# 84th Putnam contest 2023 problem B6 with solution

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## 0.1. Problem B6

The following is Problem B6 on the 84th Putnam contest 2023 in the form I proposed it (which is slightly easier than what made it on the contest, because the answer is provided).

**Problem 1.** Let $m$ be a positive integer. Let $n$ be either $2m$ or $2m-1$.

Let $A$ be the $n \times n$-matrix $\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$, where $a_{i,j}$ is the number of all pairs $(x, y)$ of nonnegative integers satisfying $xi + yj = n$.

Prove that $\det A = (-1)^{m-1} \cdot 2m$.

(For instance, for $m = 3$ and $n = 5$, we have $A = \begin{pmatrix} 6 & 3 & 2 & 2 & 2 \\ 3 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}$.)

**Remark 0.1.** This is likely connected to gcd-matrices and Möbius inversion.

*Solution to Problem 1.* By assumption, $n$ is either $2m$ or $2m-1$. Hence, $m$ is either $n/2$ or $(n+1)/2$. Thus, it easily follows that $0 \leq m \leq n$ and $m - 1 < n/2$.

We shall use the *Iverson bracket notation*: If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall mean $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

We set $[n] := \{1, 2, \ldots, n\}$ and $[0, n] := \{0, 1, \ldots, n\}$.

If $M$ is any matrix, then we will use the notation $M_{i,j}$ for the entry of $M$ in the $i$-th row and the $j$-th column. Thus, in particular, the entries of our matrix $A$ are $A_{i,j} = a_{i,j}$ for all $i, j \in [n]$.

We shall consider $n \times (n+1)$-matrices whose rows are indexed by $1, 2, \ldots, n$ and whose columns are indexed by $0, 1, \ldots, n$. Likewise, we shall consider

$(n+1) \times n$-matrices whose rows are indexed by $0, 1, \ldots, n$ and whose columns are indexed by $1, 2, \ldots, n$.

If $M$ is an $n \times (n+1)$-matrix, and $j \in [0, n]$, then $M^{\sim j}$ shall denote the $n \times n$-matrix obtained from $M$ by removing the $j$-th column. Likewise, if $N$ is an $(n+1) \times n$-matrix, and $i \in [0, n]$, then $N_{\sim i}$ shall denote the $n \times n$-matrix obtained from $N$ by removing the $i$-th column. The Cauchy–Binet formula (or, more precisely, a particular case thereof) says that if $M$ is an $n \times (n+1)$-matrix and $N$ is an $(n+1) \times n$-matrix, then

$$\det(MN) = \sum_{k=0}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right). \tag{1}$$

Now, let $M$ be the $n \times (n+1)$-matrix $([i \mid j])_{i \in [n],\ j \in [0,n]}$ (so that its $(i,j)$-entry is 1 if $i \mid j$ and 0 otherwise). Let $N$ be the $(n+1) \times n$-matrix $([j \mid n-i])_{i \in [0,n],\ j \in [n]}$. For example, for $n = 8$, we have

$$M = \begin{array}{cc} & \begin{array}{ccccccccc} j=0 & j=1 & j=2 & j=3 & j=4 & j=5 & j=6 & j=7 & j=8 \end{array} \\ \begin{array}{c} i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \\ i=7 \\ i=8 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

and

$$N = \begin{array}{cc} & \begin{array}{cccccccc} j=1 & j=2 & j=3 & j=4 & j=5 & j=6 & j=7 & j=8 \end{array} \\ \begin{array}{c} i=0 \\ i=1 \\ i=2 \\ i=3 \\ i=4 \\ i=5 \\ i=6 \\ i=7 \\ i=8 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}..$$

Now, it is easy to see that

$$A = MN. \tag{2}$$

(Indeed, for each $i, j \in [n]$, we have

$$\begin{aligned}
(MN)_{i,j} &= \sum_{k=0}^{n} M_{i,k} N_{k,j} \qquad \text{(by the definition of matrix multiplication)} \\
&= \sum_{k=0}^{n} \underbrace{[i \mid k] \, [j \mid n-k]}_{=[i|k \text{ and } j|n-k]} \qquad \text{(by the definitions of } M \text{ and } N\text{)} \\
&= \sum_{k=0}^{n} [i \mid k \text{ and } j \mid n-k] \\
&= (\text{number of } k \in [0,n] \text{ satisfying } i \mid k \text{ and } j \mid n-k) \\
&= (\text{number of pairs } (x,y) \text{ of nonnegative integers satisfying } xi + yj = n) \\
&\qquad \left(\begin{array}{c}\text{by a simple bijection, explained in} \\ \text{the footnote following this computation}\end{array}\right) \\
&= a_{i,j} \qquad (\text{by the definition of } a_{i,j}) \\
&= A_{i,j}.
\end{aligned}$$

[1] Thus, $MN = A$, so that $A = MN$.)

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[1]Let me justify the equality

$$\begin{aligned}
&(\text{number of } k \in [0,n] \text{ satisfying } i \mid k \text{ and } j \mid n-k) \\
&= (\text{number of pairs } (x,y) \text{ of nonnegative integers satisfying } xi + yj = n)
\end{aligned}$$

that was used above. Clearly, it suffices to find a bijection

from the set $\mathfrak{A} := \{k \in [0,n] \text{ satisfying } i \mid k \text{ and } j \mid n-k\}$
to the set $\mathfrak{B} := \{\text{pairs } (x,y) \text{ of nonnegative integers satisfying } xi + yj = n\}$.

We can construct such a bijection as follows:

- For each $k \in \mathfrak{A}$, we have $(k/i,\ (n-k)/j) \in \mathfrak{B}$ (since $k \in \mathfrak{A}$ entails $k \in [0,n]$ and $i \mid k$ and $j \mid n-k$ and therefore $k/i \in \mathbb{N}$ and $(n-k)/j \in \mathbb{N}$, and of course we have $(k/i)\, i + ((n-k)/j)\, j = k + (n-k) = n$). Thus, we can define a map
$$\begin{aligned} \phi : \mathfrak{A} &\to \mathfrak{B}, \\ k &\mapsto (k/i,\ (n-k)/j). \end{aligned}$$
- For each $(x,y) \in \mathfrak{B}$, we have $xi \in \mathfrak{A}$ (in fact, $(x,y) \in \mathfrak{B}$ entails $x, y \in \mathbb{N}$ and $xi + yj = n$; now we clearly have $i \mid xi$ and $j \mid yj = \underbrace{xi + yj}_{=n} - xi = n - xi$ and finally $xi \in [0,n]$ (since $xi \geq 0$ and $xi \leq xi + yj = n$); but these three facts combined yield $xi \in \mathfrak{A}$). Thus, we can define a map
$$\begin{aligned} \psi : \mathfrak{B} &\to \mathfrak{A}, \\ (x,y) &\mapsto xi. \end{aligned}$$
- The maps $\phi$ and $\psi$ are easily seen to be mutually inverse (indeed, $\psi \circ \phi = \mathrm{id}$ is obvious; and $\phi \circ \psi = \mathrm{id}$ can be proved as follows: if $(x,y) \in \mathfrak{B}$, then $xi + yj =$

Now, from (2), we obtain

$$\det A = \det(MN) = \sum_{k=0}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \qquad (3)$$

(by (1)). It remains to manage the right hand side. We will need several claims:

*Claim 1:* Let $k \in [0, n]$ satisfy $k > \dfrac{n}{2}$. Then,

$$\det\left(M^{\sim k}\right) = (-1)^{k+1}.$$

*Proof of Claim 1.* We have $k > \dfrac{n}{2}$, so that $2k > n$. Hence, the only multiples of $k$ in the set $[0, n]$ are $0$ and $k$. In other words, we have $[k \mid j] = 0$ for all $j \in [0, n]$ other than $0$ and $k$, whereas $[k \mid j] = 1$ holds for $j = 0$ and for $j = k$.

Thus, the $k$-th row of the matrix $M$ has only two entries distinct from $0$: namely, a $1$ in the $0$-st column, and a $1$ in the $k$-th column. Therefore, the $k$-th row of the matrix $M^{\sim k}$ is $(1, 0, 0, \ldots, 0)$ (since the $1$ in the $k$-th column disappears when we remove this column). Expanding $\det\left(M^{\sim k}\right)$ along this row, we thus obtain $\det\left(M^{\sim k}\right) = (-1)^{k+1} \cdot \det Q$, where $Q$ is the result of removing the $0$-th column and the $k$-th row from $M^{\sim k}$. But it is easy to see that the matrix $Q$ is upper-triangular with its diagonal entries all being equal to $1$ (since $[i \mid j] = 0$ if $i > j$, and since $[i \mid j] = 1$ if $i = j$). Therefore, its determinant is $\det Q = 1$. Thus, $\det\left(M^{\sim k}\right) = (-1)^{k+1} \cdot \underbrace{\det Q}_{=1} = (-1)^{k+1}$. This proves Claim 1. □

*Claim 2:* Assume that $n = 2m$. Then,

$$\det\left(M^{\sim(n/2)}\right) = 0.$$

*Proof of Claim 2.* From $n = 2m$, we see that $n$ is even. Thus, $n/2 \in [n]$ (since $n/2 = m > 0$), so that the matrix $M$ has an $(n/2)$-th row. Moreover, this $(n/2)$-th row is not the $n$-th row (since $n = 2m > 0$ and thus $n/2 < n$).

Now, it is easy to see that the $(n/2)$-th row and the $n$-th row of $M$ agree in all their entries except for the ones in the $(n/2)$-th column (since any $j \in [0, n]$ satisfies $[n/2 \mid j] = \begin{cases} 1, & \text{if } j \in \{0, n/2, n\}; \\ 0, & \text{otherwise} \end{cases}$ and $[n \mid j] = \begin{cases} 1, & \text{if } j \in \{0, n\}; \\ 0, & \text{otherwise,} \end{cases}$ and

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$n$ and thus $n - xi = yj$, so that $\phi(\psi((x, y))) = \phi(xi) = \left(xi/i,\ \underbrace{(n - xi)}_{=yj}/j\right) = (xi/i,\ yj/j) = (x, y)$, so that $\phi \circ \psi = \text{id}$ is proved). Thus, $\phi : \mathfrak{A} \to \mathfrak{B}$ is a bijection. This is exactly the bijection we need.

these two values disagree only for $j = n/2$). Since the $(n/2)$-th column is removed in $M^{\sim(n/2)}$, this entails that the $(n/2)$-th row and the $n$-th row of the matrix $M^{\sim(n/2)}$ completely agree. Hence, the matrix $M^{\sim(n/2)}$ has two equal rows, and therefore its determinant is 0. In other words, $\det\left(M^{\sim(n/2)}\right) = 0$. This proves Claim 2. □

*Claim 3:* Let $k \in [0, n]$. Then, $\det(N_{\sim k}) = (-1)^{n(n-1)/2} \cdot \det\left(M^{\sim(n-k)}\right)$.

*Proof of Claim 3.* From $M = ([i \mid j])_{i\in[n],\ j\in[0,n]}$, we see that the transpose $M^T$ of the matrix $M$ is given by $M^T = ([j \mid i])_{i\in[0,n],\ j\in[n]}$. Comparing this with $N = ([j \mid n-i])_{i\in[0,n],\ j\in[n]}$, we observe that the matrix $N$ can be obtained from the matrix $M^T$ by reversing the order of the rows, i.e., by permuting the rows using the permutation

$$\begin{aligned}[0,n] &\to [0,n], \\ i &\mapsto n-i.\end{aligned}$$

Hence, the matrix $N_{\sim k}$ is obtained from the matrix $\left(M^T\right)_{\sim(n-k)}$ in the exact same way (i.e., by reversing the order of the rows). Therefore,

$$\det(N_{\sim k}) = (-1)^{\sigma} \cdot \det\left(\left(M^T\right)_{\sim(n-k)}\right),$$

where $\sigma$ is a permutation of an $n$-element set that reverses the order of its elements. Basic combinatorics shows that the sign $(-1)^{\sigma}$ of this latter permutation equals $(-1)^{\sigma} = (-1)^{n(n-1)/2}$ (for example, because this permutations has $n(n-1)/2$ inversions). Thus, we obtain

$$\begin{aligned}\det(N_{\sim k}) &= \underbrace{(-1)^{\sigma}}_{=(-1)^{n(n-1)/2}} \cdot \det\left(\underbrace{\left(M^T\right)_{\sim(n-k)}}_{=\left(M^{\sim(n-k)}\right)^T}\right) \\ &= (-1)^{n(n-1)/2} \cdot \underbrace{\det\left(\left(M^{\sim(n-k)}\right)^T\right)}_{\substack{=\det\left(M^{\sim(n-k)}\right) \\ \text{(since } \det\left(U^T\right)=\det U \\ \text{for any matrix } U\text{)}}} = (-1)^{n(n-1)/2} \cdot \det\left(M^{\sim(n-k)}\right).\end{aligned}$$

This proves Claim 3. □

*Claim 4:* We have

$$\sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) = m.$$

*Proof of Claim 4.* Let $M'$ be the $(n+1) \times (n+1)$-matrix obtained from $M$ by attaching an extra row $(1, 1, \ldots, 1)$ at the very top of the matrix. Then, the topmost two rows of the matrix $M'$ are equal (since the top row of $M$ already is $(1, 1, \ldots, 1)$), so that $\det(M') = 0$. On the other hand, expanding the determinant of $M'$ along the topmost row, we obtain

$$\begin{aligned} \det(M') &= \sum_{k=0}^{n} (-1)^k 1 \cdot \det\left(M^{\sim k}\right) = \sum_{k=0}^{n} (-1)^k \det\left(M^{\sim k}\right) \\ &= \sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) + \sum_{k=m}^{n} (-1)^k \det\left(M^{\sim k}\right) \end{aligned} \tag{4}$$

(since $0 \leq m \leq n$). However, it is not hard to see that

$$\sum_{k=m}^{n} (-1)^k \det\left(M^{\sim k}\right) = -m, \tag{5}$$

for example by distinguishing between two cases:

- If $n = 2m$, then $m = n/2$ and thus

$$\begin{aligned} &\sum_{k=m}^{n} (-1)^k \det\left(M^{\sim k}\right) \\ &= \sum_{k=n/2}^{n} (-1)^k \det\left(M^{\sim k}\right) \\ &= (-1)^{n/2} \underbrace{\det\left(M^{\sim (n/2)}\right)}_{\substack{=0 \\ \text{(by Claim 2)}}} + \sum_{k=n/2+1}^{n} (-1)^k \underbrace{\det\left(M^{\sim k}\right)}_{\substack{=(-1)^{k+1} \\ \text{(by Claim 1)}}} \\ &= \underbrace{(-1)^{n/2} 0}_{=0} + \sum_{k=n/2+1}^{n} \underbrace{(-1)^k (-1)^{k+1}}_{=-1} = \sum_{k=n/2+1}^{n} (-1) = (n - n/2)(-1) \\ &= -n/2 = -m \qquad \text{(since } m = n/2\text{)}. \end{aligned}$$

- If $n \neq 2m$, then $n = 2m - 1$ (since $n$ is either $2m$ or $2m - 1$) and thus

$m = (n+1)/2$ and therefore

$$\begin{aligned}
&\sum_{k=m}^{n} (-1)^k \det\left(M^{\sim k}\right) \\
&= \sum_{k=(n+1)/2}^{n} (-1)^k \underbrace{\det\left(M^{\sim k}\right)}_{\substack{=(-1)^{k+1} \\ \text{(by Claim 1,} \\ \text{since } k \geq (n+1)/2 > n/2)}} \\
&= \sum_{k=(n+1)/2}^{n} \underbrace{(-1)^k (-1)^{k+1}}_{=-1} = (n - (n+1)/2 + 1)(-1) \\
&= -(n+1)/2 = -m \qquad (\text{since } m = (n+1)/2).
\end{aligned}$$

In either case, we have proved (5). Now, (4) becomes

$$\begin{aligned}
\det(M') &= \sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) + \underbrace{\sum_{k=m}^{n} (-1)^k \det\left(M^{\sim k}\right)}_{\substack{=-m \\ \text{(by (5))}}} \\
&= \sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) - m.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) - m = \det(M') = 0,$$

so that

$$\sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right) = m.$$

This proves Claim 4. □

Now, (3) becomes

$$\begin{aligned}
\det A &= \sum_{k=0}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \\
&= \sum_{k=0}^{m-1} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) + \sum_{k=m}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \qquad (6)
\end{aligned}$$

(since $0 \leq m \leq n$). However, it is easy to see that

$$\sum_{k=m}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) = \sum_{k=n-m+1}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right), \qquad (7)$$

for instance by distinguishing between two cases:

- If $n = 2m$, then $m = n/2$ and thus

$$\begin{aligned}
&\sum_{k=m}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \\
&= \sum_{k=n/2}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \\
&= \underbrace{\det\left(M^{\sim (n/2)}\right)}_{\substack{=0 \\ \text{(by Claim 2)}}} \cdot \det\left(N_{\sim (n/2)}\right) + \sum_{k=n/2+1}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) \\
&= \sum_{k=n/2+1}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right) = \sum_{k=n-m+1}^{n} \det\left(M^{\sim k}\right) \cdot \det\left(N_{\sim k}\right)
\end{aligned}$$

  (since $n/2 = n - m$ (because $n - \underbrace{m}_{=n/2} = n - n/2 = n/2$)), so that (7) holds.

- If $n \neq 2m$, then $n = 2m - 1$ (since $n$ is either $2m$ or $2m - 1$) and thus $m = n - m + 1$, so that (7) holds tautologically (since the summation bounds on both sides are equal).

In either case, we have proved (7). Now, (6) becomes

$$\det A = \sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot\det\left(N_{\sim k}\right) + \underbrace{\sum_{k=m}^{n} \det\left(M^{\sim k}\right)\cdot\det\left(N_{\sim k}\right)}_{\substack{=\sum\limits_{k=n-m+1}^{n}\det\left(M^{\sim k}\right)\cdot\det\left(N_{\sim k}\right) \\ \text{(by (7))}}}$$

$$= \sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot \underbrace{\det\left(N_{\sim k}\right)}_{\substack{=(-1)^{n(n-1)/2}\cdot\det\left(M^{\sim(n-k)}\right) \\ \text{(by Claim 3)}}}$$

$$+ \sum_{k=n-m+1}^{n} \det\left(M^{\sim k}\right)\cdot \underbrace{\det\left(N_{\sim k}\right)}_{\substack{=(-1)^{n(n-1)/2}\cdot\det\left(M^{\sim(n-k)}\right) \\ \text{(by Claim 3)}}}$$

$$= (-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot\det\left(M^{\sim(n-k)}\right)$$

$$+ (-1)^{n(n-1)/2}\sum_{k=n-m+1}^{n} \det\left(M^{\sim k}\right)\cdot\det\left(M^{\sim(n-k)}\right)$$

$$= (-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot\det\left(M^{\sim(n-k)}\right)$$

$$+ (-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim(n-k)}\right)\cdot\det\left(M^{\sim k}\right)$$

$$\left(\text{here, we have substituted } n-k \text{ for } k \text{ in the second sum}\right)$$

$$= (-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot\det\left(M^{\sim(n-k)}\right)$$

$$+ (-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot\det\left(M^{\sim(n-k)}\right)$$

$$= 2\cdot(-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot \underbrace{\det\left(M^{\sim(n-k)}\right)}_{\substack{=(-1)^{(n-k)+1} \\ \text{(by Claim 1, applied to } n-k \text{ instead of } k \\ \text{(since } k\leq m-1<n/2 \text{ and thus } n-k>n-n/2=n/2))}}$$

$$= 2\cdot(-1)^{n(n-1)/2}\sum_{k=0}^{m-1} \det\left(M^{\sim k}\right)\cdot \underbrace{(-1)^{(n-k)+1}}_{=(-1)^{n+1}(-1)^k}$$

$$= 2 \cdot \underbrace{(-1)^{n(n-1)/2} (-1)^{n+1}}_{\substack{=(-1)^{n(n-1)/2+(n+1)} \\ =(-1)^{m-1} \\ \text{(by easy modulo-2} \\ \text{computations)}}} \underbrace{\sum_{k=0}^{m-1} (-1)^k \det\left(M^{\sim k}\right)}_{\substack{=m \\ \text{(by Claim 4)}}}$$

$$= 2 \cdot (-1)^{m-1} \cdot m = (-1)^{m-1} \cdot 2m.$$

This solves the problem. □