Discrete Morse theory and the cohomology ring
Robin Forman
`https://math.rice.edu/~forman/product.ps`
version of 2000
Errata and addenda by Darij Grinberg

7. Errata and addenda

The following list contains some corrections and comments to Robin Forman's paper "Discrete Morse theory and the cohomology ring". I refer to the preprint version of 2000 of this paper (available from `https://math.rice.edu/~forman/product.ps`), but some of the errors listed below are also contained in the published version[1]. The latter error are marked with an ♠ sign.

I have only read Sections 1 and 2 of the paper completely; the corrections to the other sections thus are likely to be less than comprehensive.

- **page 2:** In the complex $\mathcal{M}^*$, the first arrow should be "$\leftarrow$" instead of "$\rightarrow$".

♠ **page 12, §1:**[2] In "sign chosen so that $\langle a, \partial V(b) \rangle = -1$", replace "$V(b)$" by "$V(a)$".

- **page 12, §1:** "if a for all simplices a" $\rightarrow$ "for all simplices a".

♠ **page 12, §1:**[3] Near the bottom of this page, you claim that every simplex a of M satisfies exactly one of the ofllowing:

  (i) a is the smaller simplex in one V-pair[4];

  (ii) a is the larger simplex in one V-pair;

  (iii) a is critical.

  This is correct, but should perhaps be justified. The nontrivial part of the proof is showing that (i) and (ii) cannot hold at the same time, i.e., that a simplex $a^{(p)} \in M$ cannot be both the smaller simplex in one V-pair $\left\{a^{(p)} < b^{(p+1)}\right\}$ and the larger simplex in another V-pair $\left\{c^{(p-1)} < a^{(p)}\right\}$ at the same time.

  So let me show this: Assume the contrary. Thus, there exists a simplex $a^{(p)} \in M$ that is both the smaller simplex in one V-pair $\left\{a^{(p)} < b^{(p+1)}\right\}$ and the larger simplex in another V-pair $\left\{c^{(p-1)} < a^{(p)}\right\}$ at the same time.

[1] Transactions of the American Mathematical Society **354**, issue 12, pp. 5063–5085.
[2] This is on page 5071 of the published version.
[3] This is on page 5071 of the published version.
[4] By "V-pair", I mean a pair of simplices that belongs to V.

Consider this simplex $a^{(p)}$ and these two pairs. Thus, $\left\{a^{(p)} < b^{(p+1)}\right\} \in V$ and $\left\{c^{(p-1)} < a^{(p)}\right\} \in V$.

Since a simplex of dimension k is just a $(k+1)$-element set, we see from $a^{(p)} < b^{(p+1)}$ that the set b contains a as a subset but its size is just 1 larger than the size of a. Therefore, $b = a \cup \{x\}$ for some element $x \notin a$. Consider this x. Similarly, from $c^{(p-1)} < a^{(p)}$, we see that $a = c \cup \{y\}$ for some element $y \notin c$. Consider this y. Note that $a \subseteq b$ (since $a^{(p)} < b^{(p+1)}$) and $y \in \{y\} \subseteq c \cup \{y\} = a$.

Now, define a simplex $d := b \setminus \{y\}$. Then, $d \subseteq b$, so that $d \in M$ (since $b \in M$, but M is a simplicial complex). Moreover, $y \in a \subseteq b$, so that the size of the set $b \setminus \{y\}$ is 1 smaller than the size of b. In other words, the size of the set d is 1 smaller than the size of b (since $d = b \setminus \{y\}$). Hence, the simplex d has dimension p (since b has dimension $p+1$). Thus, we can write d as $d^{(p)}$.

However, $a = c \cup \{y\}$, thus $c = a \setminus \{y\}$ (since $y \notin c$). Hence, $c = \underbrace{a}_{\subseteq b} \setminus \{y\} \subseteq b \setminus \{y\} = d$. In other words, $c^{(p-1)} \subseteq d^{(p)}$ (since $c = c^{(p-1)}$ and $d = d^{(p)}$). Therefore, $c^{(p-1)} < d^{(p)}$. In other words, $d^{(p)} > c^{(p-1)}$.

Also, recall that $d \subseteq b$. In other words, $d^{(p)} \subseteq b^{(p+1)}$. Hence, $d^{(p)} < b^{(p+1)}$.

We shall now show that $a = d$.

Recall that V is the set of all pairs $\left\{u^{(k)} < v^{(k+1)}\right\}$ of simplices in M satisfying $f(u) \geq f(v)$. Hence, we have $f(c) \geq f(a)$ (since $\left\{c^{(p-1)} < a^{(p)}\right\} \in V$) and $f(a) \geq f(b)$ (since $\left\{a^{(p)} < b^{(p+1)}\right\} \in V$). In other words, we have $f(a) \leq f(c)$ and $f(b) \leq f(a)$.

Now, we are in one of the following two cases:

Case 1: We have $f(d) \leq f(c)$.

Case 2: We have $f(d) > f(c)$.

Let us first consider Case 1. In this case, we have $f(d) \leq f(c)$.

Since f is a discrete Morse function, the set

$$\left\{v^{(p)} > c^{(p-1)} \mid f(v) \leq f(c)\right\}$$

has size ≤ 1 (by the definition of a discrete Morse function). Hence, any two elements of this set must be equal. Since both simplices $a^{(p)}$ and $d^{(p)}$ belong to this set (because $a^{(p)} > c^{(p-1)}$ and $f(a) \leq f(c)$ and $d^{(p)} > c^{(p-1)}$ and $f(d) \leq f(c)$), we thus conclude that these two simplices $a^{(p)}$ and $d^{(p)}$ are equal. In other words, $a = d$. Thus, we have proved $a = d$ in Case 1.

Let us now consider Case 2. In this case, we have $f(d) > f(c)$. Hence, $f(d) > f(c) \geq f(a) \geq f(b)$, so that $f(d) \geq f(b)$.

Since f is a discrete Morse function, the set

$$\left\{ v^{(p)} < b^{(p+1)} \mid f(v) \geq f(b) \right\}$$

has size ≤ 1 (by the definition of a discrete Morse function). Hence, any two elements of this set must be equal. Since both simplices $a^{(p)}$ and $d^{(p)}$ belong to this set (because $a^{(p)} < b^{(p+1)}$ and $f(a) \geq f(b)$ and $d^{(p)} < b^{(p+1)}$ and $f(d) \geq f(b)$), we thus conclude that these two simplices $a^{(p)}$ and $d^{(p)}$ are equal. In other words, $a = d$. Thus, we have proved $a = d$ in Case 2.

We now have shown that $a = d$ in both Cases 1 and 2. Thus, $a = d$ always holds. However, $y \notin d$ (since $d = b \setminus \{y\}$). But this contradicts $y \in a = d$. This contradiction shows that our assumption was false, qed.

♠ **page 13, definition of $m(s)$ for an upper gradient step:**[5] After "$m(s) = -\langle a_0, \partial b_0 \rangle \langle \partial b_0, a_1 \rangle$", I would add "$= \langle \partial V a_0, a_1 \rangle$" (since this is tacitly being used later on).

♠ **page 13, definition of $m(s)$ for a lower gradient step:**[6] After "$m(s) = -\langle \partial a_0, b_0 \rangle \langle b_0, \partial a_1 \rangle$", I would add "$= \langle V \partial a_0, a_1 \rangle$" (since this is tacitly being used later on). This equality follows from the fact that (if we WLOG assume that b_0 is oriented so that $V(b_0) = a_1$) we have $\langle b_0, \partial a_1 \rangle = \langle b_0, \partial V b_0 \rangle = -1$ and $\langle \partial a_0, b_0 \rangle = \langle V \partial a_0, V b_0 \rangle = \langle V \partial a_0, a_1 \rangle$.

- **page 15:** The first two words on this page should be "critical simplices", not "gradient paths".

♠ **page 15, Lemma 1.3 (i):**[7] Add a comma before "s_{r-1}".

♠ **page 15, Lemma 1.3 (ii):**[8] Remove the word "nontrivial".

♠ **page 16:**[9] "straighforward" → "straightforward".

♠ **page 16:**[10] The sentence following Theorem 1.4 should be part of Theorem 1.4 (in particular, it should be italicized).

♠ **page 18:**[11] You say: "The general case is not much harder.". Let me elaborate on this:

[5] This is on page 5071 of the published version.
[6] This is on page 5072 of the published version.
[7] This is on page 5072 of the published version.
[8] This is on page 5072 of the published version.
[9] This is on page 5073 of the published version.
[10] This is on page 5073 of the published version.
[11] This is on page 5075 of the published version.

We have $\Phi^\infty = \Phi^N$. Recall also that $\partial \circ \Phi = \Phi \circ \partial$; in other words, the operator Φ commutes with ∂. Hence, any power Φ^i of Φ also commutes with ∂. In other words, we have

$$\partial \circ \Phi^i = \Phi^i \circ \partial \qquad \text{for each } i \in \mathbb{N}. \tag{1}$$

For any two operators $\alpha, \beta : C_* (M, \mathbb{Z}) \to C_* (M, \mathbb{Z})$, we shall write $\alpha \simeq \beta$ (and say that α and β are *chain-homotopic*) if and only if there exists an operator $K : C_* (M, \mathbb{Z}) \to C_{*+1} (M, \mathbb{Z})$ satisfying $\beta - \alpha = \partial \circ K + K \circ \partial$. The relation $\simeq$ is an equivalence relation (this is a fundamental result and easy to check).

Now, for any $i \in \mathbb{N}$, we have

$$\begin{aligned}
&\partial \circ \left(\Phi^i \circ V\right) + \left(\Phi^i \circ V\right) \circ \partial \\
&= \underbrace{\partial \circ \Phi^i}_{\substack{=\Phi^i \circ \partial \\ \text{(by (1))}}} \circ V + \Phi^i \circ V \circ \partial \\
&= \Phi^i \circ \partial \circ V + \Phi^i \circ V \circ \partial = \Phi^i \circ \underbrace{(\partial \circ V + V \circ \partial)}_{\substack{=\Phi - 1 \\ (\text{since } \Phi = 1 + \partial \circ V + V \circ \partial)}} \\
&= \Phi^i \circ (\Phi - 1) = \underbrace{\Phi^i \circ \Phi}_{=\Phi^{i+1}} - \underbrace{\Phi^i \circ 1}_{=\Phi^i} \qquad \left(\text{since } \Phi^i \text{ is a linear map}\right) \\
&= \Phi^{i+1} - \Phi^i .
\end{aligned}$$

In other words, for any $i \in \mathbb{N}$, we have $\Phi^{i+1} - \Phi^i = \partial \circ \left(\Phi^i \circ V\right) + \left(\Phi^i \circ V\right) \circ \partial$. Hence, for any $i \in \mathbb{N}$, we have $\Phi^i \simeq \Phi^{i+1}$ (since $\Phi^{i+1} - \Phi^i = \partial \circ K + K \circ \partial$ for $K := \Phi^i \circ V$). In other words, we have the following chain of relations:

$$\Phi^0 \simeq \Phi^1 \simeq \Phi^2 \simeq \Phi^3 \simeq \cdots .$$

Since the relation $\simeq$ is an equivalence relation, we thus find $\Phi^0 \simeq \Phi^N$. In other words, $1 \simeq \Phi^\infty$ (since $\Phi^0 = 1$ and $\Phi^N = \Phi^\infty$). In other words, there exists an operator $K : C_* (M, \mathbb{Z}) \to C_{*+1} (M, \mathbb{Z})$ satisfying $\Phi^\infty - 1 = \partial \circ K + K \circ \partial$, qed.

- **page 20, §3:** "to be the dual" → "be the dual".
- **page 21:** "The proof easily adapted" → "The proof can be easily adapted".

♠ **page 21:**[12] Again, in the complex between Theorem 3.2 and Theorem 3.3, the first arrow should be "←" instead of "→".

[12] This is on page 5077 of the published version.

- **page 23:** Remove the period at the end of the last displayed equation on this page.
- **page 24, Theorem 3.9:** Add "for" before "$i = 1, 2, \ldots, k$".
- **page 25:** In the first displayed equation on this page, the "$\mathcal{L}$" on the right hand side should be an "L" (normal font, not calligraphic).

♠ **page 25:**[13] In the fourth displayed equation on this page, the comma in "$b_1^* \otimes b_2^*, \otimes \cdots \otimes b_\ell^*$" should be removed.

- **page 27, §5:** "Then 4.2" → "Then Corollary 4.2".
- **page 28, Example 2:** On the right hand side of the last displayed equation on this page, I think you are missing a factor of B^*.
- **page 29, Example 3:** Remove the period at the end of the cocomplex:

♠ **page 29, Example 3:**[14] Replace "L_P" by "L_p".

♠ **page 30, Example 4:**[15] "vertices v of G" → "vertices v of M".

♠ **page 30, Example 4:**[16] Is C supposed to mean the abelian group $\mathbb{Z}$?

[13] This is on page 5080 of the published version.
[14] This is on page 5083 of the published version.
[15] This is on page 5083 of the published version.
[16] This is on page 5083 of the published version.