A remark on polyhedral cones from packed words and from finite topologies

Darij Grinberg

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Contents

1. The main theorem

The purpose of this little note is to prove [2, Theorem 5.2] using the machinery of [1].

I shall use the notations of [1] (except that I write WQSym instead of **WQSym**). Here is a brief overview of these notations:

- We fix a field $\mathbb{K}$.
- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{>0} = \{1, 2, 3, \ldots\}$.
- For each $n \in \mathbb{N}$, we let $[n]$ denote the set $\{1, 2, \ldots, n\}$. In particular, $[0] = \varnothing$.
- A *word* means a n-tuple of positive integers for some $n \in \mathbb{N}$. In this case, the n is called the *length* of the word. A word $w = (w_1, w_2, \ldots, w_n)$ is identified with the map $[n] \to \mathbb{N}_{>0}$, $i \mapsto w_i$.
- A word $w = (w_1, w_2, \ldots, w_n)$ is said to be *packed* if and only if $\{w_1, w_2, \ldots, w_n\} = [k]$ for some $k \in \mathbb{N}$. In this case, the k is denoted by $\max w$. (Note that k is the largest entry of w if w is nonempty.)

For example, the word $(3,1,2,1,3)$ is packed (with $\max(3,1,2,1,3) = 3$), and so is the empty word $()$ (with $\max() = 0$); but the word $(3,1,3)$ is not packed.

- If w is any word, then the *packing* of w is the packed word $\operatorname{Pack} w$ obtained by replacing the smallest number that appears in w by 1 (as often as it appears), replacing the second-smallest number that appears in w by 2 (as often as it appears), and so on. More formally, it can be defined as follows: Write w as $w = (w_1, w_2, \ldots, w_n)$. Let $W = \{w_1, w_2, \ldots, w_n\}$ be the set of all entries of w, and let $m = |W|$. Let ϕ be the unique increasing bijection from W to $[m]$. Then, $\operatorname{Pack} w$ is defined to be the word $(\phi(w_1), \phi(w_2), \ldots, \phi(w_n))$.

  For example,

  $$\operatorname{Pack}(4,1,7,2,4,1) = (3,1,4,2,3,1) \qquad \text{and} \qquad \operatorname{Pack}(4,2) = (2,1).$$

  Also, $\operatorname{Pack} w = w$ for any packed word w.

- We let WQSym denote the free $\mathbb{K}$-vector space with basis $(w)_{w \text{ is a packed word}}$. We define a $\mathbb{K}$-bilinear operation . (you're reading right: our symbol for this operation is a period) on this vector space WQSym by setting

  $$f.g = \sum_{\substack{h=(h_1,h_2,\ldots,h_{n+m}) \text{ is a packed word of length } n+m; \\ \operatorname{Pack}(h_1,h_2,\ldots,h_n)=f \text{ and } \operatorname{Pack}(h_{n+1},h_{n+2},\ldots,h_{n+m})=g}} h$$

  for any two packed words f and g, where n and m are the lengths of f and g. Equipping WQSym with this operation . as multiplication, we obtain a $\mathbb{K}$-algebra with unity $()$ (the empty word). When we refer to the $\mathbb{K}$-algebra WQSym below, we shall always understand it to be equipped with this $\mathbb{K}$-algebra structure.

  For example, in WQSym, we have

  $$(1,1).(2,1) = (1,1,2,1) + (2,2,2,1) + (1,1,3,2) + (2,2,3,1) + (3,3,2,1).$$

  The $\mathbb{K}$-algebra WQSym has various further structures – such as a Hopf algebra structure, and an embedding into the ring of noncommutative formal power series (see [2, §4.3.2], where WQSym is constructed via this embedding, and where the image of a packed word u under this embedding is denoted by $\mathbf{M}_u$). We won't need this extra structure.

Let me add a few more definitions.[1]

[1] A *set composition* of a set X means a tuple $(X_1, X_2, \ldots, X_k)$ of disjoint nonempty subsets of X such that $X_1 \cup X_2 \cup \cdots \cup X_k = X$.

Definition 1.1. Let $n \in \mathbb{N}$. Let u be a packed word of length n. Let $r = \max u$. Define $B_i = u^{-1}(\{i\})$ for every $i \in [r]$. (Thus, $(B_1, B_2, \ldots, B_r)$ is a set composition of $[n]$; it is what is called the "set composition of $[n]$ encoded by u" in [2].) Now, we define a polyhedral cone K_u in $\mathbb{R}^n$ by

$$K_u = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{j=1}^{k} \sum_{i \in B_j} x_i \geq 0 \qquad \text{for all } k = 1, 2, \ldots, r \right\}.$$

Definition 1.2. For any two sets X and Y, let $\operatorname{Map}(X, Y)$ denote the set of all maps from X to Y. Define a $\mathbb{K}$-vector space $\mathfrak{M}$ by $\mathfrak{M} = \bigoplus_{n \geq 0} \operatorname{Map}(\mathbb{R}^n, \mathbb{K})$ (where each $\operatorname{Map}(\mathbb{R}^n, \mathbb{K})$ becomes a $\mathbb{K}$-vector space by pointwise addition and multiplication with scalars). We make $\mathfrak{M}$ into a $\mathbb{K}$-algebra, whose multiplication is defined as follows: For any $n \in \mathbb{N}$, any $m \in \mathbb{N}$, any $f \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K})$ and $g \in \operatorname{Map}(\mathbb{R}^m, \mathbb{K})$, we define fg to be the element of $\operatorname{Map}(\mathbb{R}^{n+m}, \mathbb{K})$ which sends every $(x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m}$ to $f(x_1, x_2, \ldots, x_n)\, g(x_{n+1}, x_{n+2}, \ldots, x_{n+m})$.

Definition 1.3. For every $n \in \mathbb{N}$ and any subset S of $\mathbb{R}^n$, we define a map $\underline{1}_S \in \operatorname{Map}(\mathbb{R}^n, \mathbb{K}) \subseteq \mathfrak{M}$ as the indicator function of S (that is, the map which sends every $s \in S$ to 1 and every $s \in \mathbb{R}^n \setminus S$ to 0).

Our goal is to show:

Theorem 1.4. The map

$$\alpha : \mathrm{WQSym} \to \mathfrak{M}, \qquad u \mapsto (-1)^{\max u} \underline{1}_{K_u}$$

is a $\mathbb{K}$-algebra homomorphism.

This is a stronger version of [2, Theorem 5.2][2], and a particular case of [2, Theorem 8.1][3].

[2]Notice that [2, Theorem 5.2] talks not about our map $\alpha : \mathrm{WQSym} \to \mathfrak{M}$, but rather about a map $\mathcal{P} \to \mathrm{WQSym}$ where $\mathcal{P}$ is a certain subquotient of $\mathfrak{M}$ (namely, the subalgebra of $\mathfrak{M}$ generated by $\underline{1}_{K_u}$, taken modulo functions with measure-zero support). These two maps are "in some sense" inverse (allowing us to derive [2, Theorem 5.2] from Theorem 1.4). I find Theorem 1.4 the more natural statement.

Notice that [2] denotes by $(\mathbf{M}_u)_{u \text{ is a packed word}}$ the basis of WQSym that we call $(u)_{u \text{ is a packed word}}$.

[3]At least, I suspect so – I have not checked all the details. I also suspect that the whole [2, Theorem 8.1] can be obtained in a similar way as we prove Theorem 1.4 below.

2. The proof

We shall prove Theorem 1.4 using a detour via the algebra $\mathbf{H_T}$ defined in [1, Chapter 2]. We shall use the following notations from [1, Chapter 2]:

- If X is a set, then a *topology* on X is defined to be a family $\mathcal{T}$ of subsets of X that satisfies the following three properties:
  - We have $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$.
  - The union of any number of sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
  - The intersection of any finite number of sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

  We will only use this concept in the case when X is finite; in this case, the difference between "any number of sets in $\mathcal{T}$" and "any finite number of sets in $\mathcal{T}$" is immaterial (since $\mathcal{T}$ itself must be finite), and therefore unions and intersections play symmetric roles in the notion of a topology on X.

  - If $\mathcal{T}$ is a topology on X, then the sets belonging to $\mathcal{T}$ are called the *open sets* of $\mathcal{T}$. The complements of these open sets (inside X) are called the *closed sets* of $\mathcal{T}$.

- If X is a set, then a *preorder* on X is defined to be a binary relation $\preccurlyeq$ on X that is reflexive and transitive (but, unlike a partial order, doesn't need to be antisymmetric). Both partial orders and equivalence relations are preorders.

- If X is a set, and if $\preccurlyeq$ is a preorder on X, then an *ideal* of $(X, \preccurlyeq)$ means a subset I of X that has the following property:
  - If $i \in I$ and $j \in X$ satisfy $i \preccurlyeq j$, then $j \in I$.

- If X is a finite set, then there is a canonical bijection between $\{$topologies on $X\}$ and $\{$preorders on $X\}$. This bijection (sometimes called the *Alexandrov correspondence*) proceeds as follows:
  - If $\preccurlyeq$ is a preorder on X, then we can define a topology $\mathcal{T}_{\preccurlyeq}$ on X by

$$\mathcal{T}_{\preccurlyeq} = \{\text{ideals of } (X, \preccurlyeq)\}.$$

  We shall denote this topology $\mathcal{T}_{\preccurlyeq}$ as the *topology corresponding to* $\preccurlyeq$.

  - If $\mathcal{T}$ is a topology on X, then we can define five binary relations $\leq_{\mathcal{T}}$, $\geq_{\mathcal{T}}$ and $\sim_{\mathcal{T}}$ on X by setting

$$(a \leq_{\mathcal{T}} b) \iff (\text{each } I \in \mathcal{T} \text{ satisfying } a \in I \text{ satisfies } b \in I);$$

$$(a \geq_{\mathcal{T}} b) \iff (\text{each } I \in \mathcal{T} \text{ satisfying } b \in I \text{ satisfies } a \in I);$$

$$(a \sim_{\mathcal{T}} b) \iff (\text{each } I \in \mathcal{T} \text{ satisfies the equivalence } (a \in I) \iff (b \in I));$$

$$(a <_{\mathcal{T}} b) \iff (a \leq_{\mathcal{T}} b \text{ but not } a \geq_{\mathcal{T}} b) \iff (a \leq_{\mathcal{T}} b \text{ but not } a \sim_{\mathcal{T}} b);$$
$$(a >_{\mathcal{T}} b) \iff (a \geq_{\mathcal{T}} b \text{ but not } a \leq_{\mathcal{T}} b) \iff (a \geq_{\mathcal{T}} b \text{ but not } a \sim_{\mathcal{T}} b).$$

The three binary relations $\leq_{\mathcal{T}}$, $\geq_{\mathcal{T}}$ and $\sim_{\mathcal{T}}$ are preorders on X, and the relation $\sim_{\mathcal{T}}$ is an equivalence relation (whence the quotient set $X/\sim_{\mathcal{T}}$ is well-defined). The relations $<_{\mathcal{T}}$ and $>_{\mathcal{T}}$ are strict partial orders. We shall refer to the relation $\leq_{\mathcal{T}}$ as the *preorder corresponding to* $\mathcal{T}$.

These assignments of a topology to a preorder and vice versa are mutually inverse: If $\preccurlyeq$ is a preorder on X, then $\leq_{\mathcal{T}_{\preccurlyeq}}$ is precisely $\preccurlyeq$. Conversely, if $\mathcal{T}$ is a topology on X, then $\mathcal{T}_{\leq_{\mathcal{T}}}$ is precisely $\mathcal{T}$.

- For each $n \in \mathbb{N}$, we let $\mathbf{T}_n$ denote the set of all topologies on the set $[n] = \{1, 2, \ldots, n\}$.
- We let $\mathbf{T}$ denote the set $\bigsqcup_{n\in\mathbb{N}} \mathbf{T}_n$.
- If f is a packed word of length $n \in \mathbb{N}$, then we define a preorder $\leq_f$ on the set $[n]$ by setting

$$\left(a \leq_f b\right) \iff (f(a) \leq f(b)).$$

Furthermore, if f is a packed word of length $n \in \mathbb{N}$, then we let $\mathcal{T}_f$ be the topology $\mathcal{T}_{\leq_f}$ corresponding to this preorder $\leq_f$. The closed sets of this topology $\mathcal{T}_f$ are the sets $f^{-1}(\{1, 2, \ldots, i\})$ for $i \in \{0, 1, \ldots, \max f\}$.

- If $P \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, then $P(+n)$ shall denote the set $\{k + n \mid k \in P\}$. (In other words, $P(+n)$ is the set P shifted right by n units on the number line.)
- If $\mathcal{T} \in \mathbf{T}_n$ and $\mathcal{S} \in \mathbf{T}_m$ are two topologies (on the sets $[n]$ and $[m]$, respectively) for some $n \in \mathbb{N}$ and $m \in \mathbb{N}$, then we define a topology $\mathcal{T}.\mathcal{S} \in \mathbf{T}_{n+m}$ on the set $[n+m]$ by

$$\mathcal{T}.\mathcal{S} = \{O \cup (P(+n)) \mid O \in \mathcal{T} \text{ and } P \in \mathcal{S}\}.$$

Thus, we have defined a binary operation . on $\mathbf{T}$. This binary operation . is associative (by [1, Proposition 3]), and the topology $\{\varnothing\} \in \mathbf{T}_0$ is its neutral element.

- We let $\mathbf{H_T}$ be the free $\mathbb{K}$-vector space with basis $\mathbf{T}$. We equip $\mathbf{H_T}$ with a multiplication . that linearly extends the operation . on $\mathbf{T}$ (that is, the restriction of the multiplication $\mathbf{H_T}$ to the basis $\mathbf{T}$ should be the operation . on $\mathbf{T}$). Thus, $\mathbf{H_T}$ becomes a $\mathbb{K}$-algebra with unity $\{\varnothing\} \in \mathbf{T}_0$.

The $\mathbb{K}$-algebra $\mathbf{H_T}$ also has the structure of a Hopf algebra, but we shall not need it, so we don't define it here.

We shall also use the following notation from [1, Chapter 4]:

- If X is a set, and if $\mathcal{T}$ is a topology on X, then we set

$$\mathcal{P}(\mathcal{T}) = \bigsqcup_{p \in \mathbb{N}} \{\text{surjective maps } f : X \to [p] \text{ such that every } c \in X \text{ and } d \in X$$
$$\text{satisfying } c \leq_{\mathcal{T}} d \text{ satisfy } f(x) \leq f(d)\}.$$

  Thus, if $X = [n]$ for some $n \in \mathbb{N}$, then all elements of $\mathcal{P}(\mathcal{T})$ are packed words of length n.

Next, we define a polyhedral cone for every $\mathcal{T} \in \mathbf{T}$:

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathcal{T} \in \mathbf{T}_n$ (that is, let $\mathcal{T}$ be a topology on the set $[n] = \{1, 2, \ldots, n\}$). Then, we define a polyhedral cone $K_{\mathcal{T}}$ in $\mathbb{R}^n$ by

$$K_{\mathcal{T}} = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in C} x_i \geq 0 \qquad \text{for all closed sets } C \text{ of } \mathcal{T} \right\}.$$

The following follows from the definitions:

Remark 2.2. Let u be a packed word. Then, $K_u = K_{\mathcal{T}_u}$, where $\mathcal{T}_u$ is as defined in [1, §2.1].

Let us define a few more things:

Definition 2.3. Let X be a finite totally ordered set, and let $\mathcal{T}$ be a topology on X. We define $\mathcal{U}(\mathcal{T})$ to be the set of all $f \in \mathcal{P}(\mathcal{T})$ having the property that any two elements i and j of X satisfying $i <_{\mathcal{T}} j$ must satisfy $f(i) < f(j)$. Notice that $\mathcal{U}(\mathcal{T}) \subseteq \mathcal{P}(\mathcal{T})$. (We can call the elements of $\mathcal{U}(\mathcal{T})$ "strictly increasing packed words" for $\mathcal{T}$.) (It can also be shown that $\mathcal{L}(\mathcal{T}) \subseteq \mathcal{U}(T)$, where $\mathcal{L}(T)$ is as defined in [1, Definition 15].)

Definition 2.4. We define a $\mathbb{K}$-linear map $U : \mathbf{H_T} \to \mathrm{WQSym}$ by

$$U(\mathcal{T}) = \sum_{f \in \mathcal{U}(\mathcal{T})} f \qquad \text{for every } \mathcal{T} \in \mathbf{T}.$$

Remark 2.5. This map U is easily seen to be the map $\Gamma_{(0,0,1)}$ in the notation of [1, Proposition 14]. Thus, U is a surjective Hopf algebra homomorphism.

Now, here is a rather trivial fact:

Proposition 2.6. The map

$$\begin{aligned} \beta : \mathbf{H_T} &\to \mathfrak{M}, \\ \mathcal{T} &\mapsto (-1)^{|[n]/\sim_{\mathcal{T}}|} \underline{1}_{K_{\mathcal{T}}} \end{aligned}$$

is a $\mathbb{K}$-algebra homomorphism from $\mathbf{H_T} = (\mathbf{H_T}, .)$ to $\mathfrak{M}$.

Proof of Proposition 2.6 (sketched). The proof boils down to the observation that if $n \in \mathbb{N}$, $m \in \mathbb{N}$, $\mathcal{T} \in \mathbf{T}_n$ and $\mathcal{S} \in \mathbf{T}_m$, then

$$K_{\mathcal{T}.\mathcal{S}} = \{ (x_1, x_2, \ldots, x_{n+m}) \in \mathbb{R}^{n+m} \mid (x_1, x_2, \ldots, x_n) \in K_{\mathcal{T}} \text{ and } (x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \in K_{\mathcal{S}} \} .$$

□

Now, we claim:

Theorem 2.7. The diagram

$$\begin{array}{ccc} \mathbf{H_T} & \xrightarrow{U} & \mathrm{WQSym} \\ & \searrow^{\beta} & \downarrow \alpha \\ & & \mathfrak{M} \end{array}$$

commutes. That is, we have $\beta = \alpha \circ U$.

Before we prove this, we introduce some more notations.

Definition 2.8. We define a $\mathbb{K}$-linear map $Z : \mathbf{H_T} \to \mathbf{H_T}$ by

$$Z(\mathcal{T}) = (-1)^{|[n]/\sim_{\mathcal{T}}|} \mathcal{T} \qquad \text{for every } n \in \mathbb{N} \text{ and } \mathcal{T} \in \mathbf{T}_n.$$

It is easy to see that Z is an involutive Hopf algebra isomorphism.

Definition 2.9. Let X be a finite totally ordered set, and let $\mathcal{T}$ be a topology on X. Let a and b be two elements of X. We define three new topologies $\mathcal{T} \looparrowleft (a \leq b)$, $\mathcal{T} \looparrowleft (a \geq b)$ and $\mathcal{T} \looparrowleft (a \sim b)$ on X as follows:

$$\begin{aligned} \mathcal{T} \looparrowleft (a \leq b) &= \{ O \in \mathcal{T} \mid (a \in O \implies b \in O) \} ; \\ \mathcal{T} \looparrowleft (a \geq b) &= \{ O \in \mathcal{T} \mid (b \in O \implies a \in O) \} ; \\ \mathcal{T} \looparrowleft (a \sim b) &= \{ O \in \mathcal{T} \mid (a \in O \iff b \in O) \} . \end{aligned}$$

(It is easy to check that these are actually topologies. Of course, $\mathcal{T} \looparrowleft (a \geq b) = \mathcal{T} \looparrowleft (b \leq a)$.)

Here comes a collection of simple properties of these three new topologies:

Lemma 2.10. Let X be a finite totally ordered set, and let $\mathcal{T}$ be a topology on X. Let a and b be two elements of X.

(a) We have

$$\left(\mathcal{T} \hookleftarrow (a \leq b)\right) \cap \left(\mathcal{T} \hookleftarrow (a \geq b)\right) = \mathcal{T} \hookleftarrow (a \sim b) \qquad \text{and} \tag{1}$$

$$\left(\mathcal{T} \hookleftarrow (a \leq b)\right) \cup \left(\mathcal{T} \hookleftarrow (a \geq b)\right) = \mathcal{T}. \tag{2}$$

(b) We have

$$\mathcal{T} \hookleftarrow (a \sim b) = \left(\mathcal{T} \hookleftarrow (a \leq b)\right) \hookleftarrow (a \geq b) = \left(\mathcal{T} \hookleftarrow (a \geq b)\right) \hookleftarrow (a \leq b).$$

(c) If $a \leq_{\mathcal{T}} b$, then $\mathcal{T} \hookleftarrow (a \leq b) = \mathcal{T}$ and $\mathcal{T} \hookleftarrow (a \sim b) = \mathcal{T} \hookleftarrow (a \geq b)$.

(d) If $b \leq_{\mathcal{T}} a$, then $\mathcal{T} \hookleftarrow (a \geq b) = \mathcal{T}$ and $\mathcal{T} \hookleftarrow (a \sim b) = \mathcal{T} \hookleftarrow (a \leq b)$.

(e) If c and d are two elements of X, then $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$ holds if and only if

$$\left(c \leq_{\mathcal{T}} d \text{ or } \left(c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d\right)\right).$$

(f) If c and d are two elements of X, then $c \leq_{\mathcal{T} \hookleftarrow (a \geq b)} d$ holds if and only if

$$\left(c \leq_{\mathcal{T}} d \text{ or } \left(c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d\right)\right).$$

(g) If c and d are two elements of X, then $c \leq_{\mathcal{T} \hookleftarrow (a \sim b)} d$ holds if and only if

$$\left(c \leq_{\mathcal{T}} d \text{ or } \left(c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d\right) \text{ or } \left(c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d\right)\right).$$

(h) If c and d are two elements of X, then $c \leq_{\mathcal{T} \hookleftarrow (a \sim b)} d$ holds if and only if

$$\left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d \text{ or } c \leq_{\mathcal{T} \hookleftarrow (a \geq b)} d\right).$$

(i) If c and d are two elements of X, then $c \leq_{\mathcal{T}} d$ holds if and only if

$$\left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d \text{ and } c \leq_{\mathcal{T} \hookleftarrow (a \geq b)} d\right).$$

(j) If c and d are two elements of X, then $c \sim_{\mathcal{T} \hookleftarrow (a \leq b)} d$ holds if and only if

$$\left(c \sim_{\mathcal{T}} d \text{ or } \left(b \leq_{\mathcal{T}} c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d \leq_{\mathcal{T}} a\right)\right).$$

(k) If c and d are two elements of X, and if we have neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$, then $c \sim_{\mathcal{T} \hookleftarrow (a \sim b)} d$ holds if and only if

$$\left(c \sim_{\mathcal{T}} d \text{ or } \left(c \sim_{\mathcal{T}} a \text{ and } d \sim_{\mathcal{T}} b\right) \text{ or } \left(c \sim_{\mathcal{T}} b \text{ and } d \sim_{\mathcal{T}} a\right)\right).$$

(l) We have

$$\mathcal{P}\left(\mathcal{T} \hookleftarrow (a \leq b)\right) \cap \mathcal{P}\left(\mathcal{T} \hookleftarrow (a \geq b)\right) = \mathcal{P}\left(\mathcal{T} \hookleftarrow (a \sim b)\right) \qquad \text{and}$$

$$\mathcal{P}\left(\mathcal{T} \hookleftarrow (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \hookleftarrow (a \geq b)\right) = \mathcal{P}\left(\mathcal{T}\right).$$

(m) Assume that neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$. Then, the three sets $\mathcal{U}\left(\mathcal{T} \hookleftarrow (a \leq b)\right)$, $\mathcal{U}\left(\mathcal{T} \hookleftarrow (a \geq b)\right)$ and $\mathcal{U}\left(\mathcal{T} \hookleftarrow (a \sim b)\right)$ are disjoint, and their union is $\mathcal{U}\left(\mathcal{T}\right)$.

(n) Assume that neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$. Then,

$$\left|X/\sim_{\mathcal{T} \hookleftarrow (a \leq b)}\right| = \left|X/\sim_{\mathcal{T} \hookleftarrow (a \geq b)}\right| = \left|X/\sim_{\mathcal{T}}\right| \qquad \text{and}$$

$$\left|X/\sim_{\mathcal{T} \hookleftarrow (a \sim b)}\right| = \left|X/\sim_{\mathcal{T}}\right| - 1.$$

Proof of Lemma 2.10 (sketched). Parts **(a)** and **(b)** are straightforward to check.

(c) Assume that $a \leq_{\mathcal{T}} b$. Then, every $O \in \mathcal{T}$ satisfies $(a \in \mathcal{T} \implies b \in \mathcal{T})$. Hence, $\mathcal{T} \looparrowleft (a \leq b) = \mathcal{T}$ by the definition of $\mathcal{T} \looparrowleft (a \leq b)$. From Lemma 2.10 **(b)**, we have $\mathcal{T} \looparrowleft (a \sim b) = \underbrace{(\mathcal{T} \looparrowleft (a \leq b))}_{=\mathcal{T}} \looparrowleft (a \geq b) = \mathcal{T} \looparrowleft (a \geq b)$. Thus, Lemma 2.10 **(c)** is proven.

(d) The proof of part **(d)** is similar to that of **(c)**.

(e) $\Longleftarrow$: Assume that $(c \leq_{\mathcal{T}} d$ or $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d))$. We need to check that $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$ holds. In other words, we need to check that every $O \in \mathcal{T} \looparrowleft (a \leq b)$ satisfying $c \in O$ satisfies $d \in O$. So let us fix an $O \in \mathcal{T} \looparrowleft (a \leq b)$ satisfying $c \in O$. We must prove that $d \in O$.

We have $O \in \mathcal{T} \looparrowleft (a \leq b) \subseteq \mathcal{T}$ (by the definition of $\mathcal{T} \looparrowleft (a \leq b)$). Thus, if $c \leq_{\mathcal{T}} d$, then $d \in O$. Hence, for the rest of this proof, we WLOG assume that we don't have $c \leq_{\mathcal{T}} d$. Thus, by assumption, we have $c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d$. Therefore, $a \in O$ (since $c \in O$ and $c \leq_{\mathcal{T}} a$). But $O \in \mathcal{T} \looparrowleft (a \leq b)$, and therefore $(a \in O \implies b \in O)$ (by the definition of $\mathcal{T} \looparrowleft (a \leq b)$), so that $b \in O$ (since $a \in O$), and thus $d \in O$ (since $b \leq_{\mathcal{T}} d$). This completes the proof of the $\Longleftarrow$ direction of Lemma 2.10 **(e)**.

$\Longrightarrow$: Assume that $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$ holds. We need to check that $(c \leq_{\mathcal{T}} d$ or $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d))$. We can WLOG assume that we don't have $c \leq_{\mathcal{T}} d$. Then, we must prove that $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d)$.

We don't have $c \leq_{\mathcal{T}} d$. Hence, there exists a $Q \in \mathcal{T}$ such that $c \in Q$ but $d \notin Q$. Consider this Q. If we had $(a \in Q \implies b \in Q)$, then Q would belong to $\mathcal{T} \looparrowleft (a \leq b)$, which would yield $d \in Q$ (since $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$ and $c \in Q$), which would contradict $d \notin Q$. Hence, we cannot have $(a \in Q \implies b \in Q)$. Thus, $a \in Q$ and $b \notin Q$.

Let $O \in \mathcal{T}$ be such that $c \in O$. We shall prove that $a \in O$. Indeed, assume the contrary. Then, $a \notin O$. Thus, $a \notin Q \cap O$, so that $(a \in Q \cap O \implies b \in Q \cap O)$. Since $Q \cap O \in \mathcal{T}$ (because $Q \in \mathcal{T}$ and $O \in \mathcal{T}$), this yields $Q \cap O \in \mathcal{T} \looparrowleft (a \leq b)$. Since we also have $c \in Q \cap O$ (since $c \in Q$ and $c \in O$), this yields $d \in Q \cap O$ (since $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$), so that $d \in Q \cap O \subseteq Q$, which contradicts $d \notin Q$. This contradiction proves that our assumption was wrong. Hence, $a \in O$ is proven. Forget now that we fixed O. Thus we have shown that $a \in O$ for every $O \in \mathcal{T}$ which satisfies $c \in O$. In other words, $c \leq_{\mathcal{T}} a$.

Let $O \in \mathcal{T}$ be such that $b \in O$. We shall prove that $d \in O$. Indeed, assume the contrary. Then, $d \notin O$. Thus, $d \notin Q \cup O$ (since $d \notin Q$ and $d \notin O$). But $b \in O \subseteq Q \cup O$, so that $(a \in Q \cup O \implies b \in Q \cup O)$. Since $Q \cup O \in \mathcal{T}$ (because $Q \in \mathcal{T}$ and $O \in \mathcal{T}$), this yields $Q \cup O \in \mathcal{T} \looparrowleft (a \leq b)$. Since we also have $c \in Q \cup O$ (since $c \in Q$), this yields $d \in Q \cup O$ (since $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$), which contradicts $d \notin Q \cup O$. This contradiction proves that our assumption was wrong. Hence, $d \in O$ is proven. Forget now that we fixed O. Thus we have shown that $d \in O$ for every $O \in \mathcal{T}$ which satisfies $b \in O$. In other words, $b \leq_{\mathcal{T}} d$.

We thus have shown that $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d)$. This completes the proof of the $\Longrightarrow$ direction of Lemma 2.10 **(e)**.

(f) The proof of part **(f)** is analogous to that of **(e)**.

(g) Let c and d be two elements of X. Then, we have the following logical equivalence:

$$\begin{aligned}
&\left(c \leq_{\mathcal{T} \hookleftarrow (a \sim b)} d\right) \\
\Longleftrightarrow\ &\left(c \leq_{(\mathcal{T} \hookleftarrow (a \leq b)) \hookleftarrow (a \geq b)} d\right) \qquad \text{(by Lemma 2.10 **(b)**)} \\
\Longleftrightarrow\ &\left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d \text{ or } \left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} b \text{ and } a \leq_{(\mathcal{T} \hookleftarrow (a \leq b))} d\right)\right) \\
&\text{(by Lemma 2.10 **(f)**)} \\
\Longleftrightarrow\ &((c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d)) \text{ or } \\
&((c \leq_{\mathcal{T}} b \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} b)) \text{ and } (a \leq_{\mathcal{T}} d \text{ or } (a \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d)))) \\
&\text{(by Lemma 2.10 **(e)**, applied to each of the three inequalities)} \\
\Longleftrightarrow\ &(c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d) \text{ or } (c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d)) \\
&\text{(after simplifying using the transitivity and reflexivity of } \leq_{\mathcal{T}}\text{)}.
\end{aligned}$$

This proves Lemma 2.10 **(g)**.

(h) This is just a rewriting of Lemma 2.10 **(g)** using parts **(e)** and **(f)**.

(i) $\Longrightarrow$: This is clear.

$\Longleftarrow$: Assume that $\left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d \text{ and } c \leq_{\mathcal{T} \hookleftarrow (a \geq b)} d\right)$. We need to show that $c \leq_{\mathcal{T}} d$. Indeed, assume the contrary.

We have $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$. Thus, Lemma 2.10 **(e)** yields that we must have $(c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d))$. Since we assumed that $c \leq_{\mathcal{T}} d$ does not hold, this yields $(c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d)$. Similarly, $(c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d)$. Thus, $c \leq_{\mathcal{T}} b \leq_{\mathcal{T}} d$, which contradicts our assumption that not $c \leq_{\mathcal{T}} d$. This contradiction completes the proof.

(j) We have $c \sim_{\mathcal{T} \hookleftarrow (a \leq b)} d$ if and only if $\left(c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d \text{ and } d \leq_{\mathcal{T} \hookleftarrow (a \leq b)} c\right)$. We can rewrite each of the two statements $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$ and $d \leq_{\mathcal{T} \hookleftarrow (a \leq b)} c$ using Lemma 2.10 **(e)**, and then simplify the result; we end up with Lemma 2.10 **(j)**.

(k) Let c and d be two elements of X. Assume that we have neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$. We have $c \sim_{\mathcal{T} \hookleftarrow (a \sim b)} d$ if and only if $\left(c \leq_{\mathcal{T} \hookleftarrow (a \sim b)} d \text{ and } d \leq_{\mathcal{T} \hookleftarrow (a \sim b)} c\right)$. We can rewrite each of the two statements $c \leq_{\mathcal{T} \hookleftarrow (a \sim b)} d$ and $d \leq_{\mathcal{T} \hookleftarrow (a \sim b)} c$ using Lemma 2.10 **(g)**, and then simplify the result (a disjunction with 9 cases, of which many can be ruled out due to the assumption that neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$); we end up with Lemma 2.10 **(k)**.

(l) *Proof of* $\mathcal{P}(\mathcal{T} \hookleftarrow (a \leq b)) \cap \mathcal{P}(\mathcal{T} \hookleftarrow (a \geq b)) = \mathcal{P}(\mathcal{T} \hookleftarrow (a \sim b))$*:* Whenever f is a surjective map $X \to [p]$ for some $p \in \mathbb{N}$, we have the following logical equivalence:

$$\begin{aligned}
&\left(f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cap \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right)\right) \\
\Longleftrightarrow\ & \left(\underbrace{\left(f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right)\right)}_{\Longleftrightarrow\left(\text{every } c\in X \text{ and } d\in X \text{ satisfying } c \leq_{\mathcal{T}\looparrowleft(a\leq b)} d \text{ satisfy } f(c)\leq f(d)\right)} \right. \\
& \left. \wedge \underbrace{\left(f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (b \leq a)\right)\right)}_{\Longleftrightarrow\left(\text{every } c\in X \text{ and } d\in X \text{ satisfying } c \leq_{\mathcal{T}\looparrowleft(b\leq a)} d \text{ satisfy } f(c)\leq f(d)\right)} \right) \\
\Longleftrightarrow\ & \left(\left(\text{every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}\looparrowleft(a\leq b)} d \text{ satisfy } f(c) \leq f(d)\right)\right. \\
& \left. \wedge \left(\text{every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}\looparrowleft(b\leq a)} d \text{ satisfy } f(c) \leq f(d)\right)\right) \\
\Longleftrightarrow\ & \left(\text{every } c \in X \text{ and } d \in X \text{ satisfying } \underbrace{\left(c \leq_{\mathcal{T}\looparrowleft(a\leq b)} d \text{ or } c \leq_{\mathcal{T}\looparrowleft(a\geq b)} d\right)}_{\substack{\Longleftrightarrow\left(c \leq_{\mathcal{T}\looparrowleft(a\sim b)} d\right) \\ \text{(by Lemma 2.10 } \textbf{(h)}\text{)}}} \right. \\
& \left. \text{satisfy } f(c) \leq f(d)\right) \\
\Longleftrightarrow\ & \left(\text{every } c \in X \text{ and } d \in X \text{ satisfying } c \leq_{\mathcal{T}\looparrowleft(a\sim b)} d \text{ satisfy } f(c) \leq f(d)\right) \\
\Longleftrightarrow\ & \left(f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \sim b)\right)\right).
\end{aligned}$$

Thus, $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cap \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right) = \mathcal{P}\left(\mathcal{T} \looparrowleft (a \sim b)\right)$ is proven.

It remains to prove $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right) = \mathcal{P}(\mathcal{T})$. We shall achieve this by proving both inclusions separately:

Proof of $\mathcal{P}(\mathcal{T}) \subseteq \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right)$*:* Let $f \in \mathcal{P}(\mathcal{T})$. We must prove that $f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right)$.

We WLOG assume that $f(a) \leq f(b)$. We shall now show that $f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right)$. This will yield that $f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right)$, and thus complete this proof of $\mathcal{P}(\mathcal{T}) \subseteq \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right)$.

Let $c \in X$ and $d \in X$ be such that $c \leq_{\mathcal{T}\looparrowleft(a\leq b)} d$. In order to prove that $f \in \mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right)$, we must now show that $f(c) \leq f(d)$.

We have $c \leq_{\mathcal{T}\looparrowleft(a\leq b)} d$. Due to Lemma 2.10 **(e)**, this yields that $\left(c \leq_{\mathcal{T}} d \text{ or } \left(c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d\right)\right)$. In the first of these cases, $f(c) \leq f(d)$ follows immediately from $f \in \mathcal{P}(\mathcal{T})$; thus, let us assume that we are in the second case. Thus, $c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d$. From $f \in \mathcal{P}(\mathcal{T})$, we thus obtain $f(c) \leq f(a)$ and $f(b) \leq f(d)$. Hence, $f(c) \leq f(a) \leq f(b) \leq f(d)$, qed.

Proof of $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right) \subseteq \mathcal{P}(\mathcal{T})$*:* We now need to show that $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \cup \mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right) \subseteq \mathcal{P}(\mathcal{T})$. To do so, it is clearly enough to prove $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \leq b)\right) \subseteq \mathcal{P}(\mathcal{T})$ and $\mathcal{P}\left(\mathcal{T} \looparrowleft (a \geq b)\right) \subseteq \mathcal{P}(\mathcal{T})$. We shall only show the first of these two relations, as the second is analogous. Let $f \in \mathcal{P}(\mathcal{T} \looparrowleft (a \leq b))$. Then, every $c \in X$ and $d \in X$ satisfying $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$ satisfy $f(c) \leq f(d)$. Hence, every $c \in X$ and $d \in X$ satisfying $c \leq_{\mathcal{T}} d$ satisfy $f(c) \leq f(d)$ (since every $c \in X$ and $d \in X$ satisfying $c \leq_{\mathcal{T}} d$ satisfy $c \leq_{\mathcal{T} \looparrowleft (a \leq b)} d$ (due to Lemma 2.10 **(e)**)). In other words, $f \in \mathcal{P}(\mathcal{T})$. Since this is proven for every $f \in \mathcal{P}(\mathcal{T} \looparrowleft (a \leq b))$, we thus conclude that $\mathcal{P}(\mathcal{T} \looparrowleft (a \leq b)) \subseteq \mathcal{P}(\mathcal{T})$.

The proof of Lemma 2.10 **(l)** is thus complete.

(m) It is clearly enough to prove the three equalities

$$\mathcal{U}(\mathcal{T} \looparrowleft (a \leq b)) = \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}; \tag{3}$$

$$\mathcal{U}(\mathcal{T} \looparrowleft (a \sim b)) = \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}; \tag{4}$$

$$\mathcal{U}(\mathcal{T} \looparrowleft (a \geq b)) = \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) > f(b)\}. \tag{5}$$

We shall only check the first two of these three equalities (since the third one is analogous to the first).

Let us first check that $a <_{\mathcal{T} \looparrowleft (a \leq b)} b$. Indeed, it is clear from the definition of $\mathcal{T} \looparrowleft (a \leq b)$ that $a \leq_{\mathcal{T} \looparrowleft (a \leq b)} b$. Thus, in order to prove that $a <_{\mathcal{T} \looparrowleft (a \leq b)} b$, we must only show that we don't have $b \leq_{\mathcal{T} \looparrowleft (a \leq b)} a$. To achieve this, we assume the contrary. Lemma 2.10 **(e)** (applied to $c = b$ and $d = a$) thus yields that $(b \leq_{\mathcal{T}} a$ or $(b \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} a))$. In either of these cases, we must have $b \leq_{\mathcal{T}} a$, which contradicts the assumption that neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$. So $a <_{\mathcal{T} \looparrowleft (a \leq b)} b$ is proven.

Next, we are going to prove (3) by showing its two inclusions separately:

Proof of $\mathcal{U}(\mathcal{T} \looparrowleft (a \leq b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$: Let $g \in \mathcal{U}(\mathcal{T} \looparrowleft (a \leq b))$. Thus, $g \in \mathcal{P}(\mathcal{T} \looparrowleft (a \leq b))$, and every two elements i and j of X satisfying $i <_{\mathcal{T} \looparrowleft (a \leq b)} j$ must satisfy $g(i) < g(j)$. Applying the latter fact to $i = a$ and $j = b$, we obtain $g(a) < g(b)$ (since $a <_{\mathcal{T} \looparrowleft (a \leq b)} b$).

Moreover, $g \in \mathcal{P}(\mathcal{T} \looparrowleft (a \leq b)) \subseteq \mathcal{P}(\mathcal{T} \looparrowleft (a \leq b)) \cup \mathcal{P}(\mathcal{T} \looparrowleft (a \geq b)) = \mathcal{P}(\mathcal{T})$ (by Lemma 2.10 **(l)**).

Let now i and j be any two elements of X satisfying $i <_{\mathcal{T}} j$. We shall show that $g(i) < g(j)$.

Indeed, $i <_{\mathcal{T}} j$, thus $i \leq_{\mathcal{T}} j$ and therefore $i \leq_{\mathcal{T} \looparrowleft (a \leq b)} j$ (due to Lemma 2.10 **(e)**). Assume (for the sake of contradiction) that $j \leq_{\mathcal{T} \looparrowleft (a \leq b)} i$. Then, $i \sim_{\mathcal{T} \looparrowleft (a \leq b)} j$, and thus (by Lemma 2.10 **(j)**, applied to $c = i$ and $d = j$) we have $(i \sim_{\mathcal{T}} j$ or $(b \leq_{\mathcal{T}} i \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} j \leq_{\mathcal{T}} a))$. But neither of these two cases can occur (since $i <_{\mathcal{T}} j$ precludes $i \sim_{\mathcal{T}} j$, and since $b \leq_{\mathcal{T}} i \leq_{\mathcal{T}} a$ contradicts our assumption that not $b \leq_{\mathcal{T}} a$). Hence, we have our contradiction. Thus, our assumption (that $j \leq_{\mathcal{T} \looparrowleft (a \leq b)} i$) was false. We therefore have $i \leq_{\mathcal{T} \looparrowleft (a \leq b)} j$ but not $j \leq_{\mathcal{T} \looparrowleft (a \leq b)} i$. In other words, $i <_{\mathcal{T} \looparrowleft (a \leq b)} j$. Thus, $g(i) < g(j)$ (since $g \in \mathcal{U}(\mathcal{T} \looparrowleft (a \leq b))$).

Now, let us forget that we fixed i and j. We thus have shown that any two elements i and j of X satisfying $i <_{\mathcal{T}} j$ satisfy $g(i) < g(j)$. In other words, $g \in \mathcal{U}(\mathcal{T})$ (since we already know that $g \in \mathcal{P}(\mathcal{T})$). Thus, g is an element of $\mathcal{U}(\mathcal{T})$ and satisfies $g(a) < g(b)$. In other words, $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$.

Since this is proven for every $g \in \mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b))$, we thus conclude that $\mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$.

Proof of $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\} \subseteq \mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b))$: Let $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$. Then, $g \in \mathcal{U}(\mathcal{T})$ and $g(a) < g(b)$. From $g \in \mathcal{U}(\mathcal{T})$, we obtain $g \in \mathcal{P}(\mathcal{T})$.

Let now $c \in X$ and $d \in X$ be such that $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$. We now aim to show that $g(c) \leq g(d)$.

Indeed, from $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$, we obtain $(c \leq_{\mathcal{T}} d$ or $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d))$ (by Lemma 2.10 **(e)**). In the first of these two cases, we obtain $g(c) \leq g(d)$ immediately (since $g \in \mathcal{P}(\mathcal{T})$), while in the second case we obtain

$$\begin{aligned} g(c) &\leq g(a) && (\text{since } c \leq_{\mathcal{T}} a \text{ and } g \in \mathcal{P}(\mathcal{T})) \\ &< g(b) \leq g(d) && (\text{since } b \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P}(\mathcal{T})). \end{aligned}$$

Thus, $g(c) \leq g(d)$ is proven in either case.

Now, let us forget that we fixed c and d. We thus have proven that $g(c) \leq g(d)$ for any $c \in X$ and $d \in X$ satisfying $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$. In other words, $g \in \mathcal{P}(\mathcal{T} \hookleftarrow (a \leq b))$.

Now, let $c \in X$ and $d \in X$ be such that $c <_{\mathcal{T} \hookleftarrow (a \leq b)} d$. We now aim to show that $g(c) < g(d)$.

Indeed, from $c <_{\mathcal{T} \hookleftarrow (a \leq b)} d$, we obtain $c \leq_{\mathcal{T} \hookleftarrow (a \leq b)} d$, and thus $(c \leq_{\mathcal{T}} d$ or $(c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d))$ (by Lemma 2.10 **(e)**). In the second of these two cases, we have

$$\begin{aligned} g(c) &\leq g(a) && (\text{since } c \leq_{\mathcal{T}} a \text{ and } g \in \mathcal{P}(\mathcal{T})) \\ &< g(b) \leq g(d) && (\text{since } b \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P}(\mathcal{T})). \end{aligned}$$

Thus, $g(c) < g(d)$ is proven in the second case. We thus WLOG assume that we are in the first case. That is, we have $c \leq_{\mathcal{T}} d$. If $c <_{\mathcal{T}} d$, then we can immediately conclude that $g(c) < g(d)$ (since $g \in \mathcal{U}(\mathcal{T})$). Hence, we WLOG assume that we don't have $c <_{\mathcal{T}} d$. Thus, $c \sim_{\mathcal{T}} d$ (since $c \leq_{\mathcal{T}} d$), so that $d \leq_{\mathcal{T}} c$. Hence, $(d \leq_{\mathcal{T}} c$ or $(d \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} c))$, so that Lemma 2.10 **(e)** (applied to d and c instead of c and d) yields $d \leq_{\mathcal{T} \hookleftarrow (a \leq b)} c$. But this contradicts $c <_{\mathcal{T} \hookleftarrow (a \leq b)} d$. Thus, we have obtained a contradiction, and our proof of $g(c) < g(d)$ is complete.

Now, let us forget that we fixed c and d. We thus have proven that $g(c) < g(d)$ for any $c \in X$ and $d \in X$ satisfying $c <_{\mathcal{T} \hookleftarrow (a \leq b)} d$. In other words, $g \in \mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b))$ (since $g \in \mathcal{P}(\mathcal{T} \hookleftarrow (a \leq b))$). Since this is proven for every $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$, we thus conclude that $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\} \subseteq \mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b))$.

Combining $\mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\}$ with $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) < f(b)\} \subseteq \mathcal{U}(\mathcal{T} \hookleftarrow (a \leq b))$, we obtain (3).

Let us next check that $a \sim_{\mathcal{T} \hookleftarrow (a \leq b)} b$. Indeed, it is clear from the definition of $\mathcal{T} \hookleftarrow (a \sim b)$ that $a \leq_{\mathcal{T} \hookleftarrow (a \sim b)} b$ and that $b \leq_{\mathcal{T} \hookleftarrow (a \sim b)} a$. Combining these, we obtain $a \sim_{\mathcal{T} \hookleftarrow (a \sim b)} b$.

Next, we are going to prove (4) by showing its two inclusions separately:

Proof of $\mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$*:* Let $g \in \mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$. Thus, $g \in \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$, and every two elements i and j of X satisfying $i <_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} j$ must satisfy $g(i) < g(j)$. We have $a \sim_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} b$ and $g \in \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$; thus, $g(a) = g(b)$.

Moreover,

$$\begin{aligned} g &\in \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b)) = \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \leq b)) \cap \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \geq b)) \\ &\qquad \text{(by Lemma 2.10 (l))} \\ &\subseteq \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \leq b)) \subseteq \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \leq b)) \cup \mathcal{P}(\mathcal{T} \leftarrow\!\!\!\!+ (a \geq b)) = \mathcal{P}(\mathcal{T}) \end{aligned}$$

(by Lemma 2.10 **(l)**).

Now, let i and j be any two elements of X satisfying $i <_{\mathcal{T}} j$. We shall show that $g(i) < g(j)$.

Indeed, $i <_{\mathcal{T}} j$, thus $i \leq_{\mathcal{T}} j$ and therefore $i \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} j$ (due to Lemma 2.10 **(g)**). Assume (for the sake of contradiction) that $j \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} i$. Then, $i \sim_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} j$, and thus (by Lemma 2.10 **(k)**, applied to $c = i$ and $d = j$) we have $(i \sim_{\mathcal{T}} j \text{ or } (i \sim_{\mathcal{T}} a \text{ and } j \sim_{\mathcal{T}} b) \text{ or } (i \sim_{\mathcal{T}} b \text{ and } j \sim_{\mathcal{T}} a))$. But neither of these three cases can occur[4]. Hence, we have our contradiction. Thus, our assumption (that $j \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} i$) was false. We therefore have $i \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} j$ but not $j \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} i$. In other words, $i <_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} j$. Thus, $g(i) < g(j)$ (since $g \in \mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$).

Now, let us forget that we fixed i and j. We thus have shown that any two elements i and j of X satisfying $i <_{\mathcal{T}} j$ satisfy $g(i) < g(j)$. In other words, $g \in \mathcal{U}(\mathcal{T})$ (since we already know that $g \in \mathcal{P}(\mathcal{T})$). Thus, g is an element of $\mathcal{U}(\mathcal{T})$ and satisfies $g(a) = g(b)$. In other words, $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$. Since this is proven for every $g \in \mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$, we thus conclude that $\mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$.

Proof of $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\} \subseteq \mathcal{U}(\mathcal{T} \leftarrow\!\!\!\!+ (a \sim b))$*:* Let $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$. Then, $g \in \mathcal{U}(\mathcal{T})$ and $g(a) = g(b)$. From $g \in \mathcal{U}(\mathcal{T})$, we obtain $g \in \mathcal{P}(\mathcal{T})$.

Let now $c \in X$ and $d \in X$ be such that $c \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} d$. We now aim to show that $g(c) \leq g(d)$.

Indeed, from $c \leq_{\mathcal{T} \leftarrow\!\!\!\!+ (a\sim b)} d$, we obtain $(c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d) \text{ or } (c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d))$ (by Lemma 2.10 **(g)**). In the first of these three cases, we obtain $g(c) \leq g(d)$ immediately (since $g \in \mathcal{P}(\mathcal{T})$). In the second case, we obtain

$$\begin{aligned} g(c) &\leq g(a) \qquad (\text{since } c \leq_{\mathcal{T}} a \text{ and } g \in \mathcal{P}(\mathcal{T})) \\ &= g(b) \leq g(d) \qquad (\text{since } b \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P}(\mathcal{T})). \end{aligned}$$

[4]Indeed, the first case ($i \sim_{\mathcal{T}} j$) is precluded by the fact that $i <_{\mathcal{T}} j$. The second case ($i \sim_{\mathcal{T}} a$ and $j \sim_{\mathcal{T}} b$) cannot occur since it would lead to $a \sim_{\mathcal{T}} i \leq_{\mathcal{T}} j \sim_{\mathcal{T}} b$, which would contradict the assumption that we have neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$. The third case ($i \sim_{\mathcal{T}} b$ and $j \sim_{\mathcal{T}} a$) cannot occur for a similar reason.

In the third case, we obtain

$$\begin{aligned} g(c) &\leq g(b) \qquad (\text{since } c \leq_{\mathcal{T}} b \text{ and } g \in \mathcal{P}(\mathcal{T})) \\ &= g(a) \leq g(d) \qquad (\text{since } a \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P}(\mathcal{T})) . \end{aligned}$$

Thus, $g(c) \leq g(d)$ is proven in either case.

Now, let us forget that we fixed c and d. We thus have proven that $g(c) \leq g(d)$ for any $c \in X$ and $d \in X$ satisfying $c \leq_{\mathcal{T} \looparrowleft (a \sim b)} d$. In other words, $g \in \mathcal{P}(\mathcal{T} \looparrowleft (a \sim b))$.

Now, let $c \in X$ and $d \in X$ be such that $c <_{\mathcal{T} \looparrowleft (a \sim b)} d$. We now aim to show that $g(c) < g(d)$.

Indeed, from $c <_{\mathcal{T} \looparrowleft (a \sim b)} d$, we obtain $c \leq_{\mathcal{T} \looparrowleft (a \sim b)} d$, and thus $(c \leq_{\mathcal{T}} d \text{ or } (c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d) \text{ or } (c \leq_{\mathcal{T}} b \text{ and } a \leq_{\mathcal{T}} d))$ (by Lemma 2.10 **(g)**). We study these three cases separately:

- Assume that we are in the first case, i.e., we have $c \leq_{\mathcal{T}} d$. Then, $c <_{\mathcal{T}} d$ (since otherwise, we would have $d \leq_{\mathcal{T}} c$, and therefore $d \leq_{\mathcal{T} \looparrowleft (a \sim b)} c$ (by Lemma 2.10 **(g)**), which would contradict $c <_{\mathcal{T} \looparrowleft (a \sim b)} d$). Hence, $g(c) < g(d)$ (since $g \in \mathcal{U}(\mathcal{T})$).

- Assume that we are in the second case, i.e., we have $(c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d)$. Then,

  $$\begin{aligned} g(c) &\leq g(a) \qquad (\text{since } c \leq_{\mathcal{T}} a \text{ and } g \in \mathcal{P}(\mathcal{T})) \\ &= g(b) \leq g(d) \qquad (\text{since } b \leq_{\mathcal{T}} d \text{ and } g \in \mathcal{P}(\mathcal{T})) . \end{aligned}$$

  If at least one of the strict inequalities $c <_{\mathcal{T}} a$ or $b <_{\mathcal{T}} d$ holds, then we can strengthen this to a strict inequality $g(c) < g(d)$ (because $g \in \mathcal{U}(\mathcal{T})$), and thus be done. Hence, we WLOG assume that none of the inequalities $c <_{\mathcal{T}} a$ or $b <_{\mathcal{T}} d$ holds. Thus, $c \sim_{\mathcal{T}} a$ and $b \sim_{\mathcal{T}} d$. Hence, $c \sim_{\mathcal{T} \looparrowleft (a \sim b)} a$ and $b \sim_{\mathcal{T} \looparrowleft (a \sim b)} d$ (by Lemma 2.10 **(k)**), so that $c \sim_{\mathcal{T} \looparrowleft (a \sim b)} a \sim_{\mathcal{T} \looparrowleft (a \sim b)} b \sim_{\mathcal{T} \looparrowleft (a \sim b)} d$, which contradicts $c <_{\mathcal{T} \looparrowleft (a \sim b)} d$. Hence, we are done in the second case as well.

- The third case is similar to the second case.

Thus, our proof of $g(c) < g(d)$ is complete in each case.

Now, let us forget that we fixed c and d. We thus have proven that $g(c) < g(d)$ for any $c \in X$ and $d \in X$ satisfying $c <_{\mathcal{T} \looparrowleft (a \sim b)} d$. In other words, $g \in \mathcal{U}(\mathcal{T} \looparrowleft (a \sim b))$ (since $g \in \mathcal{P}(\mathcal{T} \looparrowleft (a \sim b))$). Since this is proven for every $g \in \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$, we thus conclude that $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\} \subseteq \mathcal{U}(\mathcal{T} \looparrowleft (a \sim b))$.

Combining $\mathcal{U}(\mathcal{T} \looparrowleft (a \sim b)) \subseteq \{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\}$ with $\{f \in \mathcal{U}(\mathcal{T}) \mid f(a) = f(b)\} \subseteq \mathcal{U}(\mathcal{T} \looparrowleft (a \sim b))$, we obtain (4).

Now, our proof of Lemma 2.10 **(m)** is complete.

(n) If c and d are two elements of X, then $c \sim_{\mathcal{T}\hookleftarrow(a\leq b)} d$ holds if and only if

$$(c \sim_{\mathcal{T}} d \text{ or } (b \leq_{\mathcal{T}} c \leq_{\mathcal{T}} a \text{ and } b \leq_{\mathcal{T}} d \leq_{\mathcal{T}} a))$$

(according to Lemma 2.10 **(j)**). Since $(b \leq_{\mathcal{T}} c \leq_{\mathcal{T}} a$ and $b \leq_{\mathcal{T}} d \leq_{\mathcal{T}} a)$ cannot hold (because of our assumption that not $b \leq_{\mathcal{T}} a$), this simplifies as follows: If c and d are two elements of X, then $c \sim_{\mathcal{T}\hookleftarrow(a\leq b)} d$ holds if and only if $c \sim_{\mathcal{T}} d$. Thus, the equivalence relation $\sim_{\mathcal{T}\hookleftarrow(a\leq b)}$ is identical to $\sim_{\mathcal{T}}$. Hence, $\left|X/ \sim_{\mathcal{T}\hookleftarrow(a\leq b)}\right| = |X/ \sim_{\mathcal{T}}|$. Similarly, $\left|X/ \sim_{\mathcal{T}\hookleftarrow(a\geq b)}\right| = |X/ \sim_{\mathcal{T}}|$. Thus, $\left|X/ \sim_{\mathcal{T}\hookleftarrow(a\leq b)}\right| = \left|X/ \sim_{\mathcal{T}\hookleftarrow(a\geq b)}\right| = |X/ \sim_{\mathcal{T}}|$ is proven. It remains to show $\left|X/ \sim_{\mathcal{T}\hookleftarrow(a\sim b)}\right| = |X/ \sim_{\mathcal{T}}| - 1$.

Lemma 2.10 **(k)** yields the following: If c and d are two elements of X, then $c \sim_{\mathcal{T}\hookleftarrow(a\sim b)} d$ holds if and only if

$$(c \sim_{\mathcal{T}} d \text{ or } (c \sim_{\mathcal{T}} a \text{ and } d \sim_{\mathcal{T}} b) \text{ or } (c \sim_{\mathcal{T}} b \text{ and } d \sim_{\mathcal{T}} a)).$$

In other words, two elements of X are equivalent under the equivalence relation $\sim_{\mathcal{T}\hookleftarrow(a\sim b)}$ if and only if either they are equivalent under $\sim_{\mathcal{T}}$, or one of them is in the $\sim_{\mathcal{T}}$-class of a while the other is in the $\sim_{\mathcal{T}}$-class of b. Thus, when passing from the equivalence relation $\sim_{\mathcal{T}}$ to $\sim_{\mathcal{T}\hookleftarrow(a\sim b)}$, the equivalence classes of a and b get merged (and these two classes used to be separate for $\sim_{\mathcal{T}}$, because of our assumption that neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$), while all other equivalence classes stay as they were. Thus, the total number of equivalence classes decreases by 1. In other words, $\left|X/ \sim_{\mathcal{T}\hookleftarrow(a\sim b)}\right| = |X/ \sim_{\mathcal{T}}| - 1$. This completes the proof of Lemma 2.10 **(n)**. □

Lemma 2.11. Let $n \in \mathbb{N}$ and $\mathcal{T} \in \mathbf{T}_n$. Let a and b be two elements of $[n]$. Then,

$$\underline{1}_{K_{\mathcal{T}}} = \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\leq b)}} + \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\geq b)}} - \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\sim b)}}.$$

Proof of Lemma 2.11. It is clearly enough to prove that

$$K_{\mathcal{T}} = K_{\mathcal{T}\hookleftarrow(a\leq b)} \cap K_{\mathcal{T}\hookleftarrow(a\geq b)} \tag{6}$$

and

$$K_{\mathcal{T}\hookleftarrow(a\sim b)} = K_{\mathcal{T}\hookleftarrow(a\leq b)} \cup K_{\mathcal{T}\hookleftarrow(a\geq b)}. \tag{7}$$

Before we start proving these statements, let us rewrite the definition of $K_{\mathcal{S}}$ for any topology $\mathcal{S}$ on $[n]$. Namely, if O is a subset of $[n]$, then we define a subset K_O of $\mathbb{R}^n$ by

$$K_O = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i\in[n]\setminus O} x_i \geq 0 \right\}.$$

It is now clear that any topology $\mathcal{S}$ on $[n]$ satisfies

$$K_{\mathcal{S}} = \bigcap_{O \in \mathcal{S}} K_O. \tag{8}$$

(Indeed, this is just a restatement of the definition of $K_{\mathcal{S}}$, since the closed sets of $\mathcal{S}$ are the sets of the form $[n] \setminus O$ with O being an open set of $\mathcal{S}$.)

Proof of (6): From (8), we obtain $K_{\mathcal{T}} = \bigcap_{O \in \mathcal{T}} K_O$ and $K_{\mathcal{T} \looparrowleft (a \leq b)} = \bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O$ and $K_{\mathcal{T} \looparrowleft (a \geq b)} = \bigcap_{O \in \mathcal{T} \looparrowleft (a \geq b)} K_O$. Thus,

$$\begin{aligned}
\underbrace{K_{\mathcal{T} \looparrowleft (a \leq b)}}_{= \bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O} \cap \underbrace{K_{\mathcal{T} \looparrowleft (a \geq b)}}_{= \bigcap_{O \in \mathcal{T} \looparrowleft (a \geq b)} K_O} &= \left(\bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O \right) \cap \left(\bigcap_{O \in \mathcal{T} \looparrowleft (a \geq b)} K_O \right) \\
&= \bigcap_{O \in (\mathcal{T} \looparrowleft (a \leq b)) \cup (\mathcal{T} \looparrowleft (a \geq b))} K_O \\
&= \bigcap_{O \in \mathcal{T}} K_O \qquad \text{(by (2))} \\
&= K_{\mathcal{T}}.
\end{aligned}$$

This proves (6).

Proof of (7): It is easy to see that $K_{\mathcal{T} \looparrowleft (a \leq b)} \subseteq K_{\mathcal{T} \looparrowleft (a \sim b)}$ [5], and similarly $K_{\mathcal{T} \looparrowleft (a \geq b)} \subseteq K_{\mathcal{T} \looparrowleft (a \sim b)}$. Combining these two relations, we obtain $K_{\mathcal{T} \looparrowleft (a \leq b)} \cup K_{\mathcal{T} \looparrowleft (a \geq b)} \subseteq K_{\mathcal{T} \looparrowleft (a \sim b)}$. Hence, in order to prove (7), it remains to show that $K_{\mathcal{T} \looparrowleft (a \sim b)} \subseteq K_{\mathcal{T} \looparrowleft (a \leq b)} \cup K_{\mathcal{T} \looparrowleft (a \geq b)}$. So let us do this now.

Let $y \in K_{\mathcal{T} \looparrowleft (a \sim b)}$. Our goal is to show that $y \in K_{\mathcal{T} \looparrowleft (a \leq b)} \cup K_{\mathcal{T} \looparrowleft (a \geq b)}$. In fact, assume the contrary. Then, $y \notin K_{\mathcal{T} \looparrowleft (a \leq b)}$ and $y \notin K_{\mathcal{T} \looparrowleft (a \geq b)}$.

We have $y \notin K_{\mathcal{T} \looparrowleft (a \leq b)} = \bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O$ (by (8)). Hence, there exists a $P \in \mathcal{T} \looparrowleft (a \leq b)$ such that $y \notin K_P$. Similarly, using $y \notin K_{\mathcal{T} \looparrowleft (a \geq b)}$, we can see that there exists a $Q \in \mathcal{T} \looparrowleft (a \geq b)$ such that $y \notin K_Q$. Consider these P and Q.

We have $P \in \mathcal{T} \looparrowleft (a \leq b) = \{O \in \mathcal{T} \mid (a \in O \implies b \in O)\}$. Thus, $P \in \mathcal{T}$

[5] *Proof.* Indeed, (1) yields $(\mathcal{T} \looparrowleft (a \leq b)) \cap (\mathcal{T} \looparrowleft (a \geq b)) = \mathcal{T} \looparrowleft (a \sim b)$, so that $\mathcal{T} \looparrowleft (a \sim b) \subseteq \mathcal{T} \looparrowleft (a \leq b)$. Now, from (8), we obtain $K_{\mathcal{T} \looparrowleft (a \leq b)} = \bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O$ and $K_{\mathcal{T} \looparrowleft (a \sim b)} = \bigcap_{O \in \mathcal{T} \looparrowleft (a \sim b)} K_O$. Thus,

$$\begin{aligned}
K_{\mathcal{T} \looparrowleft (a \leq b)} &= \bigcap_{O \in \mathcal{T} \looparrowleft (a \leq b)} K_O \subseteq \bigcap_{O \in \mathcal{T} \looparrowleft (a \sim b)} K_O \qquad (\text{since } \mathcal{T} \looparrowleft (a \sim b) \subseteq \mathcal{T} \looparrowleft (a \leq b)) \\
&= K_{\mathcal{T} \looparrowleft (a \sim b)},
\end{aligned}$$

qed.

and $(a \in P \implies b \in P)$. But we do not have $(b \in P \implies a \in P)$ [6]. Hence, $a \notin P$ and $b \in P$ (since $(a \in P \implies b \in P)$ but not $(b \in P \implies a \in P)$).

We have thus shown that $P \in \mathcal{T}$, $a \notin P$ and $b \in P$. Similarly, we find that $Q \in \mathcal{T}$, $b \notin Q$ and $a \in Q$. Now, it is easy to see that $P \cap Q \in \mathcal{T} \hookleftarrow (a \sim b)$ [7] and $P \cup Q \in \mathcal{T} \hookleftarrow (a \sim b)$ [8].

Let us write $y \in \mathbb{R}^n$ in the form $y = (y_1, y_2, \ldots, y_n)$. We have $(y_1, y_2, \ldots, y_n) = y \notin K_P = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus P} x_i \geq 0 \right\}$. Hence, $\sum_{i \in [n] \setminus P} y_i < 0$. Similarly, from $y \notin K_Q$, we obtain $\sum_{i \in [n] \setminus Q} y_i < 0$.

We have

$$\begin{aligned}
(y_1, y_2, \ldots, y_n) = y \in K_{\mathcal{T} \hookleftarrow (a \sim b)} &= \bigcap_{O \in \mathcal{T} \hookleftarrow (a \sim b)} K_O \qquad \text{(by (8))} \\
&\subseteq K_{P \cap Q} \qquad \text{(since } P \cap Q \in \mathcal{T} \hookleftarrow (a \sim b)\text{)} \\
&= \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i \in [n] \setminus (P \cap Q)} x_i \geq 0 \right\},
\end{aligned}$$

so that $\sum_{i \in [n] \setminus (P \cap Q)} y_i \geq 0$. The same argument can be applied to $P \cup Q$ instead of $P \cap Q$, and leads to $\sum_{i \in [n] \setminus (P \cup Q)} y_i \geq 0$.

But any two subsets A and B of $[n]$ satisfy $\sum_{i \in A} y_i + \sum_{i \in B} y_i = \sum_{i \in A \cup B} y_i + \sum_{i \in A \cap B} y_i$.

[6] *Proof.* Assume the contrary. Then, $(b \in P \implies a \in P)$. Combining this with $(a \in P \implies b \in P)$, we obtain $(a \in P \iff b \in P)$. Hence, $P \in \mathcal{T} \hookleftarrow (a \sim b)$ (by the definition of $\mathcal{T} \hookleftarrow (a \sim b)$). Now, $y \in K_{\mathcal{T} \hookleftarrow (a \sim b)} = \bigcap_{O \in \mathcal{T} \hookleftarrow (a \sim b)} K_O$ (by (8)). But $\bigcap_{O \in \mathcal{T} \hookleftarrow (a \sim b)} K_O \subseteq K_P$ (since $P \in \mathcal{T} \hookleftarrow (a \sim b)$), so that $y \in \bigcap_{O \in \mathcal{T} \hookleftarrow (a \sim b)} K_O \subseteq K_P$, which contradicts $y \notin K_P$. This contradiction proves that our assumption was wrong, qed.

[7] *Proof.* From $P \in \mathcal{T}$ and $Q \in \mathcal{T}$, we infer that $P \cap Q \in \mathcal{T}$. Also, $a \notin P \cap Q$ (since $a \notin P$), so that $(a \in P \cap Q \implies b \in P \cap Q)$. Moreover, $b \notin P \cap Q$ (since $b \notin Q$), and thus $(b \in P \cap Q \implies a \in P \cap Q)$. Combined with $(a \in P \cap Q \implies b \in P \cap Q)$, this yields $(a \in P \cap Q \iff b \in P \cap Q)$. Thus, $P \cap Q$ is an element of $\mathcal{T}$ satisfying $(a \in P \cap Q \iff b \in P \cap Q)$. Hence, $P \cap Q \in \{O \in \mathcal{T} \mid (a \in O \iff b \in O)\} = \mathcal{T} \hookleftarrow (a \sim b)$, qed.

[8] *Proof.* From $P \in \mathcal{T}$ and $Q \in \mathcal{T}$, we infer that $P \cup Q \in \mathcal{T}$. Also, $b \in P \cup Q$ (since $b \in P$), so that $(a \in P \cup Q \implies b \in P \cup Q)$. Moreover, $a \in P \cup Q$ (since $a \in Q$), and thus $(b \in P \cup Q \implies a \in P \cup Q)$. Combined with $(a \in P \cup Q \implies b \in P \cup Q)$, this yields $(a \in P \cup Q \iff b \in P \cup Q)$. Thus, $P \cup Q$ is an element of $\mathcal{T}$ satisfying $(a \in P \cup Q \iff b \in P \cup Q)$. Hence, $P \cup Q \in \{O \in \mathcal{T} \mid (a \in O \iff b \in O)\} = \mathcal{T} \hookleftarrow (a \sim b)$, qed.

Applying this to $A = [n] \setminus P$ and $B = [n] \setminus Q$, we obtain

$$\sum_{i\in[n]\setminus P} y_i + \sum_{i\in[n]\setminus Q} y_i = \sum_{i\in([n]\setminus P)\cup([n]\setminus Q)} y_i + \sum_{i\in([n]\setminus P)\cap([n]\setminus Q)} y_i = \sum_{i\in[n]\setminus(P\cap Q)} y_i + \sum_{i\in[n]\setminus(P\cup Q)} y_i$$

(since $([n] \setminus P) \cup ([n] \setminus Q) = [n] \setminus (P \cap Q)$ and $([n] \setminus P) \cap ([n] \setminus Q) = [n] \setminus (P \cup Q)$). Thus,

$$\sum_{i\in[n]\setminus(P\cap Q)} y_i + \sum_{i\in[n]\setminus(P\cup Q)} y_i = \underbrace{\sum_{i\in[n]\setminus P} y_i}_{<0} + \underbrace{\sum_{i\in[n]\setminus Q} y_i}_{<0} < 0.$$

This contradicts

$$\underbrace{\sum_{i\in[n]\setminus(P\cap Q)} y_i}_{\geq 0} + \underbrace{\sum_{i\in[n]\setminus(P\cup Q)} y_i}_{\geq 0} \geq 0.$$

This contradiction proves that our assumption was wrong. Hence, $y \in K_{\mathcal{T}\looparrowleft(a\leq b)} \cup K_{\mathcal{T}\looparrowleft(a\geq b)}$. Since we have proven this for every $y \in K_{\mathcal{T}\looparrowleft(a\sim b)}$, we thus conclude that $K_{\mathcal{T}\looparrowleft(a\sim b)} \subseteq K_{\mathcal{T}\looparrowleft(a\leq b)} \cup K_{\mathcal{T}\looparrowleft(a\geq b)}$. This finishes the proof of (7).

Now that both (6) and (7) are proven, Lemma 2.11 easily follows. □

Definition 2.12. Let V be a $\mathbb{K}$-vector space. A $\mathbb{K}$-linear map $f : \mathbf{H_T} \to V$ is said to be $\mathbf{T}$-*additive* if and only if every $n \in \mathbb{N}$, every $\mathcal{T} \in \mathbf{T}_n$ and every two distinct elements a and b of $[n]$ satisfy

$$f(\mathcal{T}) = f(\mathcal{T} \looparrowleft (a \leq b)) + f(\mathcal{T} \looparrowleft (a \geq b)) - f(\mathcal{T} \looparrowleft (a \sim b)). \tag{9}$$

Proposition 2.13. Let V be a $\mathbb{K}$-vector space. Let f and g be two $\mathbf{T}$-additive $\mathbb{K}$-linear maps $\mathbf{H_T} \to V$. Assume that $f(\mathcal{T}_u) = g(\mathcal{T}_u)$ for every packed word u. Then, $f = g$.

Proof of Proposition 2.13. It is clearly enough to show that

$$f(\mathcal{T}) = g(\mathcal{T}) \qquad \text{for every } \mathcal{T} \in \mathbf{T}. \tag{10}$$

For any topology $\mathcal{T}$ on a finite set X, we let $h(\mathcal{T})$ denote the nonnegative integer $\sharp\left\{(x,y) \in X^2 \mid \text{neither } x \leq_{\mathcal{T}} y \text{ nor } y \leq_{\mathcal{T}} x\right\}$. We shall prove (10) by strong induction over $h(\mathcal{T})$. So we fix some $\mathcal{T} \in \mathbf{T}$, and we want to prove (10), assuming that every $\mathcal{S} \in \mathbf{T}$ satisfying $h(\mathcal{S}) < h(\mathcal{T})$ satisfies

$$f(\mathcal{S}) = g(\mathcal{S}). \tag{11}$$

Let $n \in \mathbb{N}$ be such that $\mathcal{T} \in \mathbf{T}_n$. If there exist no two elements a and b of $[n]$ satisfying neither $a \leq_{\mathcal{T}} b$ nor $b \leq_{\mathcal{T}} a$, then we have $\mathcal{T} = \mathcal{T}_u$ for some packed word u, and this u satisfies $f(\mathcal{T}_u) = g(\mathcal{T}_u)$ (due to the assumption of the proposition); thus, (10) follows immediately (since $\mathcal{T} = \mathcal{T}_u$). Hence, we can WLOG assume that such two elements a and b exist. Consider these two elements. Of course, a and b are distinct.

If $\mathcal{S}$ is any of the three posets $\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b)$, $\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b)$ and $\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b)$, then $h(\mathcal{S}) < h(\mathcal{T})$ [9]. Hence, we can apply (11) to each of these three posets. We obtain

$$f(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b)) = g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b));$$
$$f(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b)) = g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b));$$
$$f(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b)) = g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b)).$$

But since f is **T**-additive, we have

$$\begin{aligned} f(\mathcal{T}) &= \underbrace{f(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b))}_{=g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b))} + \underbrace{f(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b))}_{=g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b))} - f\underbrace{(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b))}_{=g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b))} \\ &= g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \leq b)) + g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \geq b)) - g(\mathcal{T} \leftarrow\!\!\!\!\!\!\text{P} (a \sim b)) = g(\mathcal{T}) \end{aligned}$$

(since g is **T**-additive). Thus, (10) is proven, and the induction step is complete. □

Proof of Theorem 2.7 (sketched). We need to show that $\beta = \alpha \circ U$.

We notice that every topology $\mathcal{S}$ on $[n]$ satisfies

$$\begin{aligned} (\beta \circ Z)(\mathcal{S}) &= \beta \left(\underbrace{Z(\mathcal{S})}_{\substack{=(-1)^{|[n]/\sim_{\mathcal{S}}|}\mathcal{S} \\ \text{(by the definition of } Z)}} \right) = (-1)^{|[n]/\sim_{\mathcal{S}}|} \underbrace{\beta(\mathcal{S})}_{\substack{=(-1)^{|[n]/\sim_{\mathcal{S}}|}\underline{1}_{K_{\mathcal{S}}} \\ \text{(by the definition of } \beta)}} \\ &= \underbrace{(-1)^{|[n]/\sim_{\mathcal{S}}|}(-1)^{|[n]/\sim_{\mathcal{S}}|}}_{=1} \underline{1}_{K_{\mathcal{S}}} \\ &= \underline{1}_{K_{\mathcal{S}}} \end{aligned} \tag{12}$$

[9] This is because $\{(x,y) \in X^2 \mid \text{neither } x \leq_{\mathcal{S}} y \text{ nor } y \leq_{\mathcal{S}} x\}$ is a proper subset of $\{(x,y) \in X^2 \mid \text{neither } x \leq_{\mathcal{T}} y \text{ nor } y \leq_{\mathcal{T}} x\}$. (Proper because (a,b) or (b,a) belongs to the latter but not to the former.)

and

$$\begin{aligned}
(\alpha \circ U \circ Z)(\mathcal{S}) &= \alpha\left(U\left(\underbrace{Z(\mathcal{S})}_{\substack{=(-1)^{|[n]/\sim_{\mathcal{S}}|}\mathcal{S} \\ \text{(by the definition of } Z\text{)}}}\right)\right) \\
&= (-1)^{|[n]/\sim_{\mathcal{S}}|}\,\alpha\left(\underbrace{U(\mathcal{S})}_{\substack{=\sum\limits_{f\in\mathcal{U}(\mathcal{S})} f \\ \text{(by the definition of } U\text{)}}}\right) \\
&= (-1)^{|[n]/\sim_{\mathcal{S}}|}\sum_{f\in\mathcal{U}(\mathcal{S})}\alpha(f). \qquad (13)
\end{aligned}$$

We shall now show that both maps $\beta \circ Z : \mathbf{H_T} \to \mathrm{WQSym}$ and $\alpha \circ U \circ Z : \mathbf{H_T} \to \mathrm{WQSym}$ are $\mathbf{T}$-additive.

Proof that the map $\beta \circ Z$ is $\mathbf{T}$-additive: Let $n \in \mathbb{N}$. Let $\mathcal{T} \in \mathbf{T}_n$. Let a and b be two distinct elements of $[n]$. In order to show that $\beta \circ Z$ is $\mathbf{T}$-additive, we must prove that

$$\begin{aligned}
&(\beta \circ Z)(\mathcal{T}) \\
&= (\beta \circ Z)(\mathcal{T} \hookleftarrow (a \leq b)) + (\beta \circ Z)(\mathcal{T} \hookleftarrow (a \geq b)) - (\beta \circ Z)(\mathcal{T} \hookleftarrow (a \sim b)). \qquad (14)
\end{aligned}$$

This rewrites as follows:

$$\underline{1}_{K_{\mathcal{T}}} = \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\leq b)}} + \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\geq b)}} - \underline{1}_{K_{\mathcal{T}\hookleftarrow(a\sim b)}}$$

(because of (12)). But this is precisely the claim of Lemma 2.11. Hence, (14) is proven. We thus have shown that the map $\beta \circ Z$ is $\mathbf{T}$-additive.

Proof that the map $\alpha \circ U \circ Z$ is $\mathbf{T}$-additive: Let $n \in \mathbb{N}$. Let $\mathcal{T} \in \mathbf{T}_n$. Let a and b be two distinct elements of $[n]$. In order to show that $\alpha \circ U \circ Z$ is $\mathbf{T}$-additive, we must prove that

$$\begin{aligned}
&(\alpha \circ U \circ Z)(\mathcal{T}) \\
&= (\alpha \circ U \circ Z)(\mathcal{T} \hookleftarrow (a \leq b)) + (\alpha \circ U \circ Z)(\mathcal{T} \hookleftarrow (a \geq b)) \\
&\qquad - (\alpha \circ U \circ Z)(\mathcal{T} \hookleftarrow (a \sim b)). \qquad (15)
\end{aligned}$$

This is rather obvious if $a \leq_{\mathcal{T}} b$ [10]. Hence, for the rest of this proof, we

[10] *Proof.* Assume that $a \leq_{\mathcal{T}} b$. Then, Lemma 2.10 **(c)** yields $\mathcal{T} \hookleftarrow (a \leq b) = \mathcal{T}$ and $\mathcal{T} \hookleftarrow (a \sim b) = \mathcal{T} \hookleftarrow (a \geq b)$. Hence, (15) rewrites as

$$\begin{aligned}
&(\alpha \circ U \circ Z)(\mathcal{T}) \\
&= (\alpha \circ U \circ Z)(\mathcal{T}) + (\alpha \circ U \circ Z)(\mathcal{T} \hookleftarrow (a \geq b)) - (\alpha \circ U \circ Z)(\mathcal{T} \hookleftarrow (a \geq b)).
\end{aligned}$$

But this is obvious.

WLOG assume that we don't have $a \leq_{\mathcal{T}} b$. Similarly, we WLOG assume that we don't have $b \leq_{\mathcal{T}} a$. Now, using (13), we can rewrite the equality (15) as follows:

$$\begin{aligned}
&(-1)^{|[n]/\sim_{\mathcal{T}}|} \sum_{f \in \mathcal{U}(\mathcal{T})} \alpha(f) \\
&= (-1)^{\left|[n]/\sim_{\mathcal{T} \leftarrow \mathcal{P}(a \leq b)}\right|} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \leq b))} \alpha(f) + (-1)^{\left|[n]/\sim_{\mathcal{T} \leftarrow \mathcal{P}(a \geq b)}\right|} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \geq b))} \alpha(f) \\
&\qquad - (-1)^{\left|[n]/\sim_{\mathcal{T} \leftarrow \mathcal{P}(a \sim b)}\right|} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \sim b))} \alpha(f).
\end{aligned}$$

This can be rewritten further as

$$\begin{aligned}
&(-1)^{|[n]/\sim_{\mathcal{T}}|} \sum_{f \in \mathcal{U}(\mathcal{T})} \alpha(f) \\
&= (-1)^{|[n]/\sim_{\mathcal{T}}|} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \leq b))} \alpha(f) + (-1)^{|[n]/\sim_{\mathcal{T}}|} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \geq b))} \alpha(f) \\
&\qquad - (-1)^{|[n]/\sim_{\mathcal{T}}|-1} \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \sim b))} \alpha(f)
\end{aligned}$$

(because Lemma 2.10 **(n)** (applied to $X = [n]$) yields $\left|[n] / \sim_{\mathcal{T} \leftarrow \mathcal{P}(a \leq b)}\right| = \left|[n] / \sim_{\mathcal{T} \leftarrow \mathcal{P}(a \geq b)}\right| = |[n] / \sim_{\mathcal{T}}|$ and $\left|[n] / \sim_{\mathcal{T} \leftarrow \mathcal{P}(a \sim b)}\right| = |[n] / \sim_{\mathcal{T}}| - 1$). Upon cancelling $(-1)^{|[n]/\sim_{\mathcal{T}}|}$, this simplifies to

$$\sum_{f \in \mathcal{U}(\mathcal{T})} \alpha(f) = \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \leq b))} \alpha(f) + \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \geq b))} \alpha(f) + \sum_{f \in \mathcal{U}(\mathcal{T} \leftarrow \mathcal{P}(a \sim b))} \alpha(f).$$

But this follows immediately from Lemma 2.10 **(m)** (applied to $X = [n]$). Thus, (15) is proven. We have thus shown that $\alpha \circ U \circ Z$ is $\mathbf{T}$-additive.

Now, it is easy to see that $(\beta \circ Z)(\mathcal{T}_u) = (\alpha \circ U \circ Z)(\mathcal{T}_u)$ for every packed word u [11]. Hence, Proposition 2.13 (applied to $V = \mathfrak{M}$, $f = \beta \circ Z$ and $g = \alpha \circ U \circ Z$) yields $\beta \circ Z = \alpha \circ U \circ Z$. Since Z is an isomorphism, we can cancel Z from this equality, and obtain $\beta = \alpha \circ U$. This proves Theorem 2.7. □

[11] *Proof.* Let u be a packed word. Applying (12) to $\mathcal{S} = \mathcal{T}_u$, we obtain $(\beta \circ Z)(\mathcal{T}_u) = \underline{1}_{K_{\mathcal{T}_u}} = \underline{1}_{K_u}$ (since Remark 2.2 yields $K_{\mathcal{T}_u} = K_u$). But applying (13) to $\mathcal{S} = \mathcal{T}_u$ leads to

$$\begin{aligned}
(\alpha \circ U \circ Z)(\mathcal{T}_u) &= \underbrace{(-1)^{|[n]/\sim_{\mathcal{T}_u}|}}_{\substack{=(-1)^{\max u} \\ (\text{since } |[n]/\sim_{\mathcal{T}_u}| = \max u)}} \underbrace{\sum_{f \in \mathcal{U}(\mathcal{T}_u)} \alpha(f)}_{\substack{=\alpha(u) \\ (\text{since } \mathcal{U}(\mathcal{T}_u) = \{u\})}} \\
&= (-1)^{\max u} \underbrace{\alpha(u)}_{\substack{=(-1)^{\max u} \underline{1}_{K_u} \\ (\text{by the definition of } \alpha)}} = \underbrace{(-1)^{\max u} (-1)^{\max u}}_{=1} \underline{1}_{K_u} = \underline{1}_{K_u} \\
&= (\beta \circ Z)(\mathcal{T}_u),
\end{aligned}$$

qed.

Proof of Theorem 1.4. Theorem 2.7 yields $\beta = \alpha \circ U$. Since both β and U are $\mathbb{K}$-algebra homomorphisms, and since U is surjective, this easily yields that α is a $\mathbb{K}$-algebra homomorphism. (Indeed, let $p \in \operatorname{WQSym}$ and $q \in \operatorname{WQSym}$. Then, thanks to the surjectivity of U, there exist $\mathcal{P} \in \mathbf{H_T}$ and $\mathcal{Q} \in \mathbf{H_T}$ satisfying $p = U(\mathcal{P})$ and $q = U(\mathcal{Q})$. Consider these $\mathcal{P}$ and $\mathcal{Q}$. Since U is a $\mathbb{K}$-algebra homomorphism, we have $U(\mathcal{P}.\mathcal{Q}) = \underbrace{U(\mathcal{P})}_{=p}\underbrace{U(\mathcal{Q})}_{=q} = pq$. Now,

$$\begin{aligned}
&\alpha\left(\underbrace{p}_{=U(\mathcal{P})}\right) \cdot \alpha\left(\underbrace{q}_{=U(\mathcal{Q})}\right) \\
&= \underbrace{\alpha(U(\mathcal{P}))}_{=(\alpha\circ U)(\mathcal{P})} \cdot \underbrace{\alpha(U(\mathcal{Q}))}_{=(\alpha\circ U)(\mathcal{Q})} = \underbrace{(\alpha\circ U)}_{=\beta}(\mathcal{P}) \cdot \underbrace{(\alpha\circ U)}_{=\beta}(\mathcal{Q}) \\
&= \beta(\mathcal{P}) \cdot \beta(\mathcal{Q}) = \underbrace{\beta}_{=\alpha\circ U}(\mathcal{P}.\mathcal{Q}) \qquad \text{(since } \beta \text{ is a } \mathbb{K}\text{-algebra homomorphism)} \\
&= (\alpha\circ U)(\mathcal{P}.\mathcal{Q}) = \alpha\left(\underbrace{U(\mathcal{P}.\mathcal{Q})}_{=pq}\right) = \alpha(pq),
\end{aligned}$$

and this shows that α is a $\mathbb{K}$-algebra homomorphism.) Theorem 1.4 is proven. □

3. Application: an alternating sum identity

As an application of Theorem 1.4 we can prove the following fact, which is analogous to [3, Corollary 4.8]:

Corollary 3.1. Let $n \in \mathbb{N}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. Then,

$$\sum_{\substack{u \text{ is a packed word} \\ \text{of length } n; \\ \lambda \in K_u}} (-1)^{\max u} = \begin{cases} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

This will rely on the following equality in WQSym:

Proposition 3.2. Let ζ be the packed word (1) of length 1. Then, in WQSym, we have

$$\zeta^n = \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} u.$$

Proof sketch. Induction on n (details are left to the reader). □

Proof of Corollary 3.1. Let $\mathbb{R}_+$ denote the set of all nonnegative reals. Let $\zeta \in \operatorname{WQSym}$ be the packed word (1) of length 1.

Consider the map α from Theorem 1.4. The definition of this map α yields

$$\alpha(\zeta) = \underbrace{(-1)^{\max\zeta}}_{\substack{=-1 \\ (\text{since } \max\zeta=1)}} \underline{1}_{K_\zeta} = -\underline{1}_{K_\zeta} = -\underline{1}_{\mathbb{R}_+}$$

(since the definition of K_ζ yields $K_\zeta = \mathbb{R}_+$). Hence,

$$(\alpha(\zeta))^n = \left(-\underline{1}_{\mathbb{R}_+}\right)^n = (-1)^n \underbrace{\left(\underline{1}_{\mathbb{R}_+}\right)^n}_{\substack{=\underline{1}_{\mathbb{R}_+^n} \\ (\text{this follows easily from the} \\ \text{definition of multiplication on } \mathfrak{M})}} = (-1)^n \underline{1}_{\mathbb{R}_+^n}.$$

But Proposition 3.2 yields

$$\zeta^n = \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} u.$$

Applying the map α to both sides of this equality, we obtain

$$\alpha(\zeta^n) = \alpha\left(\sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} u\right) = \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} \underbrace{\alpha(u)}_{\substack{=(-1)^{\max u}\underline{1}_{K_u} \\ (\text{by the definition of } \alpha)}}$$
$$= \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} (-1)^{\max u} \underline{1}_{K_u}.$$

Applying both sides of this equality to λ, we obtain

$$(\alpha(\zeta^n))(\lambda) = \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} (-1)^{\max u} \underbrace{\underline{1}_{K_u}(\lambda)}_{\substack{=\begin{cases} 1, & \text{if } \lambda \in K_u; \\ 0, & \text{if } \lambda \notin K_u \end{cases} \\ (\text{by the definition of } \underline{1}_{K_u})}}$$
$$= \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n}} (-1)^{\max u} \begin{cases} 1, & \text{if } \lambda \in K_u; \\ 0, & \text{if } \lambda \notin K_u \end{cases}$$
$$= \sum_{\substack{u \text{ is a packed word} \\ \text{of length } n; \\ \lambda \in K_u}} (-1)^{\max u}.$$

Hence,

$$\begin{aligned}
\sum_{\substack{u \text{ is a packed word} \\ \text{of length } n; \\ \lambda \in K_u}} (-1)^{\max u} &= \underbrace{(\alpha(\zeta^n))}_{\substack{=(\alpha(\zeta))^n \\ \text{(since } \alpha \text{ is a } \mathbb{K}\text{-algebra} \\ \text{homomorphism)}}} (\lambda) = \underbrace{(\alpha(\zeta))^n}_{=(-1)^n \underline{1}_{\mathbb{R}^n_+}} (\lambda) \\
&= (-1)^n \underbrace{\underline{1}_{\mathbb{R}^n_+}(\lambda)}_{=\begin{cases} 1, & \text{if } \lambda \in \mathbb{R}^n_+; \\ 0, & \text{otherwise} \end{cases}} = (-1)^n \begin{cases} 1, & \text{if } \lambda \in \mathbb{R}^n_+; \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (-1)^n, & \text{if } \lambda \in \mathbb{R}^n_+; \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

(since the condition "$\lambda \in \mathbb{R}^n_+$" is equivalent to "$\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$"). This proves Corollary 3.1. □

From Corollary 3.1, we can in turn derive the precise statement of [3, Corollary 4.8]:

Corollary 3.3. Let $n \in \mathbb{N}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$. Then,

$$\sum_{\substack{u \text{ is a packed word} \\ \text{of length } n; \\ \lambda \in K_u^\circ}} (-1)^{\max u} = \begin{cases} (-1)^n, & \text{if } \lambda_1, \lambda_2, \ldots, \lambda_n > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Here, for any packed word u of length n, we define the subset K_u° of $\mathbb{R}^n$ in the same way as we defined K_u, but with the "$\geq$" sign replaced by "$>$".

Proof sketch. Pick a small $\varepsilon > 0$, and let $\lambda' := (\lambda_1 - \varepsilon, \lambda_2 - \varepsilon, \ldots, \lambda_n - \varepsilon)$. If ε has been chosen small enough (say,

$$0 < \varepsilon < \frac{1}{n} \min \left\{ \sum_{i \in I} \lambda_i \ \mid \ I \subseteq [n] \text{ satisfying } \sum_{i \in I} \lambda_i > 0 \right\}$$

), then any packed word u of length n will satisfy $\lambda \in K_u^\circ$ if and only if it satisfies $\lambda' \in K_u$, and we will have $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ if and only if $\lambda_1 - \varepsilon, \lambda_2 - \varepsilon, \ldots, \lambda_n - \varepsilon \geq 0$. Hence, Corollary 3.3 follows from Corollary 3.1 (applied to λ' and $\lambda_i - \varepsilon$ instead of λ and λ_i). □

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