbit sums (since any two orbit sums $\sum_{\sigma \in \tau_1\Gamma} \sigma$ and $\sum_{\sigma \in \tau_2\Gamma} \sigma$ can be transformed into each other by left multiplication by $\tau_1\tau_2^{-1} \in S_n \subseteq \mathcal{A}$, and therefore each of them generates the other).

---

[6]Recall the definition of a permutation module:

Let G be a finite group. Let X be a right G-set. Let $\mathbf{k}^{(X)}$ be the free **k**-module with basis X. Then, $\mathbf{k}^{(X)}$ becomes a right $\mathbf{k}[G]$-module, where the action of $\mathbf{k}[G]$ on $\mathbf{k}^{(X)}$ is given by bilinearly extending the action of G on X (that is, by the rule $\left(\sum\limits_{g \in G} \alpha_g g\right)\left(\sum\limits_{x \in X} \beta_x x\right) := \sum\limits_{g \in G} \sum\limits_{x \in X} \alpha_g \beta_x gx$).

This is called the *permutation module* corresponding to the right G-set X.

In our present setup, we apply this construction to $G = \Gamma$ and $X = S_n$.

[7]Recall the definition of a space of fixed points: If a **k**-module V is equipped with a linear right action of a group G (that is, if V is a right $\mathbf{k}[G]$-module), then its *space of fixed points* is defined to be the set $\{a \in V \mid ag = a \text{ for all } g \in G\}$. This is a **k**-submodule of V.

[8]In more details:

Let G be a finite group. Let X be a right G-set. Consider the corresponding permutation module $\mathbf{k}^{(X)}$, with its right G-action.

For each G-orbit $\mathcal{O}$ on X, we define the *orbit sum* $z_\mathcal{O} := \sum\limits_{x \in \mathcal{O}} x \in \mathbf{k}^{(X)}$. Now, the known fact that we are citing here is saying that these orbit sums $z_\mathcal{O}$ (as $\mathcal{O}$ ranges over all G-orbits on X) form a basis of the space of fixed points of $\mathbf{k}^{(X)}$ (as a **k**-module).

In [Lor18, §3.3.1, "Invariants of Permutation Representations"], this is stated for left G-actions, but the case of right G-actions is analogous.

Now, let $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. We shall now show that $\nabla_\mathbf{p}$ is one of these orbit sums we just mentioned. Indeed, let $\Omega_\mathbf{p}$ be the set of all permutations $\sigma \in S_n$ that satisfy "$\sigma(J_k) = J_k$ for each $k \in [m]$" and "$\sigma(i_{k-1}) = p_k$ for each $k \in [2, m]$". Then, the definition of $\nabla_\mathbf{p}$ can be rewritten as

$$\nabla_\mathbf{p} = \sum_{\sigma \in \Omega_\mathbf{p}} \sigma. \tag{4}$$

We shall now show that $\Omega_\mathbf{p}$ is an orbit of the right Γ-action on S_n (that is, a left coset of Γ in S_n).

First, we show that the set $\Omega_\mathbf{p}$ is nonempty. Indeed, it is easy to construct some permutation $\tau \in \Omega_\mathbf{p}$: Namely, we pick a permutation $\tau_1 \in S_{J_1}$ arbitrarily. Furthermore, for each $k \in [2, m]$, we pick a permutation $\tau_k \in S_{J_k}$ that sends $i_{k-1} \in J_k$ to $p_k \in J_k$. The m-tuple $(\tau_1, \tau_2, \ldots, \tau_m)$ then belongs to $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ and – viewed as an element of S_n via the embedding $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m} \to S_n$ – belongs to $\Omega_\mathbf{p}$.

Hence, $\Omega_\mathbf{p}$ is nonempty. Pick any $\tau \in \Omega_\mathbf{p}$. Then, $\tau(J_k) = J_k$ for each $k \in [m]$, and $\tau(i_{k-1}) = p_k$ for each $k \in [2, m]$. Moreover, these equalities remain valid if we replace τ by $\tau\omega$ for any $\omega \in \Gamma$ (because every permutation $\omega \in \Gamma$ preserves the sets $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$). Thus, for each $\omega \in \Gamma$, we have $\tau\omega \in \Omega_\mathbf{p}$ as well. In other words, $\tau\Gamma \subseteq \Omega_\mathbf{p}$.

Conversely, we claim that $\Omega_\mathbf{p} \subseteq \tau\Gamma$. Indeed, let $\sigma \in \Omega_\mathbf{p}$ be arbitrary. Then, each $k \in [m]$ satisfies $\sigma(J_k) = J_k = \tau(J_k)$, whereas each $k \in [2, m]$ satisfies $\sigma(i_{k-1}) = p_k = \tau(i_{k-1})$. Set $\omega = \tau^{-1}\sigma \in S_n$; thus, each $k \in [m]$ satisfies $\omega(J_k) = \tau^{-1}(\sigma(J_k)) = J_k$ (since we just saw that $\sigma(J_k) = \tau(J_k)$), and each $k \in [2, m]$ satisfies $\omega(i_{k-1}) = \tau^{-1}(\sigma(i_{k-1})) = i_{k-1}$ (since we just saw that $\sigma(i_{k-1}) = \tau(i_{k-1})$). Thus, the permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$. Hence, $\omega \in \Gamma$ (because if some permutation $\omega \in S_n$ preserves the intervals $J_1, J_2, \ldots, J_m$ as well as the elements $i_1, i_2, \ldots, i_{m-1}$, then ω must belong to Γ). Now, from $\omega = \tau^{-1}\sigma$, we obtain $\sigma = \tau\omega \in \tau\Gamma$ (since $\omega \in \Gamma$). Forget that we fixed σ. We thus have proved that $\sigma \in \tau\Gamma$ for each $\sigma \in \Omega_\mathbf{p}$. In other words, $\Omega_\mathbf{p} \subseteq \tau\Gamma$.

Combining this with $\tau\Gamma \subseteq \Omega_\mathbf{p}$, we obtain $\Omega_\mathbf{p} = \tau\Gamma$. Hence, $\Omega_\mathbf{p}$ is an orbit of the right Γ-action on S_n. Thus, $\sum_{\sigma \in \Omega_\mathbf{p}} \sigma$ is an orbit sum of this action. In view of (4), this means that $\nabla_\mathbf{p}$ is an orbit sum of this action. Hence, as a left $\mathcal{A}$-module, $F(Q_i)$ is generated by $\nabla_\mathbf{p}$ (since we have shown that $F(Q_i)$ is generated by any **one** of the orbit sums). This proves Lemma 3.8 **(b)**.

(c) Let

$$\vartheta := \sum_{p_\ell \in J_\ell} \nabla_{(p_2, p_3, \ldots, p_m)}. \tag{5}$$

We then must show that $\vartheta \in F_{i-1}$.

We have

$$\vartheta = \sum_{p_\ell \in J_\ell} \underbrace{\nabla_{(p_2, p_3, \ldots, p_m)}}_{\in F(Q_i)} \in F(Q_i)$$

<div align="center">

(since Lemma 3.8 **(b)** shows that $\nabla_{\mathbf{q}} \in F(Q_i)$
for any $\mathbf{q} \in J_2 \times J_3 \times \cdots \times J_m$)

</div>

(since $F(Q_i)$ is a **k**-module). On the other hand,

$$\vartheta = \sum_{p_\ell \in J_\ell} \nabla_{(p_2, p_3, \ldots, p_m)}$$

$$= \sum_{p_\ell \in J_\ell} \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m]}} \sigma \qquad \left(\text{by the definition of } \nabla_{(p_2, p_3, \ldots, p_m)}\right)$$

$$= \sum_{p_\ell \in J_\ell} \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m] \setminus \{\ell\}; \\ \sigma(i_{\ell-1}) = p_\ell}} \sigma$$

<div align="center">

$$\left(\begin{array}{c} \text{here, we have split up the} \\ \text{condition } \text{``}\sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m]\text{''} \\ \text{under the second summation sign} \\ \text{into two: one for } k \neq \ell \text{ and one for } k = \ell \end{array}\right)$$

</div>

$$= \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m] \setminus \{\ell\}; \\ \sigma(i_{\ell-1}) \in J_\ell}} \sigma \tag{6}$$

(here, we have subsumed the two summation signs into one by removing the variable p_ℓ). The condition "$\sigma(i_{\ell-1}) \in J_\ell$" under the summation sign in (6) is redundant, since it follows from the condition "$\sigma(J_k) = J_k$ for each $k \in [m]$" (indeed, the latter condition implies that $\sigma(J_\ell) = J_\ell$ and therefore $\sigma\left(\underbrace{i_{\ell-1}}_{\in J_\ell}\right) \in \sigma(J_\ell) = J_\ell$).

Hence, we can remove this condition. Thus, (6) rewrites as

$$\vartheta = \sum_{\substack{\sigma \in S_n; \\ \sigma(J_k) = J_k \text{ for each } k \in [m]; \\ \sigma(i_{k-1}) = p_k \text{ for each } k \in [2,m] \setminus \{\ell\}}} \sigma. \tag{7}$$

However, the two conditions "$\sigma(J_k) = J_k$ for each $k \in [m]$" and "$\sigma(i_{k-1}) = p_k$ for each $k \in [2,m] \setminus \{\ell\}$" under the summation sign in (7) remain unchanged if we replace σ by $\sigma s_{i_{\ell-1}}$ (since this replacement merely swaps the values of σ on $i_{\ell-1}$ and $i_{\ell-1} + 1$, but this does not break any of the two conditions[9]). Hence, the set of

---

[9] Here we use the fact that the two elements $i_{\ell-1}$ and $i_{\ell-1} + 1$ lie in the same J_k (namely, in $J_\ell = [i_{\ell-1}, i_\ell - 1]$). This is because Q_i is lacunar, so that $i_{\ell-1} < i_\ell - 1$.

the permutations σ over which we sum in (7) is fixed under right multiplication by $s_{i_{\ell-1}}$. Therefore, the whole sum is fixed under right multiplication by $s_{i_{\ell-1}}$. Because of (7), this shows that $\vartheta s_{i_{\ell-1}} = \vartheta$. Combining this with $\vartheta \in F(Q_i)$, we obtain

$$\vartheta \in \left\{ t \in F(Q_i) \mid ts_{i_{\ell-1}} = t \right\} \subseteq F_{i-1}$$

(by Lemma 3.7, applied to $k = \ell - 1$). This proves Lemma 3.8 **(c)**. $\qquad\square$

3.4. Linear algebra lemmas

We shall furthermore use two facts from linear algebra over commutative rings:

> **Lemma 3.9.** Let $s \in \mathbb{N}$. Let M and N be two free **k**-modules of rank s. Then, any surjective **k**-linear map $\rho : M \to N$ is an isomorphism.

Proof. This is a well-known folklore result, and follows easily from the known fact ("Orzech's theorem" in one of its simplest forms – see, e.g., [GR20, Exercise 2.5.18 (a)], or [Grinbe16, Corollary 0.2] for a more general result) that any surjective endomorphism of a free **k**-module of finite rank is an isomorphism. For the sake of self-containedness, let me nevertheless give a direct proof:

Let $\rho : M \to N$ be a surjective **k**-linear map. We must show that ρ is an isomorphism.

Pick bases $(e_1, e_2, \ldots, e_s)$ and $(f_1, f_2, \ldots, f_s)$ of M and N (these exist, since M and N are free of rank s). The surjectivity of ρ then shows that every basis vector f_i of N lies in the image of ρ. That is, every f_i can be written as $\rho(g_i)$ for some vector $g_i \in M$. Choose such vectors g_i, and let $\tau : N \to M$ be the **k**-linear map that sends the basis vectors $f_1, f_2, \ldots, f_s$ to $g_1, g_2, \ldots, g_s$, respectively. Then, the composition $\rho \circ \tau : N \to N$ sends each vector f_i to f_i (since $f_i \overset{\tau}{\mapsto} g_i \overset{\rho}{\mapsto} \rho(g_i) = f_i$), and thus is the identity map id_N (since $f_1, f_2, \ldots, f_s$ form a basis of N).

Now, let $A \in \mathbf{k}^{s \times s}$ be the matrix that represents the linear map $\rho : M \to N$ with respect to our bases of M and N. Likewise, let $B \in \mathbf{k}^{s \times s}$ be the matrix that represents the linear map $\tau : N \to M$ with respect to our bases of N and M. Then, AB is the matrix that represents the linear map $\rho \circ \tau : N \to N$ with respect to our basis of N. Therefore, AB is the identity matrix I_s (since $\rho \circ \tau$ is the identity map id_N, which is represented by the identity matrix I_s). Hence, $\det(AB) = \det(I_s) = 1$ and thus $1 = \det(AB) = \det A \cdot \det B$. This shows that $\det A$ is invertible (with inverse $\det B$). Hence, the matrix A is invertible (with inverse $\dfrac{1}{\det A} \operatorname{adj} A$, by the well-known identity $\det A \cdot I_n = A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A$). In other words, the **k**-linear map ρ is invertible (since it is represented by the matrix A), thus an isomorphism. This proves Lemma 3.9. $\qquad\square$

> **Lemma 3.10.** Let $V_1, V_2, \ldots, V_m$ be any **k**-modules. For each $\ell \in [m]$, let W_ℓ be a **k**-submodule of V_ℓ. For each $\ell \in [m]$, we consider the **k**-submodule
>
> $$\underbrace{V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}} \quad \text{of } V_1 \otimes V_2 \otimes \cdots \otimes V_m.$$

(a) Then, there is a canonical **k**-module isomorphism

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_m) \Big/ \sum_{\ell=1}^{m} \underbrace{(V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}}$$

$$\cong (V_1/W_1) \otimes (V_2/W_2) \otimes \cdots \otimes (V_m/W_m).$$

(b) If $V_1, V_2, \ldots, V_m$ are furthermore left modules over some **k**-algebras $A_1, A_2, \ldots, A_m$, and if $W_1, W_2, \ldots, W_m$ are their submodules (i.e., each W_ℓ is a left A_ℓ-submodule of V_ℓ), then this isomorphism is a left $A_1 \otimes A_2 \otimes \cdots \otimes A_m$-module isomorphism.

Proof. **(a)** We construct both the isomorphism and its inverse using the universal properties of tensor products and quotients:

- There is a canonical **k**-linear map

$$\Phi : V_1 \otimes V_2 \otimes \cdots \otimes V_m \to (V_1/W_1) \otimes (V_2/W_2) \otimes \cdots \otimes (V_m/W_m),$$

  sending each pure tensor $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ to $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$. This **k**-linear map Φ is easily seen to vanish on the submodule $\sum_{\ell=1}^{m} (V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)$, and thus factors through the quotient module. Hence, we obtain a **k**-linear map

$$\overline{\Phi} : (V_1 \otimes V_2 \otimes \cdots \otimes V_m) \Big/ \sum_{\ell=1}^{m} (V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)$$

$$\to (V_1/W_1) \otimes (V_2/W_2) \otimes \cdots \otimes (V_m/W_m)$$

  sending each $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$ to $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$.

- Conversely, there is a canonical **k**-linear map

$$\Psi : (V_1/W_1) \otimes (V_2/W_2) \otimes \cdots \otimes (V_m/W_m)$$

$$\to (V_1 \otimes V_2 \otimes \cdots \otimes V_m) \Big/ \sum_{\ell=1}^{m} (V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)$$

  sending each $\overline{v_1} \otimes \overline{v_2} \otimes \cdots \otimes \overline{v_m}$ to $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$. To show that this map is well-defined, we need to check that $\overline{v_1 \otimes v_2 \otimes \cdots \otimes v_m}$ depends only on the residue classes $\overline{v_i}$ rather than on the v_i themselves (this is easy: replacing v_i by v_i' with $v_i - v_i' \in W_i$ only changes $v_1 \otimes v_2 \otimes \cdots \otimes v_m$ by an element of $V_1 \otimes V_2 \otimes \cdots \otimes W_i \otimes \cdots \otimes V_m$) and that this dependence is multilinear (this is again easy).

Clearly, the maps $\overline{\Phi}$ and Ψ are mutually inverse, hence isomorphisms.

(b) This follows easily from the construction in part **(a)**. $\qquad\qquad\square$

For our specific needs, we specialize Lemma 3.10 to the case $W_1 = 0$:

Lemma 3.11. Let $V_1, V_2, \ldots, V_m$ be any **k**-modules with $m \geq 1$. For each $\ell \in [2, m]$, let W_ℓ be a **k**-submodule of V_ℓ. For each $\ell \in [2, m]$, we consider the **k**-submodule

$$\underbrace{V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}} \qquad \text{of } V_1 \otimes V_2 \otimes \cdots \otimes V_m.$$

(a) Then, there is a canonical **k**-module isomorphism

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_m) \Big/ \sum_{\ell=2}^{m} \underbrace{(V_1 \otimes V_2 \otimes \cdots \otimes W_\ell \otimes \cdots \otimes V_m)}_{\substack{\text{This means the tensor product } V_1 \otimes V_2 \otimes \cdots \otimes V_m, \\ \text{in which the } \ell\text{-th factor is replaced by } W_\ell}}$$

$$\cong V_1 \otimes (V_2 / W_2) \otimes (V_3 / W_3) \otimes \cdots \otimes (V_m / W_m).$$

(b) If $V_1, V_2, \ldots, V_m$ are furthermore left modules over some **k**-algebras $A_1, A_2, \ldots, A_m$, and if each W_ℓ with $\ell \in [2, m]$ is a left A_ℓ-submodule of V_ℓ, then this isomorphism is a left $A_1 \otimes A_2 \otimes \cdots \otimes A_m$-module isomorphism.

Proof. Apply Lemma 3.10 to $W_1 = 0$, and observe that $V_1 / 0 \cong V_1$. $\qquad \square$

3.5. Proof of Theorem 3.3

We can now prove Theorem 3.3:

Proof of Theorem 3.3. We shall use the notations of Lemma 3.8. Note that each $k \in [m]$ satisfies $J_k = [i_{k-1}, i_k - 1]$ and thus

$$|J_k| = i_k - i_{k-1} = j_k. \tag{8}$$

Explicitly, there is a bijection

$$[j_k] \to J_k,$$
$$x \mapsto i_{k-1} - 1 + x \tag{9}$$

for each $k \in [m]$.

Consider the tensor product $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$. We recall that the trivial representation $\mathcal{H}_{j_1} = \mathbf{k}$ has a 1-element basis (1), while each natural representation $\mathcal{N}_{j_k}$ has basis $(e_p)_{p \in [j_k]} = (e_1, e_2, \ldots, e_{j_k})$. However, by abuse of notation, we shall rename the latter basis of $\mathcal{N}_{j_k}$ as $(e_p)_{p \in J_k} = (e_{i_{k-1}}, e_{i_{k-1}+1}, \ldots, e_{i_k-1})$ instead (by shifting all subscripts up by $i_{k-1} - 1$, that is, renaming each basis vector e_x as $e_{i_{k-1}-1+x}$). Note that this can be done because $j_k = i_k - i_{k-1}$.

Having renamed the basis vectors of the **k**-module $\mathcal{N}_{j_k}$, let us also replace the symmetric group S_{j_k} acting on this module accordingly. Namely, we reinterpret the

symmetric group S_{j_k} acting on $\mathcal{N}_{j_k}$ as the symmetric group S_{J_k} using the bijection (9) between the corresponding sets $[j_k]$ and J_k. Thus, the left action of S_{j_k} on $\mathcal{N}_{j_k}$ becomes a left action of S_{J_k} instead; it is still a permutation action (given on our now-renamed basis by the formula $\sigma e_p = e_{\sigma(p)}$ for each $p \in J_k$ and $\sigma \in S_{J_k}$). With these reinterpretations, the parabolic embedding $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m} \to S_n$ becomes the usual embedding $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m} \to S_n$, which simply combines the m permutations without any need for shifting (i.e., any m-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_m) \in S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ is identified with the permutation $\sigma \in S_n$ that sends each element $x \in J_k$ to $\sigma_k(x)$ for each $k \in [m]$).

For each $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$, we have

$$
\begin{aligned}
\nabla_{\mathbf{p}} &\in F(Q_i) &&\text{(by Lemma 3.8 \textbf{(b)})} \\
&\subseteq F_i &&\text{(since } F_i = F(Q_1) + F(Q_2) + \cdots + F(Q_i)\text{)}
\end{aligned}
$$

and thus $\overline{\nabla_{\mathbf{p}}} \in F_i/F_{i-1}$ (where $\overline{\nabla_{\mathbf{p}}}$ denotes the residue class of $\nabla_{\mathbf{p}} \in F_i$ in the quotient F_i/F_{i-1}). Hence, we can define a $\mathbf{k}$-linear map

$$
\begin{aligned}
\Phi : \mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m} &\to F_i/F_{i-1}, \\
1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m} &\mapsto \overline{\nabla_{\mathbf{p}}} \\
&\text{for any } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m.
\end{aligned}
$$

(This map is defined by linearity, since the pure tensors of the form $1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m}$ with $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$ form a basis of the $\mathbf{k}$-module $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$.) Consider this map Φ.

For each $\ell \in [2, m]$, we can consider the $\mathbf{k}$-submodule $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ of $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m}$, in which its ℓ-th factor $\mathcal{N}_{j_\ell}$ is replaced by its submodule $\mathcal{D}_{j_\ell} = \{(a, a, \ldots, a) \mid a \in \mathbf{k}\}$. We claim that the map Φ sends this submodule to 0. Indeed, this submodule is spanned by sums of the form

$$
\sum_{p_\ell \in J_\ell} 1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m}
$$

(for fixed $p_2, p_3, \ldots, p_{\ell-1}, p_{\ell+1}, \ldots, p_m$ in the respective intervals J_k) [10], and the

---

[10]*Proof.* The submodule $\mathcal{D}_{j_\ell}$ is spanned by the single vector

$$
(1, 1, \ldots, 1) = e_{i_{\ell-1}} + e_{i_{\ell-1}+1} + \cdots + e_{i_\ell - 1} = \sum_{p_\ell \in J_\ell} e_{p_\ell},
$$

and thus the tensor product $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ is spanned by the pure tensors of the form

$$
\begin{aligned}
&1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_{\ell-1}} \otimes \left(\sum_{p_\ell \in J_\ell} e_{p_\ell} \right) \otimes e_{p_{\ell+1}} \otimes \cdots \otimes e_{p_m} \\
&= \sum_{p_\ell \in J_\ell} 1 \otimes e_{p_2} \otimes e_{p_3} \otimes \cdots \otimes e_{p_m} \qquad \text{for fixed } p_2, p_3, \ldots, p_{\ell-1}, p_{\ell+1}, \ldots, p_m.
\end{aligned}
$$

map Φ sends such sums to

$$\sum_{p_\ell \in J_\ell} \overline{\nabla_{(p_2,p_3,\ldots,p_m)}} = \overline{\sum_{p_\ell \in J_\ell} \nabla_{(p_2,p_3,\ldots,p_m)}} = 0_{F_i/F_{i-1}},$$

since Lemma 3.8 **(c)** shows that $\sum\limits_{p_\ell \in J_\ell} \nabla_{(p_2,p_3,\ldots,p_m)} \in F_{i-1}$.

Thus, the **k**-linear map

$$\Phi : \mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m} \to F_i/F_{i-1}$$

sends all the **k**-submodules $\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m}$ for $\ell \in [2,m]$ to 0. By linearity, we can thus conclude that Φ also sends their sum $\sum\limits_{\ell=2}^{m} \left(\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m} \right)$ to 0. Therefore, Φ factors through the quotient **k**-module

$$\left(\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{N}_{j_m} \right) \Big/ \sum_{\ell=2}^{m} \left(\mathcal{H}_{j_1} \otimes \mathcal{N}_{j_2} \otimes \mathcal{N}_{j_3} \otimes \cdots \otimes \mathcal{D}_{j_\ell} \otimes \cdots \otimes \mathcal{N}_{j_m} \right)$$

$$\cong \mathcal{H}_{j_1} \otimes \left(\mathcal{N}_{j_2}/\mathcal{D}_{j_2} \right) \otimes \left(\mathcal{N}_{j_3}/\mathcal{D}_{j_3} \right) \otimes \cdots \otimes \left(\mathcal{N}_{j_m}/\mathcal{D}_{j_m} \right)$$

$$\left(\begin{array}{c} \text{by Lemma 3.11 (a), applied to } V_1 = \mathcal{H}_{j_1} \text{ and } V_\ell = \mathcal{N}_{j_\ell} \text{ for } \ell > 1 \\ \text{and } W_\ell = \mathcal{D}_{j_\ell} \text{ for } \ell > 1 \end{array} \right)$$

$$= \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \qquad \left(\text{since } \mathcal{N}_p/\mathcal{D}_p = \mathcal{Z}_p \text{ for each } p > 0 \right).$$

Thus, we obtain a **k**-linear map

$$\overline{\Phi} : \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \to F_i/F_{i-1},$$
$$1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \mapsto \overline{\nabla_{\mathbf{p}}}$$
$$\text{for any } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m.$$

Consider this map $\overline{\Phi}$. Using Lemma 3.8 **(a)**, it is easy to see that this map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant[11], and thus is a left **k** $\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m} \right]$-module morphism.

---

[11]*Proof.* Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$ be any m-tuple, and let $\mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m$. We shall show that

$$\overline{\Phi} \left(\tau \cdot \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right) \right) = \tau \cdot \overline{\Phi} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right).$$

By linearity, this will entail that the map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant (since elements of the form $1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}$ span $\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}$).

Indeed, as we mentioned at the beginning of our proof, we regard each S_{j_k} as S_{J_k}, so that the permutations $\tau_1, \tau_2, \ldots, \tau_m$ act not on the sets $[j_1], [j_2], \ldots, [j_m]$ but rather on the sets $J_1, J_2, \ldots, J_m$. The embedding of $S_{J_1} \times S_{J_2} \times \cdots \times S_{J_m}$ into S_n is the usual one, so that our m-tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ is equated with the permutation $\tau \in S_n$ given by

$$\tau(x) = \tau_k(x) \qquad \text{for each } k \in [m] \text{ and } x \in J_k. \tag{10}$$

But the definition of an induction product yields

$$
\begin{aligned}
&\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} \\
&= \mathrm{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \right) \\
&= \underbrace{\mathbf{k}[S_n]}_{=\mathcal{A}} \otimes_{\mathbf{k}[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \right) \qquad \text{(by (1))} \\
&= \mathcal{A} \otimes_{\mathbf{k}[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \right).
\end{aligned}
$$

Hence, we can define a left $\mathcal{A}$-module morphism

$$
\Psi : \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} \to F_i / F_{i-1},
$$
$$
a \otimes_{\mathbf{k}[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]} v \mapsto a \cdot \overline{\Phi}(v)
$$

(this is well-defined, since $\overline{\Phi} : \mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m} \to F_i / F_{i-1}$ is a left $\mathbf{k}[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]$-module morphism). Explicitly, Ψ is given by

$$
\begin{aligned}
&\Psi \left(a \otimes_{\mathbf{k}[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}]} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right) \right) \\
&= a \cdot \overline{\Phi} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right) \\
&= a \cdot \overline{\nabla_{\mathbf{p}}} \qquad \text{for any } a \in \mathcal{A} \text{ and } \mathbf{p} = (p_2, p_3, \ldots, p_m) \in J_2 \times J_3 \times \cdots \times J_m
\end{aligned}
$$

(by the definition of $\overline{\Phi}$). Hence, using Lemma 3.8 **(b)**, it is easy to see that this map Ψ is surjective[12].

We now know that Ψ is a surjective left $\mathcal{A}$-module morphism from $\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ to F_i / F_{i-1}. We shall now show that Ψ is an isomorphism.

As in Lemma 3.8 **(a)**, we set

$$
\tau \mathbf{p} := (\tau_2(p_2), \tau_3(p_3), \ldots, \tau_m(p_m)) = (\tau(p_2), \tau(p_3), \ldots, \tau(p_m)).
$$

Now, we have $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ and thus

$$
\begin{aligned}
\tau \cdot \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right) &= \tau_1 1 \otimes \tau_2 \overline{e_{p_2}} \otimes \tau_3 \overline{e_{p_3}} \otimes \cdots \otimes \tau_m \overline{e_{p_m}} \\
&= 1 \otimes \overline{e_{\tau_2(p_2)}} \otimes \overline{e_{\tau_3(p_3)}} \otimes \cdots \otimes \overline{e_{\tau_m(p_m)}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\overline{\Phi} \left(\tau \cdot \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right) \right) \\
&= \overline{\Phi} \left(1 \otimes \overline{e_{\tau_2(p_2)}} \otimes \overline{e_{\tau_3(p_3)}} \otimes \cdots \otimes \overline{e_{\tau_m(p_m)}} \right) \\
&= \overline{\nabla_{\tau \mathbf{p}}} \qquad \text{(by the definition of } \overline{\Phi}, \text{ since } \tau \mathbf{p} = (\tau_2(p_2), \tau_3(p_3), \ldots, \tau_m(p_m))) \\
&= \overline{\tau \nabla_{\mathbf{p}}} \qquad \text{(since Lemma 3.8 (a) yields } \nabla_{\tau \mathbf{p}} = \tau \nabla_{\mathbf{p}}) \\
&= \tau \cdot \overline{\nabla_{\mathbf{p}}} = \tau \cdot \overline{\Phi} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right)
\end{aligned}
$$

(since the definition of $\overline{\Phi}$ yields $\overline{\nabla_{\mathbf{p}}} = \overline{\Phi} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}} \right)$). This is precisely what we wanted to show. Hence, we have proved that the map $\overline{\Phi}$ is $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}$-equivariant.

[12]*Proof.* The map Ψ is left $\mathcal{A}$-linear. Hence, its image is a left $\mathcal{A}$-submodule of F_i / F_{i-1}.

From Lemma 3.5, we know that the **k**-module F_i/F_{i-1} is free of rank

$$\frac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod_{k=2}^{m}(j_k - 1).$$

But the **k**-module

$$\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} = \mathrm{Ind}_{S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}}^{S_n} \left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right)$$

is also free of rank[13]

$$\underbrace{\frac{|S_n|}{\left|S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right|}}_{=\frac{n!}{j_1!j_2!\cdots j_m!}} \cdot \underbrace{\dim\left(\mathcal{H}_{j_1} \otimes \mathcal{Z}_{j_2} \otimes \mathcal{Z}_{j_3} \otimes \cdots \otimes \mathcal{Z}_{j_m}\right)}_{\substack{=\dim(\mathcal{H}_{j_1})\cdot\prod\limits_{k=2}^{m}\dim(\mathcal{Z}_{j_k})\\=1\cdot\prod\limits_{k=2}^{m}(j_k-1)\\ \text{(since }\mathcal{H}_{j_1}\text{ is free of rank 1,}\\ \text{whereas each }\mathcal{Z}_{j_k}\text{ is free of rank }j_k-1)}}$$

$$= \frac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod_{k=2}^{m}(j_k - 1).$$

Thus, Lemma 3.9 (applied to $M = \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ and $N = F_i/F_{i-1}$ and $s = \dfrac{n!}{j_1!j_2!\cdots j_m!} \cdot \prod\limits_{k=2}^{m}(j_k - 1)$ and $\rho = \Psi$) shows that the surjective **k**-linear map

---

By the definition of F_i, we have

$$F_i = F(Q_1) + F(Q_2) + \cdots + F(Q_i)$$
$$= \underbrace{F(Q_1) + F(Q_2) + \cdots + F(Q_{i-1})}_{=F_{i-1}} + F(Q_i) = F_{i-1} + F(Q_i).$$

Hence, the composition of canonical maps

$$F(Q_i) \overset{\text{inclusion}}{\longrightarrow} F_i \overset{\text{projection}}{\longrightarrow} F_i/F_{i-1} \tag{11}$$

is surjective.

But Lemma 3.8 **(b)** shows that the left $\mathcal{A}$-module $F(Q_i)$ is generated by a single element of the form $\nabla_{\mathbf{p}}$. Hence, the quotient $\mathcal{A}$-module F_i/F_{i-1} is generated by a single element of the form $\overline{\nabla_{\mathbf{p}}}$ (since the map (11) is surjective). But any such element of the form $\overline{\nabla_{\mathbf{p}}}$ lies in the image of Ψ (since we have $\overline{\nabla_{\mathbf{p}}} = \Psi\left(1 \otimes_{\mathbf{k}\left[S_{j_1} \times S_{j_2} \times \cdots \times S_{j_m}\right]} \left(1 \otimes \overline{e_{p_2}} \otimes \overline{e_{p_3}} \otimes \cdots \otimes \overline{e_{p_m}}\right)\right)$ when $\mathbf{p} = (p_2, p_3, \ldots, p_m)$). Thus, the image of Ψ must contain a generator of F_i/F_{i-1}, and thus must be the entire $\mathcal{A}$-module F_i/F_{i-1} (since this image is a left $\mathcal{A}$-submodule of F_i/F_{i-1}). In other words, Ψ is surjective.

[13]Here, we are denoting the rank of a free **k**-module V by $\dim V$, and we are using the fact that an induced representation $\mathrm{Ind}_H^G V$ is free of rank $\dfrac{|G|}{|H|} \cdot \dim V$ (as a **k**-module) whenever V is free (as a **k**-module). (The latter fact is an easy consequence of the fact that $\mathbf{k}[G]$ is a free right $\mathbf{k}[H]$-module of rank $\dfrac{|G|}{|H|}$.)

$\Psi : \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m} \to F_i / F_{i-1}$ must be an isomorphism. Since Ψ is a left $\mathcal{A}$-module morphism, we thus conclude that Ψ is a left $\mathcal{A}$-module isomorphism. Therefore, $F_i / F_{i-1} \cong \mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}$ as left $\mathcal{A}$-modules, i.e., as S_n-representations. Hence, Theorem 3.3 is proved. $\qquad\square$

3.6. In terms of Littlewood–Richardson coefficients

In the characteristic-0 case, we can restate the claim of Theorem 3.3 in terms of Littlewood–Richardson coefficients. Let us first recount the bare minimum of symmetric function theory needed to state this.

We will use standard notations for (integer) partitions; in particular, the size of a partition λ will be denoted by $|\lambda|$. We let Par denote the set of all partitions. We let Λ be the ring of symmetric functions over $\mathbb{Z}$ (not over **k**); we refer to [GR20, §2.1] or [Sag01, §4.3] for its definition[14]. To each partition λ corresponds a special symmetric function $s_\lambda \in \Lambda$ called the *Schur function*; see [GR20, (2.2.4)] or [Sag01, §4.4] or [Egg19, Definition 5.3] for its definition. It is well-known (see [Sag01, (4.26) and Theorem 4.9.4] or [GR20, Definition 2.5.8 and Corollary 2.6.12] or [Egg19, Theorem 10.40]) that a product $s_\mu s_\nu$ of two Schur functions (for $\mu, \nu \in$ Par) can always be written as an $\mathbb{N}$-linear combination of Schur functions – i.e., there exist coefficients $c_{\mu,\nu}^\lambda \in \mathbb{N}$ for all $\lambda, \mu, \nu \in$ Par such that every two partitions μ and ν satisfy

$$s_\mu s_\nu = \sum_{\lambda \in \text{Par}} c_{\mu,\nu}^\lambda s_\lambda. \tag{12}$$

These coefficients $c_{\mu,\nu}^\lambda$ are known as the *Littlewood–Richardson coefficients*. More generally, if $\mu_1, \mu_2, \ldots, \mu_k$ are any k partitions, then we can write the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ in the form

$$s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = \sum_{\lambda \in \text{Par}} c_{\mu_1,\mu_2,\ldots,\mu_k}^\lambda s_\lambda \tag{13}$$

with coefficients $c_{\mu_1,\mu_2,\ldots,\mu_k}^\lambda \in \mathbb{N}$. These "*k-Littlewood–Richardson coefficients*" $c_{\mu_1,\mu_2,\ldots,\mu_k}^\lambda$ are, in fact, easily computed by recursion using the standard Littlewood–Richardson coefficients $c_{\mu,\nu}^\lambda$: Namely, for $k = 0$, we have

$$c^\lambda = \delta_{\lambda,\varnothing} \qquad \text{(Kronecker delta)} ;$$

for $k = 1$, we have

$$c_\mu^\lambda = \delta_{\lambda,\mu} \qquad \text{(Kronecker delta)} ;$$

and for any higher k, we have

$$c_{\mu_1,\mu_2,\ldots,\mu_k}^\lambda = \sum_{\nu \in \text{Par}} c_{\mu_1,\mu_2,\ldots,\mu_{k-1}}^\nu c_{\nu,\mu_k}^\lambda$$

---

[14]Note that [Sag01, §4.3] uses $\mathbb{C}$ as the base ring, but everything works for any base ring.

(since the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ can be computed as $\left(s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{k-1}} \right) s_{\mu_k}$).

Note that any Schur function s_λ is homogeneous of degree $|\lambda|$. Hence, a Littlewood–Richardson coefficient $c_{\mu,\nu}^\lambda$ is always 0 unless $|\lambda| = |\mu| + |\nu|$. Thus, we can rewrite the equality (12) as

$$s_\mu s_\nu = \sum_{\substack{\lambda \in \text{Par}; \\ |\lambda| = |\mu| + |\nu|}} c_{\mu,\nu}^\lambda s_\lambda. \tag{14}$$

Likewise, we can rewrite (13) as

$$s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = \sum_{\substack{\lambda \in \text{Par}; \\ |\lambda| = |\mu_1| + |\mu_2| + \cdots + |\mu_k|}} c_{\mu_1, \mu_2, \ldots, \mu_k}^\lambda s_\lambda. \tag{15}$$

We note that there is a second Littlewood–Richardson rule ([Sag01, Theorem 4.9.2], [GR20, (2.6.4)], [Egg19, Theorem 10.40]) that decomposes a skew Schur function $s_{\lambda/\mu}$ into an $\mathbb{N}$-linear combination of (straight) Schur functions s_ν as follows:

$$s_{\lambda/\mu} = \sum_{\nu \in \text{Par}} c_{\mu,\nu}^\lambda s_\nu. \tag{16}$$

The formula (13) can also be viewed as a particular case of that second rule, since the product $s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k}$ can be written as the skew Schur function $s_{\mu_1 * \mu_2 * \cdots * \mu_k}$ corresponding to the skew shape $\mu_1 * \mu_2 * \cdots * \mu_k$ obtained by attaching the Young diagrams of $\mu_1, \mu_2, \ldots, \mu_k$ to each other along their northeastern/southwestern corners (see [RW84, §1] for the precise definition; this claim follows from [Egg19, Proposition 5.9]; cf. also [Sta24, Figure 7.2]). Thus, a k-Littlewood–Richardson coefficient $c_{\mu_1, \mu_2, \ldots, \mu_k}^\lambda$ can actually be rewritten as a (regular) Littlewood–Richardson coefficient using (16): If we write the skew shape $\mu_1 * \mu_2 * \cdots * \mu_k$ as α/β, then

$$c_{\mu_1, \mu_2, \ldots, \mu_k}^\lambda = c_{\beta, \lambda}^\alpha. \tag{17}$$

The same Littlewood–Richardson coefficients govern the decomposition of induction products of Specht modules into Specht modules in characteristic 0. Namely, if $\mathbf{k}$ is a field of characteristic 0, and if μ and ν are two partitions of respective sizes i and j, then

$$\mathcal{S}^\mu * \mathcal{S}^\nu \cong \bigoplus_{\substack{\lambda \in \text{Par}; \\ |\lambda| = i + j}} \left(\mathcal{S}^\lambda \right)^{\oplus c_{\mu,\nu}^\lambda} \tag{18}$$

as S_{i+j}-modules[15]. Indeed, this follows from the Schur function equality (14) using the Frobenius characteristic map [Sag01, Theorem 4.7.4] (in fact, this map – or, rather, its inverse – sends Schur functions s_λ to Specht modules $\mathcal{S}^\lambda$, while sending products of symmetric functions to induction products of representations[16]).

---

[15] The notation $V^{\oplus k}$ means the direct sum $V \oplus V \oplus \cdots \oplus V$ of k copies of V.

[16] For products with two factors, this is proved in [Sag01, Theorem 4.7.4] (using characters and Frobenius reciprocity). For products with k factors, it follows from the two-factor case using Proposition 2.3.

Likewise, if $\mathbf{k}$ is a field of characteristic 0, and if $\mu_1, \mu_2, \ldots, \mu_k$ are k partitions of respective sizes $i_1, i_2, \ldots, i_k$, then

$$\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k} \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = i_1 + i_2 + \cdots + i_k}} \left(\mathcal{S}^\lambda \right)^{\oplus c^\lambda_{\mu_1, \mu_2, \ldots, \mu_k}} \tag{19}$$

as $S_{i_1 + i_2 + \cdots + i_k}$-modules. When $\mathbf{k}$ is a field of characteristic $\neq 0$, or, more generally, just any commutative ring, then the isomorphisms (18) and (19) still hold in a weaker form, where the direct sums are replaced by filtrations whose subquotients are Specht modules of the form $\mathcal{S}^\lambda$ (see [Cla91, Theorem 9.7] for the more general problem of finding such filtrations of skew Specht modules[17]).

Now, Theorem 3.3 gives rise to the following decomposition of F_i / F_{i-1}:

> **Corollary 3.12.** Assume that $\mathbf{k}$ is a field of characteristic 0.
> Let $i \in [f_{n+1}]$. Consider the lacunar subset Q_i of $[n-1]$. Write the set $Q_i \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n + 1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. (Note that $j_1 \geq 0$ and $j_2, j_3, \ldots, j_m > 1$ and $j_1 + j_2 + \cdots + j_m = n$; this follows from Lemma 3.4 (applied to $I = Q_i$).)
> Let Par_n be the set of all partitions of n. Then,
>
> $$F_i / F_{i-1} \cong \bigoplus_{\lambda \in \mathrm{Par}_n} \left(\mathcal{S}^\lambda \right)^{\oplus c^\lambda_{(j_1), \, (j_2-1,1), \, (j_3-1,1), \, \ldots, \, (j_m-1,1)}}$$
>
> (where the subscripts in $c^\lambda_{(j_1), \, (j_2-1,1), \, (j_3-1,1), \, \ldots, \, (j_m-1,1)}$ are the partition (j_1) followed by the partitions $(j_k - 1, 1)$ for all $k \in [2, m]$).

Proof. The set $Q_i \cup \{n+1\}$ is lacunar (since Q_i is lacunar, and since $Q_i \subseteq [n-1]$ ensures that $n + 1$ is larger than any element of Q_i by at least 2). In other words, every $k > 1$ satisfies $i_k > i_{k-1} + 1$. Hence, every $k > 1$ satisfies $j_k > 1$ (since $j_k = i_k - i_{k-1}$) and therefore

$$\mathcal{Z}_{j_k} \cong \mathcal{S}^{(j_k-1,1)} \tag{20}$$

(by Proposition 2.1 **(b)**, since j_k is invertible in $\mathbf{k}$).

Also, we have $\mathcal{H}_p \cong \mathcal{S}^{(p)}$ for each $p \in \mathbb{N}$ (since both $\mathcal{H}_p$ and $\mathcal{S}^{(p)}$ are trivial 1-dimensional representations of S_p). Thus, $\mathcal{H}_{j_1} \cong \mathcal{S}^{(j_1)}$.

---

[17]Indeed, the induction product $\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k}$ is easily seen to be isomorphic to the skew Specht module $\mathcal{S}^{\alpha/\beta}$, where $\alpha/\beta = \mu_1 * \mu_2 * \cdots * \mu_k$. Thus, [Cla91, Theorem 9.7] (applied to this skew diagram α/β) yields a Specht series (i.e., a filtration whose subquotients are Specht modules of the form $\mathcal{S}^\lambda$) for $\mathcal{S}^{\mu_1} * \mathcal{S}^{\mu_2} * \cdots * \mathcal{S}^{\mu_k}$.

Now, Theorem 3.3 yields

$$F_i / F_{i-1} \cong \underbrace{\mathcal{H}_{j_1} * \mathcal{Z}_{j_2} * \mathcal{Z}_{j_3} * \cdots * \mathcal{Z}_{j_m}}_{\substack{\text{the first factor is an } \mathcal{H}, \\ \text{while all others are } \mathcal{Z}\text{'s}}}$$

$$\cong \mathcal{S}^{(j_1)} * \mathcal{S}^{(j_2-1,1)} * \mathcal{S}^{(j_3-1,1)} * \cdots * \mathcal{S}^{(j_m-1,1)}$$

$$\left(\begin{array}{c} \text{since } \mathcal{H}_{j_1} \cong \mathcal{S}^{(j_1)}, \text{ and since } \mathcal{Z}_{j_k} \cong \mathcal{S}^{(j_k-1,1)} \text{ for each } k > 1 \\ \text{(by (20))} \end{array} \right)$$

$$\cong \bigoplus_{\substack{\lambda \in \text{Par}; \\ |\lambda| = j_1 + j_2 + \cdots + j_m}} \left(\mathcal{S}^\lambda \right)^{\oplus c^\lambda_{(j_1), \, (j_2-1,1), \, (j_3-1,1), \, \ldots, \, (j_m-1,1)}} \qquad \text{(by (19))}$$

$$= \bigoplus_{\substack{\lambda \in \text{Par}; \\ |\lambda| = n}} \left(\mathcal{S}^\lambda \right)^{\oplus c^\lambda_{(j_1), \, (j_2-1,1), \, (j_3-1,1), \, \ldots, \, (j_m-1,1)}} \qquad \text{(since } j_1 + j_2 + \cdots + j_m = n)$$

$$= \bigoplus_{\lambda \in \text{Par}_n} \left(\mathcal{S}^\lambda \right)^{\oplus c^\lambda_{(j_1), \, (j_2-1,1), \, (j_3-1,1), \, \ldots, \, (j_m-1,1)}} .$$

This proves Corollary 3.12. □

4. The Specht module spectrum

4.1. The theorem

We need a few more notations from [GriLaf22]. For any subset I of $[n]$, we define the following:

- We let $\widehat{I}$ be the set $\{0\} \cup I \cup \{n+1\}$. We shall refer to $\widehat{I}$ as the *enclosure* of I.

  For example, if $n = 5$, then $\widehat{\{2,3\}} = \{0,2,3,6\}$.

- For any $\ell \in [n]$, we let $m_{I,\ell}$ be the number

  $$\left(\text{smallest element of } \widehat{I} \text{ that is } \geq \ell \right) - \ell \in [0, n+1-\ell] \subseteq [0,n].$$

  For example, if $n = 6$ and $I = \{2,5\}$, then

  $$(m_{I,1}, \ m_{I,2}, \ m_{I,3}, \ m_{I,4}, \ m_{I,5}, \ m_{I,6}) = (1, \ 0, \ 2, \ 1, \ 0, \ 1).$$

  We note that an $\ell \in [n]$ satisfies $m_{I,\ell} = 0$ if and only if $\ell \in \widehat{I}$ (or, equivalently, $\ell \in I$).

We recall that any partition λ of n gives rise to an S_n-representation called the Specht module $\mathcal{S}^\lambda$. If λ is a partition of n, and if $a \in \mathcal{A}$, then the action of a on $\mathcal{S}^\lambda$ (that is, the **k**-linear map $\mathcal{S}^\lambda \to \mathcal{S}^\lambda$, $w \mapsto aw$) will be denoted by $L_\lambda(a)$.

Definition 4.1. Let λ be a partition of n. Let I be a lacunar subset of $[n-1]$. Write the set $I \cup \{n+1\}$ as $\{i_1 < i_2 < \cdots < i_m\}$, so that $i_m = n+1$. Furthermore, set $i_0 := 1$. Set $j_k := i_k - i_{k-1}$ for each $k \in [m]$. (Note that Lemma 3.4 shows that $j_1 \geq 0$ and $j_2, j_3, \ldots, j_m > 1$, hence $j_2, j_3, \ldots, j_m \geq 2$.)

The m-Littlewood–Richardson coefficient $c^{\lambda}_{(j_1),\ (j_2-1,1),\ (j_3-1,1),\ \ldots,\ (j_m-1,1)}$ (as defined in (13), where the subscripts are the partition (j_1) followed by the partitions $(j_k - 1, 1)$ for all $k \in [2, m]$) will then be denoted by c^{λ}_I.

As we recall from Corollary 3.12, if $I = Q_i$ for some $i \in [f_{n+1}]$, then this coefficient c^{λ}_I is the multiplicity of the Specht module $\mathcal{S}^{\lambda}$ in the left $\mathcal{A}$-module F_i / F_{i-1} when $\mathbf{k}$ is a field of characteristic 0. Indeed, we can rewrite Corollary 3.12 as follows using Definition 4.1:

Corollary 4.2. Assume that $\mathbf{k}$ is a field of characteristic 0.

Let $i \in [f_{n+1}]$. Let Par_n be the set of all partitions of n. Then,

$$F_i / F_{i-1} \cong \bigoplus_{\lambda \in \text{Par}_n} \left(\mathcal{S}^{\lambda} \right)^{\oplus c^{\lambda}_{Q_i}}.$$

We shall now state our first main theorem:

Theorem 4.3. Let $\mathbf{k}$ be any field. Let λ be a partition of n. Let $\omega_1, \omega_2, \ldots, \omega_n \in \mathbf{k}$. For each subset I of $[n]$, we set

$$\omega_I := \omega_1 m_{I,1} + \omega_2 m_{I,2} + \cdots + \omega_n m_{I,n} \in \mathbf{k}.$$

Then:

(a) The eigenvalues of the operator $L_{\lambda}(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ on the Specht module $\mathcal{S}^{\lambda}$ are the elements

$$\omega_I \text{ for all lacunar subsets } I \subseteq [n-1] \text{ satisfying } c^{\lambda}_I \neq 0,$$

and their respective algebraic multiplicities are the c^{λ}_I in the generic case (i.e., if no two I's produce the same ω_I; otherwise the multiplicities of colliding eigenvalues should be added together).

(b) If all these ω_I (for all lacunar subsets $I \subseteq [n-1]$ satisfying $c^{\lambda}_I \neq 0$) are distinct, then $L_{\lambda}(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ is diagonalizable.

(c) We have

$$\prod_{\substack{I \subseteq [n-1] \text{ is lacunar;} \\ c^{\lambda}_I \neq 0}} \left(L_{\lambda}(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_I \, \text{id}_{\mathcal{S}^{\lambda}} \right) = 0.$$

To prove this theorem, we will need a further theorem, which "maps" the Fibonacci filtration from $\mathcal{A}$ to a given Specht module $\mathcal{S}^\lambda$:

Theorem 4.4. Let $\mathbf{k}$ be any field of characteristic 0. Let λ be a partition of n. Then, there exists a filtration

$$0 = F^\lambda_{f_{n+1}} \subseteq F^\lambda_{f_{n+1}-1} \subseteq F^\lambda_{f_{n+1}-2} \subseteq \cdots \subseteq F^\lambda_2 \subseteq F^\lambda_1 \subseteq F^\lambda_0 = \mathcal{S}^\lambda$$

(note the "backward" indexing!) of the Specht module $\mathcal{S}^\lambda$ by left $\mathcal{T}$-submodules with the following four properties:

1. Each subquotient $F^\lambda_{i-1}/F^\lambda_i$ (for $i \in [f_{n+1}]$) has dimension $c^\lambda_{Q_i}$ as a $\mathbf{k}$-vector space (see Definition 4.1 for the meaning of c^λ_I).

2. In particular, an $i \in [f_{n+1}]$ satisfies $F^\lambda_{i-1} = F^\lambda_i$ if and only if $c^\lambda_{Q_i} = 0$.

3. On each subquotient $F^\lambda_{i-1}/F^\lambda_i$ (for $i \in [f_{n+1}]$), each element $t_\ell \in \mathcal{T}$ (for $\ell \in [n]$) acts as multiplication by the scalar $m_{Q_i,\ell}$.

4. More generally, on each subquotient $F^\lambda_{i-1}/F^\lambda_i$ (for $i \in [f_{n+1}]$), each element $P(t_1, t_2, \ldots, t_n) \in \mathcal{T}$ (where P is a polynomial in n non-commuting indeterminates over $\mathbf{k}$) acts as multiplication by the scalar $P(m_{Q_i,1}, m_{Q_i,2}, \ldots, m_{Q_i,n})$.

4.2. Lemmas about S_n-representations

In order to prove Theorem 4.3 and Theorem 4.4, we need a few lemmas about left $\mathcal{A}$-modules (i.e., representations of S_n). We begin with something basic and well-known (see, e.g., [EGH+11, last paragraph of §5.13]):

Proposition 4.5. Assume that $\mathbf{k}$ is a field of characteristic 0. Let λ and μ be two partitions of n. Then, the Specht modules $\mathcal{S}^\lambda$ and $\mathcal{S}^\mu$ satisfy

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \begin{cases} \mathbf{k}, & \text{if } \lambda = \mu; \\ 0, & \text{if } \lambda \neq \mu \end{cases} \qquad \text{as } \mathbf{k}\text{-vector spaces.}$$

(Here and in the following, "$\operatorname{Hom}_{\mathcal{A}}$" always stands for the set of left $\mathcal{A}$-module morphisms. This is always a $\mathbf{k}$-vector space, but usually not an $\mathcal{A}$-module on any side.)

For the sake of completeness, we give a proof of this proposition in the Appendix (Section A).

Lemma 4.6. Assume that $\mathbf{k}$ is a field of characteristic 0. Let λ be a partition of n. Define the contravariant functor $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^{\lambda}\right)$ from the category of left $\mathcal{A}$-modules to the category of $\mathbf{k}$-vector spaces that is given by

$$X \mapsto \operatorname{Hom}_{\mathcal{A}}\left(X, \mathcal{S}^{\lambda}\right) \qquad \text{on objects}$$

and likewise on morphisms. (This is a contravariant Hom functor.)

This contravariant functor $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^{\lambda}\right)$ is exact (i.e., respects exact sequences).

Proof. The $\mathbf{k}$-algebra $\mathcal{A} = \mathbf{k}\left[S_n\right]$ is semisimple (by Maschke's theorem, since $\mathbf{k}$ is a field of characteristic 0). Hence, every short exact sequence of left $\mathcal{A}$-modules is split. Consequently, any Hom functor from the category of left $\mathcal{A}$-modules is exact[18]. Thus, the contravariant Hom functor $\operatorname{Hom}_{\mathcal{A}}\left(-, \mathcal{S}^{\lambda}\right)$ is exact. This proves Lemma 4.6. $\square$

Note that the Specht module $\mathcal{S}^{\lambda}$ in Lemma 4.6 could be replaced by any left $\mathcal{A}$-module, but we will use $\mathcal{S}^{\lambda}$ only. The same applies to the following lemma:

Lemma 4.7. Let λ be a partition of n. Let J be a left $\mathcal{A}$-submodule of $\mathcal{A}$ (that is, a left ideal of $\mathcal{A}$). Then, there is a canonical $\mathbf{k}$-vector space isomorphism

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}/J, \ \mathcal{S}^{\lambda}\right) \to \left\{v \in \mathcal{S}^{\lambda} \ \mid \ Jv = 0\right\},$$
$$f \mapsto f\left(\overline{1_{\mathcal{A}}}\right).$$

Proof. The left $\mathcal{A}$-module morphisms from $\mathcal{A}/J$ to $\mathcal{S}^{\lambda}$ can be identified with the left $\mathcal{A}$-module morphisms from $\mathcal{A}$ to $\mathcal{S}^{\lambda}$ that vanish on J. Thus, we obtain a $\mathbf{k}$-vector

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[18]We remind the reader of the proof of this fact: Let $\mathcal{F}$ be a contravariant Hom functor from the category of left $\mathcal{A}$-modules (the case of a covariant Hom functor is analogous). We must show that $\mathcal{F}$ is exact.

Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of left $\mathcal{A}$-modules. Then, this sequence is split (since every short exact sequence of left $\mathcal{A}$-modules is split), and thus is isomorphic to the obvious exact sequence $0 \to X \to X \oplus Z \to Z \to 0$. The Hom functor $\mathcal{F}$ sends the latter sequence to another short exact sequence $0 \to \mathcal{F}(X) \to \mathcal{F}(X) \oplus \mathcal{F}(Z) \to \mathcal{F}(Z) \to 0$ (since Hom functors respect finite direct sums). But $\mathcal{F}$ is a functor; thus, $\mathcal{F}$ sends any two isomorphic complexes to two isomorphic complexes. Hence, because the two short exact sequences $0 \to X \to Y \to Z \to 0$ and $0 \to X \to X \oplus Z \to Z \to 0$ are isomorphic, their images under $\mathcal{F}$ must also be isomorphic. In other words, the image of the short exact sequence $0 \to X \to Y \to Z \to 0$ under $\mathcal{F}$ is isomorphic to the image of $0 \to X \to X \oplus Z \to Z \to 0$ under $\mathcal{F}$. But the latter image is exact, as we have shown. Hence, the former image is also exact.

Thus, $\mathcal{F}$ sends the original short exact sequence $0 \to X \to Y \to Z \to 0$ to a short exact sequence. Since this is true for any short exact sequence $0 \to X \to Y \to Z \to 0$, we thus have shown that the functor $\mathcal{F}$ respects short exact sequences. In other words, $\mathcal{F}$ is exact, qed.

space isomorphism

$$\Phi : \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}/J, \ \mathcal{S}^{\lambda} \right) \to \left\{ g \in \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}, \ \mathcal{S}^{\lambda} \right) \ \mid \ g \left(J \right) = 0 \right\},$$
$$f \mapsto \left(\mathcal{A} \to \mathcal{S}^{\lambda}, \ a \mapsto f \left(\overline{a} \right) \right).$$

However, recall the well-known **k**-vector space isomorphism $\operatorname{Hom}_{A} \left(A, M \right) \cong M$ that holds for any **k**-algebra A and any left A-module M. Thus, in particular, the left $\mathcal{A}$-module morphisms from $\mathcal{A}$ to $\mathcal{S}^{\lambda}$ can be identified with the elements of $\mathcal{S}^{\lambda}$ via the **k**-vector space isomorphism

$$\Psi : \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}, \ \mathcal{S}^{\lambda} \right) \to \mathcal{S}^{\lambda},$$
$$g \mapsto g \left(1_{\mathcal{A}} \right).$$

This latter isomorphism has the property that a left $\mathcal{A}$-module morphism $g \in \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}, \ \mathcal{S}^{\lambda} \right)$ satisfies $g \left(J \right) = 0$ if and only if its image $\Psi \left(g \right)$ satisfies $J \cdot \Psi \left(g \right) = 0$ (since $g \left(J \right) = g \left(J \cdot 1_{\mathcal{A}} \right) = J \cdot \underbrace{g \left(1_{\mathcal{A}} \right)}_{= \Psi(g)} = J \cdot \Psi \left(g \right)$). Thus, Ψ can be restricted to a **k**-vector space isomorphism

$$\Psi' : \left\{ g \in \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}, \ \mathcal{S}^{\lambda} \right) \ \mid \ g \left(J \right) = 0 \right\} \to \left\{ v \in \mathcal{S}^{\lambda} \ \mid \ Jv = 0 \right\},$$
$$g \mapsto g \left(1_{\mathcal{A}} \right).$$

The composition $\Psi' \circ \Phi$ is thus a **k**-vector space isomorphism

$$\operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}/J, \ \mathcal{S}^{\lambda} \right) \to \left\{ v \in \mathcal{S}^{\lambda} \ \mid \ Jv = 0 \right\},$$
$$f \mapsto f \left(\overline{1_{\mathcal{A}}} \right).$$

This is clearly canonical in J, so that Lemma 4.7 is proved. $\qquad\square$

4.3. The proofs

We are now ready to prove Theorem 4.4 and Theorem 4.3, in this order.

Proof of Theorem 4.4. For each $i \in [0, f_{n+1}]$, we define a subset F_i^{λ} of $\mathcal{S}^{\lambda}$ by

$$F_i^{\lambda} := \left\{ v \in \mathcal{S}^{\lambda} \ \mid \ F_i v = 0 \right\}.$$

This subset F_i^{λ} is actually a left $\mathcal{T}$-submodule of $\mathcal{S}^{\lambda}$ (since Proposition 3.2 shows that F_i is a right $\mathcal{T}$-submodule of $\mathcal{A}$) [19]. Moreover, because of

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{f_{n+1}} = \mathbf{k} \left[S_n \right] = \mathcal{A},$$

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[19]*Proof.* To show that F_i^{λ} is closed under addition, we observe that any $v, w \in F_i^{\lambda}$ satisfy $v + w \in F_i^{\lambda}$

we have

$$\mathcal{S}^\lambda = F_0^\lambda \supseteq F_1^\lambda \supseteq F_2^\lambda \supseteq \cdots \supseteq F_{f_{n+1}}^\lambda = 0.$$

This is a left $\mathcal{T}$-module filtration of $\mathcal{S}^\lambda$, albeit written backwards. We can rewrite it as

$$0 = F_{f_{n+1}}^\lambda \subseteq F_{f_{n+1}-1}^\lambda \subseteq F_{f_{n+1}-2}^\lambda \subseteq \cdots \subseteq F_2^\lambda \subseteq F_1^\lambda \subseteq F_0^\lambda = \mathcal{S}^\lambda.$$

Our goal is to show that this filtration satisfies the four properties 1, 2, 3 and 4 claimed in Theorem 4.4.

For this purpose, we fix $i \in [f_{n+1}]$. First, we shall show property 3. We must show that each element $t_\ell \in \mathcal{T}$ acts on $F_{i-1}^\lambda / F_i^\lambda$ as multiplication by the scalar $m_{Q_i,\ell}$. So we let $\ell \in [n]$ and $\overline{v} \in F_{i-1}^\lambda / F_i^\lambda$ (with $v \in F_{i-1}^\lambda$) be arbitrary. We must show that $t_\ell \overline{v} = m_{Q_i,\ell} \overline{v}$ in $F_{i-1}^\lambda / F_i^\lambda$.

We have $v \in F_{i-1}^\lambda$. In other words, $v \in \mathcal{S}^\lambda$ and $F_{i-1}v = 0$ (by the definition of F_{i-1}^λ).

Theorem 3.1 **(c)** yields $F_i \cdot (t_\ell - m_{Q_i,\ell}) \subseteq F_{i-1}$. Hence,

$$\underbrace{F_i \cdot (t_\ell - m_{Q_i,\ell})}_{\subseteq F_{i-1}} v \subseteq F_{i-1}v = 0,$$

so that $F_i \cdot (t_\ell - m_{Q_i,\ell}) v = 0$. In other words, $(t_\ell - m_{Q_i,\ell}) v \in F_i^\lambda$ (by the definition of F_i^λ). In other words, $t_\ell v - m_{Q_i,\ell} v \in F_i^\lambda$. In other words, $\overline{t_\ell v} = \overline{m_{Q_i,\ell} v}$ in $F_{i-1}^\lambda / F_i^\lambda$. In other words, $t_\ell \overline{v} = m_{Q_i,\ell} \overline{v}$ in $F_{i-1}^\lambda / F_i^\lambda$. Thus, the proof of property 3 is complete.

Property 4 follows immediately from property 3.

Let us now prove property 1.

Note that F_i is a left $\mathcal{A}$-submodule of $\mathcal{A}$ (by Proposition 3.2). Lemma 4.7 shows that whenever J is a left $\mathcal{A}$-submodule of $\mathcal{A}$ (that is, a left ideal of $\mathcal{A}$), there is a canonical **k**-vector space isomorphism

$$\operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}/J, \ \mathcal{S}^\lambda \right) \to \left\{ v \in \mathcal{S}^\lambda \ \mid \ Jv = 0 \right\},$$
$$f \mapsto f \left(\overline{1_{\mathcal{A}}} \right).$$

Thus, we have

$$\left\{ v \in \mathcal{S}^\lambda \ \mid \ Jv = 0 \right\} \cong \operatorname{Hom}_{\mathcal{A}} \left(\mathcal{A}/J, \ \mathcal{S}^\lambda \right) \tag{21}$$

(because $v, w \in F_i^\lambda$ entails $F_i v = 0$ and $F_i w = 0$ and thus $F_i (v + w) \subseteq \underbrace{F_i v}_{=0} + \underbrace{F_i w}_{=0} = 0$, so that

$F_i (v + w) = 0$ and therefore $v + w \in F_i^\lambda$).

To show that F_i^λ is closed under the left $\mathcal{T}$-action, we observe that any $t \in \mathcal{T}$ and $v \in F_i^\lambda$ satisfy $tv \in F_i^\lambda$ (because $v \in F_i^\lambda$ entails $F_i v = 0$; but we have $F_i t \subseteq F_i$ since F_i is a right $\mathcal{T}$-submodule of $\mathcal{A}$; thus $F_i (tv) = \underbrace{F_i t}_{\subseteq F_i} v \subseteq F_i v = 0$ and therefore $F_i (tv) = 0$, so that $tv \in F_i^\lambda$).

An even simpler argument shows that $0 \in F_i^\lambda$.

canonically for each left $\mathcal{A}$-submodule J of $\mathcal{A}$. Now, the definition of F_i^λ yields

$$F_i^\lambda = \left\{ v \in \mathcal{S}^\lambda \mid F_i v = 0 \right\} \cong \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{A}/F_i, \, \mathcal{S}^\lambda \right) \tag{22}$$

canonically (by (21)) and similarly

$$F_{i-1}^\lambda \cong \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{A}/F_{i-1}, \, \mathcal{S}^\lambda \right). \tag{23}$$

However, Lemma 4.6 shows that the contravariant functor $\mathrm{Hom}_{\mathcal{A}} \left(-, \mathcal{S}^\lambda \right)$ from the category of left $\mathcal{A}$-modules to the category of **k**-vector spaces is exact. Hence, applying this contravariant functor to the exact sequence

$$0 \to F_i/F_{i-1} \to \mathcal{A}/F_{i-1} \to \mathcal{A}/F_i \to 0$$

of left $\mathcal{A}$-modules, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{A}/F_i, \, \mathcal{S}^\lambda \right) \to \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{A}/F_{i-1}, \, \mathcal{S}^\lambda \right) \to \mathrm{Hom}_{\mathcal{A}} \left(F_i/F_{i-1}, \, \mathcal{S}^\lambda \right) \to 0$$

of **k**-vector spaces. In view of (23) and (22), we can rewrite this latter exact sequence as

$$0 \to F_i^\lambda \to F_{i-1}^\lambda \to \mathrm{Hom}_{\mathcal{A}} \left(F_i/F_{i-1}, \, \mathcal{S}^\lambda \right) \to 0.$$

The arrow $F_i^\lambda \to F_{i-1}^\lambda$ here is the canonical inclusion (since the isomorphisms in (23) and (22) are the canonical ones), and thus we obtain

$$\mathrm{Hom}_{\mathcal{A}} \left(F_i/F_{i-1}, \, \mathcal{S}^\lambda \right) \cong F_{i-1}^\lambda / F_i^\lambda \tag{24}$$

from the exactness of our sequence.

However, Corollary 4.2 says that

$$F_i/F_{i-1} \cong \bigoplus_{\nu \in \mathrm{Par}_n} (\mathcal{S}^\nu)^{\oplus c_{Q_i}^\nu}. \tag{25}$$

Hence,

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{A}} \left(F_i/F_{i-1}, \, \mathcal{S}^\lambda \right) &\cong \mathrm{Hom}_{\mathcal{A}} \left(\bigoplus_{\nu \in \mathrm{Par}_n} (\mathcal{S}^\nu)^{\oplus c_{Q_i}^\nu}, \, \mathcal{S}^\lambda \right) \\
&\cong \bigoplus_{\nu \in \mathrm{Par}_n} \left(\mathrm{Hom}_{\mathcal{A}} \left(\mathcal{S}^\nu, \mathcal{S}^\lambda \right) \right)^{\oplus c_{Q_i}^\nu}
\end{aligned}
\tag{26}
$$

(since Hom functors respect finite direct sums).

But each $\nu \in \mathrm{Par}_n$ satisfies

$$\mathrm{Hom}_{\mathcal{A}} \left(\mathcal{S}^\nu, \mathcal{S}^\lambda \right) \cong \begin{cases} \mathbf{k}, & \text{if } \nu = \lambda; \\ 0, & \text{if } \nu \neq \lambda \end{cases} \qquad \text{(by Proposition 4.5)}$$

and thus

$$\left(\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^{\nu},\mathcal{S}^{\lambda}\right)\right)^{\oplus c^{\nu}_{Q_i}} \cong \left(\begin{cases} \mathbf{k}, & \text{if } \nu = \lambda; \\ 0, & \text{if } \nu \neq \lambda \end{cases}\right)^{\oplus c^{\nu}_{Q_i}} \cong \begin{cases} \mathbf{k}^{\oplus c^{\nu}_{Q_i}}, & \text{if } \nu = \lambda; \\ 0, & \text{if } \nu \neq \lambda. \end{cases}$$

Thus, we can rewrite (26) as

$$\operatorname{Hom}_{\mathcal{A}}\left(F_i/F_{i-1},\, \mathcal{S}^{\lambda}\right) \cong \bigoplus_{\nu \in \operatorname{Par}_n} \begin{cases} \mathbf{k}^{\oplus c^{\nu}_{Q_i}}, & \text{if } \nu = \lambda; \\ 0, & \text{if } \nu \neq \lambda \end{cases} = \mathbf{k}^{\oplus c^{\lambda}_{Q_i}}.$$

Comparing this with (24), we see that

$$F^{\lambda}_{i-1}/F^{\lambda}_i \cong \mathbf{k}^{\oplus c^{\lambda}_{Q_i}}.$$

Thus, the $\mathbf{k}$-vector space $F^{\lambda}_{i-1}/F^{\lambda}_i$ has dimension $c^{\lambda}_{Q_i}$. This proves property 1.

Property 2 follows immediately from property 1 (since $F^{\lambda}_{i-1} = F^{\lambda}_i$ is equivalent to $\dim\left(F^{\lambda}_{i-1}/F^{\lambda}_i\right) = 0$). Hence, our proof of Theorem 4.4 is complete. $\qquad\square$

Proof of Theorem 4.3. In the following, the word "dimension" will always refer to the dimension of a $\mathbf{k}$-vector space, even if some other module structures are present. Thus, in particular, the dimension of a $\mathcal{T}$-module X will mean the dimension of X as a $\mathbf{k}$-vector space.

(a) Let us first assume that $\mathbf{k}$ is a field of characteristic 0. We shall later extend this to the general case.

Theorem 4.4 shows that there exists a filtration

$$0 = F^{\lambda}_{f_{n+1}} \subseteq F^{\lambda}_{f_{n+1}-1} \subseteq F^{\lambda}_{f_{n+1}-2} \subseteq \cdots \subseteq F^{\lambda}_2 \subseteq F^{\lambda}_1 \subseteq F^{\lambda}_0 = \mathcal{S}^{\lambda} \qquad (27)$$

of the Specht module $\mathcal{S}^{\lambda}$ by left $\mathcal{T}$-submodules with the four properties 1, 2, 3 and 4 stated in Theorem 4.4. Consider this filtration. Fix any basis $(v_1, v_2, \ldots, v_s)$ of $\mathcal{S}^{\lambda}$ that conforms with this filtration (i.e., a basis that begins with a basis of $F^{\lambda}_{f_{n+1}-1}$, then extends it to a basis of $F^{\lambda}_{f_{n+1}-2}$, then extends it to a basis of $F^{\lambda}_{f_{n+1}-3}$, and so on), so that each F^{λ}_i is spanned by $v_1, v_2, \ldots, v_{j(i)}$ for some $j(i) \in [0, s]$. Note that the inclusions in (27) yield

$$0 = j(f_{n+1}) \leq j(f_{n+1} - 1) \leq j(f_{n+1} - 2) \leq \cdots \leq j(2) \leq j(1) \leq j(0) = s.$$

Note that each $i \in [f_{n+1}]$ satisfies $j(i) = \dim\left(F^{\lambda}_i\right)$ and $j(i-1) = \dim\left(F^{\lambda}_{i-1}\right)$ and thus

$$j(i-1) - j(i) = \dim\left(F^{\lambda}_{i-1}\right) - \dim\left(F^{\lambda}_i\right) = \dim\left(F^{\lambda}_{i-1}/F^{\lambda}_i\right)$$
$$= c^{\lambda}_{Q_i} \qquad (28)$$

(by property 1 of our filtration).

The operator $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ preserves the filtration (27) (since this filtration is a filtration by left $\mathcal{T}$-submodules, but $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$). Hence, the matrix M that represents this operator with respect to the basis $(v_1, v_2, \ldots, v_s)$ is block-upper-triangular with blocks of sizes

$$j(i-1) - j(i) \qquad \text{for all } i \in [f_{n+1}]$$

(because, e.g., the fact that the operator preserves F_i^λ means that the first $j(i)$ columns of the matrix M have zeroes everywhere below the $j(i)$-th row). Moreover, property 4 of our filtration shows that the element $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$ acts as multiplication by the scalar

$$\omega_1 m_{Q_i,1} + \omega_2 m_{Q_i,2} + \cdots + \omega_n m_{Q_i,n} = \omega_{Q_i} \qquad \text{(by the definition of } \omega_{Q_i})$$

on each subquotient $F_{i-1}^\lambda / F_i^\lambda$. In other words, for each $v \in F_{i-1}^\lambda$, the vector $\overline{v} \in F_{i-1}^\lambda / F_i^\lambda$ satisfies

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \cdot \overline{v} = \omega_{Q_i} \overline{v}, \qquad (29)$$

and therefore

$$\begin{aligned}
&(L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)) (v) \\
&= (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \cdot v \\
&= \omega_{Q_i} v + \left(\text{some element of } F_i^\lambda\right) \qquad \text{(by (29))} \\
&= \omega_{Q_i} v + \left(\text{some linear combination of } v_1, v_2, \ldots, v_{j(i)}\right)
\end{aligned}$$

(since F_i^λ is spanned by $v_1, v_2, \ldots, v_{j(i)}$). We can apply this in particular to $v = v_k$ for each $k \in [j(i-1)]$ (since F_{i-1}^λ is spanned by $v_1, v_2, \ldots, v_{j(i-1)}$), and conclude that

$$\begin{aligned}
&(L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)) (v_k) \\
&= \omega_{Q_i} v_k + \left(\text{some linear combination of } v_1, v_2, \ldots, v_{j(i)}\right)
\end{aligned}$$

for each $k \in [j(i-1)]$.

Thus, the matrix M that represents the operator $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ with respect to the basis $(v_1, v_2, \ldots, v_s)$ is not only block-upper-triangular, but also has the property that its i-th diagonal block (for each $i \in [f_{n+1}]$, counted from the end) is the scalar matrix $\omega_{Q_i} \cdot I_{j(i-1)-j(i)} = \omega_{Q_i} \cdot I_{c_{Q_i}^\lambda}$ (by (28)). Consequently, the matrix M is upper-triangular, and its diagonal entries are the elements ω_{Q_i} for all $i \in [f_{n+1}]$, with each ω_{Q_i} appearing $c_{Q_i}^\lambda$ times (this means that if $c_{Q_i}^\lambda = 0$, then ω_{Q_i} does not appear at all).

Of course, this allows us to read off the eigenvalues of this matrix M, and thus of the operator $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ (since the eigenvalues of a triangular matrix are just its diagonal entries). We conclude that the eigenvalues

of $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ are the elements ω_{Q_i} for all $i \in [f_{n+1}]$, with each ω_{Q_i} appearing with algebraic multiplicity $c_{Q_i}^\lambda$. Since $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ are just the lacunar subsets of $[n-1]$, we can rewrite this as follows: The eigenvalues of $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ are the elements ω_I for all lacunar subsets $I \subseteq [n-1]$, with each ω_I appearing with algebraic multiplicity c_I^λ. We can restrict this list to those lacunar subsets $I \subseteq [n-1]$ that satisfy $c_I^\lambda \neq 0$ (since an eigenvalue ω_I that appears with algebraic multiplicity $c_I^\lambda = 0$ simply does not appear at all).

This proves Theorem 4.3 **(a)** in the case when **k** is a field of characteristic 0. It remains to extend the proof to the case when **k** is an arbitrary field. But there is a standard trick for this: We recast our result as a polynomial identity. Namely, Theorem 4.3 **(a)** is saying that

$$\underbrace{\det \left(x \, \mathrm{id}_{\mathcal{S}^\lambda} - L_\lambda \left(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \right) \right)}_{\substack{\text{This is the characteristic polynomial of the} \\ \text{endomorphism } L_\lambda(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \text{ of } \mathcal{S}^\lambda}} = \prod_{I \subseteq [n-1] \text{ lacunar}} (x - \omega_I)^{c_I^\lambda}$$

in the polynomial ring $\mathbf{k}[x]$. This is a polynomial identity in the indeterminates $x, \omega_1, \omega_2, \ldots, \omega_n$ (since the Specht module $\mathcal{S}^\lambda$ has a basis consisting of the standard polytabloids, and the action of S_n on this basis is independent of the base field **k**). Thus, knowing that this identity holds whenever **k** is a field of characteristic 0, we can immediately conclude that it holds for all fields **k** (and even all commutative rings **k**). This proves Theorem 4.3 **(a)** in the general case.

(c) Again, let us first assume that **k** is a field of characteristic 0. Recall the filtration (27) constructed in the proof of part **(a)**. As we saw in that proof, the element $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \in \mathcal{T}$ acts as multiplication by the scalar ω_{Q_i} on each subquotient $F_{i-1}^\lambda / F_i^\lambda$ of that filtration. In other words, for each $i \in [f_{n+1}]$, we have

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \overline{v} = \omega_{Q_i} \overline{v} \qquad \text{for each } \overline{v} \in F_{i-1}^\lambda / F_i^\lambda,$$

that is,

$$(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) v - \omega_{Q_i} v \in F_i^\lambda \qquad \text{for each } v \in F_{i-1}^\lambda.$$

In other words, for each $i \in [f_{n+1}]$, we have

$$\left(L_\lambda \left(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \right) - \omega_{Q_i} \, \mathrm{id}_{\mathcal{S}^\lambda} \right) F_{i-1}^\lambda \subseteq F_i^\lambda.$$

Hence, the operator

$$\prod_{i \in [f_{n+1}]} \left(L_\lambda \left(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \right) - \omega_{Q_i} \, \mathrm{id}_{\mathcal{S}^\lambda} \right) \in \mathrm{End}_{\mathbf{k}} \left(\mathcal{S}^\lambda \right)$$

[20] sends the whole $\mathcal{S}^\lambda$ to 0 (because its first factor sends $\mathcal{S}^\lambda = F_0^\lambda$ down to F_1^λ, then its second factor sends F_1^λ further down to F_2^λ, then its third factor sends F_2^λ onward

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[20]This product is well-defined (and does not depend on the order of its factors), since all its factors (being polynomials in $L_\lambda \left(\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \right)$) commute.

to F_3^λ, and so on, until the last factor sends $F_{f_{n+1}-1}^\lambda$ down to $F_{f_{n+1}}^\lambda = 0$). Moreover, for this to hold, we do not actually need all the f_{n+1} factors of this product, but rather only those factors that correspond to the numbers $i \in [f_{n+1}]$ satisfying $c_{Q_i}^\lambda \neq 0$ (because if $c_{Q_i}^\lambda = 0$, then property 2 of our filtration shows that $F_{i-1}^\lambda = F_i^\lambda$, and thus we don't need to apply the $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_{Q_i} \operatorname{id}_{\mathcal{S}^\lambda}$ factor to send us from F_{i-1}^λ down into F_i^λ). Hence, the operator

$$\prod_{\substack{i \in [f_{n+1}]; \\ c_{Q_i}^\lambda \neq 0}} \left(L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_{Q_i} \operatorname{id}_{\mathcal{S}^\lambda} \right) \in \operatorname{End}_{\mathbf{k}} \left(\mathcal{S}^\lambda \right)$$

sends the whole $\mathcal{S}^\lambda$ to 0 as well. In other words,

$$\prod_{\substack{i \in [f_{n+1}]; \\ c_{Q_i}^\lambda \neq 0}} \left(L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_{Q_i} \operatorname{id}_{\mathcal{S}^\lambda} \right) = 0.$$

Since $Q_1, Q_2, \ldots, Q_{f_{n+1}}$ are just the lacunar subsets of $[n-1]$, we can rewrite this as

$$\prod_{\substack{I \subseteq [n-1] \text{ is lacunar}; \\ c_I^\lambda \neq 0}} \left(L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) - \omega_I \operatorname{id}_{\mathcal{S}^\lambda} \right) = 0.$$

This proves Theorem 4.3 **(c)** in the case when $\mathbf{k}$ is a field of characteristic 0. Just as in our proof of part **(a)**, we can derive the general case from this case by a polynomial identity argument (treating $\omega_1, \omega_2, \ldots, \omega_n$ as indeterminates, and now considering polynomials with values in $\operatorname{End}_{\mathbf{k}} (\mathcal{S}^\lambda)$, which can be encoded as tuples of usual polynomials).

   (b) Theorem 4.3 **(c)** shows that the endomorphism $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n) \in \operatorname{End}_{\mathbf{k}} (\mathcal{S}^\lambda)$ is annihilated by the polynomial $\prod_{\substack{I \subseteq [n-1] \text{ is lacunar}; \\ c_I^\lambda \neq 0}} (x - \omega_I) \in \mathbf{k}[x]$ (meaning that the polynomial vanishes when we substitute the endomorphism for x). But it is well-known that a linear endomorphism (of a finite-dimensional $\mathbf{k}$-vector space) that is annihilated by a polynomial of the form $\prod (x - r)$ with pairwise distinct scalars r is always diagonalizable. Hence, if the ω_I in the above polynomial are pairwise distinct, then the endomorphism $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ is diagonalizable. This proves Theorem 4.3 **(b)**. $\square$

5. Final remarks

Thus we have computed the eigenvalues – and their algebraic multiplicities – for the action of any one-sided cycle shuffle $\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n$ on any Specht module $\mathcal{S}^\lambda$. With a trivial amount of work, we could extend this analysis to

the action of any element of $\mathcal{T}$ (that is, of any noncommutative polynomial in $t_1, t_2, \ldots, t_n$). This automatically allows us to identify the eigenvalues of such elements on any S_n-representation V, as long as the decomposition of V into Specht modules is known.

The proof of our result was achieved in a rather roundabout way: We did no work in the Specht modules $\mathcal{S}^\lambda$ themselves. Instead, we used a filtration of $\mathcal{A}$ (the Fibonacci filtration) whose subquotients F_i / F_{i-1} we were able to decompose into Specht modules (Theorem 3.3). Then, we "projected" this filtration onto each Specht module $\mathcal{S}^\lambda$ (Theorem 4.4) and used the semisimplicity of $\mathcal{A}$ (actually, the complete reducibility of $\mathcal{S}^\lambda$ would have sufficed) to triangularize the action of $\mathcal{T}$ on $\mathcal{S}^\lambda$. This is in contrast to other instances of similar questions, such as the recent [AFBC$^+$24], where the solution requires significant exploration of the inner life of $\mathcal{S}^\lambda$.

Our above method for proving Theorems 4.4 and 4.3 – in which we used the Fibonacci filtration to triangularize $L_\lambda (\omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n)$ – is partly generalizable:

Proposition 5.1. Let A be a **k**-algebra, and let T be a **k**-subalgebra of A. Let

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m = A \tag{30}$$

be a filtration of A by (A, T)-subbimodules. Let V be any left A-module. If B is any (A, T)-subbimodule of A, then we can define a left T-submodule

$$V^B := \{ v \in V \mid Bv = 0 \}$$

of V. Then, we have a filtration

$$0 = V^{F_m} \subseteq V^{F_{m-1}} \subseteq V^{F_{m-2}} \subseteq \cdots \subseteq V^{F_0} = V \tag{31}$$

of V by left T-submodules.

(a) Its subquotients $V^{F_{i-1}} / V^{F_i}$ can be canonically embedded into $\operatorname{Hom}_A (F_i / F_{i-1}, V)$ as left T-modules. Thus, if some element $t \in T$ acts triangularly from the right on the filtration (30) (meaning that it acts as a scalar on each subquotient F_i / F_{i-1}), then it also acts triangularly from the left on the filtration (31).

(b) If **k** is a field and the algebra A is semisimple, then these embeddings $V^{F_{i-1}} / V^{F_i} \to \operatorname{Hom}_A (F_i / F_{i-1}, V)$ are isomorphisms. Thus, in this case, knowing the dimensions of the Hom-spaces $\operatorname{Hom}_A (F_i, V)$ allows us to compute the multiplicities of eigenvalues for a triangular $t \in T$ acting on V.

Proof. This is implicit in our above proofs of Theorems 4.4 and 4.3 (where A, T and V were taken to be $\mathcal{A}$, $\mathcal{T}$ and $\mathcal{S}^\lambda$, and where the submodules V^{F_i} were called F_i^λ). $\qquad\square$

A. Omitted proofs

In this appendix, we give proofs to some folklore facts about representations of symmetric groups.

Proof of Proposition 2.1 (sketched). We will use the notations from [Gri25].

Let $(e_1, e_2, \ldots, e_n)$ be the standard basis of the natural representation $\mathcal{N}_n = \mathbf{k}^n$. The symmetric group S_n acts on it by the rule $\sigma \cdot e_i = e_{\sigma(i)}$ for all $\sigma \in S_n$ and $i \in [n]$. The subrepresentation $\mathcal{D}_n$ is then spanned by the single vector $e_1 + e_2 + \cdots + e_n = (1, 1, \ldots, 1)$.

Consider the Young diagram $Y((n-1, 1))$ of the partition $(n-1, 1)$. This Young diagram gives rise to a Young module $\mathcal{M}^{(n-1,1)} := \mathcal{M}^{Y((n-1,1))}$ and a Specht module $\mathcal{S}^{(n-1,1)} := \mathcal{S}^{Y((n-1,1))}$.

The Young module $\mathcal{M}^{(n-1,1)}$ has a basis formed by the n-tabloids of shape $Y((n-1, 1))$. We will use the short notation $\overline{k}$ for the n-tabloid of shape $Y((n-1, 1))$ that has the entry k in cell $(2, 1)$, as in [Gri25, Example 5.3.16].

The Specht module $\mathcal{S}^{(n-1,1)}$ is the submodule of $\mathcal{M}^{(n-1,1)}$ spanned by the polytabloids $\mathbf{e}_T = \overline{T(2,1)} - \overline{T(1,1)}$, or, equivalently, by the differences $\overline{k} - \overline{\ell}$ for $k \neq \ell$ in $[n]$.

We can identify the S_n-representation $\mathcal{M}^{(n-1,1)}$ with the natural representation $\mathcal{N}_n$ by the isomorphism

$$\mathcal{M}^{(n-1,1)} \to \mathcal{N}_n,$$
$$\overline{k} \mapsto e_k.$$

Then, the Specht module $\mathcal{S}^{(n-1,1)}$ is spanned by the differences $e_k - e_\ell$ for $k \neq \ell$ in $[n]$. Hence, $\mathcal{S}^{(n-1,1)}$ is the zero-sum subrepresentation of $\mathcal{N}_n$ (called $R(\mathbf{k}^n)$ in [Gri25, §4.2.5]).

(a) Let $(e_1^*, e_2^*, \ldots, e_n^*)$ be the dual basis of the standard basis $(e_1, e_2, \ldots, e_n)$ of $\mathcal{N}_n$. Thus, each $i \in [n]$ satisfies $e_i^* \in \mathcal{N}_n^*$, and each $i, j \in [n]$ satisfy $e_i^*(e_j) = \delta_{i,j}$ (Kronecker delta). Furthermore, $\sigma(e_i^*) = e_{\sigma(i)}^*$ for each $\sigma \in S_n$ and $i \in [n]$.

Now, consider the $\mathbf{k}$-linear map

$$\Phi : \mathcal{M}^{(n-1,1)} \to \mathcal{N}_n^*,$$
$$\overline{k} \mapsto e_k^* \qquad (\text{for all } k \in [n]).$$

This map Φ is invertible (since $\left(\overline{k}\right)_{k \in [n]}$ and $\left(e_k^*\right)_{k \in [n]}$ are bases of the $\mathbf{k}$-modules $\mathcal{M}^{(n-1,1)}$ and $\mathcal{N}_n^*$, respectively) and S_n-equivariant (since each $k \in [n]$ and $\sigma \in S_n$ satisfy $\sigma\left(\overline{k}\right) = \overline{\sigma(k)}$ and $\sigma(e_k^*) = e_{\sigma(k)}^*$). Hence, this map Φ is an isomorphism of S_n-representations.

Now, consider the subrepresentation $\mathcal{S}^{(n-1,1)} = \operatorname{span}_{\mathbf{k}} \left\{ \overline{k} - \overline{\ell} \mid k \neq \ell \right\}$ of $\mathcal{M}^{(n-1,1)}$. The image of this subrepresentation under Φ is

$$\Phi\left(\mathcal{S}^{(n-1,1)}\right)$$

$$= \operatorname{span}_{\mathbf{k}} \left\{ e_k^* - e_\ell^* \mid k \neq \ell \right\} \qquad \left(\text{since } \mathcal{S}^{(n-1,1)} = \operatorname{span}_{\mathbf{k}} \left\{ \overline{k} - \overline{\ell} \mid k \neq \ell \right\} \right)$$

$$= \left\{ \alpha_1 e_1^* + \alpha_2 e_2^* + \cdots + \alpha_n e_n^* \in \mathcal{N}_n^* \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0 \right\}$$

$$\left(\begin{array}{c} \text{because if } v_1, v_2, \ldots, v_n \text{ are any } n \text{ vectors in a } \mathbf{k}\text{-module } V, \\ \text{then } \operatorname{span}_{\mathbf{k}} \left\{ v_k - v_\ell \mid k \neq \ell \right\} \\ = \left\{ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0 \right\} \end{array} \right)$$

$$= \left\{ h \in \mathcal{N}_n^* \mid h \left(e_1 + e_2 + \cdots + e_n \right) = 0 \right\}$$

$$\left(\begin{array}{c} \text{since each } \mathbf{k}\text{-linear map } h \in \mathcal{N}_n^* \text{ can be written uniquely} \\ \text{as } \alpha_1 e_1^* + \alpha_2 e_2^* + \cdots + \alpha_n e_n^*, \text{ and then it} \\ \text{satisfies } \alpha_1 + \alpha_2 + \cdots + \alpha_n = h \left(e_1 + e_2 + \cdots + e_n \right) \end{array} \right)$$

$$= \left\{ h \in \mathcal{N}_n^* \mid h \left(\mathcal{D}_n \right) = 0 \right\} \qquad \left(\text{since } \mathcal{D}_n = \operatorname{span}_{\mathbf{k}} \left\{ e_1 + e_2 + \cdots + e_n \right\} \right)$$

$$\cong \left(\mathcal{N}_n / \mathcal{D}_n \right)^* \qquad \left(\begin{array}{c} \text{since the } \mathbf{k}\text{-linear maps } h \in \mathcal{N}_n^* \text{ satisfying } h \left(\mathcal{D}_n \right) = 0 \\ \text{are in canonical bijection with} \\ \text{the } \mathbf{k}\text{-linear maps } \overline{h} \in \left(\mathcal{N}_n / \mathcal{D}_n \right)^* \end{array} \right)$$

$$= \mathcal{Z}_n^* \qquad \left(\text{since } \mathcal{N}_n / \mathcal{D}_n = \mathcal{Z}_n \right),$$

and this is an isomorphism of S_n-representations. On the other hand, however, we also have $\Phi\left(\mathcal{S}^{(n-1,1)}\right) \cong \mathcal{S}^{(n-1,1)}$ as S_n-representations (since Φ is an isomorphism of S_n-representations). Thus,

$$\mathcal{S}^{(n-1,1)} \cong \Phi\left(\mathcal{S}^{(n-1,1)}\right) \cong \mathcal{Z}_n^* \qquad \text{as } S_n\text{-representations.}$$

Taking duals, we thus obtain

$$\left(\mathcal{S}^{(n-1,1)}\right)^* \cong \left(\mathcal{Z}_n^*\right)^* \qquad \text{as } S_n\text{-representations.} \tag{32}$$

But the $\mathbf{k}$-module $\mathcal{Z}_n = \mathcal{N}_n / \mathcal{D}_n = \mathbf{k}^n / \operatorname{span}_{\mathbf{k}} \left\{ (1, 1, \ldots, 1) \right\}$ has a finite basis (namely, $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_{n-1}})$, as can be easily seen from basic linear algebra). Hence, [Gri25, Proposition 5.19.22 **(b)**] shows that $\left(\mathcal{Z}_n^*\right)^* \cong \mathcal{Z}_n$ as S_n-representations. In view of this, we can rewrite (32) as follows:

$$\left(\mathcal{S}^{(n-1,1)}\right)^* \cong \mathcal{Z}_n \qquad \text{as } S_n\text{-representations.}$$

This proves Proposition 2.1 **(a)**.

(b) Assume that n is invertible in $\mathbf{k}$. Then, [Gri25, Proposition 4.2.28 **(a)**] says that $\mathbf{k}^n = R\left(\mathbf{k}^n\right) \oplus D\left(\mathbf{k}^n\right)$, where we are using the notations of [Gri25]. In our

notations, this is saying that $\mathcal{N}_n = \mathcal{S}^{(n-1,1)} \oplus \mathcal{D}_n$ (since the submodule $\mathcal{S}^{(n-1,1)} = \operatorname{span}_{\mathbf{k}} \left\{ \overline{k} - \overline{\ell} \mid k \neq \ell \right\}$ of $\mathcal{M}^{(n-1,1)}$ corresponds to the zero-sum subrepresentation $R(\mathbf{k}^n) = \operatorname{span}_{\mathbf{k}} \{e_k - e_\ell \mid k \neq \ell\}$ of $\mathbf{k}^n$). Hence, $\mathcal{S}^{(n-1,1)} \cong \mathcal{N}_n / \mathcal{D}_n = \mathcal{Z}_n$ as S_n-representations. This proves Proposition 2.1 **(b)**.

Alternatively, we can prove Proposition 2.1 **(b)** directly: Let

$$\overline{\mathrm{avg}} := \frac{1}{n} \left(\overline{1} + \overline{2} + \cdots + \overline{n} \right) = \sum_{j=1}^{n} \overline{j} \in \mathcal{M}^{(n-1,1)}.$$

The **k**-linear map

$$\varrho : \mathcal{N}_n \to \mathcal{S}^{(n-1,1)},$$
$$e_i \mapsto \overline{i} - \overline{\mathrm{avg}} \qquad \text{(for all } i \in [n])$$

is well-defined (since each $i \in [n]$ satisfies $\overline{i} - \overline{\mathrm{avg}} = \overline{i} - \sum_{j=1}^{n} \overline{j} = \frac{1}{n} \sum_{j=1}^{n} \underbrace{\left(\overline{i} - \overline{j} \right)}_{\in \mathcal{S}^{(n-1,1)}} \in$

$\mathcal{S}^{(n-1,1)}$) and S_n-equivariant (since we can easily see that $\sigma \cdot \overline{\mathrm{avg}} = \overline{\mathrm{avg}}$ for each $\sigma \in S_n$). It furthermore sends $e_1 + e_2 + \cdots + e_n$ to 0, and thus vanishes on the submodule $\mathcal{D}_n$. Hence, it factors through a **k**-linear map $\varrho' : \mathcal{N}_n / \mathcal{D}_n \to \mathcal{S}^{(n-1,1)}$. This latter map ϱ' is easily seen to be invertible (indeed, it sends the basis $(\overline{e_1}, \overline{e_2}, \ldots, \overline{e_{n-1}})$ of $\mathcal{N}_n / \mathcal{D}_n$ to the basis $\left(\overline{1} - \overline{\mathrm{avg}}, \ \overline{2} - \overline{\mathrm{avg}}, \ \ldots, \ \overline{n-1} - \overline{\mathrm{avg}} \right)$ of $\mathcal{S}^{(n-1,1)}$), and thus is an isomorphism of S_n-representations (since it is S_n-equivariant). Hence, $\mathcal{S}^{(n-1,1)} \cong \mathcal{N}_n / \mathcal{D}_n = \mathcal{Z}_n$ as S_n-representations. Proposition 2.1 **(b)** is now proved again. $\square$

Proof of Proposition 2.3. We shall use the following general facts about induced representations ([GR20, Exercise 4.1.2] and [GR20, Exercise 4.1.3], respectively[21]):

- *Transitivity of induction:* Let G be a group. Let H be a subgroup of G. Let I be a subgroup of H. Let U be a representation of I. Then,

$$\operatorname{Ind}_H^G \operatorname{Ind}_I^H U \cong \operatorname{Ind}_I^G U. \tag{33}$$

- *Monoidality of induction:* Let G_1 and G_2 be two groups. Let H_1 be a subgroup of G_1, and let H_2 be a subgroup of G_2. Let W_1 be a representation of H_1, and let W_2 be a representation of H_2. Then,

$$\operatorname{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (W_1 \otimes W_2) \cong \left(\operatorname{Ind}_{H_1}^{G_1} W_1 \right) \otimes \left(\operatorname{Ind}_{H_2}^{G_2} W_2 \right). \tag{34}$$

Now, set $G_1 := S_{n_1 + n_2 + \cdots + n_i}$ and $H_1 := S_{n_1} \times S_{n_2} \times \cdots \times S_{n_i}$ and $W_1 := U_1 \otimes U_2 \otimes \cdots \otimes U_i$ (a representation of H_1) and $G_2 := S_{n_{i+1} + n_{i+2} + \cdots + n_k}$ and $H_2 := S_{n_{i+1}} \times$

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[21]The facts are stated in [GR20] only for $\mathbf{k} = \mathbb{C}$, but the proofs work equally well for any $\mathbf{k}$.

$S_{n_{i+2}} \times \cdots \times S_{n_k}$ and $W_2 := U_{i+1} \otimes U_{i+2} \otimes \cdots \otimes U_k$ (a representation of H_2). Then, by the definition of induction products, we have

$$U_1 * U_2 * \cdots * U_i = \mathrm{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_i}}^{S_{n_1+n_2+\cdots+n_i}} (U_1 \otimes U_2 \otimes \cdots \otimes U_i) = \mathrm{Ind}_{H_1}^{G_1} W_1$$

(by the definitions of G_1 and H_1 and W_1) and likewise

$$U_{i+1} * U_{i+2} * \cdots * U_k = \mathrm{Ind}_{H_2}^{G_2} W_2.$$

Thus,

$$\underbrace{(U_1 * U_2 * \cdots * U_i)}_{= \mathrm{Ind}_{H_1}^{G_1} W_1} * \underbrace{(U_{i+1} * U_{i+2} * \cdots * U_k)}_{= \mathrm{Ind}_{H_2}^{G_2} W_2}$$

$$= \left(\mathrm{Ind}_{H_1}^{G_1} W_1 \right) * \left(\mathrm{Ind}_{H_2}^{G_2} W_2 \right)$$

$$= \mathrm{Ind}_{S_{n_1+n_2+\cdots+n_i} \times S_{n_{i+1}+n_{i+2}+\cdots+n_k}}^{S_{n_1+n_2+\cdots+n_k}} \left(\left(\mathrm{Ind}_{H_1}^{G_1} W_1 \right) \otimes \left(\mathrm{Ind}_{H_2}^{G_2} W_2 \right) \right)$$

(by the definition of an induction product)

$$= \mathrm{Ind}_{G_1 \times G_2}^{S_{n_1+n_2+\cdots+n_k}} \underbrace{\left(\left(\mathrm{Ind}_{H_1}^{G_1} W_1 \right) \otimes \left(\mathrm{Ind}_{H_2}^{G_2} W_2 \right) \right)}_{\cong \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (W_1 \otimes W_2) \atop \text{(by (34))}} \qquad \left(\begin{array}{c} \text{by the definitions} \\ \text{of } G_1 \text{ and } G_2 \end{array} \right)$$

$$\cong \mathrm{Ind}_{G_1 \times G_2}^{S_{n_1+n_2+\cdots+n_k}} \left(\mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (W_1 \otimes W_2) \right)$$

$$\cong \mathrm{Ind}_{H_1 \times H_2}^{S_{n_1+n_2+\cdots+n_k}} (W_1 \otimes W_2) \qquad \text{(by an application of (33))}$$

$$\cong \mathrm{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}}^{S_{n_1+n_2+\cdots+n_k}} (U_1 \otimes U_2 \otimes \cdots \otimes U_k)$$

(since the definitions of H_1 and H_2 yield

$$H_1 \times H_2 = (S_{n_1} \times S_{n_2} \times \cdots \times S_{n_i}) \times (S_{n_{i+1}} \times S_{n_{i+2}} \times \cdots \times S_{n_k})$$
$$\cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}$$

and

$$W_1 \otimes W_2 = (U_1 \otimes U_2 \otimes \cdots \otimes U_i) \otimes (U_{i+1} \otimes U_{i+2} \otimes \cdots \otimes U_k)$$
$$\cong U_1 \otimes U_2 \otimes \cdots \otimes U_k,$$

and both of these isomorphisms are canonical and "fit together" in that the latter isomorphism respects the action of the former groups[22]). In view of

$$U_1 * U_2 * \cdots * U_k = \mathrm{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}}^{S_{n_1+n_2+\cdots+n_k}} (U_1 \otimes U_2 \otimes \cdots \otimes U_k)$$

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[22]Alternatively,   you   can   argue   directly: Both $\mathrm{Ind}_{H_1 \times H_2}^{S_{n_1+n_2+\cdots+n_k}} (W_1 \otimes W_2)$ and

(by the definition of the induction product), we can rewrite this as

$$(U_1 * U_2 * \cdots * U_i) * (U_{i+1} * U_{i+2} * \cdots * U_k) \cong U_1 * U_2 * \cdots * U_k.$$

This proves Proposition 2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 4.5. The following proof works more generally when **k** is a commutative ring in which $n!$ is invertible.

We shall use the notations of [Gri25, Chapter 5]. Pick any n-tableau P of shape λ and any n-tableau Q of shape μ. Consider the corresponding Young symmetrizers $\mathbf{E}_P$ and $\mathbf{E}_Q$ (as defined in [Gri25, Definition 5.11.1]). From [Gri25, (228)], we have $\mathcal{S}^\lambda \cong \mathcal{A}\mathbf{E}_P$ and $\mathcal{S}^\mu \cong \mathcal{A}\mathbf{E}_Q$. Hence, $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}\mathbf{E}_P, \mathcal{A}\mathbf{E}_Q\right)$ (since $\operatorname{Hom}_{\mathcal{A}}$ is functorial).

From [Gri25, Lemma 5.11.13], we know that the coefficient of the permutation $\mathrm{id} \in S_n$ in $\mathbf{E}_P$ is 1. Thus, the single element $\mathbf{E}_P$ is **k**-linearly independent. Hence, $\mathbf{k}\mathbf{E}_P \cong \mathbf{k}$ as **k**-module.

However, [Gri25, Theorem 5.11.3] shows that $\mathbf{E}_P^2 = \dfrac{n!}{f^\lambda}\mathbf{E}_P$. This shows that the element $\widetilde{\mathbf{E}}_P := \dfrac{f^\lambda}{n!}\mathbf{E}_P \in \mathcal{A}$ is idempotent (since $\widetilde{\mathbf{E}}_P^2 = \left(\dfrac{f^\lambda}{n!}\mathbf{E}_P\right)^2 = \left(\dfrac{f^\lambda}{n!}\right)^2 \underbrace{\mathbf{E}_P^2}_{=\frac{n!}{f^\lambda}\mathbf{E}_P} =$

$\dfrac{f^\lambda}{n!}\mathbf{E}_P = \widetilde{\mathbf{E}}_P$). Moreover, $\widetilde{\mathbf{E}}_P$ is a nonzero scalar multiple of $\mathbf{E}_P$ (since the scalar $\dfrac{f^\lambda}{n!}$ is clearly nonzero). Thus, $\mathcal{A}\mathbf{E}_P = \mathcal{A}\widetilde{\mathbf{E}}_P$.

But [EGH$^+$11, Lemma 5.13.4] says that if A is a **k**-algebra and $e \in A$ is an idempotent, then $\operatorname{Hom}_A(Ae, M) \cong eM$ for any left A-module M. Applying this to $A = \mathcal{A}$ and $e = \widetilde{\mathbf{E}}_P$ and $M = \mathcal{A}\mathbf{E}_Q$, we find

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}\widetilde{\mathbf{E}}_P, \mathcal{A}\mathbf{E}_Q\right) \cong \widetilde{\mathbf{E}}_P\mathcal{A}\mathbf{E}_Q = \mathbf{E}_P\mathcal{A}\mathbf{E}_Q$$

(since $\widetilde{\mathbf{E}}_P$ is a nonzero scalar multiple of $\mathbf{E}_P$). Altogether,

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \operatorname{Hom}_{\mathcal{A}}\left(\underbrace{\mathcal{A}\mathbf{E}_P}_{=\mathcal{A}\widetilde{\mathbf{E}}_P}, \mathcal{A}\mathbf{E}_Q\right) = \operatorname{Hom}_{\mathcal{A}}\left(\mathcal{A}\widetilde{\mathbf{E}}_P, \mathcal{A}\mathbf{E}_Q\right)$$

$$\cong \mathbf{E}_P\mathcal{A}\mathbf{E}_Q. \tag{35}$$

$\operatorname{Ind}_{S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k}}^{S_{n_1+n_2+\cdots+n_k}}(U_1 \otimes U_2 \otimes \cdots \otimes U_k)$ can be written as

$$\mathbf{k}\left[S_{n_1+n_2+\cdots+n_k}\right] \otimes (U_1 \otimes U_2 \otimes \cdots \otimes U_k)$$

modulo the relation

$$\sigma\left(\sigma_1 * \sigma_2 * \cdots * \sigma_k\right) \otimes (u_1 \otimes u_2 \otimes \cdots \otimes u_k) - \sigma \otimes (\sigma_1 u_1 \otimes \sigma_2 u_2 \otimes \cdots \otimes \sigma_k u_k)$$

for all $\sigma \in S_{n_1+n_2+\cdots+n_k}$ and $\sigma_i \in S_{n_i}$ and $u_i \in U_i$.

Now, if $\lambda \neq \mu$, then [Gri25, Proposition 5.11.15] (applied to $S = P$ and $T = Q$) shows that $\mathbf{E}_P \mathbf{a} \mathbf{E}_Q = 0$ for all $\mathbf{a} \in \mathcal{A}$, and thus $\mathbf{E}_P \mathcal{A} \mathbf{E}_Q = 0$. Hence, if $\lambda \neq \mu$, then (35) becomes

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \mathbf{E}_P \mathcal{A} \mathbf{E}_Q = 0. \tag{36}$$

This proves Proposition 4.5 in the case when $\lambda \neq \mu$.

Thus, we now WLOG assume that $\lambda = \mu$. Hence, $\mu = \lambda$ and thus $\mathcal{S}^\mu = \mathcal{S}^\lambda \cong \mathcal{A}\mathbf{E}_P$. Hence, the same argument that we used to prove (35) can be applied to λ and P instead of μ and Q. This results in

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\lambda\right) \cong \mathbf{E}_P \mathcal{A} \mathbf{E}_P. \tag{37}$$

However, [Gri25, Proposition 5.11.5] (applied to $T = P$) shows that each $\mathbf{a} \in \mathcal{A}$ satisfies $\mathbf{E}_P \mathbf{a} \mathbf{E}_P = \kappa \mathbf{E}_P$ for some $\kappa \in \mathbf{k}$. In other words, each $\mathbf{a} \in \mathcal{A}$ satisfies $\mathbf{E}_P \mathbf{a} \mathbf{E}_P \in \mathbf{k} \mathbf{E}_P$. In other words, $\mathbf{E}_P \mathcal{A} \mathbf{E}_P \subseteq \mathbf{k} \mathbf{E}_P$. On the other hand, each $\kappa \in \mathbf{k}$ satisfies

$$\kappa \mathbf{E}_P = \frac{\kappa f^\lambda}{n!} \cdot \underbrace{\frac{n!}{f^\lambda} \mathbf{E}_P}_{\substack{=\mathbf{E}_P^2 = \mathbf{E}_P \mathbf{1} \mathbf{E}_P \\ \in \mathbf{E}_P \mathcal{A} \mathbf{E}_P}} \in \frac{\kappa f^\lambda}{n!} \mathbf{E}_P \mathcal{A} \mathbf{E}_P \subseteq \mathbf{E}_P \mathcal{A} \mathbf{E}_P.$$

Thus, $\mathbf{k} \mathbf{E}_P \subseteq \mathbf{E}_P \mathcal{A} \mathbf{E}_P$. Combining this with $\mathbf{E}_P \mathcal{A} \mathbf{E}_P \subseteq \mathbf{k} \mathbf{E}_P$, we obtain

$$\mathbf{E}_P \mathcal{A} \mathbf{E}_P = \mathbf{k} \mathbf{E}_P.$$

Hence, (37) becomes

$$\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\lambda\right) \cong \mathbf{E}_P \mathcal{A} \mathbf{E}_P = \mathbf{k} \mathbf{E}_P \cong \mathbf{k}.$$

In other words, $\operatorname{Hom}_{\mathcal{A}}\left(\mathcal{S}^\lambda, \mathcal{S}^\mu\right) \cong \mathbf{k}$ if $\lambda = \mu$. Thus, Proposition 4.5 is proved in the case $\lambda = \mu$ as well. $\square$

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