Rook sums in the symmetric group algebra

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Abstract. Let $\mathcal{A}$ be the group algebra $\mathbf{k}\left[S_n\right]$ of the n-th symmetric group S_n over a commutative ring $\mathbf{k}$. For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)=B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w$$

of $\mathcal{A}$. We study these elements, showing in particular that their minimal polynomials factor into linear factors (with integer coefficients). We express the product $\nabla_{D,C}\nabla_{B,A}$ as a $\mathbb{Z}$-linear combination of $\nabla_{U,V}$'s.

More generally, for any two set compositions (i.e., ordered set partitions) $\mathbf{A}$ and $\mathbf{B}$ of $\{1, 2, \ldots, n\}$, we define $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{A}$ to be the sum of all permutations $w \in S_n$ that send each block of $\mathbf{A}$ to the corresponding block of $\mathbf{B}$. This generalizes $\nabla_{B,A}$. The factorization property of minimal polynomials does not extend to the $\nabla_{\mathbf{B},\mathbf{A}}$, but we describe the ideal spanned by the $\nabla_{\mathbf{B},\mathbf{A}}$ and a further ideal complementary to it. These two ideals have a "mutually annihilative" relationship, are free as $\mathbf{k}$-modules, and appear as annihilators of tensor product S_n-representations; they are also closely related to Murphy's cellular bases, Specht modules, pattern-avoiding permutations and even some algebras appearing in quantum information theory.

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Rook theory is the study of permutations w in a symmetric group S_n that avoid certain input-output pairs (given by "boards", i.e., sets of allowed input-output pairs). Research done so far – e.g., [GJW75, BCHR11, LoeRem09] – has mostly focused on the enumeration of such permutations (known as "rook placements"). In this work, we set out in a new direction: Instead of counting the rook placements for a given board, we study their sum in the symmetric group algebra $\mathbf{k}[S_n]$ over a commutative ring $\mathbf{k}$. As with any elements of such a group algebra, we can ask for their spectral and representation-theoretical properties – what ideals do they generate? how well do their minimal polynomials factor?

   Let us give a quick overview, starting with an example that does not actually fit into our theory. Let $\mathcal{A}$ be the group algebra $\mathbf{k}[S_n]$ of a symmetric group S_n. For any subset T of $[n] \times [n]$ (where $[n]$ denotes the set $\{1, 2, \ldots, n\}$ as is usual in combinatorics), we set

$$\nabla_T := \sum_{\substack{w \in S_n; \\ (i, w(i)) \in T \text{ for each } i \in [n]}} w \in \mathcal{A}.$$

In rook-theoretical terms (see, e.g., [BCHR11, §1.2]), this is the sum of the n-rook placements on the board T.

For example, if $T = \{(i,j) \in [n] \times [n] \mid i \neq j\}$, then ∇_T is the sum of all derangements in S_n, and is well-behaved in many ways: It lies in the center of $\mathcal{A} = \mathbf{k}[S_n]$ (since the set of all derangements is fixed under conjugation), and its minimal polynomial (over a field $\mathbf{k}$) factors into linear factors (this is true for any element in the center of $\mathcal{A}$, because the center of $\mathbb{Q}[S_n]$ is split semisimple; this is well-known folklore).

Even simpler examples are $T = \{(i,i) \mid i \in [n]\}$ (yielding $\nabla_T = \mathrm{id}$) and $T = [n] \times [n]$ (here, ∇_T is the sum of all permutations in S_n).

These examples are far from representative. In general, ∇_T will rarely belong to the center of $\mathbf{k}[S_n]$, nor will its minimal polynomial often factor nicely. For example, if $n = 5$ and $T = \{(i,j) \mid j \neq i+1\}$, then the minimal polynomial of ∇_T has irreducible factors of degrees 1, 4, 5 and 6 (over $\mathbb{Q}$). Thus, the behavior of ∇_T depends starkly on the structure of T.

In this paper, we will study two types of boards T. The first one has the form "$[n] \times [n]$ with a rectangle cut out". Thus, we will define the so-called *rectangular rook sums*

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) = B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w \qquad \text{in } \mathcal{A}$$

for two subsets A and B of $[n]$. (The $\widetilde{\nabla}_{B,A}$ here is the ∇_T for $T = ([n] \times [n]) \setminus (A \times ([n] \setminus B))$. The $\nabla_{B,A}$ are – strictly speaking – redundant, as they equal either $\widetilde{\nabla}_{B,A}$ or 0 depending on whether $|A| = |B|$ or not. Conversely, however, the $\widetilde{\nabla}_{B,A}$ can be expressed as sums of $\nabla_{B,A}$'s, so the two families of elements are closely related.) Section 1 is devoted to the study of these elements. We will show that they span an ideal of $\mathcal{A}$, that they satisfy an explicit multiplication rule (Theorem 1.2.2), and that their minimal polynomials factor into linear factors (Corollary 1.4.3).

In Section 2, we will generalize the $\nabla_{B,A}$ to a wider family. Namely, we define a *set decomposition* of $[n]$ to be a tuple of disjoint subsets of $[n]$ (called *blocks*) whose union is $[n]$. (Empty blocks are allowed.) Now, if $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set decompositions of $[n]$ having the same length, then we define the element

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{of } \mathcal{A}.$$

For $k = 2$, these recover the elements $\nabla_{B,A}$ studied above. In general, $\nabla_{\mathbf{B},\mathbf{A}}$ is ∇_T for a certain board T that is obtained from $[n] \times [n]$ by cutting out multiple rectangles. While these $\nabla_{\mathbf{B},\mathbf{A}}$ lack the multiplication rule and the factoring minimal polynomials of the $\nabla_{B,A}$'s (at least we are not aware of a multiplication rule), they have their own share of interesting properties. In fact, they have already appeared in the works of Canfield/Williamson [CanWil89] and Murphy [Murphy92, Murphy95], where they were used to construct bases of $\mathcal{A}$ now known as the *Murphy cellular bases*.

Unlike Murphy, we are interested in the $\nabla_{\mathbf{B},\mathbf{A}}$ for all pairs $(\mathbf{A},\mathbf{B})$ of equal-length set decompositions, not just ones that come from standard bitableaux. We consider – for any given $k \in \mathbb{N}$ – the span $\mathcal{I}_k$ of all such elements $\nabla_{\mathbf{B},\mathbf{A}}$ where $\mathbf{A},\mathbf{B}$ are two set decompositions of $[n]$ having at most k blocks each.

Here is an incomplete survey of our results in Section 2; note that several of them have been shown before, but we give new proofs. We show (in Theorem 2.4.1) that $\mathcal{I}_k$ is an ideal of $\mathcal{A}$ and a free $\mathbf{k}$-module whose rank is the number of $12 \cdots (k+1)$-avoiding permutations in S_n, whereas the residue classes of the remaining permutations in S_n form a basis of the quotient $\mathcal{A}/\mathcal{I}_k$. Moreover, we construct another ideal $\mathcal{J}_k$ of $\mathcal{A}$ that has a mutual annihilation relationship with $\mathcal{I}_k$, meaning that each of the two ideals $\mathcal{I}_k$ and $\mathcal{J}_k$ is the annihilator of the other ideal from both left and right as well as its orthogonal complement with respect to the standard dot product on $\mathcal{A}$. The simplest way to define $\mathcal{J}_k$ (at least for $k < n$) is as the two-sided ideal generated by the single element

$$\nabla_X^- := \sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus U}} (-1)^w w \in \mathcal{A}$$

(this is not how we define $\mathcal{J}_k$, but is equivalent; see Proposition 2.2.2 **(f)**). This ideal $\mathcal{J}_k$, too, is a free $\mathbf{k}$-module, as is the quotient $\mathcal{A}/\mathcal{J}_k$ (see Theorem 2.4.1). Note that the span of the rectangular rook sums $\nabla_{B,A}$ studied in Section 1 is precisely the ideal $\mathcal{I}_2$, and so we conclude that its rank is the number of 123-avoiding permutations in S_n, that is, the Catalan number C_n (Corollary 2.4.2).

So far we have not assumed anything about the commutative ring $\mathbf{k}$. If, however, $n!$ is invertible in $\mathbf{k}$, then $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ as $\mathbf{k}$-modules and $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.

The ideals $\mathcal{I}_k$ and $\mathcal{J}_k$ are not entirely new; they can be viewed as spans of subfamilies of the Murphy cellular bases (see Remark 2.4.3). However, all our proofs are independent of this fact, and use only the most elementary algebra and combinatorics.

In Subsection 2.8, we describe the ideals $\mathcal{I}_k$ and $\mathcal{J}_{n-k-1}$ as annihilators of certain left $\mathbf{k}[S_n]$-modules. Namely:

1. The ideal $\mathcal{J}_k$ is the annihilator of the tensor power $V_k^{\otimes n}$, where $V_k = \mathbf{k}^k$ is a free $\mathbf{k}$-module of rank k and where S_n acts on $V_k^{\otimes n}$ by permuting the factors (Theorem 2.8.1). This is a classical result by de Concini and Procesi [deCPro76, Theorem 4.2].

2. The ideal $\mathcal{I}_k$ is the annihilator of a sign-twisted[1] tensor power $N_n^{\otimes(n-k-1)}$ of the natural representation $N_n = \mathbf{k}^n$ of S_n, where S_n acts diagonally on the tensor power (Theorem 2.8.3, which is stated in a slightly different form, applying the sign-twist to the ideal rather than the module). This is a recent result by Bowman, Doty and Martin [BoDoMa18, Theorem 7.4 (a)].

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[1]"Sign-twisting" means that the action of any permutation $w \in S_n$ is additionally scaled by the sign $(-1)^w$ of w.

Note once again that no requirements are made on **k** here.

When $n!$ is invertible in **k**, we can characterize the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$ in a more mainstream way (Theorem 2.9.1): The ideal $\mathcal{I}_k$ consists of those elements $\mathbf{a} \in \mathcal{A}$ that annihilate all Specht modules S^λ with $\ell(\lambda) > k$, whereas the ideal $\mathcal{J}_k$ consists of those elements $\mathbf{a} \in \mathcal{A}$ that annihilate all Specht modules S^λ with $\ell(\lambda) \leq k$. This fact is responsible for the appearance of $\mathcal{J}_k$ in recent work on quantum information theory [KlStVo25, Theorem 2.2]. Moreover, it can be used to give a new proof for the classical fact (Corollary 2.9.3) that the number of permutations $w \in S_n$ that avoid $12\cdots(k+1)$ (that is, have no increasing subsequence of length $k+1$) equals the number of pairs of standard tableaux of shape λ for partitions $\lambda \vdash n$ satisfying $\ell(\lambda) \leq k$.

We end each of the two sections with an open question. In Section 1, we propose an "abstract lift" of the span of the $\nabla_{B,A}$'s. This is a nonunital **k**-algebra with basis $(\Delta_{B,A})_{A,B \subseteq [n] \text{ of equal size}}$ consisting of "abstract nablas" $\Delta_{B,A}$, which multiply by the same rules as the $\nabla_{B,A}$'s (see Theorem 1.2.2), but do not satisfy any linear dependencies. This **k**-algebra has dimension $\binom{2n}{n}$, which is $n+1$ times the dimension of the actual span of the $\nabla_{B,A}$'s; but it appears to share some of the properties of the latter. We know (Theorem 1.6.1) that it is associative, and we conjecture that it is unital when $n!$ is invertible in **k** (certainly not for general **k**). Section 2 leads us to another open question (Question 2.9.5), about a basis of the quotient ring $\mathcal{A}/(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell))$, closely related to a recent conjecture by Donkin [Donkin24, Remark 2.4].

Acknowledgments

1. Rook sums in the symmetric group algebra

1.1. Definitions

Let n be a nonnegative integer. Let $[n] := \{1, 2, \ldots, n\}$.

Fix a commutative ring **k**. (All rings and algebras are understood to be associative and unital, unless declared otherwise.)

Let S_n be the n-th symmetric group, defined as the group of all $n!$ permutations of $[n]$. Let $\mathcal{A} := \mathbf{k}[S_n]$ be its group algebra over **k**.

The *antipode* of the group algebra $\mathcal{A}$ is the **k**-linear map $\mathcal{A} \to \mathcal{A}$ that sends each permutation $w \in S_n$ to w^{-1}. We will denote this map by S. It is well-known (see,

e.g., [Grinbe25, §3.11.4]) that S is a **k**-algebra anti-automorphism and an involution (i.e., satisfies $S \circ S = \mathrm{id}$).

For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) = B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w$$

of $\mathcal{A}$. We shall refer to these elements as *rectangular rook sums*.

For instance, for $n = 4$, we have

$$\nabla_{\{2,3\},\{1,4\}} = \sum_{\substack{w \in S_4; \\ w(\{1,4\}) = \{2,3\}}} w$$

$$= \mathrm{oln}\,(2143) + \mathrm{oln}\,(2413) + \mathrm{oln}\,(3142) + \mathrm{oln}\,(3412),$$

where $\mathrm{oln}\,(i_1 i_2 \ldots i_n)$ means the permutation in S_n with one-line notation $(i_1, i_2, \ldots, i_n)$. Moreover, again for $n = 4$, we have

$$\widetilde{\nabla}_{\{1,2,3\},\{1,4\}} = \sum_{\substack{w \in S_4; \\ w(\{1,4\}) \subseteq \{1,2,3\}}} w = \sum_{\substack{w \in S_4; \\ w(1) \neq 4 \text{ and } w(4) \neq 4}} w,$$

which is a sum of altogether 12 permutations.

The following proposition collects some easy properties of rectangular rook sums:

Proposition 1.1.1. Let A and B be two subsets of $[n]$. Then:

(a) We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.

(b) We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.

(c) We have $\widetilde{\nabla}_{B,A} = \sum_{\substack{U \subseteq B; \\ |U| = |A|}} \nabla_{U,A}$.

(d) We have $\nabla_{B,A} = \nabla_{[n] \setminus B, \, [n] \setminus A}$.

(e) If $|A| = |B|$, then $\nabla_{B,A} = \widetilde{\nabla}_{B,A}$.

(f) The antipode S satisfies $S\,(\nabla_{B,A}) = \nabla_{A,B}$.

(g) The antipode S satisfies $S\left(\widetilde{\nabla}_{B,A}\right) = \widetilde{\nabla}_{[n] \setminus A, \, [n] \setminus B}$.

(h) For any $u \in S_n$, we have

$$u \nabla_{B,A} = \nabla_{u(B),A} \qquad \text{and} \qquad u \widetilde{\nabla}_{B,A} = \widetilde{\nabla}_{u(B),A}.$$

(i) For any $u \in S_n$, we have

$$\nabla_{B,A} u = \nabla_{B, u^{-1}(A)} \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} u = \widetilde{\nabla}_{B, u^{-1}(A)}.$$

Proof sketch. **(a)** A permutation $w \in S_n$ cannot satisfy $w(A) = B$ if $|A| \neq |B|$.

(b) A permutation $w \in S_n$ cannot satisfy $w(A) \subseteq B$ if $|A| > |B|$ (since $|w(A)| = |A|$).

(c) A permutation $w \in S_n$ satisfies $w(A) \subseteq B$ if and only if it satisfies $w(A) = U$ for some subset $U \subseteq B$ satisfying $|U| = |A|$. Moreover, this subset U is uniquely determined (as $w(A)$).

(d) A permutation $w \in S_n$ satisfies $w(A) = B$ if and only if it satisfies $w([n] \setminus A) = [n] \setminus B$.

(e) If $|A| = |B|$, then a permutation $w \in S_n$ satisfies $w(A) = B$ if and only if it satisfies $w(A) \subseteq B$ (since $w(A)$ has the same size as A and therefore as B as well, and thus cannot be a proper subset of B).

(f) A permutation $w \in S_n$ satisfies $w(A) = B$ if and only if its inverse w^{-1} satisfies $w^{-1}(B) = A$.

(g) A permutation $w \in S_n$ satisfies $w(A) \subseteq B$ if and only if its inverse w^{-1} satisfies $w^{-1}([n] \setminus B) \subseteq [n] \setminus A$.

(h) Let $u \in S_n$. Then, the definition of $\widetilde{\nabla}_{B,A}$ yields

$$u\widetilde{\nabla}_{B,A} = u \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} w = \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} uw$$

$$= \sum_{\substack{w \in S_n; \\ (uw)(A) \subseteq u(B)}} uw \qquad \left(\begin{array}{c} \text{since the condition } "w(A) \subseteq B" \\ \text{is equivalent to } "(uw)(A) \subseteq u(B)" \\ \text{(because } u \text{ is a bijection)} \end{array} \right)$$

$$= \sum_{\substack{w \in S_n; \\ w(A) \subseteq u(B)}} w \qquad \left(\begin{array}{c} \text{here, we have substituted } w \text{ for } uw \\ \text{in the sum} \end{array} \right)$$

$$= \widetilde{\nabla}_{u(B),A} \qquad \left(\text{by the definition of } \widetilde{\nabla}_{u(B),A} \right).$$

Similarly, $u\nabla_{B,A} = \nabla_{u(B),A}$. This proves Proposition 1.1.1 **(h)**.

(i) Let $u \in S_n$. Then, the definition of $\widetilde{\nabla}_{B,A}$ yields

$$\widetilde{\nabla}_{B,A} u = \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} wu = \sum_{\substack{w \in S_n; \\ w(A) \subseteq B}} wu$$

$$= \sum_{\substack{w \in S_n; \\ (wu)\left(u^{-1}(A)\right) \subseteq B}} wu \qquad \left(\text{since } w(A) = (wu)\left(u^{-1}(A)\right)\right)$$

$$= \sum_{\substack{w \in S_n; \\ w\left(u^{-1}(A)\right) \subseteq B}} w \qquad \left(\begin{array}{c}\text{here, we have substituted } w \text{ for } wu \\ \text{in the sum}\end{array}\right)$$

$$= \widetilde{\nabla}_{B,u^{-1}(A)} \qquad \left(\text{by the definition of } \widetilde{\nabla}_{B,u^{-1}(A)}\right).$$

Similarly, $\nabla_{B,A} u = \nabla_{B,u^{-1}(A)}$. This proves Proposition 1.1.1 **(i)**. $\qquad\square$

Proposition 1.1.1 **(c)** shows that the elements $\nabla_{B,A}$ and $\widetilde{\nabla}_{B,A}$ have the same span as B and A range over the subsets of $[n]$, or even as B ranges over all subsets of $[n]$ while A is fixed. Later (in Corollary 2.4.2), we will learn more about this span, and in particular compute its dimension.

1.2. The product rule

Parts **(h)** and **(i)** of Proposition 1.1.1 show that the span of the elements $\nabla_{B,A}$ is an ideal of $\mathcal{A}$. Hence, this span is a nonunital **k**-subalgebra of $\mathcal{A}$. It has an explicit multiplication rule, which we shall state in three different forms. First, we define an important family of integers:

> **Definition 1.2.1.** For any two subsets B and C of $[n]$, we define the positive integer
> $$\omega_{B,C} := |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|! \in \mathbb{Z}.$$

> **Theorem 1.2.2.** Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then:
>
> **(a)** We have
> $$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w.$$
>
> **(b)** We have
> $$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

(c) We have

$$\nabla_{D,C}\nabla_{B,A} = \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V|-|B \cap C|} \binom{|V|}{|B \cap C|} \tilde{\nabla}_{D,V}.$$

Before we can prove this theorem, we shall show a few lemmas from enumerative combinatorics. The first lemma explains the appearance of the numbers $\omega_{B,C}$:

Lemma 1.2.3. Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Fix a permutation $w \in S_n$. Let Q_w denote the set of all pairs $(u, v) \in S_n \times S_n$ satisfying $u(C) = D$ and $v(A) = B$ and $uv = w$. Then:

(a) If $|w(A) \cap D| \neq |B \cap C|$, then $|Q_w| = 0$.

(b) If $|w(A) \cap D| = |B \cap C|$, then $|Q_w| = \omega_{B,C}$.

Proof. **(a)** Assume that $|w(A) \cap D| \neq |B \cap C|$.

Let $(u, v) \in Q_w$. Thus, $(u, v) \in S_n \times S_n$ and $u(C) = D$ and $v(A) = B$ and $uv = w$ (by the definition of Q_w). Now, from $w = uv$, we obtain $w(A) = (uv)(A) = u(v(A))$, so that

$$\underbrace{w(A)}_{=u(v(A))} \cap \underbrace{D}_{=u(C)} = u\left(\underbrace{v(A)}_{=B}\right) \cap u(C) = u(B) \cap u(C) = u(B \cap C)$$

(since u is a permutation). Thus, $|w(A) \cap D| = |u(B \cap C)| = |B \cap C|$ (again since u is a permutation). This contradicts $|w(A) \cap D| \neq |B \cap C|$.

Thus, we have found a contradiction for each $(u, v) \in Q_w$. Hence, there exists no $(u, v) \in Q_w$. Thus, $|Q_w| = 0$. This proves Lemma 1.2.3 **(a)**.

(b) Assume that $|w(A) \cap D| = |B \cap C|$.

Set $k := |B \cap C|$. Thus, by assumption, we have $|w(A) \cap D| = |B \cap C| = k$. Furthermore, set $p := |B \setminus C|$ and $q := |C \setminus B|$.

Since w is a permutation, we have $|w(A)| = |A| = |B|$. Thus,

$$|w(A) \setminus D| = \underbrace{|w(A)|}_{=|B|} - \underbrace{|w(A) \cap D|}_{=|B \cap C|} = |B| - |B \cap C| = |B \setminus C| = p$$

and

$$|D \setminus w(A)| = \underbrace{|D|}_{=|C|} - \underbrace{|w(A) \cap D|}_{=|B \cap C|} = |C| - |B \cap C| = |C \setminus B| = q.$$

Moreover, the set $[n] \setminus (B \cup C)$ consists of those elements of $[n]$ that belong to none of the three disjoint subsets $B \cap C$, $B \setminus C$ and $C \setminus B$. Thus,

$$|[n] \setminus (B \cup C)| = \underbrace{|[n]|}_{=n} - \underbrace{|B \cap C|}_{=k} - \underbrace{|B \setminus C|}_{=p} - \underbrace{|C \setminus B|}_{=q} = n - k - p - q.$$

Now, we want to compute $|Q_w|$. In other words, we want to count the elements of Q_w. These elements are the pairs $(u, v) \in S_n \times S_n$ satisfying $u(C) = D$ and $v(A) = B$ and $uv = w$. Clearly, such a pair (u, v) must satisfy $v = u^{-1}w$ (since $uv = w$), and thus is uniquely determined by its first entry u. Moreover, the requirement $v(A) = B$ on this pair is equivalent to $u(B) = w(A)$, because of the following chain of equivalences:

$$
\begin{aligned}
(v(A) = B) &\iff \left(\left(u^{-1}w \right) (A) = B \right) & \left(\text{since } v = u^{-1}w \right) \\
&\iff \left(u^{-1}(w(A)) = B \right) \\
&\iff (w(A) = u(B)) \iff (u(B) = w(A)).
\end{aligned}
$$

Hence, the elements (u, v) of Q_w are in one-to-one correspondence with the permutations $u \in S_n$ satisfying $u(C) = D$ and $u(B) = w(A)$. Let us call such permutations *nice*. Thus, a nice permutation $u \in S_n$ must send the subset C to D and send the subset B to $w(A)$. Hence, it must send the three subsets

$$
B \cap C, \qquad B \setminus C, \qquad C \setminus B
$$

of $[n]$ to the three subsets

$$
w(A) \cap D, \qquad w(A) \setminus D, \qquad D \setminus w(A),
$$

respectively. Moreover, any permutation $u \in S_n$ that does so is nice (since $B = (B \cap C) \cup (B \setminus C)$ and $C = (B \cap C) \cup (C \setminus B)$). Thus, we can construct a nice permutation $u \in S_n$ as follows:

1. Choose the values of u on the k elements of $B \cap C$ in such a way that these values all belong to $w(A) \cap D$ and are distinct. This can be done in $k!$ ways, since both $|B \cap C|$ and $|w(A) \cap D|$ equal k.

2. Choose the values of u on the p elements of $B \setminus C$ in such a way that these values all belong to $w(A) \setminus D$ and are distinct. This can be done in $p!$ ways, since both $|B \setminus C|$ and $|w(A) \setminus D|$ equal p.

3. Choose the values of u on the q elements of $C \setminus B$ in such a way that these values all belong to $D \setminus w(A)$ and are distinct. This can be done in $q!$ ways, since both $|C \setminus B|$ and $|D \setminus w(A)|$ equal q.

4. Choose the values of u on the remaining $n - k - p - q$ elements of $[n]$ in such a way that these values are distinct and have not been chosen yet. This can be done in $(n - k - p - q)!$ ways, since we have $n - k - p - q$ elements of $[n]$ left that have not been chosen yet.

This process can be done in $k! \cdot p! \cdot q! \cdot (n - k - p - q)!$ ways. Thus, the number of nice permutations $u \in S_n$ is $k! \cdot p! \cdot q! \cdot (n - k - p - q)!$. But we know that $|Q_w|$ is

the number of nice permutations $u \in S_n$ (since the elements (u, v) of Q_w are in one-to-one correspondence with the nice permutations $u \in S_n$). Hence, we conclude that

$$|Q_w| = \underbrace{k}_{=|B \cap C|}! \cdot \underbrace{p}_{=|B \setminus C|}! \cdot \underbrace{q}_{=|C \setminus B|}! \cdot \underbrace{(n - k - p - q)!}_{=|[n] \setminus (B \cup C)|}$$
$$= |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|!$$
$$= \omega_{B,C}.$$

This proves Lemma 1.2.3 **(b)**. □

The next lemma is a simple and elementary binomial identity:

Lemma 1.2.4. Let $n, k \in \mathbb{N}$. Then,
$$\sum_{r=0}^{n} (-1)^{r-k} \binom{n}{r} \binom{r}{k} = \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{else.} \end{cases}$$

Proof. It is well-known that each $m \in \mathbb{Z}$ satisfies

$$\sum_{i=0}^{m} (-1)^{i} \binom{m}{i} = \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{else.} \end{cases} \tag{1}$$

(Indeed, this is obvious for $m < 0$, since the left hand side is an empty sum in this case. In the case $m \geq 0$, it is proved in [Grinbe19, Proposition 1.3.28].)

For each $r \in \mathbb{N}$, we have

$$\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k} \tag{2}$$

(by the trinomial revision formula [Grinbe19, Proposition 1.3.35], applied to $a = r$ and $b = k$). Hence,

$$\sum_{r=0}^{n} (-1)^{r-k} \underbrace{\binom{n}{r} \binom{r}{k}}_{\substack{= \binom{n}{k} \binom{n-k}{r-k} \\ \text{(by (2))}}}$$
$$= \sum_{r=0}^{n} (-1)^{r-k} \binom{n}{k} \binom{n-k}{r-k} = \binom{n}{k} \sum_{r=0}^{n} (-1)^{r-k} \binom{n-k}{r-k}$$
$$= \binom{n}{k} \sum_{i=-k}^{n-k} (-1)^{i} \binom{n-k}{i} \tag{3}$$

(here, we have substituted i for $r - k$ in the sum). But the two sums

$$\sum_{i=-k}^{n-k} (-1)^i \binom{n-k}{i} \qquad \text{and} \qquad \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i}$$

differ only in their addends for $i < 0$ (if they differ at all), and all these addends are 0 (since $i < 0$ entails $\binom{n-k}{i} = 0$ and thus $(-1)^i \underbrace{\binom{n-k}{i}}_{=0} = 0$) and thus do not affect the sums. Hence, these two sums are equal. In other words,

$$
\begin{aligned}
\sum_{i=-k}^{n-k} (-1)^i \binom{n-k}{i} &= \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \\
&= \begin{cases} 1, & \text{if } n - k = 0; \\ 0, & \text{else} \end{cases} \qquad \text{(by (1), applied to } m = n - k\text{)} \\
&= \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

(since the equation $n - k = 0$ is equivalent to $n = k$). Substituting this into (3), we obtain

$$
\begin{aligned}
\sum_{r=0}^{n} (-1)^{r-k} \binom{n}{r} \binom{r}{k} = \binom{n}{k} \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{else} \end{cases} &= \begin{cases} \binom{n}{k} \cdot 1, & \text{if } n = k; \\ \binom{n}{k} \cdot 0, & \text{else} \end{cases} \\
&= \begin{cases} \binom{n}{k}, & \text{if } n = k; \\ 0, & \text{else} \end{cases} = \begin{cases} 1, & \text{if } n = k; \\ 0, & \text{else} \end{cases}
\end{aligned}
$$

(since $\binom{n}{k} = \binom{k}{k} = 1$ when $n = k$). This proves Lemma 1.2.4. $\qquad \square$

Lemma 1.2.5. Let $k \in \mathbb{N}$. Let Z be any finite set. Then,

$$\sum_{U \subseteq Z} (-1)^{|U|-k} \binom{|U|}{k} = \begin{cases} 1, & \text{if } |Z| = k; \\ 0, & \text{else.} \end{cases}$$

Proof. Any subset $U \subseteq Z$ has size $|U| \in \{0, 1, \ldots, |Z|\}$. Thus, we can break up the

sum on the left hand side as follows:

$$\sum_{U \subseteq Z} (-1)^{|U|-k} \binom{|U|}{k} = \sum_{r=0}^{|Z|} \underbrace{\sum_{\substack{U \subseteq Z; \\ |U|=r}} (-1)^{|U|-k} \binom{|U|}{k}}_{\substack{=(-1)^{r-k} \binom{r}{k} \\ \text{(since } |U|=r)}}$$

$$= \sum_{r=0}^{|Z|} \underbrace{\sum_{\substack{U \subseteq Z; \\ |U|=r}} (-1)^{r-k} \binom{r}{k}}_{\substack{=\binom{|Z|}{r} \cdot (-1)^{r-k} \binom{r}{k} \\ \text{(since there are } \binom{|Z|}{r} \text{ many} \\ r\text{-element subsets } U \text{ of } Z)}}$$

$$= \sum_{r=0}^{|Z|} \binom{|Z|}{r} \cdot (-1)^{r-k} \binom{r}{k} = \sum_{r=0}^{|Z|} (-1)^{r-k} \binom{|Z|}{r} \binom{r}{k}$$

$$= \begin{cases} 1, & \text{if } |Z| = k; \\ 0, & \text{else} \end{cases}$$

(by Lemma 1.2.4, applied to $n = |Z|$). This proves Lemma 1.2.5. $\qquad\square$

Proof of Theorem 1.2.2. **(a)** The definition of a rectangular rook sum shows that

$$\nabla_{D,C} = \sum_{\substack{u \in S_n; \\ u(C)=D}} u \qquad \text{and} \qquad \nabla_{B,A} = \sum_{\substack{v \in S_n; \\ v(A)=B}} v.$$

Multiplying these two equalities, we find

$$\nabla_{D,C} \nabla_{B,A} = \left(\sum_{\substack{u \in S_n; \\ u(C)=D}} u \right) \left(\sum_{\substack{v \in S_n; \\ v(A)=B}} v \right) = \sum_{\substack{(u,v) \in S_n \times S_n; \\ u(C)=D \text{ and } v(A)=B}} uv$$

$$= \sum_{w \in S_n} |Q_w| \, w, \tag{4}$$

where Q_w (for a given permutation $w \in S_n$) is defined as the set of all pairs $(u, v) \in$

$S_n \times S_n$ satisfying $u(C) = D$ and $v(A) = B$ and $uv = w$. Now, (4) becomes

$$\nabla_{D,C} \nabla_{B,A} = \sum_{w \in S_n} |Q_w| w$$

$$= \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} \underbrace{|Q_w|}_{\substack{= \omega_{B,C} \\ \text{(by Lemma 1.2.3 (b))}}} w + \sum_{\substack{w \in S_n; \\ |w(A) \cap D| \neq |B \cap C|}} \underbrace{|Q_w|}_{\substack{= 0 \\ \text{(by Lemma 1.2.3 (a))}}} w$$

$$= \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} \omega_{B,C} w = \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w.$$

This proves Theorem 1.2.2 **(a)**.

(b) We have

$$\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \underbrace{\nabla_{U,V}}_{\substack{= \sum\limits_{\substack{w \in S_n; \\ w(V) = U}} w \\ \text{(by the definition of } \nabla_{U,V})}}$$

$$= \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \sum_{\substack{w \in S_n; \\ w(V) = U}} w$$

$$= \sum_{w \in S_n} \left(\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|; \\ w(V) = U}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|} \right) w. \tag{5}$$

Now, let $w \in S_n$ be arbitrary. We shall simplify the sum

$$\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U| = |V|; \\ w(V) = U}} (-1)^{|U| - |B \cap C|} \binom{|U|}{|B \cap C|}.$$

Indeed, we observe that

- the condition "$|U| = |V|$" under the summation sign is redundant (since it follows from $w(V) = U$ because w is a permutation);

- the set V in this sum is uniquely determined by U via the condition "$w(V) = U$" (since w is a permutation), and thus can be simply replaced by $w^{-1}(U)$ instead of being summed over.

Thus, we can rewrite the sum as follows:

$$\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|; \\ w(V)=U}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

$$= \sum_{\substack{U \subseteq D; \\ w^{-1}(U) \subseteq A}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

$$= \sum_{\substack{U \subseteq D; \\ U \subseteq w(A)}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

$$\left(\begin{array}{c} \text{here, we rewrote the condition } ``w^{-1}(U) \subseteq A" \\ \text{under the summation sign as } ``U \subseteq w(A)" \end{array} \right)$$

$$= \sum_{U \subseteq w(A) \cap D} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

$$= \begin{cases} 1, & \text{if } |w(A) \cap D| = |B \cap C|; \\ 0, & \text{else} \end{cases} \tag{6}$$

(by Lemma 1.2.5, applied to $Z = w(A) \cap D$ and $k = |B \cap C|$).

Forget that we fixed w. We thus have proved (6) for each $w \in S_n$. Thus, (5) becomes

$$\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}$$

$$= \sum_{w \in S_n} \left[\underbrace{\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|; \\ w(V)=U}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}}_{\substack{= \begin{cases} 1, & \text{if } |w(A) \cap D| = |B \cap C|; \\ 0, & \text{else} \end{cases} \\ \text{(by (6))}}} \right] w$$

$$= \sum_{w \in S_n} \begin{cases} 1, & \text{if } |w(A) \cap D| = |B \cap C|; \\ 0, & \text{else} \end{cases} w$$

$$= \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w.$$

Multiplying this equality by $\omega_{B,C}$, we find

$$\omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \nabla_{U,V}$$

$$= \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A)\cap D|=|B\cap C|}} w = \nabla_{D,C} \nabla_{B,A} \qquad \text{(by Theorem 1.2.2 (a))}.$$

Thus, Theorem 1.2.2 **(b)** is proved.

(c) Theorem 1.2.2 **(b)** yields

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \nabla_{U,V}$$

$$= \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \nabla_{U,V}$$

$$\text{(due to the } |U| = |V| \text{ condition under the summation sign)}$$

$$= \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \underbrace{\sum_{\substack{U \subseteq D; \\ |U|=|V|}} \nabla_{U,V}}_{\substack{=\widetilde{\nabla}_{D,V} \\ \text{(by Proposition 1.1.1 (c))}}}$$

$$= \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \widetilde{\nabla}_{D,V}.$$

This proves Theorem 1.2.2 **(c)**. $\qquad\qquad\square$

1.3. The D-filtration

We shall next derive some nilpotency-type consequences from the multiplication rule.

For the rest of this section, we fix a subset D of $[n]$. We define[2]

$$\mathcal{F}_k := \text{span}\left\{ \widetilde{\nabla}_{D,C} \mid C \subseteq [n] \text{ with } |C| \le k \right\}$$

for each $k \in \mathbb{Z}$. Of course, $\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0$. It is easy to see that $\mathcal{F}_0$ is spanned by $\widetilde{\nabla}_{D,\varnothing} = \nabla_{\varnothing,\varnothing} = \sum_{w \in S_n} w$. (Note, however, that $\bigcup_{k \in \mathbb{Z}} \mathcal{F}_k$ is usually a proper subset of $\mathbf{k}[S_n]$, so that the $\mathcal{F}_k$ do not form a decreasing filtration of $\mathbf{k}[S_n]$.)

---

[2]Here and in the following, "span" always means "$\text{span}_{\mathbf{k}}$" (that is, a $\mathbf{k}$-linear span).

Definition 1.3.1. For any subset $C \subseteq [n]$ and any $k \in \mathbb{N}$, we define the integer

$$\delta_{D,C,k} := \sum_{\substack{B \subseteq D; \\ |B| = k}} \omega_{B,C} (-1)^{k - |B \cap C|} \binom{k}{|B \cap C|} \in \mathbb{Z}.$$

Now, we note the following:

Proposition 1.3.2. Let $C \subseteq [n]$ satisfy $|C| = |D|$. Let $k \in \mathbb{N}$. Then,

$$(\nabla_{D,C} - \delta_{D,C,k}) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

Proof. By the definition of $\mathcal{F}_k$, it suffices to show that

$$(\nabla_{D,C} - \delta_{D,C,k}) \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1} \tag{7}$$

for each $A \subseteq [n]$ with $|A| \leq k$.

To prove this, we fix $A \subseteq [n]$ with $|A| \leq k$. Then, Proposition 1.1.1 **(c)** yields

$$\widetilde{\nabla}_{D,A} = \sum_{\substack{U \subseteq D; \\ |U| = |A|}} \nabla_{U,A} = \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{B,A}. \tag{8}$$

Multiplying this equality by $\nabla_{D,C}$ from the left, we obtain

$$\nabla_{D,C} \widetilde{\nabla}_{D,A} = \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{D,C} \nabla_{B,A}. \tag{9}$$

However, for each subset $B \subseteq D$ satisfying $|B| = |A|$, we can use Theorem 1.2.2 **(c)** to obtain

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V| - |B \cap C|} \binom{|V|}{|B \cap C|} \widetilde{\nabla}_{D,V}$$

$$= \sum_{V \subseteq A} \omega_{B,C} (-1)^{|V| - |B \cap C|} \binom{|V|}{|B \cap C|} \widetilde{\nabla}_{D,V}$$

$$= \sum_{\substack{V \subseteq A; \\ V \neq A}} \omega_{B,C} (-1)^{|V| - |B \cap C|} \binom{|V|}{|B \cap C|} \underbrace{\widetilde{\nabla}_{D,V}}_{\substack{\in \mathcal{F}_{k-1} \\ \text{(since } V \subseteq A \text{ and } V \neq A \\ \text{entail } |V| < |A| \leq k \text{ and} \\ \text{thus } |V| \leq k-1)}}$$

$$+ \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \widetilde{\nabla}_{D,A}$$

$$\left(\begin{array}{c} \text{here, we have split off the} \\ \text{addend for } V = A \text{ from the sum} \end{array} \right)$$

$$\equiv \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \widetilde{\nabla}_{D,A} \mod \mathcal{F}_{k-1}. \tag{10}$$

Recall that $|A| \leq k$. Hence, we are in one of the following two cases:

Case 1: We have $|A| = k$.

Case 2: We have $|A| < k$.

Let us first consider Case 1. In this case, we have $|A| = k$. Hence, (9) becomes

$$
\begin{aligned}
\nabla_{D,C} \widetilde{\nabla}_{D,A} &= \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{D,C} \nabla_{B,A} \\
&\equiv \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \widetilde{\nabla}_{D,A} \qquad \text{(by (10))} \\
&= \underbrace{\sum_{\substack{B \subseteq D; \\ |B| = k}} \omega_{B,C} (-1)^{k - |B \cap C|} \binom{k}{|B \cap C|}}_{= \delta_{D,C,k}} \widetilde{\nabla}_{D,A} \qquad \text{(since } |A| = k) \\
&= \delta_{D,C,k} \widetilde{\nabla}_{D,A} \bmod \mathcal{F}_{k-1}.
\end{aligned}
$$

In other words, $\nabla_{D,C} \widetilde{\nabla}_{D,A} - \delta_{D,C,k} \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. In other words, $(\nabla_{D,C} - \delta_{D,C,k}) \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. Hence, (7) is proved in Case 1.

Let us now consider Case 2. In this case, we have $|A| < k$. Hence, $|A| \leq k - 1$, so that $\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. In other words, $\widetilde{\nabla}_{D,A} \equiv 0 \bmod \mathcal{F}_{k-1}$. Now, (9) becomes

$$
\begin{aligned}
\nabla_{D,C} \widetilde{\nabla}_{D,A} &= \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{D,C} \nabla_{B,A} \\
&\equiv \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \underbrace{\widetilde{\nabla}_{D,A}}_{\equiv 0 \bmod \mathcal{F}_{k-1}} \qquad \text{(by (10))} \\
&\equiv 0 \bmod \mathcal{F}_{k-1}.
\end{aligned}
$$

Hence,

$$
(\nabla_{D,C} - \delta_{D,C,k}) \widetilde{\nabla}_{D,A} = \underbrace{\nabla_{D,C} \widetilde{\nabla}_{D,A}}_{\equiv 0 \bmod \mathcal{F}_{k-1}} - \delta_{D,C,k} \underbrace{\widetilde{\nabla}_{D,A}}_{\equiv 0 \bmod \mathcal{F}_{k-1}} \equiv 0 - 0 = 0 \bmod \mathcal{F}_{k-1}.
$$

In other words, $(\nabla_{D,C} - \delta_{D,C,k}) \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. Hence, (7) is proved in Case 2.

We have now proved (7) in both Cases 1 and 2. Thus, (7) always holds, and Proposition 1.3.2 is proved. $\qquad \square$

Definition 1.3.3. Let $\alpha = (\alpha_C)_{C \subseteq [n]; \ |C| = |D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then, we set

$$
\nabla_{D,\alpha} := \sum_{\substack{C \subseteq [n]; \\ |C| = |D|}} \alpha_C \nabla_{D,C} \in \mathcal{A}.
$$

Furthermore, for each $k \in \mathbb{N}$, we set

$$\delta_{D,\alpha,k} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \delta_{D,C,k} \in \mathbf{k}.$$

Proposition 1.3.4. Let $\alpha = (\alpha_C)_{C \subseteq [n]; \ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Let $k \in \mathbb{N}$. Then,

$$(\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \, \mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

Proof. Proposition 1.3.2 yields $(\nabla_{D,C} - \delta_{D,C,k}) \, \mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ for each $C \subseteq [n]$ satisfying $|C| = |D|$. Multiply this relation by α_C and sum up over all $C \subseteq [n]$ satisfying $|C| = |D|$. The result is Proposition 1.3.4. $\qquad\square$

Proposition 1.3.5. Let $\alpha = (\alpha_C)_{C \subseteq [n]; \ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then, for each integer $m \geq -1$, we have

$$\left(\prod_{k=0}^{m} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \mathcal{F}_m = 0.$$

Proof. Induction on m. The *base case* is obvious, since $\mathcal{F}_{-1} = 0$. The *induction step* (from $m-1$ to m) uses Proposition 1.3.4 as follows:

$$\left(\prod_{k=0}^{m} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \mathcal{F}_m = \left(\prod_{k=0}^{m-1} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \underbrace{(\nabla_{D,\alpha} - \delta_{D,\alpha,m}) \, \mathcal{F}_m}_{\substack{\subseteq \mathcal{F}_{m-1} \\ \text{(by Proposition 1.3.4)}}}$$

$$\subseteq \left(\prod_{k=0}^{m-1} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \mathcal{F}_{m-1} = 0$$

and thus $\left(\prod_{k=0}^{m} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \mathcal{F}_m = 0$. Thus, Proposition 1.3.5 is proved. $\qquad\square$

1.4. The triangularity theorem

We can now state our main theorem (still using Definition 1.3.3):

Theorem 1.4.1. Let D be a subset of $[n]$. Let $\alpha = (\alpha_C)_{C \subseteq [n]; \ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then,

$$\left(\prod_{k=0}^{|D|} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \nabla_{D,\alpha} = 0.$$

Proof. For each subset C of $[n]$ satisfying $|C| = |D|$, we have $\nabla_{D,C} = \widetilde{\nabla}_{D,C}$ (by Proposition 1.1.1 **(e)**) and thus $\nabla_{D,C} = \widetilde{\nabla}_{D,C} \in \mathcal{F}_{|D|}$ (since $|C| = |D| \leq |D|$). Thus, $\nabla_{D,\alpha} \in \mathcal{F}_{|D|}$ as well (since $\nabla_{D,\alpha}$ is a **k**-linear combination of such $\nabla_{D,C}$'s). Hence,

$$\left(\prod_{k=0}^{|D|} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \nabla_{D,\alpha} = \left(\prod_{k=0}^{|D|} (\nabla_{D,\alpha} - \delta_{D,\alpha,k}) \right) \mathcal{F}_{|D|} = 0$$

(by Proposition 1.3.5, applied to $m = |D|$). This proves Theorem 1.4.1. $\qquad\square$

Using the antipode S of $\mathcal{A}$, we can obtain a reflected version of Theorem 1.4.1:

Theorem 1.4.2. Let D be a subset of $[n]$. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ be a family of scalars in **k** indexed by the $|D|$-element subsets of $[n]$. Set

$$\nabla_{\alpha,D} := \sum_{\substack{C \subseteq [n];\\ |C|=|D|}} \alpha_C \nabla_{C,D} \in \mathcal{A}.$$

Then,

$$\left(\prod_{k=0}^{|D|} (\nabla_{\alpha,D} - \delta_{D,\alpha,k}) \right) \nabla_{\alpha,D} = 0.$$

Proof. The antipode S is a **k**-algebra anti-homomorphism, and sends $\nabla_{D,\alpha}$ to $\nabla_{\alpha,D}$ (since it sends $\nabla_{D,C}$ to $\nabla_{C,D}$ for each C). Thus, Theorem 1.4.2 follows easily by applying the antipode to Theorem 1.4.1. (Note that we don't have to reverse the order of factors in the product, since all these factors commute with each other.) $\quad\square$

Corollary 1.4.3. Let B and D be two subsets of $[n]$. For each $k \in \mathbb{N}$, we set

$$\widetilde{\delta}_{D,B,k} := \sum_{\substack{C \subseteq B;\\ |C|=|D|}} \delta_{D,C,k} \in \mathbb{Z}.$$

Then,

$$\left(\prod_{k=0}^{|D|} \left(\widetilde{\nabla}_{B,D} - \widetilde{\delta}_{D,B,k} \right) \right) \widetilde{\nabla}_{B,D} = 0.$$

Proof. Define a family $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ of scalars in **k** by setting

$$\alpha_C = \begin{cases} 1, & \text{if } C \subseteq B; \\ 0, & \text{if } C \not\subseteq B \end{cases} \qquad \text{for each } C \subseteq [n].$$

Then, Proposition 1.1.1 **(c)** yields

$$\widetilde{\nabla}_{B,D} = \sum_{\substack{U \subseteq B;\\ |U|=|D|}} \nabla_{U,D} = \sum_{\substack{C \subseteq B;\\ |C|=|D|}} \nabla_{C,D} = \sum_{\substack{C \subseteq [n];\\ |C|=|D|}} \alpha_C \nabla_{C,D} = \nabla_{\alpha,D},$$

where $\nabla_{\alpha,D}$ is defined as in Theorem 1.4.2. Hence, Corollary 1.4.3 follows from Theorem 1.4.2, once we realize that the $\delta_{D,\alpha,k}$ from Theorem 1.4.2 is precisely the $\widetilde{\delta}_{D,B,k}$. $\qquad\square$

Corollary 1.4.3 shows that the element $\widetilde{\nabla}_{B,D}$ has a minimal polynomial that factors entirely into linear factors. Moreover, there are at most $|D| + 2$ factors, and one of them is X or else there are at most $|D| + 1$ of them.

> **Question 1.4.4.** Can we simplify the formula for $\widetilde{\delta}_{D,B,k}$?

1.5. A table of minimal polynomials

For any subsets A and B of $[n]$, we let $\kappa_{A,B}$ be the sum (in $\mathbb{Z}[S_n]$) of all permutations $w \in S_n$ that satisfy $w(A) \cap B = \varnothing$ (that is, $w(a) \notin B$ for all $a \in A$). Then, $\kappa_{A,B}$ is simply $\widetilde{\nabla}_{[n]\setminus B,\ A}$. Thus, Corollary 1.4.3 shows that the element $\kappa_{A,B}$ has a minimal polynomial that factors into at most $|A| + 2$ factors. Note that these factors will sometimes have multiplicities (e.g., the case of $n = 6$ and $a = 3$ and $b = 2$ and $c = 1$).

Let us collect a table of these minimal polynomials. We observe that the minimal polynomial of $\kappa_{B,A}$ depends only on the three numbers $a := |A|$, $b := |B|$ and $c := |A \cap B|$ (since any two pairs (A, B) that agree in these three numbers can be obtained from each other by the action of some permutation $\sigma \in S_n$, and therefore the corresponding elements $\kappa_{B,A}$ are conjugate to each other in $\mathcal{A}$). Hence, we can rename $\kappa_{B,A}$ as $\kappa_{a,b,c}$.

We also note that $\kappa_{a,b,c} = 0$ if $c > a$ or $b > a$ or $a + b > n$. Hence, we only need to consider the cases $a, b \in [0, n]$ and $a + b \le n$ and $c \in [0,\ \min\{a,b\}]$.

Moreover, $\kappa_{B,A}$ is the antipode of $\kappa_{A,B}$ (by Proposition 1.1.1 **(g)**), and the antipode preserves minimal polynomials. Thus, we only need to consider the case $a \le b$.

This being said, here is a table of minpols (= minimal polynomials) of $\kappa_{a,b,c}$'s produced by SageMath:

Let $n = 1$.
For $b = 0$, the minpol is $x - 1$.

Let $n = 2$.
For $b = 0$, the minpol is $(x - 2)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $x - 1$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 1)(x + 1)$.

Let $n = 3$.
For $b = 0$, the minpol is $(x - 6)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 4)(x - 1)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 4)x(x + 2)$.

For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 2)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 2)x(x + 1)$.

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Let $n = 4$.
For $b = 0$, the minpol is $(x - 24)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 18)(x - 2)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 18)x(x + 6)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 12)(x - 4)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 12)x(x + 4)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x - 6)x$.
For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x - 6)x(x + 2)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x - 4)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x - 4)(x + 2)x^2$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x - 4)x(x + 4)$.

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Let $n = 5$.
For $b = 0$, the minpol is $(x - 120)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 96)(x - 6)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 96)x(x + 24)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 72)(x - 12)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 72)x(x + 18)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x - 48)(x - 18)x$.
For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x - 48)x(x + 12)$.
For $a = 4$ and $b = 1$ and $c = 0$, the minpol is $(x - 24)x$.
For $a = 4$ and $b = 1$ and $c = 1$, the minpol is $(x - 24)x(x + 6)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x - 36)(x - 16)(x - 4)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x - 36)x(x + 4)$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x - 36)(x - 12)x(x + 24)$.
For $a = 3$ and $b = 2$ and $c = 0$, the minpol is $(x - 12)x$.
For $a = 3$ and $b = 2$ and $c = 1$, the minpol is $(x - 12)(x - 2)x(x + 4)$.
For $a = 3$ and $b = 2$ and $c = 2$, the minpol is $(x - 12)(x - 4)x(x + 8)$.

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Let $n = 6$.
For $b = 0$, the minpol is $(x - 720)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 600)(x - 24)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 600)x(x + 120)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 480)(x - 48)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 480)x(x + 96)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x - 360)(x - 72)x$.
For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x - 360)x(x + 72)$.
For $a = 4$ and $b = 1$ and $c = 0$, the minpol is $(x - 240)(x - 96)x$.

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For $a = 4$ and $b = 1$ and $c = 1$, the minpol is $(x - 240)x(x + 48)$.
For $a = 5$ and $b = 1$ and $c = 0$, the minpol is $(x - 120)x$.
For $a = 5$ and $b = 1$ and $c = 1$, the minpol is $(x - 120)x(x + 24)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x - 288)(x - 72)(x - 8)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x - 288)x(x + 12)(x + 36)$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x - 288)(x - 48)x(x + 144)$.
For $a = 3$ and $b = 2$ and $c = 0$, the minpol is $(x - 144)(x - 72)(x - 24)x$.
For $a = 3$ and $b = 2$ and $c = 1$, the minpol is $(x - 144)(x + 16)x^2$.
For $a = 3$ and $b = 2$ and $c = 2$, the minpol is $(x - 144)(x - 24)x(x + 72)$.
For $a = 4$ and $b = 2$ and $c = 0$, the minpol is $(x - 48)x$.
For $a = 4$ and $b = 2$ and $c = 1$, the minpol is $(x - 48)(x - 12)x(x + 12)$.
For $a = 4$ and $b = 2$ and $c = 2$, the minpol is $(x - 48)(x - 8)x(x + 24)$.
For $a = 3$ and $b = 3$ and $c = 0$, the minpol is $(x - 36)x$.
For $a = 3$ and $b = 3$ and $c = 1$, the minpol is $(x - 36)(x - 12)x(x + 4)(x + 12)$.
For $a = 3$ and $b = 3$ and $c = 2$, the minpol is $(x - 36)(x - 12)x(x + 4)(x + 12)$.
For $a = 3$ and $b = 3$ and $c = 3$, the minpol is $(x - 36)x(x + 36)$.

1.6. Aside: The abstract Nabla-algebra

We take a tangent and address a question that is suggested by Theorem 1.2.2 **(b)** but takes us out of the symmetric group algebra $\mathcal{A}$. Namely, let us see what happens if we take the multiplication rule in Theorem 1.2.2 **(b)** literally while forgetting what the $\nabla_{B,A}$ are.

Theorem 1.6.1. For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Thus, we have introduced altogether $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ symbols $\Delta_{B,A}$. Let $\mathcal{D}$ be the free **k**-module with basis $(\Delta_{B,A})_{A,B \subseteq [n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

$$\Delta_{D,C}\Delta_{B,A} := \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|} \Delta_{U,V}.$$

(Recall Definition 1.2.1, which defines the $\omega_{B,C}$ here.) Then, $\mathcal{D}$ becomes a nonunital **k**-algebra.

Proof omitted due to excessive ugliness.

Question 1.6.2. The above proof idea is clearly in bad taste. There should be a more conceptual proof that identifies $\mathcal{D}$ as some existing (nonunital) **k**-algebra (what nonunital **k**-algebra has dimension $\binom{2n}{n}$ over **k** ?) or at least with a

subquotient of a such.

Example 1.6.3. Let $n = 1$. Then, the **k**-module $\mathcal{D}$ in Theorem 1.6.1 has basis (u, v) with $u = \Delta_{\varnothing,\varnothing}$ and $v = \Delta_{\{1\},\{1\}}$. The multiplication on $\mathcal{D}$ defined ibidem is given by

$$uu = uv = vu = u, \qquad vv = v.$$

Thus, the nonunital **k**-algebra $\mathcal{D}$ is isomorphic to the **k**-algebra $\mathbf{k}[x] / (x^2 - x)$, and therefore has a unity (namely, v).

Example 1.6.4. Let $n = 2$. Then, the **k**-module $\mathcal{D}$ in Theorem 1.6.1 has basis $(u, v_{11}, v_{12}, v_{21}, v_{22}, w)$ with $u = \Delta_{\varnothing,\varnothing}$ and $v_{ij} = \Delta_{\{i\},\{j\}}$ and $w = \Delta_{[2],[2]}$. The multiplication on $\mathcal{D}$ defined ibidem is given by

$$uu = uw = wu = 2u, \qquad uv_{ij} = v_{ij}u = u,$$
$$v_{dc}v_{ba} = u - v_{da} \qquad \text{if } b \neq c;$$
$$v_{dc}v_{ba} = v_{da} \qquad \text{if } b = c,$$
$$v_{ij}w = v_{i1} + v_{i2}, \qquad wv_{ij} = v_{1j} + v_{2j},$$
$$ww = 2w.$$

This nonunital **k**-algebra $\mathcal{D}$ has a unity if and only if 2 is invertible in **k**. This unity is $\dfrac{1}{4}(v_{11} + v_{22} - v_{12} - v_{21} + 2w)$.

Example 1.6.5. Let $n = 3$. Then, the **k**-module $\mathcal{D}$ has a basis consisting of $\dbinom{6}{3} = 20$ vectors of the form $\Delta_{B,A}$ with $A, B \subseteq [3]$ satisfying $|A| = |B|$. Its multiplication turns it into a nonunital algebra. When 3! is invertible in **k**, this algebra has a unity, namely

$$\frac{1}{18}\sum_{i=1}^{3}\Delta_{\{i\},\{i\}} - \frac{1}{36}\sum_{i \neq j}\Delta_{\{i\},\{j\}} + \frac{1}{6}\sum_{i<j}\Delta_{\{i,j\},\{i,j\}} - \frac{1}{12}\sum_{i \neq j \neq k \neq i}\Delta_{\{i,j\},\{i,k\}} + \frac{1}{6}\Delta_{[3],[3]}.$$

Question 1.6.6. Does the **k**-algebra $\mathcal{D}$ in Theorem 1.6.1 have a unity if $n!$ is invertible in **k** ? (I suspect that the answer is "yes". This has been checked with SageMath for all $n \leq 5$.)

Question 1.6.7. What does the representation theory of $\mathcal{D}$ look like?

Using SageMath, we computed some data for small n and for $\mathbf{k} = \mathbb{Q}$:

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\dim \mathcal{D}$	6	20	70	252
$\dim Z(\mathcal{D})$	3	4	5	6
$\dim J(\mathcal{D})$	3	5	39	84
Cartan invs	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix}$	

(where $Z(\mathcal{D})$ and $J(\mathcal{D})$ denote the center and the Jacobson radical of $\mathcal{D}$, respectively, and where "Cartan invs" means the matrix of Cartan invariants). This shows that the algebra $\mathcal{D}$ for $n \geq 2$ is far from semisimple. For example, for $n = 2$, it has (over $\mathbb{Q}$) a 3-dimensional Jacobson radical spanned by the vectors $u - v_{12} - v_{21}$, $v_{12} - v_{21}$ and $v_{11} - v_{22}$. But $\mathcal{D}$ is not "too nilpotent" either; in particular, the semisimple quotient of $\mathcal{D}$ for $n = 3$ is not commutative.

Question 1.6.8. Does the center of $\mathcal{D}$ always have dimension $n + 1$ when $n!$ is invertible in $\mathbf{k}$?

2. Row-to-row sums in the symmetric group algebra

2.1. Definitions

As we recall, n is a nonnegative integer and $\mathbf{k}$ a commutative ring. We work in the group algebra $\mathcal{A} = \mathbf{k}[S_n]$ of the symmetric group S_n.

2.1.1. Row-to-row sums and the $\nabla_{\mathbf{B},\mathbf{A}}$

A *set decomposition* of a set U shall mean a tuple $(U_1, U_2, \ldots, U_k)$ of disjoint subsets of U such that $U_1 \cup U_2 \cup \cdots \cup U_k = U$. The subsets $U_1, U_2, \ldots, U_k$ are called the *blocks* of this set decomposition $(U_1, U_2, \ldots, U_k)$. The number k of these blocks is called the *length* of this set decomposition. The length of a set decomposition $\mathbf{U}$ is called $\ell(\mathbf{U})$.

A *set composition* of a set U shall mean a set decomposition of U whose blocks are all nonempty. Clearly, any set decomposition of U can be transformed into a set composition of U by removing all empty blocks.

For instance, $(\{1,3\}, \varnothing, \{2\})$ is a set decomposition of $[3]$, but not a set composition (due to the presence of $\varnothing$). It has length 3. Removing the block $\varnothing$ from it yields the set composition $(\{1,3\}, \{2\})$ of $[3]$, whose length is 2.

Let $\mathrm{SD}(n)$ denote the set of all set decompositions of $[n]$.

Let $\mathrm{SC}(n)$ denote the set of all set compositions of $[n]$. Clearly, $\mathrm{SC}(n) \subseteq \mathrm{SD}(n)$.

If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set decompositions of $[n]$ having the same length, then we define the element

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{of } \mathcal{A}. \tag{11}$$

This will be called a *row-to-row sum*. It has been denoted $(\mathbf{A} \to \mathbf{B})$ in Canfield's and Williamson's work [CanWil89], and also is the $q = 1$ particular case of the "Murphy element" x_{st} of the Hecke algebra studied in [Murphy92, §3] and [Murphy95, §4] (if we encode $\mathbf{A}$ and $\mathbf{B}$ as row-standard tableaux, not necessarily of partition shape).

For instance, if $n = 4$ and $\mathbf{A} = (\{3\}, \{1,2\}, \{4\})$ and $\mathbf{B} = (\{1\}, \{2,4\}, \{3\})$, then

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum_{\substack{w \in S_4; \\ w(\{3\}) = \{1\}; \\ w(\{1,2\}) = \{2,4\}; \\ w(\{4\}) = \{3\}}} w = \mathrm{oln}(2413) + \mathrm{oln}(4213),$$

where $\mathrm{oln}(i_1 i_2 \ldots i_n)$ means the permutation in S_n with one-line notation $(i_1, i_2, \ldots, i_n)$.

We observe some easy properties of row-to-row sums:

Proposition 2.1.1. Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ be two set decompositions of $[n]$ having the same length. Then:

(a) We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless each $i \in [k]$ satisfies $|A_i| = |B_i|$.

(b) The element $\nabla_{\mathbf{B},\mathbf{A}}$ does not change if we permute the blocks of $\mathbf{A}$ and the blocks of $\mathbf{B}$ using the same permutation. In other words, for any permutation $\sigma \in S_k$, we have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$, where $\mathbf{A}\sigma := \left(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(k)} \right)$ and $\mathbf{B}\sigma := \left(B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(k)} \right)$.

(c) The element $\nabla_{\mathbf{B},\mathbf{A}}$ does not change if we remove empty blocks from $\mathbf{A}$ and from $\mathbf{B}$, provided that these blocks are in the same positions in both $\mathbf{A}$ and $\mathbf{B}$.

(d) The antipode S of $\mathcal{A}$ satisfies $S\left(\nabla_{\mathbf{B},\mathbf{A}} \right) = \nabla_{\mathbf{A},\mathbf{B}}$.

Proof. **(a)** Assume that not every $i \in [k]$ satisfies $|A_i| = |B_i|$. Then, there exists no permutation $w \in S_n$ that satisfies ($w(A_i) = B_i$ for all i) (since the injectivity of such a permutation w would imply $|w(A_i)| = |A_i|$ and thus $|A_i| = |w(A_i)| = |B_i|$ because of $w(A_i) = B_i$). Hence, the sum $\sum\limits_{\substack{w \in S_n; \\ w(A_i)=B_i \text{ for all } i}} w$ in (11) is empty and thus equals 0. Therefore, (11) shows that $\nabla_{\mathbf{B},\mathbf{A}} = 0$. This proves Proposition 2.1.1 **(a)**.

(b) Let $\sigma \in S_k$. Then, (11) yields

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum\limits_{\substack{w \in S_n; \\ w(A_i)=B_i \text{ for all } i}} w \qquad \text{and} \tag{12}$$

$$\nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma} = \sum\limits_{\substack{w \in S_n; \\ w\left(A_{\sigma(i)}\right)=B_{\sigma(i)} \text{ for all } i}} w. \tag{13}$$

But σ is a permutation of $[k]$; thus the condition "$w(A_i) = B_i$ for all i" is equivalent to "$w\left(A_{\sigma(i)} \right) = B_{\sigma(i)}$ for all i". Hence, the right hand sides of (12) and (13) are equal. Thus, so are the left hand sides. In other words, $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$. This proves Proposition 2.1.1 **(b)**.

(c) Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$. Assume that both $\mathbf{A}$ and $\mathbf{B}$ have an empty block in the same position – i.e., there exists some $r \in [k]$ such that $A_r = \varnothing$ and $B_r = \varnothing$. Consider this r. Let $\mathbf{A}' := (A_1, A_2, \ldots, A_{r-1}, A_{r+1}, A_{r+2}, \ldots, A_k)$ and $\mathbf{B}' := (B_1, B_2, \ldots, B_{r-1}, B_{r+1}, B_{r+2}, \ldots, B_k)$ be the set decompositions obtained from $\mathbf{A}$ and $\mathbf{B}$ by removing the empty blocks A_r and B_r. We must show that $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}',\mathbf{A}'}$.

Essentially, this is obvious from the definition of $\nabla_{\mathbf{B},\mathbf{A}}$: The empty blocks A_r and B_r satisfy $w(A_r) = B_r$ for any permutation $w \in S_n$ (since $w(\varnothing) = \varnothing$). Hence, the condition "$w(A_i) = B_i$ for all i" in (11) is tautologically satisfied for $i = r$. Thus, we can replace "for all i" by "for all $i \neq r$" in (11) without changing the sum. But this gives us precisely $\nabla_{\mathbf{B}',\mathbf{A}'}$. Hence, $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}',\mathbf{A}'}$. This proves Proposition 2.1.1 **(c)**.

(d) Recall that S is a **k**-linear map sending each permutation $w \in S_n$ to w^{-1}. Hence, applying S to the equality (11), we obtain

$$S\left(\nabla_{\mathbf{B},\mathbf{A}}\right) = \sum_{\substack{w \in S_n; \\ w(A_i)=B_i \text{ for all } i}} w^{-1} = \sum_{\substack{w \in S_n; \\ w^{-1}(B_i)=A_i \text{ for all } i}} w^{-1}$$

(here, we have rewritten the condition "$w\left(A_i\right) = B_i$" in the form "$w^{-1}\left(B_i\right) = A_i$" (which is equivalent, since w is a permutation)). Thus,

$$S\left(\nabla_{\mathbf{B},\mathbf{A}}\right) = \sum_{\substack{w \in S_n; \\ w^{-1}(B_i)=A_i \text{ for all } i}} w^{-1} = \sum_{\substack{w \in S_n; \\ w(B_i)=A_i \text{ for all } i}} w$$

$$\left(\begin{array}{c} \text{here, we have substituted } w \text{ for } w^{-1} \text{ in the sum,} \\ \text{since the map } S_n \to S_n, \ w \mapsto w^{-1} \text{ is a bijection} \end{array}\right)$$

$$= \nabla_{\mathbf{A},\mathbf{B}} \qquad \text{(by the definition of } \nabla_{\mathbf{A},\mathbf{B}}\text{).}$$

This proves Proposition 2.1.1 **(d)**. $\qquad\square$

Moreover, these row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ generalize the rectangular rook sums $\nabla_{B,A}$ from Section 1:

Proposition 2.1.2. Let A and B be two subsets of $[n]$. Define the two set decompositions $\mathbf{A} := (A, \ [n] \setminus A)$ and $\mathbf{B} := (B, \ [n] \setminus B)$ of $[n]$. Then, $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{B,A}$.

Proof. By the definition of $\nabla_{\mathbf{B},\mathbf{A}}$, we have

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum_{\substack{w \in S_n; \\ w(A)=B; \\ w([n]\setminus A)=[n]\setminus B}} w. \tag{14}$$

However, the condition "$w\left([n] \setminus A\right) = [n] \setminus B$" under the sum here is redundant, since it follows from "$w\left(A\right) = B$" when w is a permutation. Thus, we can remove this condition. Hence, (14) simplifies to

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum_{\substack{w \in S_n; \\ w(A)=B}} w = \nabla_{B,A}$$

(by the definition of $\nabla_{B,A}$). This proves Proposition 2.1.2. $\qquad\square$

Remark 2.1.3. The row-to-row sums can also be rewritten using colorings instead of set (de)compositions. Namely, a *coloring* of $[n]$ means a map $f : [n] \to C$ to some set C. If $C = [k]$ for some $k \in \mathbb{N}$, then such a coloring f can be regarded as a set decomposition of $[n]$ of length k, where the i-th block is $f^{-1}\left(i\right)$ for each $i \in [k]$. The image $f\left(j\right)$ of an element $j \in [n]$ under a coloring $f : [n] \to C$ is called the *color* of j (under f). Now, the row-to-row sum $\nabla_{g,f}$ corresponding to two colorings f and g of $[n]$ is the sum of all permutations $w \in S_n$ that satisfy $g \circ w = f$. (This is a "preservation of colors" condition.)

Remark 2.1.4. Let $u \in S_n$ be any permutation. Let **A** be the set composition $(\{1\}, \{2\}, \ldots, \{n\})$ of $[n]$, and let **B** be the set composition $(\{u(1)\}, \{u(2)\}, \ldots, \{u(n)\})$ of $[n]$. Then, $\nabla_{\mathbf{B},\mathbf{A}} = u$. Thus, the row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ in general are not as special as their particular cases the rectangular rook sums $\nabla_{B,A}$. In particular, the minimal polynomials of general row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ cannot be factored into linear factors over $\mathbb{Z}$.

The symmetric group S_n acts on the set $\text{SD}(n) = \{\text{set decompositions of } [n]\}$ (from the left) by the rule

$$w(B_1, B_2, \ldots, B_k) = (w(B_1), w(B_2), \ldots, w(B_k))$$
$$\text{for all } w \in S_n \text{ and all } (B_1, B_2, \ldots, B_k) \in \text{SD}(n).$$

The subset $\text{SC}(n)$ of $\text{SD}(n)$ is preserved under this S_n-action; thus S_n acts on $\text{SC}(n)$ as well.

The action of S_n on $\text{SD}(n)$ we just defined allows us to rewrite the equality (11) as follows:

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w\mathbf{A} = \mathbf{B}}} w \tag{15}$$

for any two set decompositions $\mathbf{A}, \mathbf{B} \in \text{SD}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B})$. More importantly, the row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ transform in a very simple way under this action:

Proposition 2.1.5. Let $u, v \in S_n$ be any permutations. Let $\mathbf{A}, \mathbf{B} \in \text{SD}(n)$ be any two set decompositions satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B})$. Then,

$$u \nabla_{\mathbf{B},\mathbf{A}} v = \nabla_{u\mathbf{B}, v^{-1}\mathbf{A}}. \tag{16}$$

Proof. Set $k = \ell(\mathbf{A}) = \ell(\mathbf{B})$, and write the set decompositions **A** and **B** as $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$. Then, $u\mathbf{B} = (u(B_1), u(B_2), \ldots, u(B_k))$ and $v^{-1}\mathbf{A} = (v^{-1}(A_1), v^{-1}(A_2), \ldots, v^{-1}(A_k))$. Hence, the definition of row-to-

row sums yields

$$
\nabla_{u\mathbf{B}, v^{-1}\mathbf{A}} = \sum_{\substack{w \in S_n; \\ w\left(v^{-1}(A_i)\right) = u(B_i) \text{ for all } i}} w
$$

$$
= \sum_{\substack{w \in S_n; \\ (uwv)\left(v^{-1}(A_i)\right) = u(B_i) \text{ for all } i}} uwv \qquad \left(\begin{array}{c} \text{here, we have substituted } uwv \\ \text{for } w \text{ in the sum} \end{array} \right)
$$

$$
= \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} uwv \qquad \left(\begin{array}{c} \text{since the condition} \\ \text{`` } (uwv)\left(v^{-1}(A_i)\right) = u(B_i) \text{ ''} \\ \text{is equivalent to ``} w(A_i) = B_i \text{''} \end{array} \right)
$$

$$
= u \underbrace{\sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w}_{=\nabla_{\mathbf{B},\mathbf{A}}} v = u \nabla_{\mathbf{B},\mathbf{A}} v.
$$

This proves Proposition 2.1.5. $\qquad \square$

2.1.2. Antisymmetrizers and the ∇_U^-

The sign of a permutation $w \in S_n$ shall be denoted by $(-1)^w$.

For each subset U of $[n]$, we define the element

$$
\nabla_U^- := \sum_{\substack{w \in S_n; \\ w(i) = i \text{ for all } i \in [n] \setminus U}} (-1)^w \, w \in \mathcal{A}. \tag{17}
$$

This is called the *antisymmetrizer* of U (aka the *U-sign-integral* in the language of [Grinbe25, Definition 3.7.1]). Note that $\nabla_U^- = 1$ if $|U| \leq 1$. Another way to rephrase the definition of ∇_U^- is

$$
\nabla_U^- := \sum_{w \in S_U} (-1)^w \, w \in \mathcal{A}, \tag{18}
$$

where S_U denotes the symmetric group on the set U (embedded into S_n in the default way: each permutation $w \in S_U$ is extended to a permutation of $[n]$ by letting it fix all elements of $[n] \setminus U$).

The antisymmetrizers ∇_U^- interact nicely with the permutations $v \in S_n$:

Proposition 2.1.6. Let $v \in S_n$ be a permutation. Let U be any subset of $[n]$. Then,

$$
v \nabla_U^- = \nabla_{v(U)}^- v. \tag{19}
$$

Proof. The definition of $\nabla_{v(U)}^-$ yields

$$
\nabla_{v(U)}^- = \sum_{\substack{w \in S_n; \\ w(i) = i \text{ for all } i \in [n] \setminus v(U)}} (-1)^w \, w. \tag{20}
$$

The permutation $v \in S_n$ is a bijection from $[n]$ to $[n]$, and thus induces a group isomorphism

$$S_{[n]} \to S_{[n]},$$
$$w \mapsto vwv^{-1}.$$

Of course, this isomorphism is just conjugation by v in the group $S_{[n]} = S_n$. But since it is induced by the bijection $v : [n] \to [n]$, we immediately see from functoriality that it sends the permutations $w \in S_n$ satisfying "$w(i) = i$ for all $i \in [n] \setminus U$" to the permutations $w \in S_n$ satisfying "$w(i) = i$ for all $i \in [n] \setminus v(U)$". Thus, the latter permutations are the images of the former permutations under the map $w \mapsto vwv^{-1}$. Hence, we can substitute vwv^{-1} for w in the sum on the right hand side of (20). We thus obtain

$$\sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus v(U)}} (-1)^w w = \sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus U}} \underbrace{(-1)^{vwv^{-1}}}_{\substack{=(-1)^w \\ \text{(since conjugation} \\ \text{preserves the sign} \\ \text{of a permutation)}}} vwv^{-1}$$

$$= \sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus U}} (-1)^w vwv^{-1}$$

$$= v \underbrace{\sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus U}} (-1)^w w v^{-1}}_{\substack{=\nabla_U^- \\ \text{(by the definition of } \nabla_U^-)}} v^{-1} = v\nabla_U^- v^{-1}.$$

In view of (20), this rewrites as $\nabla_{v(U)}^- = v\nabla_U^- v^{-1}$. In other words, $\nabla_{v(U)}^- v = v\nabla_U^-$. This proves Proposition 2.1.6. $\qquad\square$

The following fact will also be useful:

Proposition 2.1.7. Let U and V be two subsets of $[n]$ such that $V \subseteq U$. Then, $\nabla_U^- \mathcal{A} \subseteq \nabla_V^- \mathcal{A}$.

Proof. For any two distinct elements $p, q \in [n]$, let $t_{p,q}$ be the transposition in S_n that swaps p with q. Then, a well-known formula ([Grinbe25, Lemma 3.11.6]) says that every $X \subseteq [n]$ and every $x \in X$ satisfy

$$\nabla_X^- = \nabla_{X \setminus \{x\}}^- \left(1 - \sum_{y \in X \setminus \{x\}} t_{y,x}\right).$$

Hence, every $X \subseteq [n]$ and every $x \in X$ satisfy

$$\nabla_X^- \mathcal{A} = \nabla_{X \setminus \{x\}}^- \underbrace{\left(1 - \sum_{y \in X \setminus \{x\}} t_{y,x} \right) \mathcal{A}}_{\subseteq \mathcal{A}}$$

$$\subseteq \nabla_{X \setminus \{x\}}^- \mathcal{A}. \tag{21}$$

Now, $V \subseteq U$ shows that the set V can be obtained from U by removing some elements (possibly none, if $V = U$). In other words, $V = U \setminus \{u_1, u_2, \ldots, u_m\}$ for some distinct elements $u_1, u_2, \ldots, u_m \in U$. Using these elements, we have

$$\nabla_U^- \mathcal{A} \subseteq \nabla_{U \setminus \{u_1\}}^- \mathcal{A} \qquad \text{(by (21))}$$
$$\subseteq \nabla_{(U \setminus \{u_1\}) \setminus \{u_2\}}^- \mathcal{A} \qquad \text{(by (21))}$$
$$\subseteq \nabla_{((U \setminus \{u_1\}) \setminus \{u_2\}) \setminus \{u_3\}}^- \mathcal{A} \qquad \text{(by (21))}$$
$$\subseteq \cdots$$
$$\subseteq \nabla_{(((U \setminus \{u_1\}) \setminus \{u_2\}) \setminus \cdots) \setminus \{u_m\}}^- \mathcal{A} \qquad \text{(by (21))}$$
$$= \nabla_V^- \mathcal{A}$$

(since $(((U \setminus \{u_1\}) \setminus \{u_2\}) \setminus \cdots) \setminus \{u_m\} = U \setminus \{u_1, u_2, \ldots, u_m\} = V$). This proves Proposition 2.1.7. $\qquad \square$

2.2. The two ideals

This all was easy. Let us now move towards deeper waters. Recall that the notation "span" always means a **k**-linear span.

Definition 2.2.1. Let $k \in \mathbb{N}$. We define two **k**-submodules $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ by

$$\mathcal{I}_k := \text{span} \left\{ \nabla_{\mathbf{B}, \mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \text{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\}$$

and

$$\mathcal{J}_k := \mathcal{A} \cdot \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}.$$

Note that the set $\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\}$ is empty when $k \geq n$, since no subsets of $[n]$ have size larger than n. The span of an empty set is the zero submodule $\{0\}$.

Proposition 2.2.2. Let $k \in \mathbb{N}$. Then:

(a) Both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$. ("Ideal" always means "two-sided ideal".)

(b) We have

$$\mathcal{J}_k = \mathcal{A} \cdot \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\}$$
$$= \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}.$$

(c) The antipode S satisfies $S(\mathcal{I}_k) = \mathcal{I}_k$ and $S(\mathcal{J}_k) = \mathcal{J}_k$.

(d) We have

$$\mathcal{I}_k = \text{span} \left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \text{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\}.$$

(e) We have

$$\mathcal{J}_k = \mathcal{A} \cdot \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}$$
$$= \mathcal{A} \cdot \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\}$$
$$= \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}.$$

(f) If X is any subset of $[n]$ having size $k+1$, then

$$\mathcal{J}_k = \mathcal{A} \cdot \nabla_X^- \cdot \mathcal{A}.$$

(g) We have

$$\mathcal{I}_k = \text{span} \left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \text{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) = k \right\}.$$

Proof. **(a)** Clearly, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ (since $\mathcal{J}_k$ has the form $\mathcal{J}_k = \mathcal{A}\mathcal{X}\mathcal{A}$ for some **k**-submodule $\mathcal{X} \subseteq \mathcal{A}$). It remains to show that so is $\mathcal{I}_k$.

But
$$\mathcal{I}_k = \text{span} \left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \text{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\}.$$

Hence, $\mathcal{I}_k$ is a **k**-submodule of $\mathcal{A}$. Moreover, (16) shows that for any two set compositions $\mathbf{A}, \mathbf{B} \in \text{SC}(n)$ with $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ and any two permutations $u, v \in S_n$, we have $u \nabla_{\mathbf{B},\mathbf{A}} v = \nabla_{u\mathbf{B}, v^{-1}\mathbf{A}} \in \mathcal{I}_k$ (since $v^{-1}\mathbf{A}$ and $u\mathbf{B}$ are again two set compositions in $\text{SC}(n)$ and satisfy $\ell(v^{-1}\mathbf{A}) = \ell(u\mathbf{B}) = \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$). By linearity, this shows that $\mathcal{A}\mathcal{I}_k\mathcal{A} \subseteq \mathcal{I}_k$ (since $\mathcal{A}$ is spanned by the permutations $w \in S_n$, whereas $\mathcal{I}_k$ is spanned by the $\nabla_{\mathbf{B},\mathbf{A}}$ for $\mathbf{A}, \mathbf{B} \in \text{SC}(n)$ with $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$). In other words, $\mathcal{I}_k$ is an ideal of $\mathcal{A}$. Thus, the proof of part **(a)** is complete.

(b) Since $\mathcal{A} = \operatorname{span} S_n$, we have

$$
\begin{aligned}
&\mathcal{A} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \operatorname{span} S_n \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \operatorname{span} \left\{ v \nabla_U^- \mid v \in S_n, \text{ while } U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \operatorname{span} \left\{ \nabla_{v(U)}^- v \mid v \in S_n, \text{ while } U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&\qquad\qquad \text{(by Proposition 2.1.6)} \\
&= \operatorname{span} \left\{ \nabla_U^- v \mid v \in S_n, \text{ while } U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } U \text{ for } v(U), \text{ since } v \in S_n \\ \text{permutes the subsets of } [n] \text{ having size } k+1 \end{array} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A} \\
&= \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \operatorname{span} S_n \\
&= \operatorname{span} \left\{ \nabla_U^- v \mid v \in S_n, \text{ while } U \text{ is a subset of } [n] \text{ having size } k+1 \right\}.
\end{aligned}
$$

The right hand sides of these two equalities are equal; hence, so are their left hand sides. In other words,

$$
\begin{aligned}
&\mathcal{A} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}.
\end{aligned}
$$

Now, the definition of $\mathcal{J}_k$ yields

$$
\begin{aligned}
\mathcal{J}_k &= \mathcal{A} \cdot \underbrace{\operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}}_{\substack{=\mathcal{A} \cdot \operatorname{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\ \text{(by the preceding sentence)}}} \\
&= \underbrace{\mathcal{A} \cdot \mathcal{A}}_{=\mathcal{A}} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \mathcal{A} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \\
&= \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}.
\end{aligned}
$$

Thus, part **(b)** is proved.

(c) The equality $S(\mathcal{I}_k) = \mathcal{I}_k$ follows from Proposition 2.1.1 **(d)**. It remains to prove $S(\mathcal{J}_k) = \mathcal{J}_k$.

Let $\mathcal{X}_k = \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\}$. Then, the definition of $\mathcal{J}_k$ rewrites as $\mathcal{J}_k = \mathcal{A} \mathcal{X}_k \mathcal{A}$. But we have $S\left(\nabla_U^-\right) = \nabla_U^-$ for each $U \subseteq [n]$ (see, e.g., [Grinbe25, Example 3.11.13 **(c)**]). Since the map S is **k**-linear, we thus conclude that $S(\mathcal{X}_k) = \mathcal{X}_k$ (since $\mathcal{X}_k = \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\}$).

Now, from $\mathcal{J}_k = \mathcal{A}\mathcal{X}_k\mathcal{A}$, we obtain

$$S(\mathcal{J}_k) = S(\mathcal{A}\mathcal{X}_k\mathcal{A}) = \underbrace{S(\mathcal{A})}_{=\mathcal{A}}\underbrace{S(\mathcal{X}_k)}_{=\mathcal{X}_k}\underbrace{S(\mathcal{A})}_{=\mathcal{A}} \qquad \left(\begin{array}{c} \text{since } S \text{ is a } \mathbf{k}\text{-algebra} \\ \text{anti-automorphism} \end{array} \right)$$

$$= \mathcal{A}\mathcal{X}_k\mathcal{A} = \mathcal{J}_k.$$

This completes the proof of part **(c)**.

(d) Any set composition of $[n]$ is a set decomposition of $[n]$. In other words, $\mathrm{SC}(n) \subseteq \mathrm{SD}(n)$. Thus,

$$\mathcal{I}_k \subseteq \mathrm{span}\left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A},\mathbf{B} \in \mathrm{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \le k \right\}.$$

It remains to prove the converse inclusion. To this purpose, we must show that if $\mathbf{A}$ and $\mathbf{B}$ are set decompositions of $[n]$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \le k$, then the row-to-row sum $\nabla_{\mathbf{B},\mathbf{A}}$ belongs to $\mathcal{I}_k$. So let us show this.

Let $\mathbf{A}$ and $\mathbf{B}$ be set decompositions of $[n]$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \le k$. We must prove that the row-to-row sum $\nabla_{\mathbf{B},\mathbf{A}}$ belongs to $\mathcal{I}_k$.

If any of the blocks of $\mathbf{A}$ is empty while the corresponding block of $\mathbf{B}$ is not, then this is clear, since Proposition 2.1.1 **(a)** yields $\nabla_{\mathbf{B},\mathbf{A}} = 0 \in \mathcal{I}_k$. A similar argument applies if any of the blocks of $\mathbf{B}$ is empty while the corresponding block of $\mathbf{A}$ is not. In the remaining case, the empty blocks of $\mathbf{A}$ appear at the same positions as the empty blocks of $\mathbf{B}$. Removing all these empty blocks from both $\mathbf{A}$ and $\mathbf{B}$, we obtain two set compositions $\mathbf{A}', \mathbf{B}'$ of $[n]$ satisfying $\ell(\mathbf{A}') = \ell(\mathbf{B}') \le \ell(\mathbf{A}) = \ell(\mathbf{B})$ and $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}',\mathbf{A}'}$ (by Proposition 2.1.1 **(c)**). But the definition of $\mathcal{I}_k$ yields $\nabla_{\mathbf{B}',\mathbf{A}'} \in \mathcal{I}_k$ (since $\mathbf{A}', \mathbf{B}' \in \mathrm{SC}(n)$ and $\ell(\mathbf{A}') = \ell(\mathbf{B}') \le \ell(\mathbf{A}) = \ell(\mathbf{B}) \le k$). Thus, $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}',\mathbf{A}'} \in \mathcal{I}_k$. This completes the proof of part **(d)**.

(e) The definition of $\mathcal{J}_k$ yields

$$\mathcal{J}_k = \mathcal{A} \cdot \mathrm{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}$$
$$\subseteq \mathcal{A} \cdot \mathrm{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}$$

(since any subset of size $k+1$ has size $> k$). Let us now prove the converse inclusion. Since $\mathcal{J}_k$ is an ideal of $\mathcal{A}$, it suffices to show that $\nabla_U^- \in \mathcal{J}_k$ whenever U is a subset of $[n]$ having size $> k$. So let U be a subset of $[n]$ having size $> k$. Then, U has size $\ge k+1$. Hence, U has a subset V of size $k+1$. Consider this V. Proposition 2.1.7 yields $\nabla_U^- \mathcal{A} \subseteq \nabla_V^- \mathcal{A}$. But the definition of $\mathcal{J}_k$ shows that $\nabla_V^- \in \mathcal{J}_k$ (since V is a subset of $[n]$ having size $k+1$), and thus we have $\nabla_V^- \mathcal{A} \subseteq \mathcal{J}_k\mathcal{A} \subseteq \mathcal{J}_k$ (since $\mathcal{J}_k$ is an ideal of $\mathcal{A}$). Hence, $\nabla_U^- = \nabla_U^- \cdot \underbrace{1}_{\in \mathcal{A}} \in \nabla_U^- \mathcal{A} \subseteq \nabla_V^- \mathcal{A} \subseteq \mathcal{J}_k$. So we have shown that $\nabla_U^- \in \mathcal{J}_k$ whenever U is a subset of $[n]$ having size $> k$. This proves

$$\mathcal{A} \cdot \mathrm{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A} \subseteq \mathcal{J}_k$$

(since $\mathcal{J}_k$ is an ideal of $\mathcal{A}$). Combining this with the inclusion

$$\mathcal{J}_k \subseteq \mathcal{A} \cdot \mathrm{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}$$

(which we have already proved), we obtain

$$\mathcal{J}_k = \mathcal{A} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}.$$

Similarly, we can show

$$\mathcal{J}_k = \mathcal{A} \cdot \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\}$$

and

$$\mathcal{J}_k = \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\} \cdot \mathcal{A}$$

(using Proposition 2.2.2 **(b)** as a starting point). Thus, part **(e)** is proved.

(f) Let X be any subset of $[n]$ having size $k + 1$. Then, from $\mathcal{A} = \operatorname{span} S_n$, we obtain

$$\mathcal{A} \cdot \nabla_X^- = \operatorname{span} \left\{ v \nabla_X^- \mid v \in S_n \right\} = \operatorname{span} \left\{ \nabla_{v(X)}^- v \mid v \in S_n \right\}$$

(since Proposition 2.1.6 yields $v \nabla_X^- = \nabla_{v(X)}^- v$). Hence, again using $\mathcal{A} = \operatorname{span} S_n$, we obtain

$$\underbrace{\mathcal{A} \cdot \nabla_X^-}_{= \operatorname{span} \left\{ \nabla_{v(X)}^- v \mid v \in S_n \right\}} \cdot \underbrace{\mathcal{A}}_{= \operatorname{span} S_n}$$

$$= \operatorname{span} \left\{ \nabla_{v(X)}^- v \mid v \in S_n \right\} \cdot \operatorname{span} S_n$$

$$= \operatorname{span} \left\{ \nabla_{v(X)}^- vw \mid v \in S_n \text{ and } w \in S_n \right\}$$

$$= \operatorname{span} \left\{ \nabla_{v(X)}^- u \mid v \in S_n \text{ and } u \in S_n \right\}$$

$$\qquad \text{(here, we have substituted } u \text{ for } vw\text{)}$$

$$= \operatorname{span} \left\{ \nabla_U^- u \mid U \text{ is a subset of } [n] \text{ having size } k + 1, \text{ and } u \in S_n \right\}$$

$$\left(\begin{array}{c} \text{here, we have substituted } U \text{ for } v(X), \\ \text{since each subset } U \text{ of } [n] \text{ having size } k + 1 \\ \text{can be written as } v(X) \text{ for some } v \in S_n \end{array} \right)$$

$$= \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k + 1 \right\} \cdot \underbrace{\operatorname{span} S_n}_{= \mathcal{A}}$$

$$= \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k + 1 \right\} \cdot \mathcal{A}$$

$$= \mathcal{J}_k \qquad \text{(by part **(b)**)} .$$

This proves part **(f)**.

(g) This will follow from part **(d)**, once we have proved the equality

$$\left\{ \nabla_{\mathbf{B}, \mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\}$$
$$= \left\{ \nabla_{\mathbf{B}, \mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) = k \right\} .$$

So let us prove this equality. The right hand side here is clearly a subset of the left hand side. It remains to show the reverse inclusion (i.e., that the left hand side is a subset of the right hand side). In other words, it remains to check that each $\nabla_{\mathbf{B},\mathbf{A}}$ with $\mathbf{A}, \mathbf{B} \in \mathrm{SD}\,(n)$ satisfying $\ell\,(\mathbf{A}) = \ell\,(\mathbf{B}) \leq k$ can be rewritten in the form $\nabla_{\mathbf{B}',\mathbf{A}'}$ for some $\mathbf{A}', \mathbf{B}' \in \mathrm{SD}\,(n)$ satisfying $\ell\,(\mathbf{A}') = \ell\,(\mathbf{B}') = k$. But this is easy: Set $m := \ell\,(\mathbf{A}) = \ell\,(\mathbf{B}) \leq k$, and write the set decompositions $\mathbf{A}$ and $\mathbf{B}$ in the form $\mathbf{A} = (A_1, A_2, \ldots, A_m)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_m)$; then set

$$\mathbf{A}' := \left(A_1, A_2, \ldots, A_m, \underbrace{\varnothing, \varnothing, \ldots, \varnothing}_{k-m \text{ empty sets}} \right) \qquad \text{and}$$

$$\mathbf{B}' := \left(B_1, B_2, \ldots, B_m, \underbrace{\varnothing, \varnothing, \ldots, \varnothing}_{k-m \text{ empty sets}} \right)$$

(this is allowed since $m \leq k$). Then, $\mathbf{A}'$ and $\mathbf{B}'$ are set decompositions in $\mathrm{SD}\,(n)$ satisfying $\ell\,(\mathbf{A}') = \ell\,(\mathbf{B}') = k$ and $\nabla_{\mathbf{B}',\mathbf{A}'} = \nabla_{\mathbf{B},\mathbf{A}}$ (the latter follows from Proposition 2.1.1 **(c)**, since $\mathbf{A}$ and $\mathbf{B}$ can be obtained from $\mathbf{A}'$ and $\mathbf{B}'$ by removing the $k - m$ empty blocks at the end). Thus, the proof of part **(g)** is complete. $\qquad\square$

The ideals $\mathcal{J}_k$ have been studied several times. In particular, the ideal $\mathcal{J}_k$ is the kernel of the map in [deCPro76, Theorem 4.2] (where our k and n have been renamed as n and m). Also, the ideal $\mathcal{J}_2$ has recently appeared in quantum information theory as the ideal $\widetilde{\mathcal{I}}_n^{\mathrm{swap}}$ in [BCEHK23, Lemma 3.4].

2.3. Annihilators and the bilinear form

If $\mathcal{B}$ is any subset of $\mathcal{A}$, then we define the two subsets

$$\mathrm{LAnn}\,\mathcal{B} := \{a \in \mathcal{A} \mid ab = 0 \text{ for all } b \in \mathcal{B}\} \qquad \text{and}$$
$$\mathrm{RAnn}\,\mathcal{B} := \{a \in \mathcal{A} \mid ba = 0 \text{ for all } b \in \mathcal{B}\}$$

of $\mathcal{A}$. We call them the *left annihilator* and the *right annihilator* of $\mathcal{B}$, respectively. These annihilators $\mathrm{LAnn}\,\mathcal{B}$ and $\mathrm{RAnn}\,\mathcal{B}$ are always **k**-submodules of $\mathcal{A}$, even when $\mathcal{B}$ is not.

Moreover, we define the **k**-bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathbf{k},$$

which sends the pair (u, v) to $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$ for any two permutations $u, v \in S_n$.

This is the standard nondegenerate symmetric bilinear form on $\mathcal{A} = \mathbf{k}\,[S_n]$ known from representation theory. We shall refer to this form as the *dot product*.

If $\mathcal{B}$ is any subset of $\mathcal{A}$, then we define the subset

$$\mathcal{B}^{\perp} := \{a \in \mathcal{A} \mid \langle a, b \rangle = 0 \text{ for all } b \in \mathcal{B}\}$$

of $\mathcal{A}$. This is called the *orthogonal complement* of $\mathcal{B}$ in $\mathcal{A}$. Note that it does not change if we replace $\langle a, b \rangle$ by $\langle b, a \rangle$ in its definition, since the form $\langle \cdot, \cdot \rangle$ is symmetric. Note also that $\mathcal{B}^{\perp}$ is always a **k**-submodule of $\mathcal{A}$, even when $\mathcal{B}$ is not.

Definition 2.3.1. Let $k \in \mathbb{N}$.

(a) Let $w \in S_n$ be a permutation. We say that w *avoids* $12 \cdots (k+1)$ if there exists no $(k+1)$-element subset U of $[n]$ such that the restriction $w \mid_U$ is increasing (i.e., if there exist no $k+1$ elements $i_1 < i_2 < \cdots < i_{k+1}$ of $[n]$ such that $w(i_1) < w(i_2) < \cdots < w(i_{k+1})$).

(b) We let $\mathrm{Av}_n(k+1)$ denote the set of all permutations $w \in S_n$ that avoid $12 \cdots (k+1)$.

This notion of "avoiding $12 \cdots (k+1)$" is taken from the theory of pattern avoidance (see, e.g., [Bona22, Chapters 4–5]).

2.4. The main theorem

We now arrive at one of our main results, which will be proved in Subsection 2.6:

Theorem 2.4.1. Let $k \in \mathbb{N}$. Then:

(a) We have $\mathcal{I}_k = \mathcal{J}_k^{\perp} = \mathrm{LAnn}\, \mathcal{J}_k = \mathrm{RAnn}\, \mathcal{J}_k$.

(b) We have $\mathcal{J}_k = \mathcal{I}_k^{\perp} = \mathrm{LAnn}\, \mathcal{I}_k = \mathrm{RAnn}\, \mathcal{I}_k$.

(c) The **k**-module $\mathcal{I}_k$ is free of rank $|\mathrm{Av}_n(k+1)|$.

(d) The **k**-module $\mathcal{J}_k$ is free of rank $|S_n \setminus \mathrm{Av}_n(k+1)|$.

(e) The **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$. (Here, $\overline{w}$ denotes the projection of $w \in \mathcal{A}$ onto the quotient $\mathcal{A}/\mathcal{I}_k$.)

(f) The **k**-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w \in \mathrm{Av}_n(k+1)}$. (Here, $\overline{w}$ denotes the projection of $w \in \mathcal{A}$ onto the quotient $\mathcal{A}/\mathcal{J}_k$.)

(g) Assume that $n!$ is invertible in **k**. Then, $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as **k**-module. Moreover, $\mathcal{I}_k$ and $\mathcal{J}_k$ are nonunital subalgebras of $\mathcal{A}$ that have unities and satisfy $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as **k**-algebras.[3]

Furthermore, the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ have representation-theoretical meanings:

- They are the annihilators of certain tensor power representations of S_n: see Theorem 2.8.1 and Theorem 2.8.3.

- If $n!$ is invertible in $\mathbf{k}$ (for example, if $\mathbf{k}$ is a field of characteristic 0), then they are furthermore certain subproducts of the Artin–Wedderburn decomposition of $\mathcal{A}$: see Theorem 2.9.1.

Once Theorem 2.4.1 is proved, we will easily obtain the following corollary regarding the $k = 2$ case:

Corollary 2.4.2.

(a) We have

$$\mathcal{I}_2 = \text{span}\left\{ \nabla_{B,A} \mid A, B \subseteq [n] \right\}. \tag{22}$$

(b) The $\mathbf{k}$-module $\mathcal{I}_2$ is free of rank $|\text{Av}_n(3)|$, which is the Catalan number C_n.

See Subsection 2.6 for the proof of this corollary.

Remark 2.4.3. The results in this section have a significant overlap with existing literature, particularly with prior work on the Murphy cellular bases and on annihilators of Young permutation modules. The latter will be discussed in Subsection 2.8. Let us briefly outline the former, insofar as it provides alternative proofs for parts of Theorem 2.4.1. Note that this remark is included for context only and will only be used in the proof of Theorem 2.9.6.

The Murphy cellular bases first appeared in the paper [CanWil89] by Canfield and Williamson; then, Murphy ([Murphy92] and [Murphy95]) generalized them to the Hecke algebra. The easiest way to define them (in $\mathcal{A} = \mathbf{k}[S_n]$) is as follows (following the notations of [Grinbe25, §6.8]): An *n-bitableau* shall mean a triple (λ, U, V), where λ is a partition of n and where U and V are two tableaux of shape λ with entries $1, 2, \ldots, n$ each. Given an n-bitableau (λ, U, V), we define two elements

$$\nabla_{V,U}^{\text{Row}} := \sum_{\substack{w \in S_n; \\ wU \text{ is row-equivalent to } V}} w \qquad \text{and}$$

$$\nabla_{V,U}^{-\text{Col}} := \sum_{\substack{w \in S_n; \\ wU \text{ is column-equivalent to } V}} (-1)^w w \qquad \text{of } \mathcal{A}.$$

---

[3]The formulation "nonunital subalgebras that have unities" may sound paradoxical. But it is literally true: A nonunital subalgebra of $\mathcal{A}$ can have a unity but still be nonunital **as a subalgebra** because its unity is not the unity of $\mathcal{A}$. This is precisely the situation that the nonunital subalgebras $\mathcal{I}_k$ and $\mathcal{J}_k$ find themselves in.

An n-bitableau (λ, U, V) is said to be *standard* if both tableaux U and V are standard. We let $\mathrm{SBT}(n)$ be the set of all standard n-bitableaux, while $\mathrm{BT}(n)$ denotes the set of all n-bitableaux (standard or not). Then, both families

$$\left(\nabla_{V,U}^{\mathrm{Row}} \right)_{(\lambda,U,V) \in \mathrm{SBT}(n)} \qquad \text{and} \qquad \left(\nabla_{V,U}^{-\mathrm{Col}} \right)_{(\lambda,U,V) \in \mathrm{SBT}(n)}$$

are bases of $\mathcal{A}$ (see [CanWil89, Theorems 3.12, 3.14 and 6.13] or [Grinbe25, Theorem 6.8.14]). These are the so-called *Murphy cellular bases*.

These bases have been used to define ideals (left, right, two-sided) of $\mathcal{A}$. In particular, for any $k \in \mathbb{N}$, we can define the four spans

$$\mathcal{F}_{\mathrm{std},\mathrm{len}\leq k}^{\mathrm{Row}} := \mathrm{span}\left\{ \nabla_{V,U}^{\mathrm{Row}} \mid (\lambda, U, V) \in \mathrm{SBT}(n) \text{ and } \ell(\lambda) \leq k \right\};$$

$$\mathcal{F}_{\mathrm{std},\mathrm{len}> k}^{-\mathrm{Col}} := \mathrm{span}\left\{ \nabla_{V,U}^{-\mathrm{Col}} \mid (\lambda, U, V) \in \mathrm{SBT}(n) \text{ and } \ell(\lambda) > k \right\};$$

$$\mathcal{F}_{\mathrm{all},\mathrm{len}\leq k}^{\mathrm{Row}} := \mathrm{span}\left\{ \nabla_{V,U}^{\mathrm{Row}} \mid (\lambda, U, V) \in \mathrm{BT}(n) \text{ and } \ell(\lambda) \leq k \right\};$$

$$\mathcal{F}_{\mathrm{all},\mathrm{len}> k}^{-\mathrm{Col}} := \mathrm{span}\left\{ \nabla_{V,U}^{-\mathrm{Col}} \mid (\lambda, U, V) \in \mathrm{BT}(n) \text{ and } \ell(\lambda) > k \right\}.$$

According to [Grinbe25, Theorem 6.8.47], these are actually just two spans: we have

$$\mathcal{F}_{\mathrm{std},\mathrm{len}\leq k}^{\mathrm{Row}} = \mathcal{F}_{\mathrm{all},\mathrm{len}\leq k}^{\mathrm{Row}} = \left(\mathcal{F}_{\mathrm{std},\mathrm{len}> k}^{-\mathrm{Col}} \right)^{\perp}$$

and

$$\mathcal{F}_{\mathrm{std},\mathrm{len}> k}^{-\mathrm{Col}} = \mathcal{F}_{\mathrm{all},\mathrm{len}> k}^{-\mathrm{Col}} = \left(\mathcal{F}_{\mathrm{std},\mathrm{len}\leq k}^{\mathrm{Row}} \right)^{\perp},$$

and furthermore, both spans $\mathcal{F}_{\mathrm{std},\mathrm{len}\leq k}^{\mathrm{Row}} = \mathcal{F}_{\mathrm{all},\mathrm{len}\leq k}^{\mathrm{Row}}$ and $\mathcal{F}_{\mathrm{std},\mathrm{len}> k}^{-\mathrm{Col}} = \mathcal{F}_{\mathrm{all},\mathrm{len}> k}^{-\mathrm{Col}}$ are two-sided ideals of $\mathcal{A}$.

Now, we claim that these ideals $\mathcal{F}_{\mathrm{std},\mathrm{len}\leq k}^{\mathrm{Row}}$ and $\mathcal{F}_{\mathrm{std},\mathrm{len}> k}^{-\mathrm{Col}}$ are precisely our $\mathcal{I}_k$ and $\mathcal{J}_k$, respectively. Indeed:

- Our ideal $\mathcal{I}_k$ is spanned by the row-to-row sums of the form $\nabla_{\mathbf{B},\mathbf{A}}$, where $\mathbf{A} = (A_1, A_2, \ldots, A_p)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_p)$ are set compositions of $[n]$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$. By Proposition 2.1.1 **(a)**, we can restrict ourselves to those pairs $(\mathbf{A}, \mathbf{B})$ of set compositions that satisfy $|A_i| = |B_i|$ for all i (since all other pairs produce row-to-row sums equal to 0). Such pairs $(\mathbf{A}, \mathbf{B})$ can be encoded as pairs of "weird-shaped" tableaux A, B, both of shape α, where $\alpha = (|A_1|, |A_2|, \ldots, |A_p|)$ is the list of sizes of the blocks of $\mathbf{A}$ (or equivalently $\mathbf{B}$), in the simplest possible way: Let

  $$(\text{i-th row of } A) = (\text{list of elements of } A_i \text{ (in increasing order)}) \qquad \text{and}$$
  $$(\text{i-th row of } B) = (\text{list of elements of } B_i \text{ (in increasing order)})$$

for each $i \in [p]$. We can furthermore find a permutation $\sigma \in S_p$ such that $\left| A_{\sigma(1)} \right| \geq \left| A_{\sigma(2)} \right| \geq \cdots \geq \left| A_{\sigma(p)} \right|$. By permuting the blocks of $\mathbf{A}$ and of $\mathbf{B}$ using this permutation $\sigma \in S_p$ (so that $\nabla_{\mathbf{B},\mathbf{A}}$ stays unchanged), we can furthermore ensure that α is a partition (of length $\ell(\alpha) = p \leq k$). Then, (α, A, B) is an n-bitableau (although usually not a standard one), and we have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{B,A}^{\mathrm{Row}}$.

Hence, each nonzero row-to-row sum $\nabla_{\mathbf{B},\mathbf{A}}$ with $\mathbf{A}, \mathbf{B} \in \mathrm{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ can be rewritten as $\nabla_{B,A}^{\mathrm{Row}}$ for some n-bitableau $(\lambda, A, B) \in \mathrm{BT}(n)$ with $\ell(\lambda) \leq k$. Conversely, any $\nabla_{B,A}^{\mathrm{Row}}$ can be rewritten as $\nabla_{\mathbf{B},\mathbf{A}}$ by reversing the above construction (let $\mathbf{A}$ be the set composition of $[n]$ whose i-th block is the set of all entries of the i-th row of A, and likewise construct $\mathbf{B}$ from B). Thus,

$$\left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\} \setminus \{0\}$$
$$= \left\{ \nabla_{B,A}^{\mathrm{Row}} \mid (\lambda, A, B) \in \mathrm{BT}(n) \text{ with } \ell(\lambda) \leq k \right\} \setminus \{0\}.$$

Therefore,

$$\begin{aligned}
\mathcal{I}_k &= \mathrm{span}\left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\} \\
&= \mathrm{span}\left\{ \nabla_{B,A}^{\mathrm{Row}} \mid (\lambda, A, B) \in \mathrm{BT}(n) \text{ with } \ell(\lambda) \leq k \right\} \\
&= \mathrm{span}\left\{ \nabla_{V,U}^{\mathrm{Row}} \mid (\lambda, U, V) \in \mathrm{BT}(n) \text{ and } \ell(\lambda) \leq k \right\} \\
&= \mathcal{F}_{\mathrm{all,len} \leq k}^{\mathrm{Row}} = \mathcal{F}_{\mathrm{std,len} \leq k}^{\mathrm{Row}}.
\end{aligned} \tag{23}$$

- Define a **k**-submodule $\mathcal{U}_k$ of $\mathcal{A}$ by

$$\mathcal{U}_k := \mathrm{span}\left\{ \nabla_X^- \mid X \text{ is a subset of } [n] \text{ with size } |X| > k \right\}.$$

Then, [Grinbe25, Proposition 6.8.53] says that

$$\mathcal{F}_{\mathrm{std,len}>k}^{-\,\mathrm{Col}} = \mathcal{A}\mathcal{U}_k = \mathcal{U}_k\mathcal{A} = \mathcal{A}\mathcal{U}_k\mathcal{A}.$$

But Proposition 2.2.2 **(e)** yields

$$\begin{aligned}
\mathcal{J}_k &= \mathcal{A} \cdot \mathrm{span}\underbrace{\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k \right\}}_{=\left\{ \nabla_X^- \mid X \text{ is a subset of } [n] \text{ with size } |X|>k \right\}} \cdot \mathcal{A} \\
&= \mathcal{A} \cdot \underbrace{\mathrm{span}\left\{ \nabla_X^- \mid X \text{ is a subset of } [n] \text{ with size } |X| > k \right\}}_{=\mathcal{U}_k} \cdot \mathcal{A} \\
&= \mathcal{A}\mathcal{U}_k\mathcal{A}.
\end{aligned}$$

Comparing these two equalities, we find

$$\mathcal{J}_k = \mathcal{F}_{\mathrm{std,len}>k}^{-\,\mathrm{Col}}. \tag{24}$$

With the equalities (23) and (24) in hand, we can easily derive some parts of Theorem 2.4.1 from known properties of the Murphy cellular bases. In particular, (23) shows that

$$\mathcal{I}_k = \mathcal{F}_{\mathrm{std,len}\leq k}^{\mathrm{Row}} = \mathrm{span}\left\{\nabla_{V,U}^{\mathrm{Row}} \mid (\lambda, U, V) \in \mathrm{SBT}(n) \text{ and } \ell(\lambda) \leq k\right\};$$

thus, $\mathcal{I}_k$ is spanned by the family $\left(\nabla_{V,U}^{\mathrm{Row}}\right)_{(\lambda,U,V)\in\mathrm{SBT}(n) \text{ and } \ell(\lambda)\leq k}$, which is a subfamily of the Murphy cellular basis $\left(\nabla_{V,U}^{\mathrm{Row}}\right)_{(\lambda,U,V)\in\mathrm{SBT}(n)}$ of $\mathcal{A}$. Hence, $\mathcal{I}_k$ is a direct addend of $\mathcal{A}$ as a **k**-module, and both $\mathcal{I}_k$ and $\mathcal{A}/\mathcal{I}_k$ are free **k**-modules, with respective bases $\left(\nabla_{V,U}^{\mathrm{Row}}\right)_{(\lambda,U,V)\in\mathrm{SBT}(n) \text{ and } \ell(\lambda)\leq k}$ and $\left(\overline{\nabla_{V,U}^{\mathrm{Row}}}\right)_{(\lambda,U,V)\in\mathrm{SBT}(n) \text{ and } \ell(\lambda)>k}$. Likewise, we can get analogous claims about $\mathcal{J}_k$ and $\mathcal{A}/\mathcal{J}_k$ from (24). Using (23) and (24), we can furthermore rewrite the equality $\mathcal{F}_{\mathrm{std,len}\leq k}^{\mathrm{Row}} = \left(\mathcal{F}_{\mathrm{std,len}>k}^{-\mathrm{Col}}\right)^{\perp}$ (which is part of [Grinbe25, Theorem 6.8.47]) as $\mathcal{I}_k = \mathcal{J}_k^{\perp}$, and similarly we obtain $\mathcal{J}_k = \mathcal{I}_k^{\perp}$.

However, not all of Theorem 2.4.1 follows this easily from this approach. Anyway, we shall now take a more elementary point of view.

2.5. Lemmas for the proof of the main theorem

First, however, we prepare for the proof of Theorem 2.4.1. We start with a litany of lemmas. Our first lemma is a collection of basic properties of the bilinear form $\langle \cdot, \cdot \rangle$:

Lemma 2.5.1.

(a) Let $\mathrm{coeff}_1 : \mathcal{A} \to \mathbf{k}$ be the map that sends each element **a** of $\mathcal{A} = \mathbf{k}[S_n]$ to the coefficient of the identity permutation $\mathrm{id} \in S_n$ in **a**. In other words, $\mathrm{coeff}_1 : \mathcal{A} \to \mathbf{k}$ is the **k**-linear map that sends the permutation $\mathrm{id} \in S_n$ to 1 while sending any non-identity permutation $w \in S_n$ to 0.

Then, the bilinear form $\langle \cdot, \cdot \rangle$ can be expressed as follows: For any $a, b \in \mathcal{A}$, we have

$$\langle a, b \rangle = \mathrm{coeff}_1(S(a)\,b) \tag{25}$$
$$= \mathrm{coeff}_1(b\,S(a)) \tag{26}$$
$$= \mathrm{coeff}_1(S(b)\,a) \tag{27}$$
$$= \mathrm{coeff}_1(a\,S(b)). \tag{28}$$

(b) The bilinear form $\langle \cdot, \cdot \rangle$ is S-invariant: That is, for all $a, b \in \mathcal{A}$, we have

$$\langle a, b \rangle = \langle S(a), S(b) \rangle. \tag{29}$$

(c) Let $\mathcal{B}$ be any subset of $\mathcal{A}$. Then, $(S(\mathcal{B}))^{\perp} = S(\mathcal{B}^{\perp})$.

Proof. **(a)** This is [Grinbe25, Proposition 6.8.17].

(b) Let $a, b \in \mathcal{A}$. Then, $S(S(a)) = a$ (since $S \circ S = \text{id}$). But (25) (applied to $S(a)$ and $S(b)$ instead of a and b) yields

$$\langle S(a), S(b) \rangle = \text{coeff}_1 \left(\underbrace{S(S(a))}_{=a} S(b) \right) = \text{coeff}_1(aS(b)) = \langle a, b \rangle$$

(by (28)). In other words, $\langle a, b \rangle = \langle S(a), S(b) \rangle$. This proves (29) and thus Lemma 2.5.1 **(b)**.

(c) By the definition of $(S(\mathcal{B}))^{\perp}$, we have

$$
\begin{aligned}
(S(\mathcal{B}))^{\perp} &= \{a \in \mathcal{A} \mid \langle a, b \rangle = 0 \text{ for all } b \in S(\mathcal{B})\} \\
&= \{a \in \mathcal{A} \mid \langle a, c \rangle = 0 \text{ for all } c \in S(\mathcal{B})\} \\
&= \{a \in \mathcal{A} \mid \langle a, S(b) \rangle = 0 \text{ for all } b \in \mathcal{B}\} \quad (30)
\end{aligned}
$$

(since $S(\mathcal{B}) = \{S(b) \mid b \in \mathcal{B}\}$). On the other hand,

$$
\begin{aligned}
S\left(\mathcal{B}^{\perp}\right) &= \left\{S(a) \mid a \in \mathcal{B}^{\perp}\right\} \\
&= \{S(a) \mid a \in \mathcal{A} \text{ satisfying } \langle a, b \rangle = 0 \text{ for all } b \in \mathcal{B}\} \\
&\qquad \left(\text{since } \mathcal{B}^{\perp} = \{a \in \mathcal{A} \mid \langle a, b \rangle = 0 \text{ for all } b \in \mathcal{B}\}\right) \\
&= \{S(a) \mid a \in \mathcal{A} \text{ satisfying } \langle S(a), S(b) \rangle = 0 \text{ for all } b \in \mathcal{B}\} \\
&\qquad (\text{since } (29) \text{ yields } \langle a, b \rangle = \langle S(a), S(b) \rangle) \\
&= \{a \mid a \in \mathcal{A} \text{ satisfying } \langle a, S(b) \rangle = 0 \text{ for all } b \in \mathcal{B}\}
\end{aligned}
$$

(here, we have substituted a for $S(a)$ in our set, since $S : \mathcal{A} \to \mathcal{A}$ is a bijection). In other words,

$$S\left(\mathcal{B}^{\perp}\right) = \{a \in \mathcal{A} \mid \langle a, S(b) \rangle = 0 \text{ for all } b \in \mathcal{B}\}.$$

Comparing this with (30), we obtain $(S(\mathcal{B}))^{\perp} = S(\mathcal{B}^{\perp})$. This proves Lemma 2.5.1 **(c)**. $\qquad\square$

The next lemma connects orthogonal spaces with left/right annihilators:

Lemma 2.5.2. Let $\mathcal{B}$ be a left ideal of $\mathcal{A}$. Then:

(a) We have $\mathcal{B}^{\perp} = \text{LAnn}(S(\mathcal{B}))$.

(b) If $S(\mathcal{B}) = \mathcal{B}$, then $\mathcal{B}^{\perp} = \text{LAnn} \, \mathcal{B} = \text{RAnn} \, \mathcal{B}$.

Proof. **(a)** Define the **k**-linear map $\text{coeff}_1 : \mathcal{A} \to \mathbf{k}$ as in Lemma 2.5.1 **(a)**. It is easy to see that each $a \in \mathcal{A}$ and each $w \in S_n$ satisfy the equality

$$\text{coeff}_1\left(aw^{-1}\right) = (\text{the coefficient of } w \text{ in } a) \tag{31}$$

(indeed, if we write a as $\sum\limits_{u \in S_n} \alpha_u u$ with $\alpha_u \in \mathbf{k}$, then both sides of this equality equal α_w).

Let $a \in \text{LAnn}\,(S\,(\mathcal{B}))$. Then, $aS\,(b) = 0$ for all $b \in \mathcal{B}$. Hence, (28) yields $\langle a, b \rangle =$
$\text{coeff}_1\left(\underbrace{aS\,(b)}_{=0}\right) = \text{coeff}_1\,0 = 0$ for all $b \in \mathcal{B}$. In other words, $a \in \mathcal{B}^{\perp}$. Thus, we
have shown that $\text{LAnn}\,(S\,(\mathcal{B})) \subseteq \mathcal{B}^{\perp}$.

Conversely, let $c \in \mathcal{B}^{\perp}$. Then, $\langle c, b \rangle = 0$ for all $b \in \mathcal{B}$. Now, let $b \in \mathcal{B}$ be arbitrary. Then, for every $w \in S_n$, we have $wb \in \mathcal{B}$ (since $\mathcal{B}$ is a left ideal of $\mathcal{A}$) and therefore $\langle c, wb \rangle = 0$ (since $c \in \mathcal{B}^{\perp}$), so that

$$0 = \langle c, wb \rangle = \text{coeff}_1\,(cS\,(wb)) \qquad \text{(by (28))}$$

$$= \text{coeff}_1\left(cS\,(b)\,w^{-1}\right) \qquad \left(\text{since } S\,(wb) = S\,(b)\underbrace{S\,(w)}_{=w^{-1}} = S\,(b)\,w^{-1}\right)$$

$$= (\text{the coefficient of } w \text{ in } cS\,(b)) \qquad (\text{by (31), applied to } a = cS\,(b)).$$

Since this holds for each $w \in S_n$, we thus have shown that all coefficients of $cS\,(b)$ equal 0. In other words, $cS\,(b) = 0$. Since this holds for each $b \in \mathcal{B}$, we conclude that $c \in \text{LAnn}\,(S\,(\mathcal{B}))$. Thus, we have shown that $\mathcal{B}^{\perp} \subseteq \text{LAnn}\,(S\,(\mathcal{B}))$. Combining this with $\text{LAnn}\,(S\,(\mathcal{B})) \subseteq \mathcal{B}^{\perp}$, we obtain $\mathcal{B}^{\perp} = \text{LAnn}\,(S\,(\mathcal{B}))$. Thus, Lemma 2.5.2 **(a)** is proved.

(b) Assume that $S\,(\mathcal{B}) = \mathcal{B}$. Then, Lemma 2.5.2 **(a)** yields $\mathcal{B}^{\perp} = \text{LAnn}\,(S\,(\mathcal{B})) = \text{LAnn}\,\mathcal{B}$ (since $S\,(\mathcal{B}) = \mathcal{B}$). Furthermore, Lemma 2.5.1 **(c)** yields $(S\,(\mathcal{B}))^{\perp} = S\,(\mathcal{B}^{\perp})$. In view of $S\,(\mathcal{B}) = \mathcal{B}$, we can rewrite this as $\mathcal{B}^{\perp} = S\,(\mathcal{B}^{\perp})$. However, S is a **k**-algebra anti-automorphism. Thus, $\text{RAnn}\,(S\,(\mathcal{B})) = S\,(\text{LAnn}\,\mathcal{B})$. In view of $S\,(\mathcal{B}) = \mathcal{B}$, we can rewrite this as $\text{RAnn}\,\mathcal{B} = S\left(\underbrace{\text{LAnn}\,\mathcal{B}}_{=\mathcal{B}^{\perp}}\right) = S\,(\mathcal{B}^{\perp}) = \mathcal{B}^{\perp}$. Thus, $\mathcal{B}^{\perp} = \text{RAnn}\,\mathcal{B}$. Combined with $\mathcal{B}^{\perp} = \text{LAnn}\,\mathcal{B}$, this completes the proof of Lemma 2.5.2 **(b)**. $\qquad\square$

Lemma 2.5.3. Let $\mathcal{M}$ be a free **k**-module with a basis $(m_i)_{i \in I}$. Let J and K be two disjoint subsets of I such that $J \cup K = I$. Let $\mathcal{N}$ be a **k**-submodule of $\mathcal{M}$ such that the quotient module $\mathcal{M}/\mathcal{N}$ has a basis $(\overline{m_i})_{i \in J}$. (Here, as usual, $\overline{m}$ denotes the projection of any vector $m \in \mathcal{M}$ onto the quotient $\mathcal{M}/\mathcal{N}$.)
Then:

(a) The **k**-module $\mathcal{N}$ is free of rank $|K|$.

(b) There exists a **k**-linear projection $\pi : \mathcal{M} \to \mathcal{N}$ (that is, a **k**-linear map $\pi : \mathcal{M} \to \mathcal{N}$ such that $\pi \mid_{\mathcal{N}} = \mathrm{id}$).

Proof. For each $k \in K$, the vector $\overline{m_k} \in \mathcal{M}/\mathcal{N}$ can be written as a **k**-linear combination of the family $(\overline{m_i})_{i \in J}$ (since this family is a basis of $\mathcal{M}/\mathcal{N}$). In other words, there exist coefficients $c_{k,j}$ for all $k \in K$ and $j \in J$ such that each $k \in K$ satisfies

$$\overline{m_k} = \sum_{j \in J} c_{k,j} \overline{m_j}. \tag{32}$$

Consider these $c_{k,j}$. Now, let us set

$$v_k := m_k - \sum_{j \in J} c_{k,j} m_j \tag{33}$$

for each $k \in K$. This element v_k belongs to $\mathcal{N}$ (since $\overline{v_k} = \overline{m_k - \sum_{j \in J} c_{k,j} m_j} = \overline{m_k} - \sum_{j \in J} c_{k,j} \overline{m_j} = \overline{0}$ by (32)). Thus, $(v_k)_{k \in K}$ is a family of vectors in $\mathcal{N}$. This family is easily seen to be **k**-linearly independent[4]. Moreover, it spans the **k**-module $\mathcal{N}$ (this, too, is not hard to see[5]). Thus, the family $(v_k)_{k \in K}$ is a basis of $\mathcal{N}$. Therefore, $\mathcal{N}$ is free of rank $|K|$. This proves Lemma 2.5.3 **(a)**.

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[4]*Proof.* Let $(\lambda_k)_{k \in K} \in \mathbf{k}^K$ be a family of scalars such that $\sum\limits_{k \in K} \lambda_k v_k = 0$. We must show that all λ_k are 0.

We have

$$0 = \sum_{k \in K} \lambda_k v_k = \sum_{k \in K} \lambda_k \left(m_k - \sum_{j \in J} c_{k,j} m_j \right) \qquad \text{(by (33))}$$

$$= \sum_{k \in K} \lambda_k m_k - \sum_{k \in K} \lambda_k \sum_{j \in J} c_{k,j} m_j$$

$$= \sum_{k \in K} \lambda_k m_k - \sum_{j \in J} \left(\sum_{k \in K} \lambda_k c_{k,j} \right) m_j. \tag{34}$$

Note that there is no overlap between the m_k's in the first sum here and the m_j's in the second sum, since the sets J and K are disjoint. Since the family $(m_i)_{i \in I}$ is **k**-linearly independent (being a basis of $\mathcal{M}$), we thus conclude from (34) that all the coefficients λ_k and $\sum\limits_{k \in K} \lambda_k c_{k,j}$ are 0. Hence, in particular, all λ_k are 0. Qed.

[5]*Proof.* Let $w \in \mathcal{N}$. We must show that $w \in \mathrm{span}\{v_k \mid k \in K\}$.

First, we expand $w \in \mathcal{N} \subseteq \mathcal{M}$ in the basis $(m_i)_{i \in I}$ of $\mathcal{M}$ as follows:

$$w = \sum_{i \in I} w_i m_i \qquad \text{with } w_i \in \mathbf{k}.$$

(b) Recall that $v_k \in \mathcal{N}$ for each $k \in K$. Let $\pi : \mathcal{M} \to \mathcal{N}$ be the **k**-linear map that sends each basis element m_i of $\mathcal{M}$ to

$$\begin{cases} v_i, & \text{if } i \in K; \\ 0, & \text{if } i \in J. \end{cases}$$

This is well-defined (since $(m_i)_{i \in I}$ is a basis of $\mathcal{M}$, and since each $i \in I$ belongs to exactly one of the sets K and J). It is easy to see that this map π sends v_k to v_k for each $k \in K$ (because applying π to the right hand side of (33) kills all m_j with $j \in J$ but sends m_k to v_k). Thus, by linearity, we obtain $\pi \mid_{\mathcal{N}} = \mathrm{id}$ (since $(v_k)_{k \in K}$ is a basis of $\mathcal{N}$). Thus, π is a projection. This proves Lemma 2.5.3 **(b)**. $\qquad\square$

Since the set I is the union of its disjoint subsets J and K, we can break this sum up as follows:

$$w = \sum_{i \in J} w_i m_i + \sum_{i \in K} w_i m_i = \sum_{j \in J} w_j m_j + \sum_{k \in K} w_k \underbrace{m_k}_{\substack{= \sum\limits_{j \in J} c_{k,j} m_j + v_k \\ \text{(by (33))}}}$$

$$= \sum_{j \in J} w_j m_j + \underbrace{\sum_{k \in K} w_k \left(\sum_{j \in J} c_{k,j} m_j + v_k \right)}_{= \sum\limits_{j \in J} \sum\limits_{k \in K} w_k c_{k,j} m_j + \sum\limits_{k \in K} w_k v_k} = \sum_{j \in J} w_j m_j + \underbrace{\sum_{j \in J} \sum_{k \in K} w_k c_{k,j} m_j}_{= \sum\limits_{j \in J} \left(w_j + \sum\limits_{k \in K} w_k c_{k,j} \right) m_j} + \sum_{k \in K} w_k v_k$$

$$= \sum_{j \in J} \left(w_j + \sum_{k \in K} w_k c_{k,j} \right) m_j + \sum_{k \in K} w_k v_k. \tag{35}$$

Projecting this onto $\mathcal{M} / \mathcal{N}$, we obtain

$$\overline{w} = \overline{\sum_{j \in J} \left(w_j + \sum_{k \in K} w_k c_{k,j} \right) m_j + \sum_{k \in K} w_k v_k}$$

$$= \sum_{j \in J} \left(w_j + \sum_{k \in K} w_k c_{k,j} \right) \overline{m_j} + \sum_{k \in K} w_k \underbrace{\overline{v_k}}_{\substack{=0 \\ (\text{since } v_k \in \mathcal{N})}} = \sum_{j \in J} \left(w_j + \sum_{k \in K} w_k c_{k,j} \right) \overline{m_j}.$$

Hence,

$$\sum_{j \in J} \left(w_j + \sum_{k \in K} w_k c_{k,j} \right) \overline{m_j} = \overline{w} = 0 \qquad \left(\text{since } w \in \mathcal{N} \right).$$

Since the family $(\overline{m_i})_{i \in J}$ is **k**-linearly independent (being a basis of $\mathcal{N}$), we thus conclude that all the coefficients $w_j + \sum\limits_{k \in K} w_k c_{k,j}$ here are 0. Thus, (35) becomes

$$w = \sum_{j \in J} \underbrace{\left(w_j + \sum_{k \in K} w_k c_{k,j} \right)}_{=0} m_j + \sum_{k \in K} w_k v_k = \sum_{k \in K} w_k v_k \in \mathrm{span} \left\{ v_k \mid k \in K \right\},$$

qed.

Lemma 2.5.4. Let $k \in \mathbb{N}$. Then, $\mathcal{I}_k \mathcal{J}_k = \mathcal{J}_k \mathcal{I}_k = 0$.

Proof. Let us first show that $\mathcal{I}_k \mathcal{J}_k = 0$. Indeed, we have

$$\mathcal{I}_k = \operatorname{span} \left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \right\}$$

and

$$\mathcal{J}_k = \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}$$

(by Proposition 2.2.2 **(b)**). Thus, in order to prove that $\mathcal{I}_k \mathcal{J}_k = 0$, it suffices to show that $\nabla_{\mathbf{B},\mathbf{A}} \nabla_U^- = 0$ for all set compositions $\mathbf{A}, \mathbf{B} \in \mathrm{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ and all subsets U of $[n]$ having size $k+1$. So let us show this. We fix two set compositions $\mathbf{A}, \mathbf{B} \in \mathrm{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ and a subset U of $[n]$ having size $k+1$. We must show that $\nabla_{\mathbf{B},\mathbf{A}} \nabla_U^- = 0$.

We have $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k < k+1 = |U|$ (since U has size $k+1$). In other words, $\mathbf{A}$ has fewer blocks than U has elements. Hence, by the pigeonhole principle, there exist two distinct elements u and v of U that belong to the same block of $\mathbf{A}$. Pick such u and v. Let $\tau \in S_U$ be the transposition that swaps u and v. Then, $\nabla_U^- = (1 - \tau) q$ for some $q \in \mathbf{k}[S_U]$ (since $\langle \tau \rangle = \{1, \tau\}$ is a subgroup of the group S_U and since the permutation τ is odd[6]). Consider this q. But (16) (applied to $u = \mathrm{id}$ and $v = \tau$) yields

$$\nabla_{\mathbf{B},\mathbf{A}} \tau = \nabla_{\mathbf{B}, \tau^{-1}\mathbf{A}} = \nabla_{\mathbf{B},\mathbf{A}}$$

(since $\tau^{-1}\mathbf{A} = \mathbf{A}$ (because u and v belong to the same block of $\mathbf{A}$, and thus the transposition τ preserves each block of $\mathbf{A}$)). Hence,

$$\nabla_{\mathbf{B},\mathbf{A}} \underbrace{\nabla_U^-}_{=(1-\tau)q} = \underbrace{\nabla_{\mathbf{B},\mathbf{A}}(1-\tau)}_{\substack{=\nabla_{\mathbf{B},\mathbf{A}} - \nabla_{\mathbf{B},\mathbf{A}}\tau \\ =0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}}\tau = \nabla_{\mathbf{B},\mathbf{A}})}} q = 0.$$

This completes our proof of $\mathcal{I}_k \mathcal{J}_k = 0$.

It remains to prove that $\mathcal{J}_k \mathcal{I}_k = 0$. This can be done similarly, but can also be derived from $\mathcal{I}_k \mathcal{J}_k = 0$ easily: Since S is an algebra anti-automorphism, we have

$$S(\mathcal{I}_k \mathcal{J}_k) = \underbrace{S(\mathcal{J}_k)}_{\substack{=\mathcal{J}_k \\ (\text{by Proposition 2.2.2 (c)})}} \underbrace{S(\mathcal{I}_k)}_{\substack{=\mathcal{I}_k \\ (\text{by Proposition 2.2.2 (c)})}} = \mathcal{J}_k \mathcal{I}_k.$$

Thus, $\mathcal{J}_k \mathcal{I}_k = S(\mathcal{I}_k \mathcal{J}_k) = 0$ (since $\mathcal{I}_k \mathcal{J}_k = 0$). The proof of Lemma 2.5.4 is thus complete. $\square$

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[6]In more details: We have $\nabla_U^- = (1 - \tau) \nabla_U^{\mathrm{even}}$, where $\nabla_U^{\mathrm{even}} = \sum\limits_{\substack{w \in S_U; \\ (-1)^w = 1}} w$. This follows from [Grinbe25, Proposition 3.7.4 **(d)**], applied to $X = U$ and $i = u$ and $j = v$ (our τ is thus the $t_{i,j}$ of [Grinbe25, Proposition 3.7.4 **(d)**]).

The following purely combinatorial lemma is a variant of the Erdös–Szekeres theorem:

Lemma 2.5.5. Let $k \in \mathbb{N}$. Let $v \in \mathrm{Av}_n \left(k+1\right)$. Then, there exists a set decomposition $\mathbf{A} = \left(A_1, A_2, \ldots, A_k\right)$ of $[n]$ such that all restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, $\ldots$, $v \mid_{A_k}$ are decreasing.

Proof. Let $\mathbf{v}$ be the sequence $\left(v\left(1\right), v\left(2\right), \ldots, v\left(n\right)\right)$. Then, $\mathbf{v}$ has no increasing subsequence of length $k+1$ (because $v \in \mathrm{Av}_n \left(k+1\right)$). Thus, $\mathbf{v}$ has no increasing subsequence of length $> k$. Hence, the length of any nonempty increasing subsequence of $\mathbf{v}$ must be some $i \in [k]$.

For each $i \in [k]$, we define a set

$$A_i := \{ j \in [n] \mid \text{the longest increasing subsequence of } \mathbf{v} \\ \text{ending with } v\left(j\right) \text{ has length } i \} .$$

These k sets $A_1, A_2, \ldots, A_k$ are clearly disjoint, and their union is $[n]$ (since the length of any nonempty increasing subsequence of $\mathbf{v}$ must be some $i \in [k]$). In other words, $\mathbf{A} := \left(A_1, A_2, \ldots, A_k\right)$ is a set decomposition of $[n]$. It remains to show that all restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, $\ldots$, $v \mid_{A_k}$ are decreasing.

To do this, we assume the contrary. Thus, there exists some $i \in [k]$ such that $v \mid_{A_i}$ is not decreasing. Consider this i. Then, there exist two elements $p < q$ of A_i such that $v\left(p\right) < v\left(q\right)$. Consider these p and q. The longest increasing subsequence of $\mathbf{v}$ ending with $v\left(q\right)$ has length i (since $q \in A_i$). Now, consider the longest increasing subsequence of $\mathbf{v}$ ending with $v\left(p\right)$. This subsequence, too, has length i (since $p \in A_i$), and thus can be written as $\left(v\left(p_1\right) < v\left(p_2\right) < \cdots < v\left(p_i\right)\right)$ with $p_i = p$ (since it ends with $v\left(p\right)$). By appending $v\left(q\right)$ to it, we obtain an increasing subsequence $\left(v\left(p_1\right) < v\left(p_2\right) < \cdots < v\left(p_i\right) < v\left(q\right)\right)$ of $\mathbf{v}$ (since $p_i = p$ and thus $v\left(p_i\right) = v\left(p\right) < v\left(q\right)$) that ends with $v\left(q\right)$ and has length $i + 1$. But this contradicts the fact that the longest increasing subsequence of $\mathbf{v}$ ending with $v\left(q\right)$ has length i. This contradiction shows that our assumption was false. Thus, all restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, $\ldots$, $v \mid_{A_k}$ are decreasing. This completes the proof of Lemma 2.5.5. $\square$

In the following few lemmas, we will use the lexicographic order on S_n. This is a total order on S_n, defined by setting

$$\left(u < v\right) \iff \left(\text{the smallest } i \in [n] \text{ satisfying } u\left(i\right) \neq v\left(i\right) \right. \\ \left. \text{satisfies } u\left(i\right) < v\left(i\right)\right) \qquad \text{for all } u, v \in S_n.$$

Lemma 2.5.6. Let $k \in \mathbb{N}$. Let $\mathbf{A} = \left(A_1, A_2, \ldots, A_k\right)$ be a set decomposition of $[n]$. Let $v \in S_n$ be a permutation such that all restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, $\ldots$, $v \mid_{A_k}$ are decreasing. Let $w \in S_n$ be a further permutation such that all $i \in [k]$ satisfy $w\left(A_i\right) = v\left(A_i\right)$. Then, $w \leq v$ in lexicographic order.

Proof. Intuitively, this is clear: The condition "all $i \in [k]$ satisfy $w(A_i) = v(A_i)$" shows that w can be obtained from v by permuting the values of v on A_1, permuting the values of v on A_2, and so on. But any such permutation (unless it is the identity) decreases v in lexicographic order (because the restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, ..., $v \mid_{A_k}$ are decreasing, and thus any nontrivial permutation "puts elements in a more natural order"). Hence, $w \leq v$.

Here is a rigorous proof: If $w = v$, then the claim is obvious. So we WLOG assume that $w \neq v$. Then, there exists some $p \in [n]$ such that $w(p) \neq v(p)$. Consider the **smallest** such p. Then,

$$w(r) = v(r) \qquad \text{for each } r < p. \tag{36}$$

Since $\mathbf{A}$ is a set decomposition of $[n]$, there exists some $i \in [k]$ such that $p \in A_i$. Consider this i.

We have $p \in A_i$, thus $w(p) \in w(A_i) = v(A_i)$ (by assumption). In other words, $w(p) = v(r)$ for some $r \in A_i$. Consider this r.

If we had $r < p$, then (36) would yield $w(r) = v(r) = w(p)$, which would yield $r = p$ (since w is injective), which would contradict $r < p$. Hence, $r < p$ is impossible. Thus, $r \geq p$.

Moreover, $w(p) \neq v(p)$ and thus $v(p) \neq w(p) = v(r)$, so that $p \neq r$. Combining this with $r \geq p$, we obtain $r > p$. Since the restriction $v \mid_{A_i}$ is decreasing (by assumption), this entails $v(r) < v(p)$ (since r and p belong to A_i). Thus, $w(p) = v(r) < v(p)$.

So we have shown that the smallest $p \in [n]$ that satisfies $w(p) \neq v(p)$ must satisfy $w(p) < v(p)$. In other words, $w < v$ in lexicographic order. Hence, $w \leq v$ follows, and Lemma 2.5.6 is proved. $\qquad\square$

Lemma 2.5.7. Let $k \in \mathbb{N}$. Then, the quotient **k**-module $\mathcal{A}/\mathcal{I}_k$ is spanned by the family $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$.

Proof. Clearly, $\mathcal{A}/\mathcal{I}_k$ is spanned by the $\overline{u}$ for $u \in S_n$. Hence, it suffices to prove that

$$\overline{u} \in \mathrm{span}\left((\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \right) \qquad \text{for each } u \in S_n. \tag{37}$$

To prove this, we proceed by induction on u in lexicographic order. Thus, we fix a permutation $v \in S_n$, and we assume (as the induction hypothesis) that (37) holds for every $u < v$ in lexicographic order. We must now prove (37) for $u = v$.

If $v \in S_n \setminus \mathrm{Av}_n(k+1)$, then this is trivial. Thus, we WLOG assume that $v \notin S_n \setminus \mathrm{Av}_n(k+1)$. Hence, $v \in \mathrm{Av}_n(k+1)$. Therefore, by Lemma 2.5.5, there exists a set decomposition $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ of $[n]$ such that all restrictions $v \mid_{A_1}$, $v \mid_{A_2}$, ..., $v \mid_{A_k}$ are decreasing. Consider this set decomposition $\mathbf{A} = (A_1, A_2, \ldots, A_k)$. Define a further set decomposition $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ of $[n]$ by $\mathbf{B} = v\mathbf{A}$ (using the action of S_n on $\mathrm{SD}(n)$), that is, by

$$B_i := v(A_i) \qquad \text{for each } i \in [k].$$

Thus, $v \in S_n$ is a permutation satisfying $v(A_i) = B_i$ for all i. Hence, the row-to-row sum

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w$$

contains the permutation v as one of its addends. All its remaining addends are permutations w that satisfy $w < v$ in lexicographic order (by Lemma 2.5.6[7]). Thus, all the addends of $\nabla_{\mathbf{B},\mathbf{A}}$ except for v are lexicographically smaller than v. Hence,

$$\nabla_{\mathbf{B},\mathbf{A}} = v + (\text{some permutations } w < v).$$

Therefore,

$$v = \nabla_{\mathbf{B},\mathbf{A}} - (\text{some permutations } w < v). \tag{38}$$

But the set decompositions $\mathbf{A}, \mathbf{B} \in \text{SD}(n)$ satisfy $\ell(\mathbf{A}) = \ell(\mathbf{B}) = k$. Hence, Proposition 2.2.2 **(d)** yields $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$. Thus, projecting the equality (38) onto the quotient $\mathcal{A}/\mathcal{I}_k$, we obtain

$$\overline{v} = \underbrace{\overline{\nabla_{\mathbf{B},\mathbf{A}}}}_{\substack{=0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k)}} - \overline{(\text{some permutations } w < v)}$$

$$= - \underbrace{\overline{(\text{some permutations } w < v)}}_{\substack{\in \text{span}\left((\overline{w})_{w \in S_n \backslash \text{Av}_n(k+1)}\right) \\ (\text{by our induction hypothesis})}} \in \text{span}\left((\overline{w})_{w \in S_n \backslash \text{Av}_n(k+1)}\right).$$

In other words, (37) holds for $u = v$. This completes the induction. Thus, Lemma 2.5.7 is proved. $\qquad\square$

> **Lemma 2.5.8.** Let $v \in S_n$ be a permutation. Let U be a subset of $[n]$ such that the restriction $v \mid_U$ is increasing. Let $w \in S_n$ be a permutation. Assume that w agrees with v on all elements outside of U (that is, we have $w(i) = v(i)$ for each $i \in [n] \setminus U$). Then, $w \geq v$ in lexicographic order.

Proof. Essentially, this is because w can be obtained from v by permuting the values of v on U (since w agrees with v on all elements outside of U), and any such permutation increases v in lexicographic order (because the restriction $v \mid_U$ is increasing).

Here is a rigorous proof: If $w = v$, then the claim is obvious. So we WLOG assume that $w \neq v$. Then, there exists some $p \in [n]$ such that $w(p) \neq v(p)$. Consider the **smallest** such p. Then,

$$w(r) = v(r) \qquad \text{for each } r < p. \tag{39}$$

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[7]In more detail: We must show that any permutation $w \in S_n$ that satisfies $(w(A_i) = B_i \text{ for all } i)$ and is distinct from v must satisfy $w < v$ in lexicographic order. So let w be such a permutation. Then, $w(A_i) = B_i = v(A_i)$ for each $i \in [k]$. Hence, Lemma 2.5.6 shows that $w \leq v$ is lexicographic order. Since w is distinct from v, we thus obtain $w < v$.

From $w(p) \neq v(p)$, we obtain $p \notin [n] \setminus U$ (since $w(i) = v(i)$ for each $i \in [n] \setminus U$). Thus, $p \in U$.

But the permutations w and v agree on all elements outside of U. Hence, in particular, $w([n] \setminus U) = v([n] \setminus U)$. However, w and v are permutations of $[n]$, so we have $w([n] \setminus U) = [n] \setminus w(U)$ and $v([n] \setminus U) = [n] \setminus v(U)$, and therefore

$$[n] \setminus w(U) = w([n] \setminus U) = v([n] \setminus U) = [n] \setminus v(U).$$

Since $w(U)$ and $v(U)$ are subsets of $[n]$, we can take complements in this equality, and conclude that $w(U) = v(U)$.

From $p \in U$, we obtain $w(p) \in w(U) = v(U)$. In other words, $w(p) = v(r)$ for some $r \in U$. Consider this r. If we had $r < p$, then (39) would yield $w(r) = v(r) = w(p)$, which would entail $r = p$ (since w is injective), and this would contradict $r < p$. Thus we cannot have $r < p$. Hence, $r \geq p$. But we also have $w(p) \neq v(p)$, hence $v(p) \neq w(p) = v(r)$ and thus $p \neq r$. Combined with $r \geq p$, this shows that $r > p$. Since the restriction $v \mid_U$ is increasing, we thus conclude that $v(r) > v(p)$ (since r and p belong to U). Thus, $w(p) = v(r) > v(p)$.

So we have shown that the smallest $p \in [n]$ that satisfies $w(p) \neq v(p)$ must satisfy $w(p) > v(p)$. In other words, $w > v$ in lexicographic order. Hence, $w \geq v$ follows, and Lemma 2.5.8 is proved. $\square$

Lemma 2.5.9. Let $k \in \mathbb{N}$. Then, the quotient **k**-module $\mathcal{A}/\mathcal{J}_k$ is spanned by the family $(\overline{w})_{w \in \mathrm{Av}_n(k+1)}$.

Proof. Clearly, $\mathcal{A}/\mathcal{J}_k$ is spanned by the $\overline{u}$ for $u \in S_n$. Hence, it suffices to prove that

$$\overline{u} \in \mathrm{span}\left((\overline{w})_{w \in \mathrm{Av}_n(k+1)}\right) \qquad \text{for each } u \in S_n. \tag{40}$$

To prove this, we proceed by induction on u in reverse lexicographic order. Thus, we fix a permutation $v \in S_n$, and we assume (as the induction hypothesis) that (40) holds for every $u > v$ in lexicographic order. We must now prove (40) for $u = v$.

If $v \in \mathrm{Av}_n(k+1)$, then this is trivial. Thus, we WLOG assume that $v \notin \mathrm{Av}_n(k+1)$. Hence, $v \in S_n \setminus \mathrm{Av}_n(k+1)$. Therefore, there exists a $(k+1)$-element subset U of $[n]$ such that the restriction $v \mid_U$ is increasing. Consider this U. Thus, the sum

$$v \nabla_U^- = \sum_{\substack{w \in S_n \text{ agrees with } v \text{ on} \\ \text{all elements outside of } U}} \pm w$$

contains the permutation v as one of its addends. All its remaining addends have the form $\pm w$ where the permutation $w \in S_n$ satisfies $w > v$ in lexicographic order (because if $w \in S_n$ agrees with v on all elements outside of U but is distinct from v, then Lemma 2.5.8 yields $w \geq v$ and therefore $w > v$). Hence,

$$v \nabla_U^- = v \pm (\text{some permutations } w > v).$$

Therefore,

$$v = v \nabla_U^- \pm (\text{some permutations } w > v). \tag{41}$$

But the definition of $\mathcal{J}_k$ yields $v\nabla_U^- \in \mathcal{J}_k$. Thus, projecting the equality (41) onto the quotient $\mathcal{A}/\mathcal{J}_k$, we obtain

$$\overline{v} = \underbrace{\overline{v\nabla_U^-}}_{\substack{=0 \\ \text{(since } v\nabla_U^- \in \mathcal{J}_k\text{)}}} \pm \overline{\text{(some permutations } w > v)}$$

$$= -\underbrace{\overline{\text{(some permutations } w > v)}}_{\substack{\in \text{span}\left((\overline{w})_{w \in \text{Av}_n(k+1)}\right) \\ \text{(by our induction hypothesis)}}} \in \text{span}\left((\overline{w})_{w \in \text{Av}_n(k+1)}\right).$$

In other words, (40) holds for $u = v$. This completes the induction. Thus, Lemma 2.5.9 is proved. $\qquad\square$

> **Lemma 2.5.10.** Let $k \in \mathbb{N}$. Let $(\alpha_w)_{w \in \text{Av}_n(k+1)} \in \mathbf{k}^{\text{Av}_n(k+1)}$ be a family of scalars satisfying
>
> $$\sum_{w \in \text{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k^\perp.$$
>
> Then, $\alpha_w = 0$ for all $w \in \text{Av}_n(k+1)$.

Proof. Assume the contrary. Thus, there exist some $w \in \text{Av}_n(k+1)$ such that $\alpha_w \neq 0$. Let v be the lexicographically smallest such w. Thus, $\alpha_v \neq 0$, but

$$\alpha_w = 0 \qquad \text{for every } w \in \text{Av}_n(k+1) \text{ satisfying } w < v. \tag{42}$$

As in the proof of Lemma 2.5.7, we can construct set decompositions $\mathbf{A}, \mathbf{B} \in \text{SD}(n)$ such that $\ell(\mathbf{A}) = \ell(\mathbf{B}) = k$ and $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$ and

$$\nabla_{\mathbf{B},\mathbf{A}} = v + (\text{some permutations } w < v) \tag{43}$$

hold. Consider these $\mathbf{A}$ and $\mathbf{B}$. The $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle$ has the property that for every $w \in S_n$ and every $\mathbf{a} \in \mathcal{A}$, the coefficient of w in $\mathbf{a}$ is $\langle w, \mathbf{a} \rangle$. Hence, the equality (43) shows that

$$\langle v, \nabla_{\mathbf{B},\mathbf{A}} \rangle = 1 \tag{44}$$

and

$$\langle w, \nabla_{\mathbf{B},\mathbf{A}} \rangle = 0 \qquad \text{for each permutation } w > v. \tag{45}$$

From $\sum_{w \in \text{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k^\perp$ and $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$, we conclude that

$$\left\langle \sum_{w \in \text{Av}_n(k+1)} \alpha_w w, \ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle = 0.$$

Thus,

$$0 = \left\langle \sum_{w \in \text{Av}_n(k+1)} \alpha_w w, \ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle$$

$$= \sum_{w \in \text{Av}_n(k+1)} \underbrace{\alpha_w}_{\substack{=0 \text{ if } w<v \\ (\text{by } (42))}} \underbrace{\langle w, \ \nabla_{\mathbf{B},\mathbf{A}} \rangle}_{\substack{=0 \text{ if } w>v \\ (\text{by } (45))}}$$

$$\underbrace{}_{\substack{=0 \text{ if } w \neq v \\ (\text{since } w \neq v \text{ entails } w<v \text{ or } w>v)}}$$

$$= \alpha_v \underbrace{\langle v, \ \nabla_{\mathbf{B},\mathbf{A}} \rangle}_{\substack{=1 \\ (\text{by } (44))}} = \alpha_v \neq 0,$$

which is absurd. This completes the proof of Lemma 2.5.10. $\qquad\square$

Lemma 2.5.11. Let $k \in \mathbb{N}$. Let $(\alpha_w)_{w \in S_n \setminus \text{Av}_n(k+1)} \in \mathbf{k}^{S_n \setminus \text{Av}_n(k+1)}$ be a family of scalars satisfying

$$\sum_{w \in S_n \setminus \text{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^{\perp}.$$

Then, $\alpha_w = 0$ for all $w \in S_n \setminus \text{Av}_n(k+1)$.

Proof. Assume the contrary. Thus, there exist some $w \in S_n \setminus \text{Av}_n(k+1)$ such that $\alpha_w \neq 0$. Let v be the lexicographically largest such w. Thus, $\alpha_v \neq 0$, but

$$\alpha_w = 0 \qquad \text{for every } w \in S_n \setminus \text{Av}_n(k+1) \text{ satisfying } w > v. \qquad (46)$$

As in the proof of Lemma 2.5.9, we can construct a $(k+1)$-element subset U of $[n]$ such that $v\nabla_U^- \in \mathcal{J}_k$ and

$$v\nabla_U^- = v \pm (\text{some permutations } w > v). \qquad (47)$$

Consider this U. The $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle$ has the property that for every $w \in S_n$ and every $\mathbf{a} \in \mathcal{A}$, the coefficient of w in $\mathbf{a}$ is $\langle w, \mathbf{a} \rangle$. Hence, the equality (47) shows that

$$\langle v, \ v\nabla_U^- \rangle = 1 \qquad (48)$$

and

$$\langle w, \ v\nabla_U^- \rangle = 0 \qquad \text{for each permutation } w < v. \qquad (49)$$

From $\sum_{w \in S_n \setminus \text{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^{\perp}$ and $v\nabla_U^- \in \mathcal{J}_k$, we conclude that

$$\left\langle \sum_{w \in S_n \setminus \text{Av}_n(k+1)} \alpha_w w, \ v\nabla_U^- \right\rangle = 0.$$

Thus,

$$
\begin{aligned}
0 &= \left\langle \sum_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \alpha_w w, \; v\nabla_U^- \right\rangle \\
&= \sum_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \underbrace{\alpha_w}_{\substack{=0 \text{ if } w>v \\ \text{(by (46))}}} \underbrace{\left\langle w, \; v\nabla_U^- \right\rangle}_{\substack{=0 \text{ if } w<v \\ \text{(by (49))}}} \\
&\underbrace{}_{\substack{=0 \text{ if } w \neq v \\ \text{(since } w\neq v \text{ entails } w<v \text{ or } w>v)}} \\
&= \alpha_v \underbrace{\left\langle v, \; v\nabla_U^- \right\rangle}_{\substack{=1 \\ \text{(by (48))}}} = \alpha_v \neq 0,
\end{aligned}
$$

which is absurd. This completes the proof of Lemma 2.5.11. $\qquad\square$

Lemma 2.5.12. Let $k \in \mathbb{N}$. Then, the **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$.

Proof. The family $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$ spans this **k**-module $\mathcal{A}/\mathcal{I}_k$, as we know from Lemma 2.5.7. It remains to prove that it is **k**-linearly independent.

Let $(\alpha_w)_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \in \mathbf{k}^{S_n \setminus \mathrm{Av}_n(k+1)}$ be a family of scalars satisfying

$$
\sum_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \alpha_w \overline{w} = 0. \tag{50}
$$

We thus need to show that $\alpha_w = 0$ for all $w \in S_n \setminus \mathrm{Av}_n(k+1)$.

However, (50) means that $\sum_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k$. But Lemma 2.5.4 yields $\mathcal{I}_k \mathcal{J}_k = 0$. Thus, $\mathcal{I}_k \subseteq \mathrm{LAnn}\, \mathcal{J}_k$. However, Proposition 2.2.2 **(c)** yields $S(\mathcal{J}_k) = \mathcal{J}_k$. Furthermore, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ (by Proposition 2.2.2 **(a)**), hence a left ideal of $\mathcal{A}$. Thus, Lemma 2.5.2 **(b)** (applied to $\mathcal{B} = \mathcal{J}_k$) yields $\mathcal{J}_k^\perp = \mathrm{LAnn}(\mathcal{J}_k) = \mathrm{RAnn}(\mathcal{J}_k)$. Thus,

$$
\sum_{w \in S_n \setminus \mathrm{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k \subseteq \mathrm{LAnn}\, \mathcal{J}_k = \mathcal{J}_k^\perp.
$$

Lemma 2.5.11 thus yields that $\alpha_w = 0$ for all $w \in S_n \setminus \mathrm{Av}_n(k+1)$. This completes the proof of Lemma 2.5.12. $\qquad\square$

Lemma 2.5.13. Let $k \in \mathbb{N}$. Then, the **k**-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w \in \mathrm{Av}_n(k+1)}$.

Proof. Analogous to the proof of Lemma 2.5.12. (Of course, use Lemma 2.5.9 and Lemma 2.5.10 instead of Lemma 2.5.7 and Lemma 2.5.11 now.) $\qquad\square$

Lemma 2.5.14. Let $k \in \mathbb{N}$. Then,

$$\mathcal{I}_k = \mathcal{J}_k^\perp = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k.$$

Proof. Proposition 2.2.2 **(c)** yields $S(\mathcal{J}_k) = \mathcal{J}_k$. Furthermore, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ (by Proposition 2.2.2 **(a)**). Thus, Lemma 2.5.2 **(b)** (applied to $\mathcal{B} = \mathcal{J}_k$) yields $\mathcal{J}_k^\perp = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k$. Thus, it remains to prove that $\mathcal{I}_k = \mathcal{J}_k^\perp$.

Lemma 2.5.4 yields $\mathcal{I}_k \mathcal{J}_k = 0$. Thus, $\mathcal{I}_k \subseteq \operatorname{LAnn} \mathcal{J}_k = \mathcal{J}_k^\perp$. Thus, we only need to show that $\mathcal{J}_k^\perp \subseteq \mathcal{I}_k$.

Let $a \in \mathcal{J}_k^\perp$. We must prove that $a \in \mathcal{I}_k$.

Lemma 2.5.7 shows that the quotient module $\mathcal{A}/\mathcal{I}_k$ is spanned by the family $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}$. Hence, the projection $\overline{a} \in \mathcal{A}/\mathcal{I}_k$ can be written as a **k**-linear combination of this family. In other words, we can write $\overline{a}$ as

$$\overline{a} = \sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w \overline{w} \tag{51}$$

for some family $(\alpha_w)_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \in \mathbf{k}^{S_n \setminus \operatorname{Av}_n(k+1)}$ of scalars. Consider this family. We can rewrite (51) as

$$a - \sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k \subseteq \mathcal{J}_k^\perp.$$

Since $a \in \mathcal{J}_k^\perp$, this yields $\sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^\perp$. By Lemma 2.5.11, we thus conclude that $\alpha_w = 0$ for all $w \in S_n \setminus \operatorname{Av}_n(k+1)$. Thus, (51) rewrites as $\overline{a} = \sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} 0\overline{w} = 0$, so that $a \in \mathcal{I}_k$. This completes our proof of Lemma 2.5.14. $\quad\square$

Lemma 2.5.15. Let $k \in \mathbb{N}$. Then,

$$\mathcal{J}_k = \mathcal{I}_k^\perp = \operatorname{LAnn} \mathcal{I}_k = \operatorname{RAnn} \mathcal{I}_k.$$

Proof. Analogous to the proof of Lemma 2.5.14. (Of course, use Lemma 2.5.9 and Lemma 2.5.10 instead of Lemma 2.5.7 and Lemma 2.5.11 now.) $\quad\square$

Lemma 2.5.16. Assume that $n!$ is invertible in **k**. Let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Assume that there exists a **k**-linear projection $\pi : \mathcal{A} \to \mathcal{I}$. Then, $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ that has a unity.

Proof. Note that $\mathcal{I}$ is an ideal of $\mathcal{A}$, thus a left ideal of $\mathcal{A}$, hence a left $\mathcal{A}$-submodule of $\mathcal{A}$. Moreover, $|S_n| = n!$ is invertible in **k**. Hence, the standard proof of the Maschke theorem (via averaging the projection π over S_n) yields that there exists a

k-linear projection $\pi' : \mathcal{A} \to \mathcal{I}$ that is a left $\mathcal{A}$-module homomorphism[8]. Consider this π'.

Let $e := \pi'(1) \in \mathcal{I}$. Then, we claim that

$$ue = u \qquad \text{for each } u \in \mathcal{I}. \tag{52}$$

[*Proof of (52):* Let $u \in \mathcal{I}$. Then, $\pi'(u) = u$ (since π' is a projection). However, π' is a left $\mathcal{A}$-module homomorphism. Thus, $\pi'(u1) = u \underbrace{\pi'(1)}_{=e} = ue$. Since $u1 = u$, we can rewrite this as $\pi'(u) = ue$. Hence, $ue = \pi'(u) = u$. This proves (52).]

Clearly, $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ (since $\mathcal{I}$ is an ideal of $\mathcal{A}$). From (52), we see that this algebra $\mathcal{I}$ has a right unity (namely, e). A similar argument (using right instead of left $\mathcal{A}$-modules) yields that $\mathcal{I}$ has a left unity. Thus, a standard argument shows that $\mathcal{I}$ has a unity (since any binary operation that has a left neutral element and a right neutral element has a neutral element). Thus, Lemma 2.5.16 is proved. $\qquad\square$

> **Lemma 2.5.17.** Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathcal{A}$ such that $\mathcal{I} = \operatorname{LAnn}\mathcal{J}$ and $\mathcal{J} = \operatorname{LAnn}\mathcal{I}$. Assume that $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ that has a unity. Then, $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as **k**-module. Moreover, $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ that have unities and satisfy $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as **k**-algebras.

Proof. Clearly, $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ (since any ideal of $\mathcal{A}$ is a nonunital subalgebra).

From $\mathcal{I} = \operatorname{LAnn}\mathcal{J}$, we obtain $\mathcal{I}\mathcal{J} = 0$. Similarly, $\mathcal{J}\mathcal{I} = 0$.

We have assumed that $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ that has a unity. Let $1_{\mathcal{I}}$ denote its unity.

Set $g := 1 - 1_{\mathcal{I}}$. Then, each $u \in \mathcal{I}$ satisfies

$$gu = (1 - 1_{\mathcal{I}})u = u - \underbrace{1_{\mathcal{I}}u}_{\substack{=u \\ \text{(since } 1_{\mathcal{I}} \text{ is the} \\ \text{unity of } \mathcal{I})}} = u - u = 0.$$

In other words, $g \in \operatorname{LAnn}\mathcal{I} = \mathcal{J}$. Moreover, each $v \in \mathcal{J}$ satisfies $1_{\mathcal{I}}v = 0$ (since $\underbrace{1_{\mathcal{I}}}_{\in \mathcal{I}} \underbrace{v}_{\in \mathcal{J}} \in \mathcal{I}\mathcal{J} = 0$) and thus

$$\underbrace{g}_{=1-1_{\mathcal{I}}} v = (1 - 1_{\mathcal{I}})v = v - \underbrace{1_{\mathcal{I}}v}_{=0} = v.$$

---

[8]Explicitly, π' can be constructed as follows:

$$\pi'(a) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma\pi\left(\sigma^{-1}a\right) \qquad \text{for each } a \in \mathcal{A}.$$

For a concrete reference, see [Grinbe25, Theorem 4.4.14]. Note that the existence of a **k**-linear projection from $\mathcal{A}$ onto $\mathcal{I}$ is equivalent to saying that $\mathcal{I}$ is a direct addend of $\mathcal{A}$ as a **k**-module; furthermore, the same holds when each appearance of "**k**-" is replaced by "left $\mathcal{A}$-".

Hence, g is a left unity of the algebra $\mathcal{J}$ (since $g \in \mathcal{J}$). A similar computation shows that g is a right unity of $\mathcal{J}$. Hence, g is a unity of $\mathcal{J}$. We shall thus rename g as $1_{\mathcal{J}}$ now. Of course, this shows that the nonunital algebra $\mathcal{J}$ has a unity.

Moreover, each $u \in \mathcal{I} \cap \mathcal{J}$ satisfies $u \in \mathcal{I} \cap \mathcal{J} \subseteq \mathcal{J}$ and therefore

$$u = \underbrace{u}_{\in \mathcal{I} \cap \mathcal{J} \subseteq \mathcal{I}} \underbrace{1_{\mathcal{J}}}_{\in \mathcal{J}} \qquad \text{(since } 1_{\mathcal{J}} \text{ is the unity of } \mathcal{J})$$
$$\in \mathcal{I} \mathcal{J} = 0$$

and thus $u = 0$. In other words, $\mathcal{I} \cap \mathcal{J} = 0$. Furthermore, each $a \in \mathcal{A}$ satisfies

$$a = 1_{\mathcal{I}} a + \underbrace{(1 - 1_{\mathcal{I}})}_{=g=1_{\mathcal{J}}} a = \underbrace{1_{\mathcal{I}} a}_{\substack{\in \mathcal{I} \\ \text{(since } \mathcal{I} \text{ is an ideal} \\ \text{and } 1_{\mathcal{I}} \in \mathcal{I})}} + \underbrace{1_{\mathcal{J}} a}_{\substack{\in \mathcal{J} \\ \text{(since } \mathcal{J} \text{ is an ideal} \\ \text{and } 1_{\mathcal{J}} \in \mathcal{J})}} \in \mathcal{I} + \mathcal{J}.$$

This shows that $\mathcal{I} + \mathcal{J} = \mathcal{A}$. Combining this with $\mathcal{I} \cap \mathcal{J} = 0$, we conclude that $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as **k**-module. Hence, the **k**-linear map

$$\mathcal{I} \times \mathcal{J} \to \mathcal{A},$$
$$(i, j) \mapsto i + j$$

is a **k**-module isomorphism. This isomorphism furthermore respects the multiplication (since $\mathcal{I} \mathcal{J} = \mathcal{J} \mathcal{I} = 0$, and thus every $(i, j), (i', j') \in \mathcal{I} \times \mathcal{J}$ satisfy $(i + j)(i' + j') = ii' + \underbrace{ij'}_{\substack{=0 \\ \text{(since } \mathcal{I} \mathcal{J} = 0)}} + \underbrace{ji'}_{\substack{=0 \\ \text{(since } \mathcal{J} \mathcal{I} = 0)}} + jj' = ii' + jj'$), and thus is a nonunital **k**-algebra isomorphism. Hence, it must also respect the unity (since it is an isomorphism), and thus is a **k**-algebra isomorphism. We thus conclude that $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as **k**-algebras. This completes the proof of Lemma 2.5.17. $\qquad \square$

> **Lemma 2.5.18.** Assume that $n!$ is invertible in **k**. Let J be a subset of S_n. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathcal{A}$ such that $\mathcal{I} = \operatorname{LAnn} \mathcal{J}$ and $\mathcal{J} = \operatorname{LAnn} \mathcal{I}$. Assume that the family $(\overline{w})_{w \in J}$ is a basis of the **k**-module $\mathcal{A}/\mathcal{I}$. Then, $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as **k**-module. Moreover, $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ that have unities and satisfy $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as **k**-algebras.

Proof. Clearly, $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ (since any ideal of $\mathcal{A}$ is a nonunital subalgebra).

Let $K := S_n \setminus J$. Thus, J and K are two disjoint subsets of S_n such that $J \cup K = S_n$. Moreover, $(w)_{w \in S_n}$ is a basis of the **k**-module $\mathcal{A}$, whereas $(\overline{w})_{w \in J}$ is a basis of the **k**-module $\mathcal{A}/\mathcal{I}$. Hence, Lemma 2.5.3 **(b)** (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{I}$ and $I = S_n$ and $(m_i)_{i \in I} = (w)_{w \in S_n}$) yields that there exists a **k**-linear projection $\pi : \mathcal{A} \to \mathcal{I}$ (that is, a **k**-linear map $\pi : \mathcal{A} \to \mathcal{I}$ such that $\pi \mid_{\mathcal{I}} = \operatorname{id}$). Hence, Lemma 2.5.16 shows that $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ that has a unity.

Thus, Lemma 2.5.17 shows that $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as **k**-module, and furthermore, it shows that $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ that have unities and satisfy $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as **k**-algebras. This proves Lemma 2.5.18. $\qquad \square$

2.6. Proof of the main theorem

We can now prove Theorem 2.4.1 and Corollary 2.4.2 by combining what we have shown so far:

Proof of Theorem 2.4.1. **(a)** This is just Lemma 2.5.14.

(b) This is just Lemma 2.5.15.

(c) Lemma 2.5.12 yields that the **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$. Hence, Lemma 2.5.3 **(a)** (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{I}_k$ and $I = S_n$ and $J = S_n \setminus \mathrm{Av}_n(k+1)$ and $K = \mathrm{Av}_n(k+1)$ and $(m_i)_{i \in I} = (w)_{w \in S_n}$) yields that the **k**-module $\mathcal{I}_k$ is free of rank $|\mathrm{Av}_n(k+1)|$. This proves Theorem 2.4.1 **(c)**.

(d) This is proved similarly to part **(c)**, but using Lemma 2.5.13 instead of Lemma 2.5.12. (This time, Lemma 2.5.3 **(a)** must be applied to $J = \mathrm{Av}_n(k+1)$ and $K = S_n \setminus \mathrm{Av}_n(k+1)$.)

(e) This is just Lemma 2.5.12.

(f) This is just Lemma 2.5.13.

(g) Proposition 2.2.2 **(a)** yields that both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$. Theorem 2.4.1 **(a)** yields $\mathcal{I}_k = \mathrm{LAnn}\, \mathcal{J}_k$. Theorem 2.4.1 **(b)** yields $\mathcal{J}_k = \mathrm{LAnn}\, \mathcal{I}_k$. Clearly, $S_n \setminus \mathrm{Av}_n(k+1)$ and $\mathrm{Av}_n(k+1)$ are two disjoint subsets of S_n such that $(S_n \setminus \mathrm{Av}_n(k+1)) \cup \mathrm{Av}_n(k+1) = S_n$. Theorem 2.4.1 **(e)** says that the **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$. Hence, Lemma 2.5.18 (applied to $J = S_n \setminus \mathrm{Av}_n(k+1)$ and $\mathcal{I} = \mathcal{I}_k$ and $\mathcal{J} = \mathcal{J}_k$) yields that $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as **k**-module, and moreover, $\mathcal{I}_k$ and $\mathcal{J}_k$ are nonunital subalgebras of $\mathcal{A}$ that have unities and satisfy $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as **k**-algebras. This proves Theorem 2.4.1 **(g)**. $\square$

Proof of Corollary 2.4.2. **(a)** Proposition 2.2.2 **(g)** (applied to $k = 2$) yields

$$\mathcal{I}_2 = \mathrm{span}\left\{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) = 2 \right\}$$
$$= \mathrm{span}\left\{ \nabla_{(B,[n]\setminus B),(A,[n]\setminus A)} \mid A, B \subseteq [n] \right\} \tag{53}$$

(since the set decompositions $\mathbf{A} \in \mathrm{SD}(n)$ with $\ell(\mathbf{A}) = 2$ are precisely the pairs of the form $(A, [n] \setminus A)$ for $A \subseteq [n]$). But Proposition 2.1.2 allows us to rewrite the $\nabla_{(B,[n]\setminus B),(A,[n]\setminus A)}$ on the right hand side here as $\nabla_{B,A}$. Thus, (53) rewrites as $\mathcal{I}_2 = \mathrm{span}\left\{ \nabla_{B,A} \mid A, B \subseteq [n] \right\}$. This proves part **(a)**.

(b) Theorem 2.4.1 **(c)** (applied to $k = 2$) shows that the **k**-module $\mathcal{I}_2$ is free of rank $|\mathrm{Av}_n(3)|$. But it is known (see, e.g., [Bona22, Corollary 4.8]) that $|\mathrm{Av}_n(3)| = C_n$. Corollary 2.4.2 **(b)** follows from these two observations. $\square$

2.7. Opposite avoidance

Parts of Theorem 2.4.1 can be transformed into "twin" forms by replacing the relevant permutations with their complements (i.e., multiplying them with the permutation with one-line notation $(n, n-1, \ldots, 1)$). To state this, we need an analogue of Definition 2.3.1:

Definition 2.7.1. Let $k \in \mathbb{N}$.

(a) Let $w \in S_n$ be a permutation. We say that w *avoids* $(k+1) k \cdots 1$ if there exists no $(k+1)$-element subset U of $[n]$ such that the restriction $w \mid_U$ is decreasing (i.e., if there exist no $k+1$ elements $i_1 < i_2 < \cdots < i_{k+1}$ of $[n]$ such that $w(i_1) > w(i_2) > \cdots > w(i_{k+1})$).

(b) We let $\mathrm{Av}'_n(k+1)$ denote the set of all permutations $w \in S_n$ that avoid $(k+1) k \cdots 1$.

Proposition 2.7.2. Let $w_0 \in S_n$ be the permutation that sends each $i \in [n]$ to $n+1-i$. Let $k \in \mathbb{N}$ and $w \in S_n$. Then, $w \in \mathrm{Av}'_n(k+1)$ if and only if $w_0 w \in \mathrm{Av}_n(k+1)$.

Proof. Given a subset U of $[n]$, it is clear that the restriction $(w_0 w) \mid_U$ is increasing if and only if the corresponding restriction $w \mid_U$ is decreasing (since the permutation w_0 is strictly decreasing). Hence, $w_0 w$ avoids $12 \cdots (k+1)$ if and only if w avoids $(k+1) k \cdots 1$. Thus, Proposition 2.7.2 follows. $\square$

The set $\mathrm{Av}'_n(k+1)$ allows us to easily obtain a "twin" to parts **(c)**–**(f)** of Theorem 2.4.1:

Corollary 2.7.3. Let $k \in \mathbb{N}$. Then:

(a) The **k**-module $\mathcal{I}_k$ is free of rank $\left| \mathrm{Av}'_n(k+1) \right|$.

(b) The **k**-module $\mathcal{J}_k$ is free of rank $\left| S_n \setminus \mathrm{Av}'_n(k+1) \right|$.

(c) The **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}'_n(k+1)}$. (Here, $\overline{w}$ denotes the projection of $w \in \mathcal{A}$ onto the quotient $\mathcal{A}/\mathcal{I}_k$.)

(d) The **k**-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w \in \mathrm{Av}'_n(k+1)}$. (Here, $\overline{w}$ denotes the projection of $w \in \mathcal{A}$ onto the quotient $\mathcal{A}/\mathcal{J}_k$.)

Proof. **(a)** Proposition 2.7.2 shows that the map

$$S_n \to S_n,$$
$$w \mapsto w_0 w$$

restricts to a bijection from $\mathrm{Av}'_n(k+1)$ to $\mathrm{Av}_n(k+1)$. Hence,

$$\left| \mathrm{Av}_n(k+1) \right| = \left| \mathrm{Av}'_n(k+1) \right|. \tag{54}$$

But Theorem 2.4.1 **(c)** shows that the **k**-module $\mathcal{I}_k$ is free of rank $\left| \mathrm{Av}_n(k+1) \right| = \left| \mathrm{Av}'_n(k+1) \right|$. This proves Corollary 2.7.3 **(a)**.

(b) Analogous to Corollary 2.7.3 **(a)**, but using Theorem 2.4.1 **(d)** instead of Theorem 2.4.1 **(c)**.

(c) Proposition 2.7.2 shows that a permutation $w \in S_n$ satisfies $w \in \mathrm{Av}'_n(k+1)$ if and only if $w_0 w \in \mathrm{Av}_n(k+1)$. Thus, the contrapositive also holds: A permutation $w \in S_n$ satisfies $w \notin \mathrm{Av}'_n(k+1)$ if and only if $w_0 w \notin \mathrm{Av}_n(k+1)$. Therefore, the map

$$S_n \setminus \mathrm{Av}'_n(k+1) \to S_n \setminus \mathrm{Av}_n(k+1),$$
$$w \mapsto w_0 w \tag{55}$$

is a bijection. The inverse of this map must also send each w to $w_0 w$ (because w_0 is an involution, i.e., we have $w_0 w_0 = \mathrm{id}$). Thus, this inverse is the map

$$S_n \setminus \mathrm{Av}_n(k+1) \to S_n \setminus \mathrm{Av}'_n(k+1),$$
$$w \mapsto w_0 w. \tag{56}$$

As a consequence, this latter map is a bijection.

Theorem 2.4.1 **(e)** shows that the **k**-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$. But $\mathcal{I}_k$ is a left ideal of $\mathcal{A}$ and thus fixed under left multiplication by w_0. Hence, the map

$$\mathcal{A}/\mathcal{I}_k \to \mathcal{A}/\mathcal{I}_k,$$
$$\overline{\mathbf{a}} \mapsto \overline{w_0 \mathbf{a}}$$

is well-defined and is an automorphism of the **k**-module $\mathcal{A}/\mathcal{I}_k$ (being invertible because $w_0^2 = \mathrm{id}$). Applying this map to the basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$ of $\mathcal{A}/\mathcal{I}_k$, we thus obtain a new basis $(\overline{w_0 w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$ of $\mathcal{A}/\mathcal{I}_k$ (since the image of a basis under a **k**-module isomorphism is again a basis). But the latter basis $(\overline{w_0 w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$ can be reindexed as $(\overline{w})_{w \in S_n \setminus \mathrm{Av}'_n(k+1)}$, since the map (56) is a bijection. Hence, we have shown that $(\overline{w})_{w \in S_n \setminus \mathrm{Av}'_n(k+1)}$ is a basis of the **k**-module $\mathcal{A}/\mathcal{I}_k$. This proves part **(c)**.

(d) Analogous to Corollary 2.7.3 **(c)**, but using Theorem 2.4.1 **(f)** instead of Theorem 2.4.1 **(e)**. $\qquad\square$

2.8. $\mathcal{J}_k$ and $\mathcal{I}_{n-k-1}$ as annihilators of tensor modules

We shall now discuss how the ideals $\mathcal{J}_k$ and $\mathcal{I}_k$ (more precisely, $\mathcal{I}_{n-k-1}$, but this is just a matter of indexing) can be interpreted as annihilators of certain left $\mathcal{A}$-modules. This breaks no new ground, but rather recovers results by de Concini and Procesi [deCPro76, Theorem 4.2] and Bowman, Doty and Martin [BoDoMa22]; we hope that our elementary approach makes these results more accessible.

2.8.1. $\mathcal{J}_k$ as annihilator of $V_k^{\otimes n}$ (action on places)

Recall that $\mathcal{A} = \mathbf{k}[S_n]$. Thus, the representations of S_n over $\mathbf{k}$ are precisely the left $\mathcal{A}$-modules. If M is any left $\mathcal{A}$-module, then $\operatorname{Ann} M$ shall denote its annihilator, i.e., the ideal

$$\{a \in \mathcal{A} \mid aM = 0\} = \{a \in \mathcal{A} \mid am = 0 \text{ for all } m \in M\}$$

of $\mathcal{A}$. Note that if M is a left ideal of $\mathcal{A}$, then this annihilator $\operatorname{Ann} M$ is just its left annihilator $\operatorname{LAnn} M$.

For each $k \in \mathbb{N}$, we consider the free $\mathbf{k}$-module V_k with basis $(e_1, e_2, \ldots, e_k)$. On its n-th tensor power $V_k^{\otimes n}$, the symmetric group S_n acts by permuting the tensor factors:

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$
$$\text{for all } \sigma \in S_n \text{ and } v_1, v_2, \ldots, v_n \in V_k.$$

Thus, $V_k^{\otimes n}$ is a left $\mathcal{A}$-module for each $k \in \mathbb{N}$. This action of $\mathcal{A}$ (or of S_n) is called *action on places* or *action by place permutation*.

The $\mathbf{k}$-module $V_k^{\otimes n}$ can also be identified with the $\mathbf{k}$-module of homogeneous polynomials of degree n in k noncommutative indeterminates $x_1, x_2, \ldots, x_k$ (via the isomorphism that sends each pure tensor $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ to the noncommutative monomial $x_{i_1} x_{i_2} \cdots x_{i_n}$). Under this identification, the action of S_n permutes the order of factors.

Now we shall prove the following result of de Concini and Procesi [deCPro76, Theorem 4.2]:

Theorem 2.8.1. Let $k \in \mathbb{N}$. Then,

$$\mathcal{J}_k = \operatorname{Ann}\left(V_k^{\otimes n}\right).$$

Proof. We say that two set decompositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$ are *equal-shaped* if they satisfy $\ell(\mathbf{A}) = \ell(\mathbf{B})$ and if each block of $\mathbf{B}$ has the same size as the corresponding block of $\mathbf{A}$. In other words, two set decompositions $(A_1, A_2, \ldots, A_p)$ and $(B_1, B_2, \ldots, B_q)$ of $[n]$ are *equal-shaped* if and only if $p = q$ and $|A_i| = |B_i|$ for each $i \in [p] = [q]$.

An equivalent characterization of equal-shapedness is the following: Two set decompositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$ are equal-shaped if and only if there exists some $w \in S_n$ such that $w\mathbf{A} = \mathbf{B}$. (Indeed, the "if" part is obvious, whereas the "only if" part is easily shown by choosing a permutation $w \in S_n$ that sends each block of $\mathbf{A}$ to the respective block of $\mathbf{B}$.)

Recall that $\operatorname{SD}(n)$ is the set of all set decompositions of $[n]$. For each $\mathbf{A} \in \operatorname{SD}(n)$, we define a $\mathbf{k}$-submodule

$$\mathcal{I}_{\mathbf{A}} := \operatorname{span}\left\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{B} \in \operatorname{SD}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B})\right\}$$

of $\mathcal{A}$. This **k**-submodule $\mathcal{I}_{\mathbf{A}}$ is actually a left ideal of $\mathcal{A}$, since (16) yields

$$w \nabla_{\mathbf{B},\mathbf{A}} = \nabla_{w\mathbf{B},\mathbf{A}} \tag{57}$$

for all $w \in S_n$ and $\mathbf{B} \in \text{SD}(n)$ with $\ell(\mathbf{A}) = \ell(\mathbf{B})$. It is easy to see that as a **k**-module, $\mathcal{I}_{\mathbf{A}}$ has a basis consisting of the elements $\nabla_{\mathbf{B},\mathbf{A}}$, where $\mathbf{B}$ ranges over all those set decompositions of $[n]$ that are equal-shaped to $\mathbf{A}$. (Indeed, these elements span $\mathcal{I}_{\mathbf{A}}$ because Proposition 2.1.1 **(a)** shows that all other $\nabla_{\mathbf{B},\mathbf{A}}$'s are 0. And they are linearly independent because they are nonzero and have disjoint supports, i.e., because they don't have any addends in common.)

It is furthermore clear from Proposition 2.2.2 **(d)** that

$$\mathcal{I}_k = \sum_{\substack{\mathbf{A} \in \text{SD}(n); \\ \ell(\mathbf{A}) \leq k}} \mathcal{I}_{\mathbf{A}}. \tag{58}$$

Hence,

$$\text{Ann}(\mathcal{I}_k) = \bigcap_{\substack{\mathbf{A} \in \text{SD}(n); \\ \ell(\mathbf{A}) \leq k}} \text{Ann}(\mathcal{I}_{\mathbf{A}}) \tag{59}$$

(since the annihilator of a sum of $\mathcal{A}$-submodules is the intersection of their individual annihilators).

On the other hand, let $\mathbf{A} = (A_1, A_2, \ldots, A_m) \in \text{SD}(n)$ be a set decomposition. Then, we can define a diagram $D(\mathbf{A})$ of size n (see [Grinbe25, Definition 5.1.2] for the notion of a diagram) by setting

$$D(\mathbf{A}) := \bigcup_{i=1}^{m} \{(i,1),\, (i,2),\, \ldots,\, (i,|A_i|)\}.$$

This is almost a Young diagram, except that its shape is the weak composition $(|A_1|, |A_2|, \ldots, |A_m|)$ instead of a partition. We consider the Young module $\mathcal{M}^{D(\mathbf{A})}$, which is a left $\mathcal{A}$-module (i.e., an S_n-representation) that (as a **k**-module) has basis $\{n\text{-tabloids of shape } D(\mathbf{A})\}$ (see [Grinbe25, §5.3] for the relevant definitions). If $\overline{T}$ is an n-tabloid of shape $D(\mathbf{A})$, and if $i \in [m]$, then $\text{Row}(i, \overline{T})$ shall denote the set of all entries in the i-th row of $\overline{T}$. Conversely, if $\overline{T}$ is an n-tabloid of shape $D(\mathbf{A})$, and if $k \in [n]$, then $r_{\overline{T}}(k)$ shall denote the number of the row of $\overline{T}$ that contains the entry k.

It is easy to see that if $\ell(\mathbf{A}) \leq k$, then the **k**-linear map

$$\omega_{\mathbf{A}} : \mathcal{M}^{D(\mathbf{A})} \to V_k^{\otimes n},$$
$$\overline{T} \mapsto e_{r_{\overline{T}}(1)} \otimes e_{r_{\overline{T}}(2)} \otimes \cdots \otimes e_{r_{\overline{T}}(n)}$$

is an injective left $\mathcal{A}$-module morphism[9].

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[9]Indeed,

On the other hand, the **k**-linear map

$$\gamma_{\mathbf{A}} : \mathcal{M}^{D(\mathbf{A})} \to \mathcal{I}_{\mathbf{A}},$$
$$\overline{T} \mapsto \nabla_{\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right),\, \mathbf{A}}$$

is a left $\mathcal{A}$-module morphism as well[10]. Next we shall show that this morphism $\gamma_{\mathbf{A}}$ is invertible. Indeed, recall that the **k**-module $\mathcal{I}_{\mathbf{A}}$ has a basis consisting of the elements $\nabla_{\mathbf{B},\mathbf{A}}$, where **B** ranges over all those set decompositions of $[n]$ that are equal-shaped to **A**. In other words,

$$\left(\nabla_{\mathbf{B},\mathbf{A}} \right)_{\mathbf{B} \text{ is a set decomposition of } [n] \text{ that is equal-shaped to } \mathbf{A}} \tag{60}$$

is a basis of the **k**-module $\mathcal{I}_{\mathbf{A}}$. But there is a bijection

$$\{n\text{-tabloids of shape } D\left(\mathbf{A}\right)\}$$
$$\to \{\text{set decompositions } \mathbf{B} \text{ of } [n] \text{ that are equal-shaped to } \mathbf{A}\}$$

that sends each n-tabloid $\overline{T}$ to $\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right)$ (indeed, the lengths of the rows of $D\left(\mathbf{A}\right)$ are the sizes of the respective blocks of **A**, and thus are exactly the right size to fit the blocks of a set decomposition **B** that is equal-shaped to **A**). Hence, we can reindex our basis (60) of $\mathcal{I}_{\mathbf{A}}$ as

$$\left(\nabla_{\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right),\, \mathbf{A}} \right)_{\overline{T} \text{ is an } n\text{-tabloid of shape } D(\mathbf{A})}. \tag{61}$$

Now, the **k**-linear map $\gamma_{\mathbf{A}}$ sends the basis $\left(\overline{T} \right)_{\overline{T} \text{ is an } n\text{-tabloid of shape } D(\mathbf{A})}$ of $\mathcal{M}^{D(\mathbf{A})}$ to the basis (61) of $\mathcal{I}_{\mathbf{A}}$ (by the definition of $\gamma_{\mathbf{A}}$). Thus, it is invertible (since any **k**-linear

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- this map is a left $\mathcal{A}$-module morphism because any n-tabloid $\overline{T}$ and any permutation $w \in S_n$ satisfy $r_{\overline{wT}}(i) = r_{\overline{T}}\left(w^{-1}(i)\right)$ for all $i \in [n]$;

- this map is injective because an n-tabloid $\overline{T}$ of a given shape (in our case, $D\left(\mathbf{A}\right)$) is uniquely determined by the n-tuple $\left(r_{\overline{T}}(1), r_{\overline{T}}(2), \dots, r_{\overline{T}}(n) \right)$.

[10]Indeed, for any n-tabloid $\overline{T}$ of shape $D\left(\mathbf{A}\right)$ and any permutation $w \in S_n$, we have

$$\gamma_{\mathbf{A}}\left(w\overline{T}\right) = \gamma_{\mathbf{A}}\left(\overline{wT}\right) = \nabla_{\left(\text{Row}\left(1,\overline{wT}\right),\, \text{Row}\left(2,\overline{wT}\right),\, \dots,\, \text{Row}\left(m,\overline{wT}\right) \right),\, \mathbf{A}}$$

$$= \nabla_{w\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right),\, \mathbf{A}}$$

$$\begin{pmatrix} \text{since } \text{Row}\left(i,\overline{wT}\right) = w\left(\text{Row}\left(i,\overline{T}\right) \right) \text{ for each } i \in [m], \\ \text{and thus } \left(\text{Row}\left(1,\overline{wT}\right),\, \text{Row}\left(2,\overline{wT}\right),\, \dots,\, \text{Row}\left(m,\overline{wT}\right) \right) \\ = w\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right) \end{pmatrix}$$

$$= w \underbrace{\nabla_{\left(\text{Row}\left(1,\overline{T}\right),\, \text{Row}\left(2,\overline{T}\right),\, \dots,\, \text{Row}\left(m,\overline{T}\right) \right),\, \mathbf{A}}}_{=\gamma_{\mathbf{A}}\left(\overline{T}\right)} \qquad \text{(by (57))}$$

$$= w\gamma_{\mathbf{A}}\left(\overline{T}\right).$$

map that sends a basis of its domain to a basis of its target must be invertible). Therefore, it has an inverse $\gamma_{\mathbf{A}}^{-1} : \mathcal{I}_{\mathbf{A}} \to \mathcal{M}^{D(\mathbf{A})}$, which is also a left $\mathcal{A}$-module isomorphism (since $\gamma_{\mathbf{A}}$ is a left $\mathcal{A}$-module morphism).

Composing this inverse with the injective left $\mathcal{A}$-module morphism $\omega_{\mathbf{A}} : \mathcal{M}^{D(\mathbf{A})} \to V_k^{\otimes n}$ (when $\ell(\mathbf{A}) \le k$), we obtain the injective left $\mathcal{A}$-module morphism

$$\psi_{\mathbf{A}} := \omega_{\mathbf{A}} \circ \gamma_{\mathbf{A}}^{-1} : \mathcal{I}_{\mathbf{A}} \to V_k^{\otimes n}.$$

Forget that we fixed $\mathbf{A}$. We thus have found an injective left $\mathcal{A}$-module morphism

$$\psi_{\mathbf{A}} : \mathcal{I}_{\mathbf{A}} \to V_k^{\otimes n}$$

for each set decomposition $\mathbf{A} \in \mathrm{SD}(n)$ with $\ell(\mathbf{A}) \le k$.

It is easy to see that

$$V_k^{\otimes n} = \sum_{\substack{\mathbf{A} \in \mathrm{SD}(n); \\ \ell(\mathbf{A}) \le k}} \psi_{\mathbf{A}}(\mathcal{I}_{\mathbf{A}}) \tag{62}$$

(indeed, each basis vector $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$ of $V_k^{\otimes n}$ can be written as the image of an appropriate n-tabloid $\overline{T}$ under the map $\omega_{\mathbf{A}}$ (where $\mathbf{A} \in \mathrm{SD}(n)$ is the set decomposition $(A_1, A_2, \ldots, A_k)$ defined by $A_j := \{p \in [n] \mid i_p = j\}$), and thus also as an image under $\omega_{\mathbf{A}} \circ \gamma_{\mathbf{A}}^{-1} = \psi_{\mathbf{A}}$). Hence,

$$\mathrm{Ann}(V_k^{\otimes n}) = \bigcap_{\substack{\mathbf{A} \in \mathrm{SD}(n); \\ \ell(\mathbf{A}) \le k}} \mathrm{Ann}(\psi_{\mathbf{A}}(\mathcal{I}_{\mathbf{A}})) \tag{63}$$

(since the annihilator of a sum of $\mathcal{A}$-submodules is the intersection of their individual annihilators). But each $\mathbf{A} \in \mathrm{SD}(n)$ satisfying $\ell(\mathbf{A}) \le k$ must satisfy $\psi_{\mathbf{A}}(\mathcal{I}_{\mathbf{A}}) \cong \mathcal{I}_{\mathbf{A}}$ as left $\mathcal{A}$-modules (since the left $\mathcal{A}$-module morphism $\psi_{\mathbf{A}}$ is injective) and thus $\mathrm{Ann}(\psi_{\mathbf{A}}(\mathcal{I}_{\mathbf{A}})) = \mathrm{Ann}(\mathcal{I}_{\mathbf{A}})$. Hence, (63) rewrites as

$$\mathrm{Ann}(V_k^{\otimes n}) = \bigcap_{\substack{\mathbf{A} \in \mathrm{SD}(n); \\ \ell(\mathbf{A}) \le k}} \mathrm{Ann}(\mathcal{I}_{\mathbf{A}}).$$

Comparing this with (59), we obtain

$$\mathrm{Ann}(V_k^{\otimes n}) = \mathrm{Ann}(\mathcal{I}_k) = \mathrm{LAnn}(\mathcal{I}_k)$$

(since $\mathrm{Ann}\, M = \mathrm{LAnn}\, M$ for any left ideal M of $\mathcal{A}$). Since Theorem 2.4.1 **(b)** yields $\mathcal{J}_k = \mathrm{LAnn}(\mathcal{I}_k)$, we can rewrite this as $\mathrm{Ann}(V_k^{\otimes n}) = \mathcal{J}_k$. This proves Theorem 2.8.1. $\qquad \square$

Corollary 2.8.2. Let $k \in \mathbb{N}$. Consider the action of $\mathcal{A}$ on $V_k^{\otimes n}$ as a **k**-algebra morphism $\rho_k : \mathcal{A} \to \mathrm{End}_{\mathbf{k}}(V_k^{\otimes n})$. Then, the image $\rho_k(\mathcal{A})$ of this map ρ_k has two bases $(\rho_k(w))_{w \in \mathrm{Av}_n(k+1)}$ and $(\rho_k(w))_{w \in \mathrm{Av}'_n(k+1)}$ (as a **k**-module).

Proof. We have $\operatorname{Ker} \rho_k = \operatorname{Ann}\left(V_k^{\otimes n}\right) = \mathcal{J}_k$ by Theorem 2.8.1. Hence, by the first isomorphism theorem, there is a **k**-algebra isomorphism $\mathcal{A}/\mathcal{J}_k \to \rho_k(\mathcal{A})$ that sends each residue class $\overline{a}$ to $\rho_k(a)$. Hence, we only need to show that the quotient **k**-module $\mathcal{A}/\mathcal{J}_k$ has two bases $(\overline{w})_{w \in \operatorname{Av}_n(k+1)}$ and $(\overline{w})_{w \in \operatorname{Av}'_n(k+1)}$. But this is Theorem 2.4.1 **(f)** and Corollary 2.7.3 **(d)**. $\square$

Theorem 2.8.1 is [deCPro76, Theorem 4.2], and also appears in [Donkin24, Example 2.11]. It is also a particular case of Härterich's [Harter99, §3]. Indeed, Härterich considers a sign-twisted version of the S_n-representation $V_k^{\otimes n}$. The *sign-twist* is the **k**-linear map $T_{\operatorname{sign}} : \mathcal{A} \to \mathcal{A}$ that sends each permutation $w \in S_n$ to $(-1)^w w$. This map is a **k**-algebra automorphism (see [Grinbe25, Theorem 3.11.5]), so it transforms any S_n-representation M into a new S_n-representation M^{sign}, which is called the *sign-twist* of M (see [Grinbe25, Definition 5.18.2] for the precise definition) and which satisfies

$$\operatorname{Ann}\left(M^{\operatorname{sign}}\right) = T_{\operatorname{sign}}\left(\operatorname{Ann} M\right). \tag{64}$$

Härterich's $V^{\otimes n}$ (for $m = k$) is the sign-twist of our S_n-representation $V_k^{\otimes n}$ (since his S_n-action on $V^{\otimes n}$ is given by permuting the factors and multiplying with the sign of the permutation). Thus, the ideal $\mathcal{J} = \operatorname{ann}_{\mathcal{H}_n} V^{\otimes n}$ from [Harter99, §3] for $m = k$ and $q = 1$ is the image of our annihilator $\operatorname{Ann}\left(V_k^{\otimes n}\right)$ under T_{sign} (by (64)). Therefore, our Theorem 2.8.1 entails that this ideal $\mathcal{J}$ is spanned by the $T_{\operatorname{sign}}\left(\nabla_{\mathbf{B},\mathbf{A}}\right)$ for $\mathbf{A}, \mathbf{B} \in \operatorname{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$. This is implicitly what [Harter99, §3] (for $m = k$ and $q = 1$) is saying as well, although the latter theorem additionally picks out a basis of $\mathcal{J}$ from these spanning elements.

We also note that Corollary 2.8.2 recovers [Proces21, Theorem 3.3] and also appears in [BoDoMa22, Corollary 1.4 (i)], in [Donkin24, Example 2.11] and in [RaSaSu12, Theorem 1]. Implicitly, [BaiRai01, Lemma 8.1 and the remark after it] also boils down to our Theorem 2.8.1.

The image $\rho_k(\mathcal{A})$ of $\mathcal{A}$ that was considered in Corollary 2.8.2 is called the "k-swap algebra" in [KlStVo25, §1.3]. Thus, as we saw above, this algebra is isomorphic to the quotient algebra $\mathcal{A}/\mathcal{J}_k$.

2.8.2. $T_{\operatorname{sign}}\left(\mathcal{I}_{n-k-1}\right)$ as annihilator of $N_n^{\otimes k}$ (action on entries)

An interesting counterpart to Theorem 2.8.1 was recently proved by Bowman, Doty and Martin ([BoDoMa18, Theorem 7.4 (a)]; see also [Donkin24, Example 2.13]). We can prove this using our methods, too. Let us first restate this result in our language:

Recall that the *sign-twist* is the **k**-linear map $T_{\operatorname{sign}} : \mathcal{A} \to \mathcal{A}$ that sends each permutation $w \in S_n$ to $(-1)^w w$. This is a **k**-algebra automorphism of $\mathcal{A}$.

Consider a free **k**-module N_n with basis $(e_1, e_2, \ldots, e_n)$, and let the group S_n act on it by permuting the basis vectors (that is, $\sigma \cdot e_i = e_{\sigma(i)}$ for all $\sigma \in S_n$ and $i \in [n]$). This S_n-representation N_n is called the *natural representation* of S_n.

Let $k \in \mathbb{N}$, and consider the k-th tensor power $N_n^{\otimes k}$ of this S_n-representation (where S_n acts diagonally, i.e., by the formula $\sigma \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sigma v_1 \otimes \sigma v_2 \otimes$

$\cdots \otimes \sigma v_k$ for all $\sigma \in S_n$ and all $v_1, v_2, \ldots, v_k \in V$). This is again an S_n-representation, i.e., a left $\mathcal{A}$-module. The action of $\mathcal{A}$ (or S_n) on it is called the *action on entries* (or *diagonal action*).

The **k**-module $N_n^{\otimes k}$ can also be identified with the **k**-module of homogeneous polynomials of degree k in n noncommutative indeterminates $x_1, x_2, \ldots, x_n$ (via the isomorphism that sends each pure tensor $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ to the noncommutative monomial $x_{i_1} x_{i_2} \cdots x_{i_k}$). Under this identification, the action of S_n permutes the indeterminates (i.e., a permutation $\sigma \in S_n$ sends each x_i to $x_{\sigma(i)}$, and acts on products factor by factor).

Now, [BoDoMa18, Theorem 7.4 (a)] characterizes the annihilator of this left $\mathcal{A}$-module $N_n^{\otimes k}$ as follows:

Theorem 2.8.3. Let $k \in \mathbb{N}$. Then,

$$T_{\text{sign}} \left(\mathcal{I}_{n-k-1} \right) = \text{Ann} \left(N_n^{\otimes k} \right).$$

Here, we understand $\mathcal{I}_m$ to mean 0 when $m < 0$ (this is consistent with Definition 2.2.1).

(Once again, this is written in terms of the Murphy cellular basis in [BoDoMa18, Theorem 7.4 (a)], but boils down to what we just said. The same claim appears in a different disguise in [Donkin24, Example 2.13].)

The proof of this theorem will be enabled by the following two lemmas:

Lemma 2.8.4. Let $a_1, a_2, \ldots, a_k \in [n]$. Then,

$$\sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w = T_{\text{sign}} \left(\nabla^-_{[n] \setminus \{a_1, a_2, \ldots, a_k\}} \right).$$

Proof. The definition of $\nabla^-_{[n] \setminus \{a_1, a_2, \ldots, a_k\}}$ yields

$$\nabla^-_{[n] \setminus \{a_1, a_2, \ldots, a_k\}} = \sum_{\substack{w \in S_n; \\ w(i) = i \text{ for all } i \in [n] \setminus ([n] \setminus \{a_1, a_2, \ldots, a_k\})}} (-1)^w w$$

$$= \sum_{\substack{w \in S_n; \\ w(i) = i \text{ for all } i \in \{a_1, a_2, \ldots, a_k\}}} (-1)^w w = \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} (-1)^w w.$$

Applying the **k**-linear map T_{sign} to this equality, we obtain

$$T_{\text{sign}} \left(\nabla^-_{[n] \setminus \{a_1, a_2, \ldots, a_k\}} \right) = \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} \underbrace{(-1)^w T_{\text{sign}}(w)}_{\substack{= w \\ (\text{since } T_{\text{sign}}(w) = (-1)^w w)}} = \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w.$$

This proves Lemma 2.8.4. $\qquad \square$

Lemma 2.8.5. Let $k \in \mathbb{N}$. For any k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in [n]^k$, we define the element

$$\nabla_{\mathbf{a}} := \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w \in \mathcal{A}. \tag{65}$$

Then,

$$T_{\text{sign}} (\mathcal{J}_{n-k-1}) = \mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}.$$

Here, we understand $\mathcal{J}_m$ to mean $\mathcal{A}$ when $m < 0$.

Proof. We are in one of the following two cases:

   Case 1: We have $k < n$.

   Case 2: We have $k \geq n$.

   Let us first consider Case 1. In this case, $k < n$, so that $n - k - 1 \in \mathbb{N}$. Hence, Proposition 2.2.2 **(b)** (applied to $n - k - 1$ instead of k) yields

$$\mathcal{J}_{n-k-1} = \mathcal{A} \cdot \text{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } n - k \right\}.$$

Applying the **k**-algebra isomorphism $T_{\text{sign}} : \mathcal{A} \to \mathcal{A}$ to this equality, we find

$$T_{\text{sign}} (\mathcal{J}_{n-k-1}) = \mathcal{A} \cdot \text{span} \left\{ T_{\text{sign}} (\nabla_U^-) \mid U \text{ is a subset of } [n] \text{ having size } n - k \right\}.$$

   However, if U is a subset of $[n]$ having size $n - k$, and if we denote the k elements of $[n] \setminus U$ by $a_1, a_2, \ldots, a_k$, then

$$
\begin{aligned}
\nabla_{(a_1, a_2, \ldots, a_k)} &= \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w && \text{(by (65))} \\
&= T_{\text{sign}} \left(\nabla_{[n] \setminus \{a_1, a_2, \ldots, a_k\}}^- \right) && \text{(by Lemma 2.8.4)} \\
&= T_{\text{sign}} (\nabla_U^-)
\end{aligned}
$$

(since the definition of $a_1, a_2, \ldots, a_k$ yields $[n] \setminus \{a_1, a_2, \ldots, a_k\} = U$) and thus

$$T_{\text{sign}} (\nabla_U^-) = \nabla_{(a_1, a_2, \ldots, a_k)} \in \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}.$$

Hence,

$$\left\{ T_{\text{sign}} (\nabla_U^-) \mid U \text{ is a subset of } [n] \text{ having size } n - k \right\} \subseteq \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}.$$

Thus,

$$
\begin{aligned}
T_{\text{sign}} (\mathcal{J}_{n-k-1}) &= \mathcal{A} \cdot \text{span} \underbrace{\left\{ T_{\text{sign}} (\nabla_U^-) \mid U \text{ is a subset of } [n] \text{ having size } n - k \right\}}_{\subseteq \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}} \\
&\subseteq \mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}. \tag{66}
\end{aligned}
$$

On the other hand, each k-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in [n]^k$ satisfies

$$\nabla_{\mathbf{a}} = \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w = T_{\text{sign}} \left(\nabla^-_{[n] \setminus \{a_1, a_2, \ldots, a_k\}} \right) \qquad \text{(by Lemma 2.8.4)}$$

$$\in \left\{ T_{\text{sign}} \left(\nabla^-_U \right) \mid U \text{ is a subset of } [n] \text{ having size } > n - k - 1 \right\}$$

(since $[n] \setminus \{a_1, a_2, \ldots, a_k\}$ is a subset of $[n]$ having size $\geq n - k > n - k - 1$). Thus,

$$\left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\} \subseteq \left\{ T_{\text{sign}} \left(\nabla^-_U \right) \mid U \text{ is a subset of } [n] \text{ having size } > n - k - 1 \right\}.$$

Hence,

$$\mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}$$
$$\subseteq \mathcal{A} \cdot \text{span} \left\{ T_{\text{sign}} \left(\nabla^-_U \right) \mid U \text{ is a subset of } [n] \text{ having size } > n - k - 1 \right\}$$
$$= T_{\text{sign}} \left(\mathcal{A} \cdot \text{span} \left\{ \nabla^-_U \mid U \text{ is a subset of } [n] \text{ having size } > n - k - 1 \right\} \right)$$
$$\text{(since } T_{\text{sign}} : \mathcal{A} \to \mathcal{A} \text{ is a } \mathbf{k}\text{-algebra isomorphism)}.$$

In view of

$$\mathcal{J}_{n-k-1} = \mathcal{A} \cdot \text{span} \left\{ \nabla^-_U \mid U \text{ is a subset of } [n] \text{ having size } > n - k - 1 \right\}$$

(by Proposition 2.2.2 **(e)**, applied to $n - k - 1$ instead of k), we can rewrite this as

$$\mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\} \subseteq T_{\text{sign}} \left(\mathcal{J}_{n-k-1} \right).$$

Combining this with (66), we obtain

$$T_{\text{sign}} \left(\mathcal{J}_{n-k-1} \right) = \mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}.$$

Thus, Lemma 2.8.5 is proved in Case 1.

Now, let us consider Case 2. In this case, we have $k \geq n$, so that $n - k - 1 < 0$ and therefore $\mathcal{J}_{n-k-1} = \mathcal{A}$ (by our convention that $\mathcal{J}_m = \mathcal{A}$ for $m < 0$). Hence, $T_{\text{sign}} \left(\mathcal{J}_{n-k-1} \right) = T_{\text{sign}} \left(\mathcal{A} \right) = \mathcal{A}$. On the other hand, the set $[n]^k$ contains the k-tuple $\mathbf{a}_0 := (1, 2, 3, \ldots, n, n, n, \ldots, n)$ (since $k \geq n$), and the corresponding element $\nabla_{\mathbf{a}_0}$ is id (because if we apply (65) to $\mathbf{a} = \mathbf{a}_0$, then the only permutation $w \in S_n$ under the sum is id). Hence, id $= \nabla_{\mathbf{a}_0} \in \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}$. Thus, the left ideal $\mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}$ of $\mathcal{A}$ contains the element id $= 1_{\mathcal{A}}$, whence it must be the whole $\mathcal{A}$. In other words,

$$\mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\} = \mathcal{A}.$$

Comparing this with $T_{\text{sign}} \left(\mathcal{J}_{n-k-1} \right) = \mathcal{A}$, we obtain

$$T_{\text{sign}} \left(\mathcal{J}_{n-k-1} \right) = \mathcal{A} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\}.$$

Thus, Lemma 2.8.5 is proved in Case 2. We have now proved Lemma 2.8.5 in both cases. $\qquad \square$

Proof of Theorem 2.8.3. We extend the definition of $\mathcal{J}_m$ to negative m by setting $\mathcal{J}_m := \mathcal{A}$ when $m < 0$.

Theorem 2.4.1 **(a)** shows that each $m \in \mathbb{N}$ satisfies $\mathcal{I}_m = \mathrm{LAnn}\, \mathcal{J}_m$. This equality also holds for negative m (since $0 = \mathrm{LAnn}\,\mathcal{A}$). Thus, each integer m satisfies $\mathcal{I}_m = \mathrm{LAnn}\, \mathcal{J}_m$. Applying T_{sign} to both sides of this, we obtain

$$T_{\mathrm{sign}}\left(\mathcal{I}_m\right) = T_{\mathrm{sign}}\left(\mathrm{LAnn}\,\mathcal{J}_m\right) = \mathrm{LAnn}\left(T_{\mathrm{sign}}\left(\mathcal{J}_m\right)\right)$$

(since T_{sign} is a **k**-algebra isomorphism). Applying this to $m = n - k - 1$, we obtain

$$T_{\mathrm{sign}}\left(\mathcal{I}_{n-k-1}\right) = \mathrm{LAnn}\left(T_{\mathrm{sign}}\left(\mathcal{J}_{n-k-1}\right)\right) = \mathrm{Ann}\left(T_{\mathrm{sign}}\left(\mathcal{J}_{n-k-1}\right)\right)$$

(since $\mathrm{LAnn}\,\mathcal{K} = \mathrm{Ann}\,\mathcal{K}$ whenever $\mathcal{K}$ is a left ideal of $\mathcal{A}$). It thus remains to show that
$$\mathrm{Ann}\left(T_{\mathrm{sign}}\left(\mathcal{J}_{n-k-1}\right)\right) = \mathrm{Ann}\left(N_n^{\otimes k}\right). \tag{67}$$

We let the symmetric group S_n act on the set $[n]^k$ of all k-tuples of elements of $[n]$ by the rule

$$w\left(a_1, a_2, \ldots, a_k\right) := \left(w\left(a_1\right), w\left(a_2\right), \ldots, w\left(a_k\right)\right)$$
$$\text{for all } w \in S_n \text{ and } \left(a_1, a_2, \ldots, a_k\right) \in [n]^k.$$

For any k-tuple $\mathbf{a} = \left(a_1, a_2, \ldots, a_k\right) \in [n]^k$, we define the element

$$\nabla_{\mathbf{a}} := \sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{a}}} w = \sum_{\substack{w \in S_n; \\ w(a_i) = a_i \text{ for all } i}} w \in \mathcal{A}.$$

This is just the element $T_{\mathrm{sign}}\left(\nabla_{[n]\setminus\{a_1, a_2, \ldots, a_k\}}^-\right)$ (by Lemma 2.8.4). Lemma 2.8.5 yields

$$T_{\mathrm{sign}}\left(\mathcal{J}_{n-k-1}\right) = \mathcal{A} \cdot \mathrm{span}\left\{\nabla_{\mathbf{a}} \;\middle|\; \mathbf{a} \in [n]^k\right\} = \sum_{\mathbf{a} \in [n]^k} \mathcal{A}\nabla_{\mathbf{a}}.$$

Hence,
$$\mathrm{Ann}\left(T_{\mathrm{sign}}\left(\mathcal{J}_{n-k-1}\right)\right) = \bigcap_{\mathbf{a} \in [n]^k} \mathrm{Ann}\left(\mathcal{A}\nabla_{\mathbf{a}}\right) \tag{68}$$

(since the annihilator of a sum of left $\mathcal{A}$-submodules is the intersection of their annihilators).

On the other hand, let us set $e_{\mathbf{a}} := e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_k} \in N_n^{\otimes k}$ for any k-tuple $\mathbf{a} = \left(a_1, a_2, \ldots, a_k\right) \in [n]^k$. Then, the family $\left(e_{\mathbf{a}}\right)_{\mathbf{a} \in [n]^k}$ is a basis of the **k**-module $N_n^{\otimes k}$, and thus also generates $N_n^{\otimes k}$ as a left $\mathcal{A}$-module. In other words,

$$N_n^{\otimes k} = \sum_{\mathbf{a} \in [n]^k} \mathcal{A}e_{\mathbf{a}}.$$

Hence,

$$\text{Ann}\left(N_n^{\otimes k}\right) = \bigcap_{\mathbf{a} \in [n]^k} \text{Ann}\left(\mathcal{A}e_{\mathbf{a}}\right) \tag{69}$$

(since the annihilator of a sum of left $\mathcal{A}$-submodules is the intersection of their annihilators).

We want to show that the left hand sides of (68) and (69) are equal (because this will yield (67)). Of course, it suffices to show that the right hand sides are equal. For this purpose, we will show that $\mathcal{A}\nabla_{\mathbf{a}} \cong \mathcal{A}e_{\mathbf{a}}$ as left $\mathcal{A}$-modules for every $\mathbf{a} \in [n]^k$ (this will suffice, since isomorphic $\mathcal{A}$-modules have the same annihilator).

But this is indeed quite easy: Let $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in [n]^k$. Then, we can define a $\mathbf{k}$-linear map

$$\phi : N_n^{\otimes k} \to \mathcal{A},$$
$$e_{\mathbf{c}} \mapsto \sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{c}}} w \qquad \text{for all } \mathbf{c} \in [n]^k$$

(this is well-defined, since $(e_{\mathbf{c}})_{\mathbf{c} \in [n]^k}$ is a basis of the $\mathbf{k}$-module $N_n^{\otimes k}$). It is easy to see that this map ϕ is S_n-equivariant (because every $u \in S_n$ and $\mathbf{c} \in [n]^k$ satisfy $ue_{\mathbf{c}} = e_{u\mathbf{c}}$ and $u \sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{c}}} w = \sum_{\substack{w \in S_n; \\ w\mathbf{a} = u\mathbf{c}}} w$) and thus left $\mathcal{A}$-linear. Moreover, it sends $e_{\mathbf{a}}$ to $\sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{a}}} w = \nabla_{\mathbf{a}}$ (by the definition of $\nabla_{\mathbf{a}}$). Hence, $\phi(\mathcal{A}e_{\mathbf{a}}) = \mathcal{A}\nabla_{\mathbf{a}}$ (since ϕ is left $\mathcal{A}$-linear).

The map ϕ itself is not usually injective. However, we claim that its restriction to $\mathcal{A}e_{\mathbf{a}}$ is injective. Indeed, recall again that $ue_{\mathbf{c}} = e_{u\mathbf{c}}$ for all $u \in S_n$ and $\mathbf{c} \in [n]^k$. In particular, $ue_{\mathbf{a}} = e_{u\mathbf{a}}$ for all $u \in S_n$. Hence, the $\mathbf{k}$-module $\mathcal{A}e_{\mathbf{a}}$ has a basis $(e_{\mathbf{c}})_{\mathbf{c} \in S_n\mathbf{a}}$ (where $S_n\mathbf{a}$ is the orbit of $\mathbf{a} \in [n]^k$ under the S_n-action on $[n]^k$) [11]. The images of all these basis vectors $e_{\mathbf{c}}$ under ϕ are the sums $\sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{c}}} w$, which are $\mathbf{k}$-linearly independent (since they are sums of disjoint sets of permutations, and each of them is nonempty because $\mathbf{c} \in S_n\mathbf{a}$ guarantees the existence of at least one $w \in S_n$ satisfying $w\mathbf{a} = \mathbf{c}$). Thus, the $\mathbf{k}$-linear map ϕ sends the basis vectors $e_{\mathbf{c}}$ of $\mathcal{A}e_{\mathbf{a}}$ to

---

[11]*Proof:* We have $\mathcal{A} = \text{span}\{u \mid u \in S_n\}$. Thus,

$$\mathcal{A}e_{\mathbf{a}} = (\text{span}\{u \mid u \in S_n\})\, e_{\mathbf{a}} = \text{span}\{ue_{\mathbf{a}} \mid u \in S_n\}$$
$$= \text{span}\underbrace{\{e_{u\mathbf{a}} \mid u \in S_n\}}_{=\{e_{\mathbf{c}} \mid \mathbf{c} \in S_n\mathbf{a}\}} \qquad (\text{since } ue_{\mathbf{a}} = e_{u\mathbf{a}} \text{ for all } u \in S_n)$$
$$= \text{span}\{e_{\mathbf{c}} \mid \mathbf{c} \in S_n\mathbf{a}\}.$$

Hence, the family $(e_{\mathbf{c}})_{\mathbf{c} \in S_n\mathbf{a}}$ spans the $\mathbf{k}$-module $\mathcal{A}e_{\mathbf{a}}$. Since this family is furthermore $\mathbf{k}$-linearly independent (being a subfamily of the basis $(e_{\mathbf{c}})_{\mathbf{c} \in [n]^k}$ of $N_n^{\otimes k}$), we thus conclude that it is a basis of $\mathcal{A}e_{\mathbf{a}}$.

---

k-linearly independent vectors in $\mathcal{A}$. Consequently, the restriction of this map ϕ to $\mathcal{A}e_{\mathbf{a}}$ is injective.

Hence, $\phi(\mathcal{A}e_{\mathbf{a}}) \cong \mathcal{A}e_{\mathbf{a}}$. In view of $\phi(\mathcal{A}e_{\mathbf{a}}) = \mathcal{A}\nabla_{\mathbf{a}}$, we can rewrite this as $\mathcal{A}\nabla_{\mathbf{a}} \cong \mathcal{A}e_{\mathbf{a}}$.

As explained above, this completes the proof of Theorem 2.8.3. $\qquad\square$

> **Corollary 2.8.6.** Let $k \in \mathbb{N}$. Consider the action of $\mathcal{A}$ on $N_n^{\otimes k}$ as a **k**-algebra morphism $\rho_k : \mathcal{A} \to \operatorname{End}_{\mathbf{k}}(N_n^{\otimes k})$. Then, the image $\rho_k(\mathcal{A})$ of this map ρ_k has two bases $(\rho_k(w))_{w \in S_n \setminus \operatorname{Av}_n(n-k)}$ and $(\rho_k(w))_{w \in S_n \setminus \operatorname{Av}'_n(n-k)}$ (as a **k**-module). Here, we understand $\operatorname{Av}_n(m)$ to be $\varnothing$ when $m < 0$.

Proof. We have $\operatorname{Ker}\rho_k = \operatorname{Ann}(N_n^{\otimes k}) = T_{\operatorname{sign}}(\mathcal{I}_{n-k-1})$ by Theorem 2.8.3. Hence, by the first isomorphism theorem, there is a **k**-algebra isomorphism $\mathcal{A}/T_{\operatorname{sign}}(\mathcal{I}_{n-k-1}) \to \rho_k(\mathcal{A})$ that sends each residue class $\bar{a}$ to $\rho_k(a)$. Hence, we only need to show that the quotient **k**-module $\mathcal{A}/T_{\operatorname{sign}}(\mathcal{I}_{n-k-1})$ has two bases $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(n-k)}$ and $(\overline{w})_{w \in S_n \setminus \operatorname{Av}'_n(n-k)}$. Upon applying the **k**-algebra automorphism T_{sign} of $\mathcal{A}$ to this statement, it takes the following simpler form: The quotient **k**-module $\mathcal{A}/\mathcal{I}_{n-k-1}$ has two bases $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(n-k)}$ and $(\overline{w})_{w \in S_n \setminus \operatorname{Av}'_n(n-k)}$ (indeed, the bases still look the same, because T_{sign} sends each permutation $w \in S_n$ to $(-1)^w w = \pm w$, and thus the bases are transformed only by scaling some of the basis vectors by -1). In order to prove this statement, we only need to apply Theorem 2.4.1 **(e)** and Corollary 2.7.3 **(c)** to $n-k-1$ instead of k (at least when $k \leq n-1$; but the other case is trivial because in that case, we have $\mathcal{I}_{n-k-1} = 0$ and $\operatorname{Av}_n(n-k) = \varnothing$ and $\operatorname{Av}'_n(n-k) = \varnothing$). $\qquad\square$

Corollary 2.8.6 appears in [Donkin24, Example 2.13] and (in a slightly restated form) in [BoDoMa22, Corollary 1.4 (ii)].

Further related results are found in [DotNym07], [BoDoMa18], [BoDoMa22], [RaSaSu12], [Donkin24] and [BaiRai01, §8].

2.9. The Specht module connection

2.9.1. $\mathcal{I}_k$ and $\mathcal{J}_k$ after Artin–Wedderburn

The following theorem discusses the representation-theoretical significance of the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$. We use some basic representation theory, including the concept of a Specht module (see, e.g., [EGHLSVY11, §5.12] or [Grinbe25, Definition 5.11.1 **(a)**][12]).

In what follows, the notation "$\lambda \vdash n$" shall always mean "λ is a partition of n". In particular, the set of all partitions of n shall be denoted by $\{\lambda \mid \lambda \vdash n\}$ or just $\{\lambda \vdash n\}$. If λ is a partition of n, then the entries of λ will be denoted by $\lambda_1, \lambda_2, \lambda_3, \ldots$, whereas the length of λ will be written as $\ell(\lambda)$.

---

[12]Note that the Specht module corresponding to a partition λ is called V_λ in [EGHLSVY11, §5.12] and is called $\mathcal{S}^\lambda$ in [Grinbe25, Definition 5.11.1 **(a)**]. The definitions are not identical, but they define isomorphic modules (because of [Grinbe25, Theorem 5.5.13 **(b)**]).

Theorem 2.9.1. Assume that $n!$ is invertible in $\mathbf{k}$. For each partition λ of n, let S^λ denote the corresponding Specht module (a left $\mathcal{A}$-module). For each $a \in \mathcal{A}$ and each partition λ of n, we let $a_\lambda \in \operatorname{End}\left(S^\lambda\right)$ denote the action of a on the Specht module S^λ.

Consider the map

$$\mathrm{AW} : \mathcal{A} \to \prod_{\lambda \vdash n} \operatorname{End}\left(S^\lambda\right),$$
$$a \mapsto (a_\lambda)_{\lambda \vdash n}.$$

This map AW is known to be a $\mathbf{k}$-algebra isomorphism. (When $\mathbf{k}$ is a field, this follows from the Artin–Wedderburn decomposition of $\mathcal{A}$, since the S^λ are the absolutely irreducible $\mathcal{A}$-modules; alternatively, this can be derived from [Ruther48, §17, Theorem 12]. For a detailed proof, see [Grinbe25, Theorem 5.14.1].)

For each subset U of $\{\lambda \mid \lambda \vdash n\}$, we consider the subproduct $\prod\limits_{\lambda \in U} \operatorname{End}\left(S^\lambda\right)$ of $\prod\limits_{\lambda \vdash n} \operatorname{End}\left(S^\lambda\right)$. This is an ideal of $\prod\limits_{\lambda \vdash n} \operatorname{End}\left(S^\lambda\right)$. The preimage of this subproduct under AW is thus an ideal of $\mathcal{A}$, and will be denoted by $\mathcal{A}_U$.

Now, let $k \in \mathbb{N}$. Then,

$$\mathcal{I}_k = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}} \qquad \text{and} \qquad \mathcal{J}_k = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}}.$$

The proof of this theorem will rely on the following general fact:

Lemma 2.9.2. Let $\mathcal{M}$ be a $\mathbf{k}$-module. Let $\mathcal{I}$ and $\mathcal{J}$ be two $\mathbf{k}$-submodules of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{I} + \mathcal{J}$. Let $\mathcal{U}$ and $\mathcal{V}$ be two $\mathbf{k}$-submodules of $\mathcal{M}$ such that $\mathcal{I} \subseteq \mathcal{U}$ and $\mathcal{J} \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = 0$. Then, $\mathcal{I} = \mathcal{U}$ and $\mathcal{J} = \mathcal{V}$.

Proof. Let $u \in \mathcal{U}$. We shall show that $u \in \mathcal{I}$.

We have $u \in \mathcal{U} \subseteq \mathcal{M} = \mathcal{I} + \mathcal{J}$. Thus, $u = i + j$ for some $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Consider these i and j. From $u = i + j$, we obtain $j = \underbrace{u}_{\in \mathcal{U}} - \underbrace{i}_{\in \mathcal{I} \subseteq \mathcal{U}} \in \mathcal{U} - \mathcal{U} \subseteq \mathcal{U}$.

Combining this with $j \in \mathcal{J} \subseteq \mathcal{V}$, we obtain $j \in \mathcal{U} \cap \mathcal{V} = 0$. In other words, $j = 0$. Hence, $u = i + \underbrace{j}_{=0} = i \in \mathcal{I}$.

Forget that we fixed u. We thus have shown that $u \in \mathcal{I}$ for each $u \in \mathcal{U}$. In other words, $\mathcal{U} \subseteq \mathcal{I}$. Combined with $\mathcal{I} \subseteq \mathcal{U}$, this yields $\mathcal{I} = \mathcal{U}$. Similarly, we can show $\mathcal{J} = \mathcal{V}$. This proves Lemma 2.9.2. $\qquad\square$

Proof of Theorem 2.9.1. For each $U \subseteq \{\lambda \mid \lambda \vdash n\}$, we have

$$
\begin{aligned}
\mathcal{A}_U &= \mathrm{AW}^{-1}\left(\text{the subproduct } \prod_{\lambda \in U} \mathrm{End}\left(S^\lambda\right) \text{ of } \prod_{\lambda \vdash n} \mathrm{End}\left(S^\lambda\right)\right) \\
&= \{a \in \mathcal{A} \mid \text{the } \lambda\text{-th entry of } \mathrm{AW}\left(a\right) \text{ is } 0 \text{ for all } \lambda \notin U\} \\
&= \{a \in \mathcal{A} \mid a_\lambda = 0 \text{ for all } \lambda \notin U\} \\
&= \left\{a \in \mathcal{A} \mid aS^\lambda = 0 \text{ for all } \lambda \notin U\right\}
\end{aligned} \tag{70}
$$

(since aS^λ is the image of $a_\lambda \in \mathrm{End}\left(S^\lambda\right)$).

We shall first prove the following two claims:

Claim 1: We have $\mathcal{I}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}}$.

Claim 2: We have $\mathcal{J}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}}$.

Proof of Claim 1. Applying (70) to $U = \{\lambda \vdash n \mid \ell\left(\lambda\right) \leq k\}$, we obtain

$$
\begin{aligned}
&\mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}} \\
&= \left\{a \in \mathcal{A} \mid aS^\lambda = 0 \text{ for all } \lambda \vdash n \text{ that don't satisfy } \ell\left(\lambda\right) \leq k\right\}.
\end{aligned}
$$

Hence, in order to prove that $\mathcal{I}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}}$, it suffices to show that all $a \in \mathcal{I}_k$ satisfy $aS^\lambda = 0$ for all partitions $\lambda \vdash n$ that don't satisfy $\ell\left(\lambda\right) \leq k$. Let us prove this.

Let $\lambda \vdash n$ be a partition that doesn't satisfy $\ell\left(\lambda\right) \leq k$. Thus, $\ell\left(\lambda\right) > k$. We must prove that all $a \in \mathcal{I}_k$ satisfy $aS^\lambda = 0$.

Since $\mathcal{I}_k = \mathrm{span}\left\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \mathrm{SC}\left(n\right) \text{ with } \ell\left(\mathbf{A}\right) = \ell\left(\mathbf{B}\right) \leq k\right\}$, it suffices to prove that $\nabla_{\mathbf{B},\mathbf{A}}S^\lambda = 0$ for any two set compositions $\mathbf{A}, \mathbf{B} \in \mathrm{SC}\left(n\right)$ with $\ell\left(\mathbf{A}\right) = \ell\left(\mathbf{B}\right) \leq k$. So let us consider two set compositions $\mathbf{A}, \mathbf{B} \in \mathrm{SC}\left(n\right)$ with $\ell\left(\mathbf{A}\right) = \ell\left(\mathbf{B}\right) \leq k$. We must prove that $\nabla_{\mathbf{B},\mathbf{A}}S^\lambda = 0$.

If T is any Young tableau of shape λ filled with the entries $1, 2, \ldots, n$ (not necessarily standard), then

- we let $\mathcal{R}\left(T\right)$ denote the row group of T (that is, the group of all permutations $\sigma \in S_n$ that preserve the rows of T as sets);

- we let $\mathbf{a}_T := \sum_{\sigma \in \mathcal{R}(T)} \sigma \in \mathcal{A}$ denote the row symmetrizer of T;

- we let $\mathcal{C}\left(T\right)$ denote the column group of T (that is, the group of all permutations $\sigma \in S_n$ that preserve the columns of T as sets);

- we let $\mathbf{b}_T := \sum_{\sigma \in \mathcal{C}(T)} \left(-1\right)^\sigma \sigma \in \mathcal{A}$ denote the column antisymmetrizer of T.

Note that $\mathbf{a}_T$ and $\mathbf{b}_T$ are called $\nabla_{\text{Row } T}$ and $\nabla^-_{\text{Col } T}$ in [Grinbe25, §5.5.1].

Let T_λ be the Young tableau of shape λ filled with the entries $1, 2, \ldots, n$ in the order "row by row, starting with the top row and proceeding down the rows".[13] We denote the elements $\mathbf{a}_{T_\lambda}$ and $\mathbf{b}_{T_\lambda}$ of $\mathcal{A}$ by $\mathbf{a}_\lambda$ and $\mathbf{b}_\lambda$. Thus, [EGHLSVY11, Theorem 5.12.2] or [Grinbe25, Proposition 5.11.19 **(c)**] shows that $S^\lambda \cong \mathcal{A}\mathbf{a}_\lambda\mathbf{b}_\lambda$ as left $\mathcal{A}$-modules[14].

Hence, $S^\lambda \cong \underbrace{\mathcal{A}\mathbf{a}_\lambda}_{\subseteq \mathcal{A}} \mathbf{b}_\lambda \subseteq \mathcal{A}\mathbf{b}_\lambda$. It thus suffices to show that $\nabla_{\mathbf{B},\mathbf{A}}\mathcal{A}\mathbf{b}_\lambda = 0$ (since $\nabla_{\mathbf{B},\mathbf{A}}S^\lambda = 0$ will then follow). In other words, it suffices to show that $\nabla_{\mathbf{B},\mathbf{A}}w\mathbf{b}_\lambda = 0$ for any $w \in S_n$. But this is not hard:

Let $w \in S_n$. Let T be the Young tableau of shape λ obtained from T_λ by applying the permutation w to each entry. (This is $w \rightharpoonup T_\lambda$ in the notations of [Grinbe25, Definition 5.3.9].) Thus, $\mathbf{b}_T = w\mathbf{b}_\lambda w^{-1}$ (by [Grinbe25, Proposition 5.5.11], applied to T_λ instead of T), so that $w\mathbf{b}_\lambda = \mathbf{b}_T w$.

The first column of the tableau T contains $\ell(\lambda)$ entries, and thus contains more than k entries (since $\ell(\lambda) > k$). Hence, at least two entries of this column belong to the same block of $\mathbf{A}$ (by the pigeonhole principle, since $\mathbf{A}$ has only $\ell(\mathbf{A}) \leq k$ blocks). Pick two such entries. Let $\tau \in S_n$ be the transposition that swaps these two entries. This transposition τ thus preserves the blocks of $\mathbf{A}$ (since it swaps two numbers from the same block), and therefore preserves $\nabla_{\mathbf{B},\mathbf{A}}$ from the right (i.e., satisfies $\nabla_{\mathbf{B},\mathbf{A}}\tau = \nabla_{\mathbf{B},\mathbf{A}}$) because of (16). On the other hand, this transposition τ swaps two entries in the first column of T, and thus belongs to the column group $\mathcal{C}(T)$ of T. Hence, $\mathbf{b}_T = (1 - \tau)\eta$ for some $\eta \in \mathcal{A}$ (since $\langle \tau \rangle = \{1, \tau\}$ is a subgroup of the column group $\mathcal{C}(T)$, and since the permutation τ is odd; see [Grinbe25, Proposition 5.5.9 **(b)**] for details). Thus,

$$\nabla_{\mathbf{B},\mathbf{A}} \underbrace{w\mathbf{b}_\lambda}_{=\mathbf{b}_T w} = \nabla_{\mathbf{B},\mathbf{A}} \underbrace{\mathbf{b}_T}_{=(1-\tau)\eta} w = \underbrace{\nabla_{\mathbf{B},\mathbf{A}}(1-\tau)}_{\substack{=\nabla_{\mathbf{B},\mathbf{A}} - \nabla_{\mathbf{B},\mathbf{A}}\tau \\ =0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}}\tau = \nabla_{\mathbf{B},\mathbf{A}})}} \eta w = 0.$$

This completes our proof of $\nabla_{\mathbf{B},\mathbf{A}}S^\lambda = 0$. Thus, as explained above, we have proved Claim 1. $\qquad\square$

Proof of Claim 2. This is fairly similar to the above proof of Claim 1. Here are the main milestones of the proof:

---

[13]For instance, if $\lambda = (4, 2)$, then $T_\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} \end{array}$.

[14]Specifically, this follows from [EGHLSVY11, Theorem 5.12.2] after observing that the a_λ and b_λ in [EGHLSVY11, Theorem 5.12.2] are our $\dfrac{1}{|\mathcal{R}(T)|}\mathbf{a}_\lambda$ and $\dfrac{1}{|\mathcal{C}(T)|}\mathbf{b}_\lambda$ (the fractions here are well-defined, since $|\mathcal{R}(T)|$ and $|\mathcal{C}(T)|$ are divisors of $n!$ and thus invertible in $\mathbf{k}$). Alternatively, this follows by applying [Grinbe25, Theorem 5.5.13 **(b)**] to $T = T_\lambda$ and noticing that $\mathbf{a}_\lambda\mathbf{b}_\lambda = \mathbf{a}_T\mathbf{b}_T = \nabla_{\text{Row } T}\nabla^-_{\text{Col } T} = \mathbf{F}_T$ in the notations of [Grinbe25, Theorem 5.5.13 **(b)**] and that the number h^λ defined in [Grinbe25, Theorem 5.5.13 **(b)**] is invertible in $\mathbf{k}$ (since it is a divisor of $n!$).

Let $\lambda \vdash n$ be a partition that satisfies $\ell(\lambda) \leq k$. We must prove that all $a \in \mathcal{J}_k$ satisfy $aS^\lambda = 0$.

Recall that $\mathcal{J}_k = \mathcal{A} \cdot \mathrm{span}\left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\}$ (by Proposition 2.2.2 **(b)**). Hence, it suffices to show that $\nabla_U^- S^\lambda = 0$ whenever U is a subset of $[n]$ having size $k+1$ (because then, $\mathcal{A} \underbrace{\nabla_U^- S^\lambda}_{=0} = 0$ will automatically follow).

So let us fix a subset U of $[n]$ having size $k+1$. Consider the Young tableau T_λ and the elements $\mathbf{a}_\lambda$ and $\mathbf{b}_\lambda$ defined as in the above proof of Claim 1. Then, $S^\lambda \cong \mathcal{A}\mathbf{a}_\lambda\mathbf{b}_\lambda$. Hence, in order to prove that $\nabla_U^- S^\lambda = 0$, it suffices to show that $\nabla_U^- \mathcal{A}\mathbf{a}_\lambda = 0$. For this, in turn, it suffices to show that $\nabla_U^- w\mathbf{a}_\lambda = 0$ for each $w \in S_n$.

So let $w \in S_n$ be arbitrary. Let T be the Young tableau of shape λ obtained from T_λ by applying the permutation w to each entry. Then, $\mathbf{a}_T = w\mathbf{a}_\lambda w^{-1}$ (by [Grinbe25, Proposition 5.5.11]), so that $w\mathbf{a}_\lambda = \mathbf{a}_T w$.

But the pigeonhole principle shows that there are two distinct elements of U that belong to the same row of our tableau T (since the set U has $k+1$ elements, but the tableau T has only $\ell(\lambda) \leq k$ rows). Let τ be the transposition that swaps these two elements. Then, $\nabla_U^- \tau = -\nabla_U^-$ (by [Grinbe25, Proposition 3.7.4 **(c)**]), but also $\mathbf{a}_T = (1+\tau)\eta$ for some $\eta \in \mathcal{A}$ (since $\langle\tau\rangle = \{1, \tau\}$ is a subgroup of the row group $\mathcal{R}(T)$; see [Grinbe25, Proposition 5.5.8 **(b)**] for details). Thus,

$$\nabla_U^- \underbrace{w\mathbf{a}_\lambda}_{=\mathbf{a}_T w} = \nabla_U^- \underbrace{\mathbf{a}_T}_{=(1+\tau)\eta} w = \underbrace{\nabla_U^-(1+\tau)}_{\substack{=\nabla_U^-+\nabla_U^-\tau\\=0\\(\text{since } \nabla_U^-\tau=-\nabla_U^-)}} \eta w = 0.$$

As we explained above, this completes the proof of Claim 2. $\qquad\square$

However, if X and Y are two disjoint subsets of $\{\lambda \mid \lambda \vdash n\}$, then $\mathcal{A}_X \cap \mathcal{A}_Y = 0$ (since the subproducts $\prod_{\lambda \in X} \mathrm{End}\left(S^\lambda\right)$ and $\prod_{\lambda \in Y} \mathrm{End}\left(S^\lambda\right)$ of $\prod_{\lambda \vdash n} \mathrm{End}\left(S^\lambda\right)$ have intersection 0). Thus,

$$\mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}} \cap \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}} = 0.$$

Meanwhile, Claim 1 and Claim 2 yield

$$\mathcal{I}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}} \qquad \text{and} \qquad \mathcal{J}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}}.$$

Furthermore, Theorem 2.4.1 **(g)** yields $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum), so that $\mathcal{A} = \mathcal{I}_k + \mathcal{J}_k$. Thus, Lemma 2.9.2 (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{I} = \mathcal{I}_k$ and $\mathcal{J} = \mathcal{J}_k$ and $\mathcal{U} = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}}$ and $\mathcal{V} = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}}$) yields

$$\mathcal{I}_k = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \leq k\}} \qquad \text{and} \qquad \mathcal{J}_k = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > k\}}.$$

This proves Theorem 2.9.1. $\qquad\square$

Note that part of Theorem 2.9.1 for $k = 3$ appears in [BCEHK23, Proposition 3.5 (a)]. Moreover, [KlStVo25, Theorem 2.2] can also be viewed as a corollary of

Theorem 2.9.1 (indeed, we already saw that the k-swap algebra from [KlStVo25, Theorem 2.2] is isomorphic to our $\mathcal{A}/\mathcal{J}_k$, which according to Theorem 2.9.1 is in turn isomorphic to $\prod_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \operatorname{End}\left(S^\lambda\right)$).

2.9.2. Application: Counting avoiding permutations

As a consequence of Theorem 2.9.1, we recover a classical enumerative result that is commonly proved using the RSK correspondence (see, e.g., [Bona22, (7.2)], [Stanle23, Corollary 7.23.12]):

> **Corollary 2.9.3.** For each partition λ of n, let f^λ be the number of standard tableaux of shape λ. Let $k \in \mathbb{N}$. Then, the number of all permutations $w \in S_n$ that avoid $12 \cdots (k+1)$ is $\sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \left(f^\lambda\right)^2 = \sum_{\substack{\lambda \vdash n; \\ \lambda_1 \leq k}} \left(f^\lambda\right)^2$.

Proof. Let $\mathbf{k} = \mathbb{Q}$. Let us use the notations of Theorem 2.9.1. Then, the standard basis theorem for Specht modules (see, e.g., [Grinbe25, Lemma 5.9.17]) says that

$$\dim\left(S^\lambda\right) = f^\lambda \tag{71}$$

for each partition λ of n.

However, Theorem 2.9.1 shows that

$$\mathcal{I}_k = \mathcal{A}_{\{\lambda \vdash n \ | \ \ell(\lambda) \leq k\}} \cong \prod_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \operatorname{End}\left(S^\lambda\right) \qquad \text{as } \mathbf{k}\text{-vector spaces}$$

(since $\mathcal{A}_{\{\lambda \vdash n \ | \ \ell(\lambda) \leq k\}}$ was defined as the preimage of $\prod_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \operatorname{End}\left(S^\lambda\right)$ under the isomorphism AW). Thus,

$$\dim\left(\mathcal{I}_k\right) = \dim\left(\prod_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \operatorname{End}\left(S^\lambda\right) \right) = \sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \underbrace{\dim\left(\operatorname{End}\left(S^\lambda\right)\right)}_{\substack{=\left(\dim\left(S^\lambda\right)\right)^2 = \left(f^\lambda\right)^2 \\ \text{(by (71))}}} = \sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \left(f^\lambda\right)^2.$$

On the other hand, Theorem 2.4.1 **(c)** yields

$$\dim\left(\mathcal{I}_k\right) = \left|\operatorname{Av}_n\left(k+1\right)\right|.$$

Comparing these two equalities, we find $\left|\operatorname{Av}_n\left(k+1\right)\right| = \sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \left(f^\lambda\right)^2$. In other words, the number of all permutations $w \in S_n$ that avoid $12 \cdots (k+1)$ is $\sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \left(f^\lambda\right)^2$.

It remains to prove that this sum also equals $\sum\limits_{\substack{\lambda \vdash n; \\ \lambda_1 \leq k}} \left(f^\lambda\right)^2$. But this is easy: The bijection $\{\lambda \vdash n\} \to \{\lambda \vdash n\}$ that sends each partition λ to its transpose λ^t swaps the roles of $\ell(\lambda)$ and λ_1 (that is, we have $\ell(\lambda^t) = \lambda_1$ and $\lambda_1^t = \ell(\lambda)$). Thus, the sums $\sum\limits_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k}} \left(f^\lambda\right)^2$ and $\sum\limits_{\substack{\lambda \vdash n; \\ \lambda_1 \leq k}} \left(f^\lambda\right)^2$ are the same up to reindexing using this bijection.

Hence, Corollary 2.9.3 is proved. $\qquad\square$

2.9.3. $\mathcal{A}/\left(\mathcal{I}_k + T_{\text{sign}}\left(\mathcal{J}_\ell\right)\right)$

Corollary 2.9.3 is not the end of the line. There is a more general result ([Schens60, Theorem 3], [Stanle71, Proposition 17.5]) saying the following:

> **Theorem 2.9.4.** Let $k, \ell \in \mathbb{N}$. Then, the number of all permutations $w \in S_n$ that avoid $12 \cdots (k+1)$ and $(\ell+1)\ell \cdots 1$ is $\sum\limits_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k \text{ and } \lambda_1 \leq \ell}} \left(f^\lambda\right)^2$.

Proof. From [Schens60, Theorem 3] or [Stanle71, Proposition 17.5], we know that for any $c, d \in \mathbb{N}$, we have

$$\begin{aligned}
&(\text{the number of permutations } w \in S_n \text{ such that} \\
&\qquad \text{the longest increasing subsequence of } w \text{ has length } c \\
&\qquad \text{and the longest decreasing subsequence of } w \text{ has length } d) \\
&= \sum\limits_{\substack{\lambda \vdash n; \\ \ell(\lambda) = d \text{ and } \lambda_1 = c}} \left(f^\lambda\right)^2.
\end{aligned}$$

Summing this over all $c \in \{0, 1, \ldots, \ell\}$ and all $d \in \{0, 1, \ldots, k\}$, we obtain

$$\begin{aligned}
&(\text{the number of permutations } w \in S_n \text{ such that} \\
&\qquad \text{the longest increasing subsequence of } w \text{ has length } \leq \ell \\
&\qquad \text{and the longest decreasing subsequence of } w \text{ has length } \leq k) \\
&= \sum\limits_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k \text{ and } \lambda_1 \leq \ell}} \left(f^\lambda\right)^2.
\end{aligned}$$

But the permutations counted on the left hand side are just the permutations in $w \in S_n$ that avoid $12 \cdots (\ell+1)$ and $(k+1)\ell \cdots 1$. Upon multiplying them by w_0 from the left, they become the permutations $w \in S_n$ that avoid $12 \cdots (k+1)$ and $(\ell+1)\ell \cdots 1$ (by Proposition 2.7.2). Thus, the number of the latter permutations is $\sum\limits_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k \text{ and } \lambda_1 \leq \ell}} \left(f^\lambda\right)^2$. This proves Theorem 2.9.4. $\qquad\square$

We may thus wonder:

Question 2.9.5. Can Theorem 2.9.4 be proved using the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$?

One way to approach this is as follows. Let $T_{\text{sign}} : \mathcal{A} \to \mathcal{A}$ be the **k**-algebra automorphism of $\mathcal{A}$ sending each permutation $w \in S_n$ to $(-1)^w w$. If $n!$ is invertible in **k**, then $\mathcal{I}_k \cap T_{\text{sign}}(\mathcal{I}_\ell)$ is a free **k**-submodule of $\mathcal{A}$ having dimension $\sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k \text{ and } \lambda_1 \leq \ell}} (f^\lambda)^2$. Thus, we could answer Question 2.9.5 by finding a **k**-module basis of $\mathcal{I}_k \cap T_{\text{sign}}(\mathcal{I}_\ell)$ indexed by permutations that avoid $12 \cdots (k+1)$ and $(\ell+1) \ell \cdots 1$. However, such a basis cannot be independent on char **k** like our above basis of $\mathcal{I}_k$ was, since (for instance) the subspace $\mathcal{I}_2 \cap T_{\text{sign}}(\mathcal{I}_2)$ for $n = 3$ has dimension 4 when $\mathbf{k} = \mathbb{Q}$ but dimension 5 when $\mathbf{k} = \mathbb{F}_2$. While this does not strictly rule out a proof using this subspace, it suggests that new methods are required, since the techniques we have used to prove Theorem 2.4.1 above cannot create a dependence on the characteristic of **k**. The dimension of the quotient **k**-module $\mathcal{A}/\left(\mathcal{J}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)$ can depend on char **k** as well (for $n = 3$ and $k = \ell = 2$, it equals 4 when $\mathbf{k} = \mathbb{Q}$ but equals 5 when $\mathbf{k} = \mathbb{F}_2$).

However, the "mixed" quotient $\mathcal{I}_k/\left(\mathcal{I}_k \cap T_{\text{sign}}(\mathcal{J}_\ell)\right) \cong \left(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)/T_{\text{sign}}(\mathcal{J}_\ell)$ behaves better. Equivalently, the quotient $\mathcal{A}/\left(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)$ has a basis of size $\text{Av}'_n(\ell+1) \setminus \text{Av}_n(k+1)$ for all **k**:

Theorem 2.9.6. Let $k, \ell \in \mathbb{N}$. Then, the family $(\overline{w})_{\text{Av}'_n(\ell+1)\setminus \text{Av}_n(k+1)}$ is a basis of the **k**-module $\mathcal{A}/\left(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)$.

We shall now outline a proof of this theorem (which might not be new; it is probably a particular case of Donkin's [Donkin24, Corollary 2.12]). However, the proof itself relies on Theorem 2.9.4, so it produces no new proof of Theorem 2.9.4. Thus, Question 2.9.5 remains open.

Outline of the proof of Theorem 2.9.6. The proof consists of two steps: **(a)** showing that the family $(\overline{w})_{\text{Av}'_n(\ell+1)\setminus \text{Av}_n(k+1)}$ spans the **k**-module $\mathcal{A}/\left(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)$, and **(b)** showing that this **k**-module is free of rank $\left|\text{Av}'_n(\ell+1) \setminus \text{Av}_n(k+1)\right|$. Once these two claims are proved, we will be easily able to conclude the proof using [Grinbe25, Lemma 5.21.9].

We begin with step **(a)**. Thus, we set out to prove that the family $(\overline{w})_{\text{Av}'_n(\ell+1)\setminus \text{Av}_n(k+1)}$ spans the **k**-module $\mathcal{A}/\left(\mathcal{I}_k + T_{\text{sign}}(\mathcal{J}_\ell)\right)$. In other words, we must prove that

$$\overline{u} \in \text{span}\left((\overline{w})_{\text{Av}'_n(\ell+1)\setminus \text{Av}_n(k+1)}\right) \qquad \text{for each } u \in S_n. \tag{72}$$

To prove this, we proceed by induction on u as in lexicographic order, as in the proof of (37) above. Thus, we let $v \in S_n$ be arbitrary, and we assume that (72) holds for all $u < v$; we must then prove that (72) holds for $u = v$. We distinguish between three cases:

1. If $v \in \mathrm{Av}'_n (\ell + 1) \setminus \mathrm{Av}_n (k + 1)$, then this claim is obvious.

2. If $v \in \mathrm{Av}_n (k + 1)$, then this claim is proved just as in our above proof of (37), since $\mathcal{I}_k \subseteq \mathcal{I}_k + T_{\mathrm{sign}} (\mathcal{J}_\ell)$.

3. If $v \notin \mathrm{Av}'_n (\ell + 1)$, then we proceed as follows: Set $v' := w_0 v$. Then, $v \notin \mathrm{Av}'_n (\ell + 1)$ entails $v' \notin \mathrm{Av}_n (\ell + 1)$ (by Proposition 2.7.2). Hence, the argument made in the proof of (40) (but applied to ℓ and v' instead of k and v) shows that

$$\overline{v'} = -\overline{(\text{some permutations } w > v')} \qquad \text{in } \mathcal{A} / \mathcal{J}_\ell.$$

Multiplying by $\overline{w_0}$ on the left, we transform this into

$$\overline{w_0 v'} = -\overline{(\text{some permutations } w_0 w \text{ with } w > v')}$$
$$= -\overline{(\text{some permutations } w' < w_0 v')} \qquad \text{in } \mathcal{A} / \mathcal{J}_\ell$$

(because multiplying by w_0 on the left flips the lexicographic order: if $x > y$ in S_n, then $w_0 x < w_0 y$). Since $w_0 v' = v$ (because $v' = w_0 v$ but w_0 is an involution), we can rewrite this as

$$\overline{v} = -\overline{(\text{some permutations } w' < v)} \qquad \text{in } \mathcal{A} / \mathcal{J}_\ell.$$

Applying the **k**-algebra automorphism T_{sign} (or, rather, the **k**-algebra isomorphism $\mathcal{A} / \mathcal{J}_\ell \to \mathcal{A} / T_{\mathrm{sign}} (\mathcal{J}_\ell)$ it induces) to this equality, we obtain

$$\overline{v} = -\overline{(\text{some permutations } w' < v)} \qquad \text{in } \mathcal{A} / T_{\mathrm{sign}} (\mathcal{J}_\ell)$$

(we have suppressed the signs $(-1)^v$ and $(-1)^{w'}$ here, since we are dealing with a linear combination anyway). Thus, this equality also holds in $\mathcal{A} / \left(\mathcal{I}_k + T_{\mathrm{sign}} (\mathcal{J}_\ell) \right)$. From here, we easily obtain (72) using the induction hypothesis.

Thus, (72) is proved, and we conclude that the family $(\overline{w})_{\mathrm{Av}'_n(\ell+1) \setminus \mathrm{Av}_n(k+1)}$ spans the **k**-module $\mathcal{A} / \left(\mathcal{I}_k + T_{\mathrm{sign}} (\mathcal{J}_\ell) \right)$.

Now we come to step **(b)** of our plan: showing that this **k**-module is free of rank $\left| \mathrm{Av}'_n (\ell + 1) \setminus \mathrm{Av}_n (k + 1) \right|$.

Here we will need the Murphy cellular bases (see Remark 2.4.3). Namely, apply-

ing (24) to ℓ instead of k, we obtain $\mathcal{J}_\ell = \mathcal{F}_{\text{std},\text{len}>\ell}^{-\,\text{Col}}$. Thus,

$$
\begin{aligned}
T_{\text{sign}}\left(\mathcal{J}_\ell\right) &= T_{\text{sign}}\left(\mathcal{F}_{\text{std},\text{len}>\ell}^{-\,\text{Col}}\right) \\
&= \text{span}\left\{ T_{\text{sign}}\left(\nabla_{V,U}^{\text{Row}}\right) \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \ell\,(\lambda) > \ell \right\} \\
&= \text{span}\left\{ \nabla_{V\mathbf{r},U\mathbf{r}}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \ell\,(\lambda) > \ell \right\} \\
&\qquad \left(
\begin{array}{c}
\text{here, } T\mathbf{r} \text{ denotes the transpose of a tableau } T, \\
\text{and we have used } T_{\text{sign}}\left(\nabla_{V,U}^{\text{Row}}\right) = \nabla_{V\mathbf{r},U\mathbf{r}}^{\text{Row}} \\
\text{(which is proved in [Grinbe25, Proposition 6.8.12])}
\end{array}
\right) \\
&= \text{span}\left\{ \nabla_{V,U}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \ell\,(\lambda^t) > \ell \right\} \\
&\qquad \left(\text{here, we have substituted } (\lambda^t, U\mathbf{r}, V\mathbf{r}) \ \text{for} \ (\lambda, U, V)\right) \\
&= \text{span}\left\{ \nabla_{V,U}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \lambda_1 > \ell \right\}
\end{aligned}
$$

(since $\ell\,(\lambda^t) = \lambda_1$ for each partition λ). Adding this to

$$
\mathcal{I}_k = \mathcal{F}_{\text{std},\text{len}\leq k}^{\text{Row}} = \text{span}\left\{ \nabla_{V,U}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \ell\,(\lambda) \leq k \right\},
$$

we find

$$
\begin{aligned}
&\mathcal{I}_k + T_{\text{sign}}\left(\mathcal{J}_\ell\right) \\
&= \text{span}\left\{ \nabla_{V,U}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \left(\ell\,(\lambda) \leq k \ \text{or} \ \lambda_1 > \ell\right) \right\} \\
&= \text{span}\left\{ \nabla_{V,U}^{\text{Row}} \ \mid \ (\lambda, U, V) \in \text{SBT}\,(n) \ \text{and} \ \lambda \in X \right\},
\end{aligned}
$$

where

$$
X := \left\{ \lambda \vdash n \ \mid \ \ell\,(\lambda) \leq k \ \text{or} \ \lambda_1 > \ell \right\}.
$$

This clearly shows that $\mathcal{I}_k + T_{\text{sign}}\left(\mathcal{J}_\ell\right)$ is a direct addend of $\mathcal{A}$ as a **k**-module (since it is spanned by a subfamily of the row Murphy basis), and is free of rank

$$
(\text{\# of } (\lambda, U, V) \in \text{SBT}\,(n) \ \text{such that} \ \lambda \in X) = \sum_{\lambda \in X} \left(f^\lambda\right)^2,
$$

whereas the quotient **k**-module $\mathcal{A} / \left(\mathcal{I}_k + T_{\text{sign}} \left(\mathcal{J}_\ell \right) \right)$ is free of rank

$$\sum_{\substack{\lambda \vdash n; \\ \lambda \notin X}} \left(f^\lambda \right)^2 = \sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) > k \text{ and } \lambda_1 \leq \ell}} \left(f^\lambda \right)^2 = \underbrace{\sum_{\substack{\lambda \vdash n; \\ \lambda_1 \leq \ell}} \left(f^\lambda \right)^2}_{\substack{= |\text{Av}_n(\ell+1)| \\ \text{(by Corollary 2.9.3,} \\ \text{applied to } \ell \text{ instead of } k)}} - \underbrace{\sum_{\substack{\lambda \vdash n; \\ \ell(\lambda) \leq k \text{ and } \lambda_1 \leq \ell}} \left(f^\lambda \right)^2}_{\substack{= |\text{Av}_n(k+1) \cap \text{Av}'_n(\ell+1)| \\ \text{(by Theorem 2.9.4)}}}$$

$$= \underbrace{\left\lfloor \text{Av}_n \left(\ell + 1 \right) \right\rfloor}_{\substack{= |\text{Av}'_n(\ell+1)| \\ \text{(by (54))}}} - \left| \text{Av}_n \left(k + 1 \right) \cap \text{Av}'_n \left(\ell + 1 \right) \right|$$

$$= \left| \text{Av}'_n \left(\ell + 1 \right) \right| - \left| \text{Av}_n \left(k + 1 \right) \cap \text{Av}'_n \left(\ell + 1 \right) \right|$$

$$= \left| \text{Av}'_n \left(\ell + 1 \right) \setminus \text{Av}_n \left(k + 1 \right) \right|.$$

Finally, we are ready for everything to come together. We have shown that the **k**-module $\mathcal{A} / \left(\mathcal{I}_k + T_{\text{sign}} \left(\mathcal{J}_\ell \right) \right)$ is free of rank $\left| \text{Av}'_n \left(\ell + 1 \right) \setminus \text{Av}_n \left(k + 1 \right) \right|$. Hence, [Grinbe25, Lemma 5.21.9] shows that every family of $\left| \text{Av}'_n \left(\ell + 1 \right) \setminus \text{Av}_n \left(k + 1 \right) \right|$ vectors that spans this **k**-module must be a basis of this **k**-module. Hence, the family $(\overline{w})_{\text{Av}'_n(\ell+1) \setminus \text{Av}_n(k+1)}$ is a basis of this **k**-module (since we know that it spans it). This completes the proof of Theorem 2.9.6. □

2.9.4. $T_{\text{sign}} \left(\mathcal{I}_{n-k-1} \right)$ after Artin–Wedderburn

We end with a brief excursion into some related work by Hamaker and Rhoades [HamRho25].

Let $k \in \mathbb{N}$. For any two k-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in [n]^k$ and $\mathbf{b} = (b_1, b_2, \ldots, b_k) \in [n]^k$, we define the element

$$\nabla_{\mathbf{b}, \mathbf{a}} := \sum_{\substack{w \in S_n; \\ w(a_i) = b_i \text{ for all } i}} w \in \mathcal{A}. \tag{73}$$

Note that the $\nabla_{\mathbf{a}}$ in Lemma 2.8.5 can thus be rewritten as $\nabla_{\mathbf{a}, \mathbf{a}}$. If we equip the set $[n]^k$ with the left S_n-action given by

$$w \left(a_1, a_2, \ldots, a_k \right) := \left(w \left(a_1 \right), w \left(a_2 \right), \ldots, w \left(a_k \right) \right)$$
$$\text{for all } w \in S_n \text{ and } \left(a_1, a_2, \ldots, a_k \right) \in [n]^k$$

(as we have already done in the proof of Theorem 2.8.3), then we can rewrite the definition (73) of $\nabla_{\mathbf{b}, \mathbf{a}}$ as

$$\nabla_{\mathbf{b}, \mathbf{a}} := \sum_{\substack{w \in S_n; \\ w\mathbf{a} = \mathbf{b}}} w \in \mathcal{A}. \tag{74}$$

Thus,

$$\nabla_{\mathbf{b}, \mathbf{a}} = 0 \qquad \text{unless } \mathbf{b} = w\mathbf{a} \text{ for some } w \in S_n. \tag{75}$$

Now, Lemma 2.8.5 can be restated as follows:

Proposition 2.9.7. Let $k \in \mathbb{N}$. Then,

$$T_{\text{sign}} (\mathcal{J}_{n-k-1}) = \text{span} \left\{ \nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k \right\}.$$

Here, we understand $\mathcal{J}_m$ to mean $\mathcal{A}$ when $m < 0$.

Proof. It is easy to see that $w \nabla_{\mathbf{b},\mathbf{a}} = \nabla_{w\mathbf{b},\mathbf{a}}$ for any $w \in S_n$ and any $\mathbf{a}, \mathbf{b} \in [n]^k$. Applying this to $\mathbf{b} = \mathbf{a}$, we obtain

$$w \nabla_{\mathbf{a}} = \nabla_{w\mathbf{a},\mathbf{a}} \tag{76}$$

for any $w \in S_n$ and any $\mathbf{a} \in [n]^k$ (since $\nabla_{\mathbf{a}} = \nabla_{\mathbf{a},\mathbf{a}}$).

But Lemma 2.8.5 yields

$$
\begin{aligned}
T_{\text{sign}} (\mathcal{J}_{n-k-1}) &= \underbrace{\mathcal{A}}_{=\text{span}\{w \mid w \in S_n\}} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\} \\
&= \text{span} \left\{ w \mid w \in S_n \right\} \cdot \text{span} \left\{ \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \right\} \\
&= \text{span} \left\{ w \nabla_{\mathbf{a}} \mid \mathbf{a} \in [n]^k \text{ and } w \in S_n \right\} \\
&= \text{span} \left\{ \nabla_{w\mathbf{a},\mathbf{a}} \mid \mathbf{a} \in [n]^k \text{ and } w \in S_n \right\} \qquad \text{(by (76))} \\
&= \text{span} \left\{ \nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k \text{ such that } \mathbf{b} = w\mathbf{a} \text{ for some } w \in S_n \right\} \\
&= \text{span} \left\{ \nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k \right\} \qquad \text{(by (75))}.
\end{aligned}
$$

This proves Proposition 2.9.7. $\qquad\square$

We conclude with the following coda to Theorem 2.9.1:

Theorem 2.9.8. Assume that $n!$ is invertible in $\mathbf{k}$. Let $k \in \mathbb{N}$. Let all notations be as in Theorem 2.9.1. Then,

$$T_{\text{sign}} (\mathcal{J}_{n-k-1}) = \mathcal{A}_{\{\lambda \vdash n \mid \lambda_1 \geq n-k\}}.$$

Here, λ_1 denotes the first entry of λ.

To prove this, we need a general fact about how T_{sign} interacts with the Artin–Wedderburn isomorphism AW:

Proposition 2.9.9. Assume that $n!$ is invertible in $\mathbf{k}$. Let all notations be as in Theorem 2.9.1. Let X be a subset of $\{\lambda \vdash n\}$. Let λ^t denote the transpose (i.e., conjugate) of any partition λ. Then,

$$T_{\text{sign}} (\mathcal{A}_X) = \mathcal{A}_{X^t},$$

where X^t denotes the subset $\{\lambda^t \mid \lambda \in X\} = \{\lambda \vdash n \mid \lambda^t \in X\}$ of $\{\lambda \vdash n\}$.

Proof of Proposition 2.9.9. Let λ be a partition of n. It is well-known (e.g., [Grinbe25, Theorem 5.18.13]) that the sign-twist of the Specht module S^λ satisfies $\left(S^\lambda\right)^{\text{sign}} \cong S^{\lambda^t}$. Thus, in turn, $S^\lambda \cong \left(S^{\lambda^t}\right)^{\text{sign}}$ (since each S_n-representation V satisfies $\left(V^{\text{sign}}\right)^{\text{sign}} = V$). Hence, for any $a \in \mathcal{A}$, we have the equivalence

$$\left(aS^\lambda = 0\right) \iff \left(a\left(S^{\lambda^t}\right)^{\text{sign}} = 0\right)$$

$$\iff \left(T_{\text{sign}}\left(a\right)S^{\lambda^t} = 0\right) \tag{77}$$

(because the definition of $\left(S^{\lambda^t}\right)^{\text{sign}}$ reveals that $a\left(S^{\lambda^t}\right)^{\text{sign}} = T_{\text{sign}}\left(a\right)S^{\lambda^t}$ as **k**-modules). Now, (70) yields

$$\mathcal{A}_X = \left\{a \in \mathcal{A} \mid aS^\lambda = 0 \text{ for all } \lambda \notin X\right\}$$

$$= \left\{a \in \mathcal{A} \mid T_{\text{sign}}\left(a\right)S^{\lambda^t} = 0 \text{ for all } \lambda \notin X\right\} \qquad \text{(by (77))}$$

$$= \left\{a \in \mathcal{A} \mid T_{\text{sign}}\left(a\right)S^\lambda = 0 \text{ for all } \lambda \notin X^t\right\}$$

$$\left(\begin{array}{c} \text{here, we have substituted } \lambda \text{ for } \lambda^t, \\ \text{since } X^t = \left\{\lambda^t \mid \lambda \in X\right\} \end{array}\right)$$

$$= T_{\text{sign}}^{-1}\left(\underbrace{\left\{a \in \mathcal{A} \mid aS^\lambda = 0 \text{ for all } \lambda \notin X^t\right\}}_{\substack{=\mathcal{A}_{X^t} \\ \text{(by (70), applied to } U=X^t\text{)}}}\right)$$

$$= T_{\text{sign}}^{-1}\left(\mathcal{A}_{X^t}\right).$$

Since T_{sign} is an isomorphism, we thus conclude that $T_{\text{sign}}\left(\mathcal{A}_X\right) = \mathcal{A}_{X^t}$. This proves Proposition 2.9.9. $\square$

Proof of Theorem 2.9.8. The case $k \geq n$ is left to the reader (both sides are 0). In the case $k < n$, we have $n - k - 1 \in \mathbb{N}$, so that we can apply Theorem 2.9.1 to $n - k - 1$ instead of k. Thus, from the last equality of Theorem 2.9.1, we find

$$\mathcal{J}_{n-k-1} = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) > n-k-1\}} = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \geq n-k\}}. \tag{78}$$

Now, let us use the notations of Proposition 2.9.9. Applying Proposition 2.9.9 to $X = \{\lambda \vdash n \mid \ell(\lambda) \geq n - k\}$, we obtain

$$T_{\text{sign}}\left(\mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \geq n-k\}}\right) = \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda) \geq n-k\}^t}$$

$$= \mathcal{A}_{\{\lambda \vdash n \mid \ell(\lambda^t) \geq n-k\}} \qquad \left(\begin{array}{c} \text{since } X^t = \left\{\lambda \vdash n \mid \lambda^t \in X\right\} \\ \text{for each } X \subseteq \{\lambda \vdash n\} \end{array}\right)$$

$$= \mathcal{A}_{\{\lambda \vdash n \mid \lambda_1 \geq n-k\}},$$

because it is well-known (see, e.g., [Grinbe25, Theorem 5.1.10 **(c)**]) that every partition λ satisfies $\ell\left(\lambda^t\right) = \lambda_1$. In light of (78), we can rewrite this as

$$T_{\text{sign}}\left(\mathcal{J}_{n-k-1}\right) = \mathcal{A}_{\{\lambda \vdash n \mid \lambda_1 \geq n-k\}}.$$

Theorem 2.9.8 is thus proved. □

Combining Theorem 2.9.8 with Proposition 2.9.7, we conclude that

$$\text{span}\left\{\nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k\right\} = \mathcal{A}_{\{\lambda \vdash n \mid \lambda_1 \geq n-k\}} \tag{79}$$

for any $k \in \mathbb{N}$, under the assumption that $n!$ is invertible in $\mathbf{k}$. This is essentially [HamRho25, Theorem 3.5]. ("Essentially" because the $\text{Loc}_k\left(\mathfrak{S}_n, \mathbb{C}\right)$ from [HamRho25] is not span $\left\{\nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k\right\}$ but rather span $\left\{\nabla_{\mathbf{b},\mathbf{a}} \mid \mathbf{a}, \mathbf{b} \in [n]^k \text{ injective}\right\}$, where a k-tuple is said to be *injective* if its k entries are distinct.) The same result appears in [ElFrPi11, Theorem 7] (where, again, the "k-cosets" are the $\nabla_{\mathbf{b},\mathbf{a}}$ for injective k-tuples $\mathbf{a}, \mathbf{b} \in [n]^k$). Note that the condition "$\lambda_1 \geq n-k$" on a partition λ of n is equivalent to "$\lambda \geq \left(n-k, 1^k\right)$ in lexicographic order", and this is the condition used in [ElFrPi11, Theorem 7].

It is perhaps also worth mentioning that [EveZoh20, Lemma 3] is saying that if $\mathbf{k}$ is a field of characteristic 0, and if $l \in [n]$, then

$$\mathcal{A}_{\{\lambda \vdash n \mid \lambda_1 < l\}} = \left(\mathcal{A}\nabla_{[l]}\mathcal{A}\right)^{\perp}, \tag{80}$$

where $\nabla_{[l]}$ is the sum of all permutations $w \in S_n$ that fix the elements $l+1, l+2, \ldots, n$. This can be derived from Theorem 2.9.1, Theorem 2.4.1 **(a)** and Proposition 2.2.2 **(f)** using Proposition 2.9.9 (and the fact that $\nabla_{[l]} = T_{\text{sign}}\left(\nabla_{[l]}^-\right)$). We leave the details to the reader.

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