

La décomposition en poids des algèbres de Hopf

Frédéric Patras

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Errata and remarks (by *Darij Grinberg*)

The following are remarks I have made while reading the above-cited paper by Frédéric Patras. I think it is an interesting and rather readable text (despite some minor typos and tersely written proofs).

Some of the below remarks are just quick corrections of minor mistakes (at least as far as I can tell; I can neither guarantee that these "mistakes" really are mistakes, nor that my "corrections" are correct!). Some others are detailed expositions of certain proofs which have been only vaguely sketched in Patras's paper. Finally, some others give alternative proofs for results in Patras's paper (sometimes inserting additional results into Patras's paper, to be used as lemmata later on).

Different remarks are separated by horizontal lines, like this:

Page 1068: I think "aux endomorphismes ψ^k " should be "aux endomorphismes Ψ^k " here.

Page 1069, Definition 1.1: There is nothing wrong here, but I think it would be helpful to notice that what Patras calls "algèbre de Hopf" is *not* the same as what modern-day algebraists call a Hopf algebra. What Patras calls "algèbre de Hopf" is a kind of super-version of a graded bialgebra (not necessarily having an antipode!); in contrast, what modern-day algebraists call a Hopf algebra is just a bialgebra with antipode. (Nevertheless, I am going to use the words "Hopf algebra" for what Patras calls "algèbre de Hopf" in the following.)

Page 1070, fifth line of this page: Here, Patras write:

"Une bigèbre graduée ou une algèbre de Hopf est *connexe* si $H_0 \cong K$."

This definition is good when K is a field, but in the general case when K is a commutative ring, it is not a reasonable definition of "connected". Since Patras, in his paper, always works over a field K , this is not a problem for him, but I still prefer the following (in my opinion, better) definition of "connected": A graded bialgebra or Hopf algebra is *connected* if and only if the map $\epsilon|_{H_0}: H_0 \rightarrow K$ is an isomorphism.

Note that, when K is a field, this definition is equivalent to Patras's definition,

because we have the following equivalence of assertions:

$$\begin{aligned}
& (H_0 \cong K \text{ as } K\text{-vector spaces}) \\
& \iff (\dim(H_0) = 1) \quad (\text{where } \dim V \text{ denotes the dimension of any } K\text{-vector space } V) \\
& \iff (\dim(\text{Ker}(\epsilon|_{H_0})) = 0) \\
& \left(\begin{array}{l} \text{since we know that the map } \epsilon|_{H_0}: H_0 \rightarrow K \text{ is surjective (because} \\ (\epsilon|_{H_0})(1) = \epsilon(1) = 1 \text{ (by the axioms of a bialgebra, since } H \text{ is a bialgebra)),} \\ \text{and thus (by the isomorphism theorem) } K \cong H_0 / \text{Ker}(\epsilon|_{H_0}), \text{ so that} \\ \dim K = \dim(H_0 / \text{Ker}(\epsilon|_{H_0})) = \dim H_0 - \dim(\text{Ker}(\epsilon|_{H_0})), \\ \text{so that } \dim H_0 = \underbrace{\dim K}_{=1} + \dim(\text{Ker}(\epsilon|_{H_0})) = 1 + \dim(\text{Ker}(\epsilon|_{H_0})), \\ \text{and therefore the equation } \dim(H_0) = 1 \text{ is equivalent to } \dim(\text{Ker}(\epsilon|_{H_0})) = 0 \end{array} \right) \\
& \iff (\text{Ker}(\epsilon|_{H_0}) = 0) \iff (\epsilon|_{H_0} \text{ is injective}) \iff (\epsilon|_{H_0} \text{ is bijective}) \\
& \left(\begin{array}{l} \text{since we know that the map } \epsilon|_{H_0}: H_0 \rightarrow K \text{ is surjective (because} \\ (\epsilon|_{H_0})(1) = \epsilon(1) = 1 \text{ (by the axioms of a bialgebra, since } H \text{ is a bialgebra)),} \\ \text{and thus this map } \epsilon|_{H_0} \text{ is injective if and only if it is bijective} \end{array} \right) \\
& \iff (\epsilon|_{H_0} \text{ is an isomorphism}).
\end{aligned}$$

Page 1070, two lines above Definition 1.2: Here, Patras writes:

"[...] l'ensemble $\mathcal{L}(H)$ des endomorphismes linéaires de H [...]"

I don't think that $\mathcal{L}(H)$ denotes the set of all linear endomorphisms of H throughout the text. It seems to me that $\mathcal{L}(H)$ indeed denotes the set of all linear endomorphisms of H when H is just a bialgebra (not graded); but when H is a graded bialgebra or an "algèbre de Hopf" (I would translate this by "Hopf algebra", but as I said, this does not mean what people nowadays mean by a "Hopf algebra"), $\mathcal{L}(H)$ denotes the set of all *graded*¹ linear endomorphisms of H .

Note that I might be wrong about this, and $\mathcal{L}(H)$ might indeed mean the set of *all* linear endomorphisms of H throughout the text. But in this case, the homomorphism ρ_n defined on page 1074 ("Notons ρ_n l'homomorphisme de restriction de $\mathcal{L}(H)$ dans $\mathcal{L}(H)_n$.")) is not a simple restriction homomorphism (i. e., it is not just given by $\rho_n(f) = f|_{\bigoplus_{i=0}^n H_i}$ for every $f \in \mathcal{L}(H)$), but instead requires a more subtle definition:

It must then be defined by

$$\left(\rho_n(f) = \sum_{i=0}^n p_i \circ f \circ p_i \quad \text{for all } f \in \mathcal{L}(H) \right),$$

where $p_i: H \rightarrow H$ denotes the map which sends every element of H to its i -th graded component (seen again as an element of H).

Page 1071, proof of Proposition 1.4: Here, Patras writes: "La deuxième partie de la proposition se ramène à établir l'égalité :

$$\Delta^{[k]} \circ \Pi^{[l]} = \Pi_{(k)}^{[l]} \circ (\Delta^{[k]})^{\otimes l},$$

¹A linear map $f: V \rightarrow W$ between graded vector spaces V and W is said to be *graded* (or *compatible with the grading*) if, for every $n \in \mathbb{N}$, it satisfies $f(V_n) \subseteq W_n$.

qui est une conséquence à peu près immédiate des axiomes de structure des bigèbres commutatives.” This is not totally precise. The identity $\Delta^{[k]} \circ \Pi^{[l]} = \Pi^{[l]} \circ (\Delta^{[k]})^{\otimes l}$ is true in any bialgebra, not only in commutative ones (it follows from the axioms of a bialgebra by a double induction over k and l). But deriving the ”deuxième partie de la proposition” from this identity requires the bialgebra to be commutative. Here are the details of this derivation: We have $(\Delta^{[k]})^{\otimes l} \circ \Delta^{[l]} = \Delta^{[lk]}$ (this holds for any coalgebra, and can be proven by induction using the coassociativity and counity axioms of a coalgebra²) and $\Pi^{[k]} \circ \Pi^{[l]}_{(k)} = \Pi^{[lk]}$ (this holds for any commutative algebra, and is easy to see - but doesn't generally hold for noncommutative algebras!), so that

$$\begin{aligned}
& \underbrace{\Psi^k}_{=I^{*k}=\Pi^{[k]} \circ I^{\otimes k} \circ \Delta^{[k]} = \Pi^{[k]} \circ \Delta^{[k]}} \quad \circ \quad \underbrace{\Psi^l}_{=I^{*l}=\Pi^{[l]} \circ I^{\otimes l} \circ \Delta^{[l]} = \Pi^{[l]} \circ \Delta^{[l]}} \\
& = \Pi^{[k]} \circ \underbrace{\Delta^{[k]} \circ \Pi^{[l]}}_{=\Pi^{[l]} \circ (\Delta^{[k]})^{\otimes l}} \circ \Delta^{[l]} = \underbrace{\Pi^{[k]} \circ \Pi^{[l]}_{(k)}}_{=\Pi^{[lk]}} \circ \underbrace{(\Delta^{[k]})^{\otimes l} \circ \Delta^{[l]}}_{=\Delta^{[lk]}} = \Pi^{[lk]} \circ \Delta^{[lk]} \\
& = \Pi^{[lk]} \circ I^{\otimes lk} \circ \Delta^{[lk]} = I^{*lk} = \Psi^{lk},
\end{aligned}$$

and this proves the second part of Proposition 1.4.

Page 1072: A typo: ”Notons Φ^k the n -ième endomorphisme” should be ”Notons Φ^k the k -ième endomorphisme”.

Page 1073, proof of Proposition 2.3: There is nothing wrong to be corrected here, but I don't find the proof of this proposition as obvious as Patras does, so let me write down this proof here:

Proof of Proposition 2.3. We can prove that any commuting $x \in M$ and $y \in M$ satisfy $\log_k x + \log_k y = \log_k(xy)$ (this follows from the well-known fact that $\log(1+X) + \log(1+Y) = \log((1+X)(1+Y))$ in the ring $K[[X, Y]]$ of formal power series, using the fact that the K -representation A is unipotent of rank k). Using this fact, we can prove (by induction over n) that every $x \in M$ and $n \in \mathbb{N}$ satisfy $n \log_k x = \log_k(x^n)$.

But for every $x \in M$, we have

$$(\rho \circ \Phi^n)(x) = \rho \left(\underbrace{\Phi^n(x)}_{=x^n} \right) = \rho(x^n)$$

²This proof can be found in [P3] (Lemma II.8 of [P3], to be precise).

and

$$\begin{aligned}
\left(\sum_{i=0}^{k-1} n^i \cdot \varepsilon^i \right) (x) &= \sum_{i=0}^{k-1} n^i \cdot \underbrace{\varepsilon^i(x)}_{\frac{(\log_k x)^i}{i!}} = \sum_{i=0}^{k-1} n^i \cdot \frac{(\log_k x)^i}{i!} \\
&= \sum_{i=0}^{k-1} \frac{(n \log_k x)^i}{i!} = \exp_k \left(\underbrace{n \log_k x}_{=\log_k(x^n)} \right) \quad (\text{by the definition of } \exp_k(n \log_k x)) \\
&= \exp_k(\log_k(x^n)) = \underbrace{(\exp_k \circ \log_k)}_{=\rho \text{ (by Lemma 2.1)}}(x^n) = \rho(x^n).
\end{aligned}$$

Hence, for every $x \in M$, we have $(\rho \circ \Phi^n)(x) = \rho(x) = \left(\sum_{i=0}^{k-1} n^i \cdot \varepsilon^i \right) (x)$. Thus,

$$\rho \circ \Phi^n = \sum_{i=0}^{k-1} n^i \cdot \varepsilon^i. \text{ This proves Proposition 2.3.}$$

Page 1074, proof of Lemma 3.1: Let me add that the same argument which Patras used to prove Lemma 3.1 can be used to prove a more general statement:

Lemma 3.11. Let $\rho'_n : \text{Hom}_K(H, H) \rightarrow \text{Hom}_K\left(\bigoplus_{i=0}^n H_i, H\right)$ be the map which takes every linear map $g \in \text{Hom}_K(H, H)$ to the restriction of g to $\bigoplus_{i=0}^n H_i$.

For every map $f \in \text{Hom}_K(H, H)$ satisfying $f(1) = 1$, we have $(\rho'_n(f - 1))^{*(n+1)} = 0$ (where 1 denotes the unity of the K -algebra $\mathcal{L}(H)$, i. e., the map $\eta \circ \epsilon$).

Proof of Lemma 3.11. Copy the proof of Lemma 3.1, replacing every occurrence of Ψ^k by f , and replacing every occurrence of ρ_n by ρ'_n . This gives a proof of Lemma 3.11.

Note that we replaced ρ_n by ρ'_n in the statement of Lemma 3.11 because we didn't want to require f to be graded. If $f \in \text{Hom}_K(H, H)$ is a graded map, then $\rho_n(f)$ is "more or less the same" as $\rho'_n(f)$ (the only difference between the maps $\rho_n(f)$ and $\rho'_n(f)$ is that the codomain of $\rho_n(f)$ is $\bigoplus_{i=0}^n H_i$, whereas the codomain of $\rho'_n(f)$ is the whole H). However, if f is not a graded map, $\rho_n(f)$ is either not defined or not identic with $\rho'_n(f)$ (depending on how ρ_n is defined: see my remark about "Page 1070, two lines above Definition 1.2" above).

Here is a very useful consequence of Lemma 3.11:

Lemma 3.12. Let $f \in \text{Hom}_K(H, H)$ be a map satisfying $f(1) = 1$. Then, for every $x \in H$, the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1)^{*n}}{n}(x)$ has only finitely many nonzero terms.

Proof of Lemma 3.12. Let $x \in H$. Since $x \in H = \bigoplus_{i \in \mathbb{N}} H_i = \bigcup_{j \in \mathbb{N}} \left(\bigoplus_{i=0}^j H_i \right)$, there exists some $j \in \mathbb{N}$ such that $x \in \bigoplus_{i=0}^j H_i$. Consider this j .

Recall that $\rho'_j : \text{Hom}_K(H, H) \rightarrow \text{Hom}_K\left(\bigoplus_{i=0}^j H_i, H\right)$ is the map which takes every linear map $g \in \text{Hom}_K(H, H)$ to the restriction of g to $\bigoplus_{i=0}^j H_i$. Hence, for every $n \in \mathbb{N}$, the map $\rho'_j((f-1)^{*n})$ is the restriction of the map $(f-1)^{*n}$ to $\bigoplus_{i=0}^j H_i$. Since $x \in \bigoplus_{i=0}^j H_i$, this yields that $(\rho'_j((f-1)^{*n}))(x) = (f-1)^{*n}(x)$ for every $n \in \mathbb{N}$. But we also have $\rho'_j((f-1)^{*n}) = (\rho'_j(f-1))^{*n}$ for every $n \in \mathbb{N}$ (since ρ'_j is a K -algebra homomorphism).

But Lemma 3.11 (applied to j instead of n) yields $(\rho'_j(f-1))^{*(j+1)} = 0$. Hence, every integer $n \geq j+1$ satisfies

$$(\rho'_j(f-1))^{*n} = (\rho'_j(f-1))^{*(j+1)+(n-(j+1))} = \underbrace{(\rho'_j(f-1))^{*(j+1)}}_{=0} * (\rho'_j(f-1))^{*(n-(j+1))} = 0.$$

Thus, every integer $n \geq j+1$ satisfies

$$\begin{aligned} (-1)^{n+1} \frac{(f-1)^{*n}}{n}(x) &= (-1)^{n+1} \frac{(f-1)^{*n}(x)}{n} = (-1)^{n+1} \frac{(\rho'_j((f-1)^{*n}))(x)}{n} \\ &\quad \text{(since } (\rho'_j((f-1)^{*n}))(x) = (f-1)^{*n}(x)\text{)} \\ &= (-1)^{n+1} \frac{(\rho'_j(f-1))^{*n}(x)}{n} \quad \text{(since } \rho'_j((f-1)^{*n}) = (\rho'_j(f-1))^{*n}\text{)} \\ &= (-1)^{n+1} \frac{0}{n} \quad \text{(since } (\rho'_j(f-1))^{*n} = 0 \text{ (due to } n \geq j+1)\text{)} \\ &= 0. \end{aligned}$$

This proves Lemma 3.12.

Page 1074: Two lines above Proposition 3.2, Patras writes: " ε_n^i est donc un morphisme de E dans $\mathcal{L}(H)_n$ ". It would be helpful to emphasize that "morphism" means a morphism of sets here, not a morphism of monoids (unless I am missing something!).

Page 1074: Four lines above Proposition 3.2, Patras writes: "Nous noterons dans la suite ε_n^i , $1 \leq i \leq n$, [...]". I think that considering the ε_n^i only for $1 \leq i \leq n$ (but not for $i = 0$) is a bad decision, since it leads to several minor mistakes afterwards. For example, the first identity on page 1077,

$$\rho_n(\Psi^\zeta) = \rho_n\left(\sum_{i=1}^n \zeta^i \cdot e^i\right),$$

is not completely correct, since the sum on the right hand is missing an $i = 0$ term, but as long as e^0 is not defined, this does not make much sense. For another example, Definition 3.7 does not uniquely define Ψ^ζ , because H is not the direct sum of all $H_n^{(i)}$ unless we allow i to be 0.

I think the simplest way to clean up this mess is to define the maps ε_n^i for all $0 \leq i \leq n$ in the same way as they are defined for all $1 \leq i \leq n$ in the text. This yields that $\varepsilon_n^0(x) = \frac{(\log_1 x)^{*0}}{0!} = \frac{1}{1} = 1$ (where 1 denotes the unity of $\mathcal{L}(H)$); this is the map $\eta \circ \epsilon$ (not the map I). Thus, in particular, $\varepsilon_n^0(I) = 1$, so that

$$\begin{aligned} e_n^0 &= \varepsilon_n^0(I) |_{H_n} && \text{(by the definition of } e_n^0\text{)} \\ &= 1 |_{H_n} = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0 \end{cases} \end{aligned}$$

Hence, for $n \neq 0$, we have $e_n^0 = 0$. This is why we don't have care about e_n^0 when $n \neq 0$. But for $n = 0$, we have $e_0^0 = 1$.

Now we have the following (in my opinion, slightly better) version of Proposition 3.2:

Proposition 3.2'. For every $n \in \mathbb{N}$ (including the case $n = 0$), we have

$$\Psi_n^k = \sum_{i=0}^n k^i \cdot e_n^i.$$

Proof of Proposition 3.2'. Let $n \in \mathbb{N}$. Then, by the definition of Ψ_n^k , we have

$$\begin{aligned} \Psi_n^k &= \Psi^k |_{H_n} = \underbrace{\left(\Psi^k |_{\bigoplus_{i=0}^n H_i} \right)}_{=\rho_n(\Psi^k)} |_{H_n} = \rho_n \left(\underbrace{\Psi^k}_{=\Phi^k(I)} \right) |_{H_n} \\ &= \rho_n(\Phi^k(I)) |_{H_n} = (\rho_n \circ \Phi^k)(I) |_{H_n} = \left(\sum_{i=0}^n k^i \cdot \varepsilon_n^i \right) (I) |_{H_n} \\ &\quad \left(\text{since } \rho_n \circ \Phi^k = \sum_{i=0}^n k^i \cdot \varepsilon_n^i \text{ by Proposition 2.3 (applied to)} \right. \\ &\quad \left. E, \mathcal{L}(H)_n, \rho_n, n+1, k \text{ instead of } M, A, \rho, k \text{ and } n \right) \\ &= \sum_{i=0}^n k^i \cdot \underbrace{\varepsilon_n^i(I) |_{H_n}}_{=e_n^i} = \sum_{i=0}^n k^i \cdot e_n^i. \\ &\quad \text{(because this is how } e_n^i \text{ was defined)} \end{aligned}$$

This proves Proposition 3.2'.

Proposition 3.2 is merely an obvious consequence of Proposition 3.2':

Proof of Proposition 3.2. For every $n > 0$, we have

$$\begin{aligned} \Psi_n^k &= \sum_{i=0}^n k^i \cdot e_n^i && \text{(by Proposition 3.2')} \\ &= k^0 \cdot \underbrace{e_n^0}_{=0 \text{ (since } n \neq 0)}} + \sum_{i=1}^n k^i \cdot e_n^i = \sum_{i=1}^n k^i \cdot e_n^i. \end{aligned}$$

This proves Proposition 3.2.

Page 1075, Definition 3.3: It would not harm to add here that e_n^i is understood to be 0 if $i > n$. (Otherwise, e_n^i would not be defined at all for $i > n$.)

Page 1075, proof of Proposition 3.4: Patras' proof of Proposition 3.4 confines itself to one sentence: "Les deux identités résultent respectivement de 1.4 et 3.2, et de la définition 2.2 des projecteurs de poids i ."

I don't think this enough, however indirectly "résultent" is meant. It is indeed easy to conclude $e^i * e^j = \binom{i+j}{i} \cdot e^{i+j}$ from 1.4, but concluding $e^i \circ e^j = \delta_j^i \cdot e^i$ from 3.2 is rather difficult. Here is how I would prove Proposition 3.4:

First, a rather standard combinatorial identity which we won't prove:

Theorem 0.1. Let $N \in \mathbb{N}$. Then, the equalities

$$\sum_{k=0}^N (-1)^k \binom{N}{k} k^\ell = 0 \quad \text{for every } \ell \in \{0, 1, \dots, N-1\} \quad (1)$$

and

$$\sum_{k=0}^N (-1)^k \binom{N}{k} k^N = (-1)^N N! \quad (2)$$

are satisfied in \mathbb{Z} .

(This Theorem 0.1 is, for example, the result of applying Theorem 1 of [DG1] to $R = \mathbb{Z}$.)

This has, as a consequence, a kind of "polynomials that are zero at all nonnegative integers must be identically zero" result for torsionfree abelian groups:

Theorem 0.2. Let R be a torsionfree abelian group. Let $n \in \mathbb{N}$. Let $(\alpha_0, \alpha_1, \dots, \alpha_n)$ and $(\beta_0, \beta_1, \dots, \beta_n)$ be two $(n+1)$ -tuples of elements of R such that every $k \in \mathbb{N}$ satisfies $\sum_{m=0}^n k^m \alpha_m = \sum_{m=0}^n k^m \beta_m$. Then, $\alpha_m = \beta_m$ for every $m \in \{0, 1, \dots, n\}$.

Proof of Theorem 0.2. We are going to prove that for every $\ell \in \{0, 1, \dots, n\}$, we have

$$\alpha_{n-\ell} = \beta_{n-\ell}. \quad (3)$$

Proof of (3). We will prove (3) by strong induction over ℓ . A strong induction does not need an induction base, so let us start with the induction step:

Induction step: Let $L \in \{0, 1, \dots, n\}$ be arbitrary. Assume that (3) is already proven for all $\ell \in \{0, 1, \dots, n\}$ satisfying $\ell < L$. Now we must prove (3) for $\ell = L$.

We have

$$\alpha_{n-\ell} = \beta_{n-\ell} \text{ for every } \ell \in \{0, 1, \dots, n\} \text{ satisfying } \ell < L \quad (4)$$

(since (3) is already proven for all $\ell \in \{0, 1, \dots, n\}$ satisfying $\ell < L$).

Let $k \in \mathbb{N}$ be arbitrary. Then,

$$\sum_{m=0}^n k^m \alpha_m = \sum_{m=0}^{n-L} k^m \alpha_m + \sum_{m=n-L+1}^n k^m \alpha_m = \sum_{m=0}^{n-L} k^m \alpha_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \alpha_{n-\ell} \quad (5)$$

(here, we substituted $n - \ell$ for m in the second sum)

and similarly

$$\sum_{m=0}^n k^m \beta_m = \sum_{m=0}^{n-L} k^m \beta_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \beta_{n-\ell}. \quad (6)$$

Subtracting (6) from (5), we get

$$\begin{aligned} \sum_{m=0}^n k^m \alpha_m - \sum_{m=0}^n k^m \beta_m &= \left(\sum_{m=0}^{n-L} k^m \alpha_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \underbrace{\alpha_{n-\ell}}_{\substack{=\beta_{n-\ell} \\ \text{(by (4),} \\ \text{since } \ell < L)}} \right) - \left(\sum_{m=0}^{n-L} k^m \beta_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \beta_{n-\ell} \right) \\ &= \left(\sum_{m=0}^{n-L} k^m \alpha_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \beta_{n-\ell} \right) - \left(\sum_{m=0}^{n-L} k^m \beta_m + \sum_{\ell=0}^{L-1} k^{n-\ell} \beta_{n-\ell} \right) \\ &= \sum_{m=0}^{n-L} k^m \alpha_m - \sum_{m=0}^{n-L} k^m \beta_m. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{m=0}^{n-L} k^m \alpha_m - \sum_{m=0}^{n-L} k^m \beta_m &= \underbrace{\sum_{m=0}^n k^m \alpha_m}_{= \sum_{m=0}^n k^m \beta_m} - \sum_{m=0}^n k^m \beta_m = \sum_{m=0}^n k^m \beta_m - \sum_{m=0}^n k^m \beta_m = 0. \end{aligned}$$

In other words,

$$\sum_{m=0}^{n-L} k^m \alpha_m = \sum_{m=0}^{n-L} k^m \beta_m. \quad (7)$$

Now, forget that we fixed k . We thus have shown (7) for every $k \in \mathbb{N}$.

Now, $L \in \{0, 1, \dots, n\}$ yields $n \geq L$, so that $n - L \geq 0$. Denote the nonnegative integer $n - L$ by N . Then, every $k \in \mathbb{N}$ satisfies

$$\sum_{m=0}^N k^m \alpha_m = \sum_{m=0}^N k^m \beta_m$$

(this is just the identity (7), rewritten using $N = n - L$). Thus,

$$\sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \alpha_m = \sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \beta_m. \quad (8)$$

But since

$$\begin{aligned}
& \sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \alpha_m \\
&= \sum_{m=0}^N \alpha_m \sum_{k=0}^N (-1)^k \binom{N}{k} k^m = \sum_{\ell=0}^N \alpha_\ell \sum_{k=0}^N (-1)^k \binom{N}{k} k^\ell \\
&\quad \text{(here, we renamed the index } m \text{ as } \ell \text{ in the first sum)} \\
&= \sum_{\ell=0}^{N-1} \alpha_\ell \underbrace{\sum_{k=0}^N (-1)^k \binom{N}{k} k^\ell}_{=0 \text{ (by (1))}} + \alpha_N \underbrace{\sum_{k=0}^N (-1)^k \binom{N}{k} k^N}_{=(-1)^N N! \text{ (by (2))}} \\
&= \underbrace{\sum_{\ell=0}^{N-1} \alpha_\ell 0}_{=0} + \alpha_N (-1)^N N! = \alpha_N (-1)^N N! = (-1)^N N! \alpha_N
\end{aligned}$$

and similarly

$$\sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \beta_m = (-1)^N N! \beta_N,$$

we have

$$\begin{aligned}
(-1)^N N! \alpha_N &= \sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \alpha_m = \sum_{k=0}^N (-1)^k \binom{N}{k} \sum_{m=0}^N k^m \beta_m && \text{(by (8))} \\
&= (-1)^N N! \beta_N,
\end{aligned}$$

so that

$$0 = (-1)^N N! \alpha_N - (-1)^N N! \beta_N = (-1)^N N! (\alpha_N - \beta_N).$$

Since $(-1)^N N!$ is a nonzero integer, this yields $0 = \alpha_N - \beta_N$ (since R is torsionfree), so that $\alpha_N = \beta_N$. Since $N = n - L$, this rewrites as $\alpha_{n-L} = \beta_{n-L}$. In other words, (3) is proven for $\ell = L$. This completes the induction step. Thus, the induction proof of (3) is complete.

Now, for every $m \in \{0, 1, \dots, n\}$, we have

$$\begin{aligned}
\alpha_m &= \alpha_{n-(n-m)} = \beta_{n-(n-m)} && \text{(by (3), applied to } \ell = n - m) \\
&= \beta_m.
\end{aligned}$$

This proves Theorem 0.2.

Proof of Proposition 3.4. We prove Proposition 3.4 in several steps.

a) For every $n \in \mathbb{N}$, every $k \in \mathbb{N}$ and every $l \in \mathbb{N}$ we have

$$\left(\sum_{i=0}^n k^i \cdot e_n^i \right) \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) = \sum_{i=0}^n (kl)^i \cdot e_n^i. \quad (9)$$

Proof of (9). Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ and $l \in \mathbb{N}$. Then, $\Psi^k \circ \Psi^l = \Psi^{kl}$ by Proposition 1.4. This yields $\Psi_n^k \circ \Psi_n^l = \Psi_n^{kl}$ (since Ψ_n^k , Ψ_n^l and Ψ_n^{kl} are just the restrictions of Ψ^k , Ψ^l and Ψ^{kl} to H_n).

Proposition 3.2' yields $\Psi_n^k = \sum_{i=0}^n k^i \cdot e_n^i$. Proposition 3.2' (applied to l instead of k) yields $\Psi_n^l = \sum_{i=0}^n l^i \cdot e_n^i = \sum_{j=0}^n l^j \cdot e_n^j$ (here we renamed the index i as j in the sum).

Proposition 3.2' (applied to kl instead of k) yields $\Psi_n^{kl} = \sum_{i=0}^n (kl)^i \cdot e_n^i$. Thus,

$$\underbrace{\left(\sum_{i=0}^n k^i \cdot e_n^i \right)}_{=\Psi_n^k} \circ \underbrace{\left(\sum_{j=0}^n l^j \cdot e_n^j \right)}_{=\Psi_n^l} = \Psi^k \circ \Psi^l = \Psi^{kl} = \sum_{i=0}^n (kl)^i \cdot e_n^i,$$

so that (9) is proven.

b) For every $n \in \mathbb{N}$, every $l \in \mathbb{N}$ and every $i \in \{0, 1, \dots, n\}$, we have

$$e_n^i \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) = l^i \cdot e_n^i. \quad (10)$$

Proof of (10). Let $n \in \mathbb{N}$ and $l \in \mathbb{N}$. For every $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{m=0}^n k^m \left(e_n^m \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) \right) &= \left(\sum_{m=0}^n k^m \cdot e_n^m \right) \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) \\ &\quad \text{(since composition of } K\text{-linear maps is } K\text{-bilinear)} \\ &= \left(\sum_{i=0}^n k^i \cdot e_n^i \right) \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) \\ &\quad \text{(here, we renamed the index } m \text{ as } i \text{ in the first sum)} \\ &= \sum_{i=0}^n (kl)^i \cdot e_n^i \quad \text{(by (9))} \\ &= \sum_{m=0}^n \underbrace{(kl)^m}_{=k^m l^m} \cdot e_n^m \quad \text{(here, we renamed the index } i \text{ as } m \text{ in the sum)} \\ &= \sum_{m=0}^n k^m l^m \cdot e_n^m. \end{aligned}$$

Thus, Theorem 0.2 (applied to $R = \text{End}_K(H_n)$, $\alpha_m = e_n^m \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right)$ and $\beta_m =$

$l^m \cdot e_n^m$) yields that $e_n^m \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) = l^m \cdot e_n^m$ for every $m \in \{0, 1, \dots, n\}$. Renaming the

index m as i in this result, we obtain: $e_n^i \circ \left(\sum_{j=0}^n l^j \cdot e_n^j \right) = l^i \cdot e_n^i$ for every $i \in \{0, 1, \dots, n\}$.

Thus, (10) is proven.

c) For every $n \in \mathbb{N}$, every $i \in \{0, 1, \dots, n\}$ and every $j \in \{0, 1, \dots, n\}$, we have

$$e_n^i \circ e_n^j = \delta_j^i \cdot e_n^i. \quad (11)$$

Proof of (11). Let $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$. For every $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{m=0}^n k^m \delta_m^i &= \sum_{m \in \{0, 1, \dots, n\}} k^m \delta_m^i = \underbrace{\sum_{\substack{m \in \{0, 1, \dots, n\}; \\ m=i}} k^m \delta_m^i}_{=k^i \delta_i^i} + \sum_{\substack{m \in \{0, 1, \dots, n\}; \\ m \neq i}} k^m \underbrace{\delta_m^i}_{=0} \\ &= k^i \underbrace{\delta_i^i}_{=1} + \underbrace{\sum_{\substack{m \in \{0, 1, \dots, n\}; \\ m \neq i}} k^m 0}_{=0} = k^i. \end{aligned}$$

Now, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{m=0}^n k^m e_n^i \circ e_n^m &= e_n^i \circ \left(\sum_{m=0}^n k^m \cdot e_n^m \right) \\ &\quad \text{(since composition of } K\text{-linear maps is } K\text{-bilinear)} \\ &= e_n^i \circ \left(\sum_{j=0}^n k^j \cdot e_n^j \right) \quad \text{(here, we renamed the index } m \text{ as } j \text{ in the sum)} \\ &= \underbrace{k^i}_{= \sum_{m=0}^n k^m \delta_m^i} \cdot e_n^i \quad \text{(by (10), applied to } k \text{ instead of } l) \\ &= \left(\sum_{m=0}^n k^m \delta_m^i \right) \cdot e_n^i = \sum_{m=0}^n k^m \delta_m^i \cdot e_n^i. \end{aligned}$$

Thus, Theorem 0.2 (applied to $R = \text{End}_K(H_n)$, $\alpha_m = e_n^i \circ e_n^m$ and $\beta_m = \delta_m^i \cdot e_n^i$) yields that $e_n^i \circ e_n^m = \delta_m^i \cdot e_n^i$ for every $m \in \{0, 1, \dots, n\}$. Renaming the index m as j in this result, we obtain: $e_n^i \circ e_n^j = \delta_j^i \cdot e_n^i$ for every $j \in \{0, 1, \dots, n\}$. Thus, (11) is proven.

d) We have

$$e_n^i \circ e_n^j = \delta_j^i \cdot e_n^i \quad \text{for every } i \in \mathbb{N}, j \in \mathbb{N} \text{ and } n \in \mathbb{N}. \quad (12)$$

Proof of (12). Let $i \in \mathbb{N}$, $j \in \mathbb{N}$ and $n \in \mathbb{N}$. We must have one of the following three cases:

Case 1: We have $i \in \{0, 1, \dots, n\}$ and $j \in \{0, 1, \dots, n\}$.

Case 2: We have $i \notin \{0, 1, \dots, n\}$.

Case 3: We have $i \in \{0, 1, \dots, n\}$ and $j \notin \{0, 1, \dots, n\}$.

In Case 1, we notice that (12) follows directly from (11).

In Case 2, we find that e_n^i is 0, and thus (12) is trivially true.

In Case 3, both e_n^j and δ_j^i are 0 (in fact, $\delta_j^i = 0$ because $i \neq j$), and thus (12) is trivially true.

Hence, we have seen that (12) in each of the three Cases 1, 2 and 3. This proves (12).

e) For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we have $e^i \circ e^j = \delta_j^i \cdot e^i$.

Proof. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}$. The maps e^i and e^j are defined as graded endomorphisms of $\mathcal{L}(H)$, whose n -th graded components are e_n^i and e_n^j (respectively) for each $n \in \mathbb{N}$. Hence, in order to prove that $e^i \circ e^j = \delta_j^i \cdot e^i$, it is enough to show that

$e_n^i \circ e_n^j = \delta_j^i \cdot e_n^i$ for every $n \in \mathbb{N}$. But this has already been shown in (12). Thus, we are done proving that $e^i \circ e^j = \delta_j^i \cdot e^i$.

f) For every $i \in \mathbb{N}$, we have

$$e^i = \frac{(\log I)^{*i}}{i!}, \quad (13)$$

where $\log I$ is to be understood as the result of applying the formal power series of the logarithm to I . (This result is well-defined, since for every $x \in H$, only finitely many terms of the formal power series $(\log I)(x)$ are nonzero.)

Proof. Let $i \in \mathbb{N}$. Also, let $n \in \mathbb{N}$ be arbitrary.

By the definition of ρ_n , every $f \in E$ such that $f(1) = 1$ satisfies

$$\begin{aligned} & \rho_n(\log f) \\ &= \rho_n \left(\sum_{m \geq 1} (-1)^{m+1} \frac{(f-1)^m}{m} \right) \\ & \quad \left(\text{since } \log f = \sum_{m \geq 1} (-1)^{m+1} \frac{(f-1)^m}{m} \text{ by the definition of the logarithm} \right) \\ &= \sum_{m \geq 1} (-1)^{m+1} \frac{(\rho_n(f) - 1)^m}{m} \quad \left(\text{since } \rho_n \text{ is a } K\text{-algebra homomorphism, and can} \right. \\ & \quad \left. \text{easily be seen to commute with reasonable infinite series} \right) \\ &= \underbrace{\sum_{\substack{m \geq 1; \\ m < n+1}}}_{= \sum_{m=1}^n} (-1)^{m+1} \frac{(\rho_n(f) - 1)^m}{m} + \sum_{\substack{m \geq 1; \\ m \geq n+1}} (-1)^{m+1} \underbrace{\frac{(\rho_n(f) - 1)^m}{m}}_{=0} \\ & \quad \left(\text{since Lemma 3.1 yields } (\rho_n(f) - 1)^{n+1} = 0, \text{ so that } (\rho_n(f) - 1)^m = 0 \text{ (since } m \geq n+1) \right) \\ &= \sum_{m=1}^n (-1)^{m+1} \frac{(\rho_n(f) - 1)^m}{m} + \underbrace{\sum_{\substack{m \geq 1; \\ m \geq n+1}}}_{=0} (-1)^{m+1} 0 = \sum_{m=1}^n (-1)^{m+1} \frac{(\rho_n(f) - 1)^m}{m} \\ &= \log_{n+1} f \quad \left(\text{since } \log_{n+1} f = \sum_{m=1}^n (-1)^{m+1} \frac{(\rho_n(f) - 1)^m}{m} \text{ by the definition of } \log_{n+1} f \right). \end{aligned}$$

Applied to $f = I$, this yields $\rho_n(\log I) = \log_{n+1} I$.

But every graded map $g : H \rightarrow H$ satisfies $g|_{H_n} = \left(g|_{\bigoplus_{i=0}^n H_i} \right)|_{H_n}$ ³ and $\rho_n(g) = g|_{\bigoplus_{i=0}^n H_i}$ (because this is how ρ_n was defined). Hence, every graded map $g : H \rightarrow H$

³In fact, if we restrict the map g to H_n , we get the same result as if we restrict the map g to $\bigoplus_{i=0}^n H_i$ first and then restrict this restriction to H_n .

satisfies $g|_{H_n} = \underbrace{\left(g|_{\bigoplus_{i=0}^n H_i} \right)}_{=\rho_n(g)}|_{H_n} = \rho_n(g)|_{H_n}$. Applied to $g = \frac{(\log I)^{*i}}{i!}$, this yields

$$\begin{aligned} & \frac{(\log I)^{*i}}{i!} |_{H_n} \\ &= \rho_n \left(\frac{(\log I)^{*i}}{i!} \right) |_{H_n} = \frac{(\rho_n(\log I))^{*i}}{i!} |_{H_n} \\ & \quad \left(\text{since } \rho_n \text{ is a } K\text{-algebra homomorphism, so that } \rho_n \left(\frac{(\log I)^{*i}}{i!} \right) = \frac{(\rho_n(\log I))^{*i}}{i!} \right) \\ &= \frac{(\log_{n+1} I)^{*i}}{i!} |_{H_n} \quad (\text{since } \rho_n(\log I) = \log_{n+1} I). \end{aligned}$$

Combined with

$$\begin{aligned} e^i |_{H_n} &= e_n^i && (\text{by the definition of } e^i) \\ &= \underbrace{\varepsilon_n^i(I)}_{(\log_{n+1} I)^{*i}} |_{H_n} = \frac{(\log_{n+1} I)^{*i}}{i!} |_{H_n}, \\ &= \frac{(\log_{n+1} I)^{*i}}{i!} \\ & \quad (\text{by the definition of } \varepsilon_n^i) \end{aligned}$$

this yields $e^i |_{H_n} = \frac{(\log_{n+1} I)^{*i}}{i!} |_{H_n} = \frac{(\log I)^{*i}}{i!} |_{H_n}$.

Now forget that we fixed $n \in \mathbb{N}$. We thus have proven that every $n \in \mathbb{N}$ satisfies $e^i |_{H_n} = \frac{(\log I)^{*i}}{i!} |_{H_n}$. Therefore, $e^i = \frac{(\log I)^{*i}}{i!}$. We have thus proven (13).

g) For every $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we have $e^i * e^j = \binom{i+j}{i} \cdot e^{i+j}$.

Proof. Let $i \in \mathbb{N}$ and $j \in \mathbb{N}$. By (13), we have $e^i = \frac{(\log I)^{*i}}{i!}$. By (13) (applied to j instead of i), we have $e^j = \frac{(\log I)^{*j}}{j!}$. By (13) (applied to $i+j$ instead of i), we have $e^{i+j} = \frac{(\log I)^{*(i+j)}}{(i+j)!}$. Now,

$$\begin{aligned} & \underbrace{e^i}_{(\log I)^{*i}} * \underbrace{e^j}_{(\log I)^{*j}} = \frac{(\log I)^{*i}}{i!} * \frac{(\log I)^{*j}}{j!} \\ &= \frac{(\log I)^{*(i+j)}}{(i+j)!} \\ &= \underbrace{\frac{(i+j)!}{i!j!}}_{\binom{i+j}{i}} \cdot \underbrace{\frac{(\log I)^{*(i+j)}}{(i+j)!}}_{=e^{i+j}} = \binom{i+j}{i} \cdot e^{i+j}, \end{aligned}$$

qed.

From the results of steps **e)** and **g)**, we conclude that Proposition 3.4 holds.

Page 1076, Corollary 3.6: Here is an alternative proof of Corollary 3.6 (without using eigenspaces).

Before we even begin proving this corollary, let us record some lemmata that could just as well have been stated (and proven) in Section 1:

a) First, an elementary lemma:

Lemma 1.5. For any two K -coalgebras C and D , any two K -algebras A and B , and any four K -linear maps $p : C \rightarrow A$, $q : C \rightarrow A$, $r : D \rightarrow B$ and $s : D \rightarrow B$, we have

$$(p \otimes r) * (q \otimes s) = (p * q) \otimes (r * s).$$

(Here, $(p \otimes r) * (q \otimes s)$ means the convolution of the two K -linear maps $p \otimes r : C \otimes D \rightarrow A \otimes B$ and $q \otimes s : C \otimes D \rightarrow A \otimes B$.)

This lemma is easy to prove (particularly if you are using Sweedler's notation, but even without it).

The following two lemmata are easy consequences of Lemma 1.5:

Lemma 1.6. For any two K -coalgebras C and D , any two K -algebras A and B , and any two K -linear maps $f : C \rightarrow A$ and $g : D \rightarrow B$, we have

$$(f \otimes 1_{\text{Hom}_K(D,B)}) * (1_{\text{Hom}_K(C,A)} \otimes g) = f \otimes g = (1_{\text{Hom}_K(C,A)} \otimes g) * (f \otimes 1_{\text{Hom}_K(D,B)}).$$

⁴ Here, both $\text{Hom}_K(D, B)$ and $\text{Hom}_K(C, A)$ are made into K -algebras by the convolution.

Lemma 1.7. For any two K -coalgebras C and D , any two K -algebras A and B , and any two K -linear maps $f : C \rightarrow A$ and $g : C \rightarrow A$, we have

$$(f \otimes 1_{\text{Hom}_K(D,B)}) * (g \otimes 1_{\text{Hom}_K(D,B)}) = (f * g) \otimes 1_{\text{Hom}_K(D,B)}.$$

Here, $\text{Hom}_K(D, B)$ is made into a K -algebra by the convolution.

By repeated application of Lemma 1.7, we get:

Lemma 1.8. (a) For any two K -coalgebras C and D , any two K -algebras A and B , any $n \in \mathbb{N}$ and any K -linear map $f : C \rightarrow A$, we have

$$(f \otimes 1_{\text{Hom}_K(D,B)})^{*n} = f^{*n} \otimes 1_{\text{Hom}_K(D,B)}.$$

Here, $\text{Hom}_K(D, B)$ is made into a K -algebra by the convolution.

(b) For any two K -coalgebras C and D , any two K -algebras A and B , any $n \in \mathbb{N}$ and any K -linear map $g : D \rightarrow B$, we have

$$(1_{\text{Hom}_K(C,A)} \otimes g)^{*n} = 1_{\text{Hom}_K(C,A)} \otimes g^{*n}.$$

Here, $\text{Hom}_K(C, A)$ is made into a K -algebra by the convolution.

⁴Here and in the following, for any K -algebra U , we denote by 1_U the unity of U .

Proof of Lemma 1.8. Lemma 1.8 **(a)** is proven by induction over n (and use of Lemma 1.7 in the induction step). Lemma 1.8 **(b)** is completely analogous to Lemma 1.8 **(a)** (the only difference is the order of the tensorands), so the proof is analogous as well. The details of these proofs are left to the reader.

b) On the other hand, we recall that for every connected graded bialgebra A and any K -linear map $f : A \rightarrow A$ satisfying $f(1) = 1$, the K -linear map $\log f : A \rightarrow A$ is well-defined. Namely, this map $\log f$ is defined by applying the formal power series of the logarithm to f ; in other words, $\log f$ is defined as the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f-1)^{*n}}{n}$ (where 1 denotes the unity of the K -algebra $\mathcal{L}(A)$, i. e., the map $\eta \circ \epsilon$).⁵

The following lemma is easy to check:

Lemma 1.9. For every connected graded bialgebra A and any two K -linear maps $f : A \rightarrow A$ and $g : A \rightarrow A$ satisfying $f(1) = g(1) = 1$ and $f * g = g * f$, we have $\log(f * g) = \log f + \log g$.

In fact, Lemma 1.9 follows from the identity $\log((1+X)(1+Y)) = \log(1+X) + \log(1+Y)$ in the ring $K[[X, Y]]$ of formal power series.

c) Next, we have:

Lemma 1.10. For any two connected graded K -bialgebras A and B and any K -linear map $f : A \rightarrow A$ satisfying $f(1) = 1$, we have $(f \otimes 1_{\mathcal{L}(B)})(1) = 1$ and

$$\log(f \otimes 1_{\mathcal{L}(B)}) = (\log f) \otimes 1_{\mathcal{L}(B)}.$$

Proof of Lemma 1.10. Checking $(f \otimes 1_{\mathcal{L}(B)})(1) = 1$ is very easy and left to the reader. Now let us prove that $\log(f \otimes 1_{\mathcal{L}(B)}) = (\log f) \otimes 1_{\mathcal{L}(B)}$:

For every $n \in \mathbb{N}$, we have $((f - 1_{\mathcal{L}(A)}) \otimes 1_{\text{Hom}_K(B, B)})^{*n} = (f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\text{Hom}_K(B, B)}$ (by Lemma 1.8 **(a)**, applied to A , B and $f - 1_{\mathcal{L}(A)}$ instead of C , D and f). Since $1_{\text{Hom}_K(B, B)} = 1_{\mathcal{L}(B)}$, this rewrites as $((f - 1_{\mathcal{L}(A)}) \otimes 1_{\mathcal{L}(B)})^{*n} = (f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\mathcal{L}(B)}$ for every $n \in \mathbb{N}$.

⁵This sum is infinite, but it still gives us a well-defined map $A \rightarrow A$, because for every $x \in A$, the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f-1)^{*n}}{n}(x)$ has only finitely many nonzero terms (by Lemma 3.12, applied to A instead of H) and thus has a well-defined value in A .

By the definition of $\log(f \otimes 1_{\mathcal{L}(B)})$, we have

$$\begin{aligned}
& \log(f \otimes 1_{\mathcal{L}(B)}) \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f \otimes 1_{\mathcal{L}(B)} - 1_{\mathcal{L}(A \otimes B)})^{*n}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{((f - 1_{\mathcal{L}(A)}) \otimes 1_{\mathcal{L}(B)})^{*n}}{n} \\
&\quad \left(\text{since } f \otimes 1_{\mathcal{L}(B)} - \underbrace{1_{\mathcal{L}(A \otimes B)}}_{=1_{\mathcal{L}(A)} \otimes 1_{\mathcal{L}(B)}} = f \otimes 1_{\mathcal{L}(B)} - 1_{\mathcal{L}(A)} \otimes 1_{\mathcal{L}(B)} = (f - 1_{\mathcal{L}(A)}) \otimes 1_{\mathcal{L}(B)} \right) \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\mathcal{L}(B)}}{n} \\
&\quad (\text{since } ((f - 1_{\mathcal{L}(A)}) \otimes 1_{\mathcal{L}(B)})^{*n} = (f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\mathcal{L}(B)} \text{ for every } n \in \mathbb{N}) \\
&= \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n}}{n} \right) \otimes 1_{\mathcal{L}(B)}.
\end{aligned}$$

On the other hand, by the definition of $\log f$, we have $\log f = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n}}{n}$, so that

$$(\log f) \otimes 1_{\mathcal{L}(B)} = \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n}}{n} \right) \otimes 1_{\mathcal{L}(B)} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\mathcal{L}(B)}}{n}$$

(here, we are using the fact that the tensor product commutes with convergent infinite sums; this can be easily checked). Thus,

$$\log(f \otimes 1_{\mathcal{L}(B)}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f - 1_{\mathcal{L}(A)})^{*n} \otimes 1_{\mathcal{L}(B)}}{n} = (\log f) \otimes 1_{\mathcal{L}(B)}.$$

This proves Lemma 1.10.

We now have:

Lemma 1.11. For any two connected graded K -bialgebras A and B , any K -linear map $f : A \rightarrow A$ satisfying $f(1) = 1$, and any K -linear map $g : B \rightarrow B$ satisfying $g(1) = 1$, we have $(f \otimes g)(1) = 1$ and

$$\log(f \otimes g) = (\log f) \otimes 1_{\mathcal{L}(B)} + 1_{\mathcal{L}(A)} \otimes (\log g).$$

Proof of Lemma 1.11. Again, checking that $(f \otimes g)(1) = 1$ is very easy. Let us now prove that $\log(f \otimes g) = (\log f) \otimes 1_{\mathcal{L}(B)} + 1_{\mathcal{L}(A)} \otimes (\log g)$.

By Lemma 1.6 (applied to $C = A$ and $D = B$), we get

$$(f \otimes 1_{\text{Hom}_K(B,B)}) * (1_{\text{Hom}_K(A,A)} \otimes g) = f \otimes g = (1_{\text{Hom}_K(A,A)} \otimes g) * (f \otimes 1_{\text{Hom}_K(B,B)}).$$

Since $1_{\text{Hom}_K(A,A)} = 1_{\mathcal{L}(A)}$ and $1_{\text{Hom}_K(B,B)} = 1_{\mathcal{L}(B)}$, this rewrites as

$$(f \otimes 1_{\mathcal{L}(B)}) * (1_{\mathcal{L}(A)} \otimes g) = f \otimes g = (1_{\mathcal{L}(A)} \otimes g) * (f \otimes 1_{\mathcal{L}(B)}).$$

By Lemma 1.10, we have $(f \otimes 1_{\mathcal{L}(B)})(1) = 1$ and $\log(f \otimes 1_{\mathcal{L}(B)}) = (\log f) \otimes 1_{\mathcal{L}(B)}$. Similarly, $(1_{\mathcal{L}(A)} \otimes g)(1) = 1$ and $\log(1_{\mathcal{L}(A)} \otimes g) = 1_{\mathcal{L}(A)} \otimes (\log g)$.

Since $(f \otimes 1_{\mathcal{L}(B)}) * (1_{\mathcal{L}(A)} \otimes g) = (1_{\mathcal{L}(A)} \otimes g) * (f \otimes 1_{\mathcal{L}(B)})$, $(f \otimes 1_{\mathcal{L}(B)})(1) = 1$ and $(1_{\mathcal{L}(A)} \otimes g)(1) = 1$, we can apply Lemma 1.9 to $f \otimes 1_{\mathcal{L}(B)}$, $1_{\mathcal{L}(A)} \otimes g$ and $A \otimes B$ instead of f , g and A . We obtain $\log((f \otimes 1_{\mathcal{L}(B)}) * (1_{\mathcal{L}(A)} \otimes g)) = \log(f \otimes 1_{\mathcal{L}(B)}) + \log(1_{\mathcal{L}(A)} \otimes g)$. Hence,

$$\begin{aligned} \log \left(\underbrace{f \otimes g}_{=(f \otimes 1_{\mathcal{L}(B)}) * (1_{\mathcal{L}(A)} \otimes g)} \right) &= \log((f \otimes 1_{\mathcal{L}(B)}) * (1_{\mathcal{L}(A)} \otimes g)) \\ &= \underbrace{\log(f \otimes 1_{\mathcal{L}(B)})}_{=(\log f) \otimes 1_{\mathcal{L}(B)}} + \underbrace{\log(1_{\mathcal{L}(A)} \otimes g)}_{=1_{\mathcal{L}(A)} \otimes (\log g)} \\ &= (\log f) \otimes 1_{\mathcal{L}(B)} + 1_{\mathcal{L}(A)} \otimes (\log g). \end{aligned}$$

This proves Lemma 1.11.

d) Now, finally, let us come to the *alternative proof of Corollary 3.6*:

Assume that H is a graded bialgebra. (The case of H being a Hopf algebra is similar, in that the same argument works but we have to interpret $H \otimes H$ as a tensor product of graded superalgebras rather than as a plain tensor product of graded algebras.)

For every $i \in \mathbb{N}$, let $e_{H \otimes H}^i$ denote the i -th "projecteur de poids i " of the graded bialgebra (or Hopf algebra) $H \otimes H$. Note that e^i still denotes the i -th "projecteur de poids i " of the graded bialgebra (or Hopf algebra) H .

By (13) (applied to $i = 1$), we have $e^1 = \frac{(\log I)^{*1}}{1!} = \frac{\log I}{1} = \log I$.

Note also that every $u \in \mathbb{N}$ satisfies

$$\begin{aligned} e^u &= \frac{(\log I)^{*u}}{u!} && \text{(by (13), applied to } i = u \text{)} \\ &= \frac{(e^1)^{*u}}{u!} && \text{(since } \log I = e^1 \text{),} \end{aligned}$$

so that

$$(e^1)^{*u} = u! \cdot e^u. \quad (14)$$

Denote by $I_{H \otimes H}$ the identity map $H \otimes H \rightarrow H \otimes H$. Then,

$$\begin{aligned} \log \left(\underbrace{I_{H \otimes H}}_{=I \otimes I} \right) &= \log(I \otimes I) = \underbrace{(\log I)}_{=e^1} \otimes 1_{\mathcal{L}(H)} + 1_{\mathcal{L}(H)} \otimes \underbrace{(\log I)}_{=e^1} \\ &\quad \text{(by Lemma 1.11, applied to } A = H, B = H, f = I \text{ and } g = I \text{)} \\ &= e^1 \otimes 1_{\mathcal{L}(H)} + 1_{\mathcal{L}(H)} \otimes e^1. \end{aligned}$$

Now, for every $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
e_{H \otimes H}^\ell &= \frac{(\log(I_{H \otimes H}))^{*\ell}}{\ell!} && \text{(by (13), applied to } \ell \text{ and } H \otimes H \text{ instead of } i \text{ and } H) \\
&= \frac{1}{\ell!} \left(\underbrace{\log(I_{H \otimes H})}_{=e^1 \otimes 1_{\mathcal{L}(H)} + 1_{\mathcal{L}(H)} \otimes e^1} \right)^{*\ell} = \frac{1}{\ell!} \underbrace{(e^1 \otimes 1_{\mathcal{L}(H)} + 1_{\mathcal{L}(H)} \otimes e^1)^{*\ell}}_{= \sum_{k=0}^{\ell} \binom{\ell}{k} (e^1 \otimes 1_{\mathcal{L}(H)})^{*k} * (1_{\mathcal{L}(H)} \otimes e^1)^{*(\ell-k)}} \\
& && \text{(by the binomial formula)} \\
&= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \underbrace{(e^1 \otimes 1_{\mathcal{L}(H)})^{*k}}_{= (e^1)^{*k} \otimes 1_{\mathcal{L}(H)}} * \underbrace{(1_{\mathcal{L}(H)} \otimes e^1)^{*(\ell-k)}}_{= 1_{\mathcal{L}(H)} \otimes (e^1)^{*(\ell-k)}} \\
& && \begin{array}{l} \text{(by Lemma 1.8 (a),} \\ \text{applied to } C=H, D=H, \\ A=H, B=H, f=e^1 \text{ and } n=k) \end{array} \quad \begin{array}{l} \text{(by Lemma 1.8 (a),} \\ \text{applied to } C=H, D=H, \\ A=H, B=H, g=e^1 \text{ and } n=\ell-k) \end{array} \\
&= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \left((e^1)^{*k} \otimes 1_{\mathcal{L}(H)} \right) * \left(1_{\mathcal{L}(H)} \otimes (e^1)^{*(\ell-k)} \right) \\
&= \frac{1}{\ell!} \sum_{k=0}^{\ell} \binom{\ell}{k} \underbrace{\left((e^1)^{*k} \otimes 1_{\text{Hom}_K(H,H)} \right) * \left(1_{\text{Hom}_K(H,H)} \otimes (e^1)^{*(\ell-k)} \right)}_{= (e^1)^{*k} \otimes (e^1)^{*(\ell-k)}} \\
& && \text{(because Lemma 1.6 (applied to } C=H, D=H, A=H, B=H, f=(e^1)^{*k} \text{ and } g=(e^1)^{*(\ell-k)}) \\
& && \text{yields } ((e^1)^{*k} \otimes 1_{\text{Hom}_K(H,H)}) * (1_{\text{Hom}_K(H,H)} \otimes (e^1)^{*(\ell-k)}) \\
& && = (e^1)^{*k} \otimes (e^1)^{*(\ell-k)} = (1_{\text{Hom}_K(H,H)} \otimes (e^1)^{*(\ell-k)}) * ((e^1)^{*k} \otimes 1_{\text{Hom}_K(H,H)}) \\
& && \text{(since } 1_{\mathcal{L}(H)} = 1_{\text{Hom}_K(H,H)}) \\
&= \frac{1}{\ell!} \sum_{k=0}^{\ell} \underbrace{\binom{\ell}{k}}_{\ell!} \underbrace{(e^1)^{*k}}_{=k! \cdot e^k} \otimes \underbrace{(e^1)^{*(\ell-k)}}_{=(\ell-k)! \cdot e^{\ell-k}} \\
& && \text{(by (14), applied to } u=k) \quad \text{(by (14), applied to } u=\ell-k) \\
&= \frac{1}{\ell!} \sum_{k=0}^{\ell} \frac{\ell!}{k! (\ell-k)!} \cdot (k! \cdot e^k) \otimes ((\ell-k)! \cdot e^{\ell-k}) = \sum_{k=0}^{\ell} e^k \otimes e^{\ell-k}. \tag{15}
\end{aligned}$$

Now, let us consider two cases:

Case 1: The bialgebra H is commutative.

Case 2: The bialgebra H is cocommutative.

Note that this is not really a case distinction, because in each of the two cases we have to prove a *different* claim: In Case 1, we have to prove that the decomposition of Theorem 3.5 endows H with a structure of a bigraded algebra, whereas in Case 2, we have to prove that the decomposition of Theorem 3.5 endows H with a structure of a bigraded coalgebra.

First let us consider Case 1. In this case, H is commutative. Thus, the map $\Pi : H \otimes H \rightarrow H$ is an algebra homomorphism. Since Π also is a coalgebra homomorphism (by the axioms of a bialgebra) and a graded map (since H is a graded bialgebra), this

yields that Π is a graded bialgebra homomorphism. Thus, since the definition of e^i was functorial with respect to H , the diagram

$$\begin{array}{ccc} H \otimes H & \xrightarrow{e_{H \otimes H}^u} & H \otimes H \\ \Pi \downarrow & & \downarrow \Pi \\ H & \xrightarrow{e^u} & H \end{array}$$

is commutative for every $u \in \mathbb{N}$. In other words,

$$\Pi \circ e_{H \otimes H}^u = e^u \circ \Pi \quad \text{for every } u \in \mathbb{N}. \quad (16)$$

Now, let $i \in \mathbb{N}$ and $j \in \mathbb{N}$ be arbitrary. By (15) (applied to $\ell = i + j$), we have $e_{H \otimes H}^{i+j} = \sum_{k=0}^{i+j} e^k \otimes e^{i+j-k}$, so that

$$\begin{aligned} e_{H \otimes H}^{i+j} \circ (e^i \otimes e^j) &= \left(\sum_{k=0}^{i+j} e^k \otimes e^{i+j-k} \right) \circ (e^i \otimes e^j) = \sum_{k=0}^{i+j} \underbrace{(e^k \otimes e^{i+j-k}) \circ (e^i \otimes e^j)}_{=(e^k \circ e^i) \otimes (e^{i+j-k} \circ e^j)} \\ &= \sum_{k=0}^{i+j} \underbrace{(e^k \circ e^i)}_{=\delta_i^k e^k} \otimes \underbrace{(e^{i+j-k} \circ e^j)}_{=\delta_j^{i+j-k} e^{i+j-k}} \\ &= \sum_{k \in \{0,1,\dots,i+j\}} \text{(by Proposition 3.4, applied to } k \text{ and } i \text{ instead of } i \text{ and } j\text{)} \quad \text{(by Proposition 3.4, applied to } i+j-k \text{ and } j \text{ instead of } i \text{ and } j\text{)} \\ &= \sum_{k \in \{0,1,\dots,i+j\}} \delta_i^k e^k \otimes \delta_j^{i+j-k} e^{i+j-k} \\ &= \underbrace{\sum_{\substack{k \in \{0,1,\dots,i+j\}; \\ k=i}} \delta_i^k e^k \otimes \delta_j^{i+j-k} e^{i+j-k}}_{=\delta_i^i e^i \otimes \delta_j^{i+j-i} e^{i+j-i} \text{ (since } i \in \{0,1,\dots,i+j\}\text{)}} + \sum_{\substack{k \in \{0,1,\dots,i+j\}; \\ k \neq i}} \underbrace{\delta_i^k e^k \otimes \delta_j^{i+j-k} e^{i+j-k}}_{=0 \text{ (since } k \neq i, \text{ so that } \delta_i^k = 0\text{)}} \\ &= \underbrace{\delta_i^i}_{=1} e^i \otimes \underbrace{\delta_j^{i+j-i}}_{=\delta_j^j=1} e^{i+j-i} + \underbrace{\sum_{\substack{k \in \{0,1,\dots,i+j\}; \\ k \neq i}} 0}_{=0} = e^i \otimes e^j + 0 = e^i \otimes e^j. \quad (17) \end{aligned}$$

Now, fix $n \in \mathbb{N}$ and $m \in \mathbb{N}$. By the definitions of $H_n^{(i)}$ and $H_m^{(j)}$, we have $H_n^{(i)} =$

$e^i(H_n)$ and $H_m^{(j)} = e^j(H_m)$. Thus,

$$\begin{aligned}
\underbrace{H_n^{(i)}}_{=e^i(H_n)} \otimes \underbrace{H_m^{(j)}}_{=e^j(H_m)} &= e^i(H_n) \otimes e^j(H_m) = \underbrace{(e^i \otimes e^j)}_{=e_{H \otimes H}^{i+j} \circ (e^i \otimes e^j)} (H_n \otimes H_m) \\
&= (e_{H \otimes H}^{i+j} \circ (e^i \otimes e^j)) (H_n \otimes H_m) = e_{H \otimes H}^{i+j} \left(\underbrace{(e^i \otimes e^j) (H_n \otimes H_m)}_{=e^i(H_n) \otimes e^j(H_m)} \right) \\
&= e_{H \otimes H}^{i+j} \left(\underbrace{e^i(H_n)}_{\subseteq H_n} \otimes \underbrace{e^j(H_m)}_{\subseteq H_m} \right) \\
&\quad \left(\begin{array}{l} \text{(since } e^i \text{ is a graded map)} \quad \text{(since } e^j \text{ is a graded map)} \end{array} \right) \\
&\subseteq e_{H \otimes H}^{i+j} (H_n \otimes H_m).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Pi(H_n^{(i)} \otimes H_m^{(j)}) &\subseteq \Pi(e_{H \otimes H}^{i+j} (H_n \otimes H_m)) = \underbrace{(\Pi \circ e_{H \otimes H}^{i+j})}_{=e^{i+j} \circ \Pi} (H_n \otimes H_m) \\
&\quad \text{(by (16), applied to } u=i+j\text{)} \\
&= (e^{i+j} \circ \Pi) (H_n \otimes H_m) = e^{i+j} \underbrace{(\Pi(H_n \otimes H_m))}_{=H_n \cdot H_m \text{ (since } \Pi \text{ is the multiplication map)}} = e^{i+j} \underbrace{(H_n \cdot H_m)}_{\subseteq H_{n+m}} \\
&\quad \text{(since } H \text{ is a graded algebra)} \\
&\subseteq e^{i+j}(H_{n+m}) = H_{n+m}^{(i+j)} \quad \left(\text{since } H_{n+m}^{(i+j)} \text{ was defined as } e^{i+j}(H_{n+m}) \right).
\end{aligned}$$

Let us now forget that we fixed i, j, n and m . We have thus proved that $\Pi(H_n^{(i)} \otimes H_m^{(j)}) \subseteq H_{n+m}^{(i+j)}$ for every $i \in \mathbb{N}, j \in \mathbb{N}, n \in \mathbb{N}$ and $m \in \mathbb{N}$. In other words, we have proved that the multiplication map Π is bigraded. Due to this (and to the trivial fact that $1_H \in H_0^{(0)}$), the decomposition of Theorem 3.5 endows H with a structure of a bigraded algebra. Thus, we have proven our claim in Case 1.

Next, let us consider Case 2. In this case, H is cocommutative. Thus, the map $\Delta : H \rightarrow H \otimes H$ is a coalgebra homomorphism. Since Δ also is an algebra homomorphism (by the axioms of a bialgebra) and a graded map (since H is a graded bialgebra), this yields that Δ is a graded bialgebra homomorphism. Thus, since the definition of e^i was functorial with respect to H , the diagram

$$\begin{array}{ccc}
H & \xrightarrow{e^u} & H \\
\Delta \downarrow & & \downarrow \Delta \\
H \otimes H & \xrightarrow{e_{H \otimes H}^u} & H \otimes H
\end{array}$$

is commutative for every $u \in \mathbb{N}$. In other words,

$$\Delta \circ e^u = e_{H \otimes H}^u \circ \Delta \quad \text{for every } u \in \mathbb{N}. \quad (18)$$

Now, fix $n \in \mathbb{N}$ and $i \in \mathbb{N}$. Since H is a graded coalgebra, we have $\Delta(H_n) \subseteq \bigoplus_{m \leq n} (H_m \otimes H_{n-m}) = \sum_{m \leq n} H_m \otimes H_{n-m}$ (since direct sums are sums).

Let us notice that the direct sum $\bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H_{n-m}^{(i-j)})$ is well-defined.⁶ Moreover, it satisfies

$$\begin{aligned} \bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H_{n-m}^{(i-j)}) &= \sum_{m \leq n, j \leq i} \left(\underbrace{H_m^{(j)}}_{=e^j(H_m)} \otimes \underbrace{H_{n-m}^{(i-j)}}_{=e^{i-j}(H_{n-m})} \right) \\ &\quad \left(\begin{array}{c} \text{(by the definition of } H_m^{(j)}) \quad \text{(by the definition of } H_{n-m}^{(i-j)}) \end{array} \right) \\ &\quad \text{(since direct sums are sums)} \\ &= \sum_{m \leq n, j \leq i} e^j(H_m) \otimes e^{i-j}(H_{n-m}). \end{aligned}$$

⁶*Proof.* We know that $H = \bigoplus_{(m,j) \in \mathbb{N}^2} H_m^{(j)}$, so that $H \otimes H = \left(\bigoplus_{(m,j) \in \mathbb{N}^2} H_m^{(j)} \right) \otimes H = \bigoplus_{(m,j) \in \mathbb{N}^2} (H_m^{(j)} \otimes H)$ (because tensor products commute with direct sums). Thus, the direct sum $\bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H)$ is well-defined (because it is a partial sum of the well-defined direct sum $\bigoplus_{(m,j) \in \mathbb{N}^2} (H_m^{(j)} \otimes H)$). Hence,

$$\left(\begin{array}{l} \text{whenever } (x_{m,j})_{m \leq n, j \leq i} \text{ is a family of elements of } H \otimes H \text{ such} \\ \text{that } \sum_{m \leq n, j \leq i} x_{m,j} = 0 \text{ and such that every } m \leq n \text{ and } j \leq i \text{ satisfy} \\ x_{m,j} \in H_m^{(j)} \otimes H, \text{ then } (x_{m,j})_{m \leq n, j \leq i} = (0)_{m \leq n, j \leq i} \end{array} \right). \quad (19)$$

As a consequence of this,

$$\left(\begin{array}{l} \text{whenever } (x_{m,j})_{m \leq n, j \leq i} \text{ is a family of elements of } H \otimes H \text{ such} \\ \text{that } \sum_{m \leq n, j \leq i} x_{m,j} = 0 \text{ and such that every } m \leq n \text{ and } j \leq i \text{ satisfy} \\ x_{m,j} \in H_m^{(j)} \otimes H_{n-m}^{(i-j)}, \text{ then } (x_{m,j})_{m \leq n, j \leq i} = (0)_{m \leq n, j \leq i} \end{array} \right)$$

(because if $(x_{m,j})_{m \leq n, j \leq i}$ is a family of elements of $H \otimes H$ such that $\sum_{m \leq n, j \leq i} x_{m,j} = 0$ and such that every $m \leq n$ and $j \leq i$ satisfy $x_{m,j} \in H_m^{(j)} \otimes H_{n-m}^{(i-j)}$, then every $m \leq n$ and $j \leq i$ satisfy $x_{m,j} \in \underbrace{H_m^{(j)} \otimes H_{n-m}^{(i-j)}}_{\subseteq H} \subseteq H_m^{(j)} \otimes H$, and thus we can apply (19) to obtain $(x_{m,j})_{m \leq n, j \leq i} = (0)_{m \leq n, j \leq i}$).

In other words, the direct sum $\bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H_{n-m}^{(i-j)})$ is well-defined, qed.

But by the definition of $H_n^{(i)}$, we have $H_n^{(i)} = e^i(H_n)$, so that

$$\begin{aligned}
\Delta(H_n^{(i)}) &= \Delta(e^i(H_n)) = \underbrace{(\Delta \circ e^i)}_{\substack{= e^i_{H \otimes H} \circ \Delta \\ \text{(by (18), applied to } u=i)}}(H_n) = (e^i_{H \otimes H} \circ \Delta)(H_n) \\
&= e^i_{H \otimes H} \left(\underbrace{\Delta(H_n)}_{\subseteq \sum_{m \leq n} H_m \otimes H_{n-m}} \right) \subseteq e^i_{H \otimes H} \left(\sum_{m \leq n} H_m \otimes H_{n-m} \right) \\
&= \left(\sum_{k=0}^i e^k \otimes e^{i-k} \right) \left(\sum_{m \leq n} H_m \otimes H_{n-m} \right) \\
&\quad \left(\text{since (15) (applied to } \ell = i) \text{ yields } e^i_{H \otimes H} = \sum_{k=0}^i e^k \otimes e^{i-k} \right) \\
&\subseteq \sum_{\substack{k=0 \\ m \leq n}}^i \sum_{m \leq n} \underbrace{(e^k \otimes e^{i-k})(H_m \otimes H_{n-m})}_{= e^k(H_m) \otimes e^{i-k}(H_{n-m})} = \sum_{m \leq n, k \leq i} e^k(H_m) \otimes e^{i-k}(H_{n-m}) \\
&= \sum_{m \leq n, k \leq i} e^k(H_m) \otimes e^{i-k}(H_{n-m}) \\
&= \sum_{m \leq n, j \leq i} e^j(H_m) \otimes e^{i-j}(H_{n-m}) \quad (\text{here, we renamed the index } k \text{ as } j \text{ in the sum}).
\end{aligned}$$

Hence,

$$\Delta(H_n^{(i)}) \subseteq \sum_{m \leq n, j \leq i} e^j(H_m) \otimes e^{i-j}(H_{n-m}) = \bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H_{n-m}^{(i-j)}).$$

Now forget that we fixed n and i . We have shown that every $n \in \mathbb{N}$ and $i \in \mathbb{N}$ satisfy $\Delta(H_n^{(i)}) \subseteq \bigoplus_{m \leq n, j \leq i} (H_m^{(j)} \otimes H_{n-m}^{(i-j)})$. In other words, the comultiplication map

Δ is bigraded. Due to this (and to the easily checked fact that the map ϵ is 0 on $H_n^{(i)}$ for all $(n, i) \neq (0, 0)$), the decomposition of Theorem 3.5 endows H with a structure of a bigraded coalgebra. Thus, we have proven our claim in Case 2.

The proof of Corollary 3.6 is now complete (since the claim of Corollary 3.6 has been proven in each of the Cases 1 and 2).

Page 1076, Definition 3.7: As explained above, this definition is only correct if i is allowed to be 0.

Page 1077, Proof of Proposition 3.8: The formulas in this proof are slightly wrong: Replace the $\sum_{i=1}^n$ sign by $\sum_{i=0}^n$. Also, replace each of the $\sum_{i,j=1}^n$ signs by $\sum_{i,j=0}^n$.

Page 1077, Proposition 3.9: Let me give some hints for the proof of this proposition.

First, in order to prove that "Les opérations caractéristiques, les opérations caractéristiques généralisées et les projecteurs de poids i sur H et H^{gr} sont alors deux à deux adjoints", the main step to make is to show that $f^* * g^* = (f * g)^*$ for any two

graded K -linear maps $f : H \rightarrow H$ and $g : H \rightarrow H$. This is easy to show (using the definition of the convolution: $f * g = \Pi \circ (f \otimes g) \circ \Delta$). Once this is shown, it yields that $(f^{*i})^* = (f^*)^{*i}$ for every graded K -linear map $f : H \rightarrow H$ and every $i \in \mathbb{N}$, and that $(\log f)^* = \log(f^*)$ whenever both sides of this equation are well-defined, etc. - and ultimately the adjointness part of Proposition 3.9.

Now to the proof that $(H^{(i)})^\perp = \bigoplus_{j \neq i} H^{*\text{gr}(j)}$:

For every $n \in \mathbb{N}$, let e^{n*} denote the "projecteur de poids i associé à $H^{*\text{gr}}$ ". We know that e^{n*} is adjoint to e^n for every $n \in \mathbb{N}$.

For every $f \in H^{*\text{gr}}$, we have the following equivalence of assertions:

$$\begin{aligned} \left(f \in \bigoplus_{j \neq i} H^{*\text{gr}(j)} \right) &\iff (e^{i*}(f) = 0) \iff (f \circ e^i = 0) \\ &\quad (\text{since } e^{i*} \text{ is adjoint to } e^i, \text{ so that } e^{i*}(f) = f \circ e^i) \\ &\iff (f(e^i(H)) = 0) \iff (f(H^{(i)}) = 0) \iff \left(f \in (H^{(i)})^\perp \right). \end{aligned}$$

Hence, $\bigoplus_{j \neq i} H^{*\text{gr}(j)} = (H^{(i)})^\perp$, thus completing our proof of Proposition 3.9.

Section 3: Let me add a lemma into this section which I am going to use further below:

Lemma 3.10. Let $n \in \mathbb{N}$ and $i \in \{0, 1, \dots, n\}$. There exists some $N \in \mathbb{N}$ and some elements $\alpha_0, \alpha_1, \dots, \alpha_N$ of K such that every graded bialgebra or Hopf algebra H satisfies

$$e_n^i = \sum_{k=0}^N \alpha_k \Psi_n^k. \quad (20)$$

(Note that we are using the notations of Section 3 here, i. e., the e_n^i and Ψ_n^k)

in Lemma 3.10 are the endomorphisms of H_n defined on page 1074.) 7

Page 1078: One line above Lemma 4.1, Patras writes: "éléments de H ". This should be "éléments x de H ".

Page 1078, proof of Lemma 4.1: I wouldn't agree that "L'inclusion $\text{Prim } H \subset H^{(1)}$ est immédiate."

Here is how I would prove that $\text{Prim } H \subset H^{(1)}$:

⁷Proof of Lemma 3.10. By the definition of e_n^i , we have

$$\begin{aligned}
 e_n^i &= \underbrace{\varepsilon_n^i(I)}_{\substack{= \frac{(\log_{n+1}(I))^{*i}}{i!} \\ \text{(by the definition of } \varepsilon_n^i)}}} \Big|_{H_n} = \frac{(\log_{n+1}(I))^{*i}}{i!} \Big|_{H_n} \\
 &= \frac{1}{i!} \left(\underbrace{\log_{n+1}(I)}_{= \sum_{m=1}^n (-1)^{m+1} \frac{(\rho_n(I) - 1)^{*m}}{m} \text{ (by the definition of } \log_{n+1})}} \right)^{*i} \Big|_{H_n} = \frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(\rho_n(I) - 1)^{*m}}{m} \right)^{*i} \Big|_{H_n} \\
 &= \rho_n \left(\frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(I - 1)^{*m}}{m} \right)^{*i} \right) \Big|_{H_n} \quad (\text{because } \rho_n \text{ is a } K\text{-algebra homomorphism}) \\
 &= \frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(I - 1)^{*m}}{m} \right)^{*i} \Big|_{H_n} \\
 &\quad \left(\begin{array}{l} \text{because any } f \in \mathcal{L}(H) \text{ satisfies } \rho_n(f) \Big|_{H_n} = f \Big|_{H_n} \\ \text{(since } \rho_n(f) \text{ is the restriction of } f \text{ to } \bigoplus_{i=0}^n H_i, \text{ and restricting this restriction} \\ \text{further to } H_n \text{ gives the same result as just restricting } f \text{ itself to } H_n) \end{array} \right)
 \end{aligned}$$

in every graded bialgebra or Hopf algebra H . But $\frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(I - 1)^{*m}}{m} \right)^{*i}$ can be seen as a polynomial (whose coefficients lie in K and don't depend on H) applied to I . In other words, there exists some $N \in \mathbb{N}$ and some appropriate elements $\alpha_0, \alpha_1, \dots, \alpha_N$ of K (which don't depend on H) such that $\frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(I - 1)^{*m}}{m} \right)^{*i} = \sum_{k=0}^N \alpha_k I^{*k}$. Consider this N and these elements $\alpha_0, \alpha_1, \dots, \alpha_N$ of K . Then, in every graded bialgebra or Hopf algebra H , we have

$$\begin{aligned}
 e_n^i &= \underbrace{\frac{1}{i!} \left(\sum_{m=1}^n (-1)^{m+1} \frac{(I - 1)^{*m}}{m} \right)^{*i}}_{= \sum_{k=0}^N \alpha_k I^{*k}} \Big|_{H_n} = \left(\sum_{k=0}^N \alpha_k I^{*k} \right) \Big|_{H_n} \\
 &= \sum_{k=0}^N \alpha_k \left(\underbrace{I^{*k}}_{=\Psi^k} \Big|_{H_n} \right) = \sum_{k=0}^N \alpha_k \underbrace{(\Psi^k \Big|_{H_n})}_{=\Psi_n^k} = \sum_{k=0}^N \alpha_k \Psi_n^k,
 \end{aligned}$$

thus proving Lemma 3.10.

a) First, a lemma:

$$\left(\begin{array}{l} \text{For any } K\text{-algebra } A \text{ and any two } K\text{-linear maps } f : H \rightarrow A \text{ and } g : H \rightarrow A \\ \text{satisfying } f(1) = g(1) = 0, \text{ we have } (f * g)(1) = 0 \text{ and } (f * g)(\text{Prim } H) = 0 \end{array} \right). \quad (21)$$

Proof of (21). Let A be a K -algebra, and let $f : H \rightarrow A$ and $g : H \rightarrow A$ be two K -linear maps satisfying $f(1) = g(1) = 0$. Let $x \in \text{Prim } H$. Then, x is primitive (by the definition of $\text{Prim } H$), so that $\Delta(x) = x \otimes 1 + 1 \otimes x$. Now, by the definition of convolution, $f * g = \Pi \circ (f \otimes g) \circ \Delta$, so that

$$\begin{aligned} (f * g)(x) &= (\Pi \circ (f \otimes g) \circ \Delta)(x) = \Pi \left((f \otimes g) \underbrace{(\Delta(x))}_{=x \otimes 1 + 1 \otimes x} \right) = \Pi \left(\underbrace{(f \otimes g)(x \otimes 1 + 1 \otimes x)}_{=f(x) \otimes g(1) + f(1) \otimes g(x)} \right) \\ &= \Pi \left(\underbrace{f(x) \otimes \underbrace{g(1)}_{=0}}_{=0} + \underbrace{f(1) \otimes g(x)}_{=0} \right) = \Pi \left(\underbrace{f(x) \otimes 0 + 0 \otimes g(x)}_{=0} \right) = 0. \end{aligned}$$

Now forget that we fixed x . We thus have shown that $(f * g)(x) = 0$ for every $x \in \text{Prim } H$. Thus, $(f * g)(\text{Prim } H) = 0$. Besides,

$$\begin{aligned} \underbrace{(f * g)}_{=\Pi \circ (f \otimes g) \circ \Delta}(1) &= (\Pi \circ (f \otimes g) \circ \Delta)(1) = \Pi \left((f \otimes g) \underbrace{(\Delta(1))}_{=1 \otimes 1} \right) = \Pi \left(\underbrace{(f \otimes g)(1 \otimes 1)}_{=f(1) \otimes g(1)} \right) \\ &= \Pi \left(\underbrace{f(1) \otimes \underbrace{g(1)}_{=0}}_{=0} \right) = \Pi(0 \otimes 0) = 0. \end{aligned}$$

Thus, (21) is proven.

b) As a consequence of the previous lemma, we can prove the next lemma:

$$\left(\begin{array}{l} \text{For any } K\text{-algebra } A, \text{ any integer } i \geq 2 \text{ and any } K\text{-linear map } f : H \rightarrow A \\ \text{satisfying } f(1) = 0, \text{ we have } f^{*i}(1) = 0 \text{ and } f^{*i}(\text{Prim } H) = 0 \end{array} \right). \quad (22)$$

Proof of (22). We will prove (22) by induction over i (where the induction base is the case $i = 2$):

Induction base: For any K -algebra A and any K -linear map $f : H \rightarrow A$ satisfying $f(1) = 0$, we have

$$\underbrace{f^{*2}}_{=f*f}(1) = (f * f)(1) = 0 \quad (\text{by (21), applied to } g = f)$$

and

$$\underbrace{f^{*2}}_{=f*f}(\text{Prim } H) = (f * f)(\text{Prim } H) = 0 \quad (\text{by (21), applied to } g = f).$$

In other words, (22) holds for $i = 2$. This completes the induction base.

Induction step: Let n be an integer ≥ 2 . Assume that (22) holds for $i = n$. We must prove that (22) also holds for $i = n + 1$.

Let A be a K -algebra, and $f : H \rightarrow A$ be a K -linear map satisfying $f(1) = 0$. We assumed that (22) holds for $i = n$. Hence, by (22) (applied to $i = n$), we get $f^{*n}(1) = 0$ and $f^{*n}(\text{Prim } H) = 0$. Since $f(1) = f^{*n}(1) = 0$, we can apply (21) to $g = f^{*n}$, and conclude that $(f * f^{*n})(1) = 0$ and $(f * f^{*n})(\text{Prim } H) = 0$. Since $f * f^{*n} = f^{*(n+1)}$, this rewrites as follows: We have $f^{*(n+1)}(1) = 0$ and $f^{*(n+1)}(\text{Prim } H) = 0$.

Forget that we fixed A and f . We thus have showed that for any K -algebra A and any K -linear map $f : H \rightarrow A$ satisfying $f(1) = 0$, we have $f^{*(n+1)}(1) = 0$ and $f^{*(n+1)}(\text{Prim } H) = 0$. In other words, we have proven that (22) holds for $i = n + 1$. This completes the induction step. Thus, the induction proof of (22) is complete.

c) Now, we will prove:

$$\left(\begin{array}{l} \text{For any } K\text{-algebra } A \text{ and any } K\text{-linear map } f : H \rightarrow A \text{ satisfying} \\ f(1) = 1, \text{ we have } (\log f) |_{\text{Prim } H} = f |_{\text{Prim } H} . \end{array} \right) \quad (23)$$

Note that, in this assertion, $\log f$ is defined by applying the formal power series of the logarithm to f ; in other words, $\log f$ is defined as the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f-1)^{*n}}{n}$ (where 1 denotes the unity of the K -algebra $\text{Hom}_K(H, A)$ ⁸, i. e., the map $\eta \circ \epsilon$). This sum is infinite, but it still gives us a well-defined map $H \rightarrow A$, because for every $x \in H$, the infinite sum $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f-1)^{*n}}{n}(x)$ has only finitely many nonzero terms⁹ and thus has a well-defined value in A .

Proof of (23). Let A be a K -algebra, and let $f : H \rightarrow A$ be a K -linear map satisfying $f(1) = 1$. Then, $(f-1)(1) = \underbrace{f(1)}_{=1} - \underbrace{1(1)}_{=1} = 1 - 1 = 0$. Hence,

$$\text{for any integer } i \geq 2, \text{ we have } (f-1)^{*i}(1) = 0 \text{ and } (f-1)^{*i}(\text{Prim } H) = 0 \quad (24)$$

(by (22), applied to $f-1$ instead of f).

Now let $x \in \text{Prim } H$. Then,

$$\text{for any integer } i \geq 2, \text{ we have } (f-1)^{*i}(x) = 0 \quad (25)$$

(since for any integer $i \geq 2$, we have $(f-1)^{*i}(\text{Prim } H) = 0$ (by (24)), but $(f-1)^{*i}(x) \in (f-1)^{*i}(\text{Prim } H)$ (since $x \in \text{Prim } H$), so that $(f-1)^{*i}(x) \in (f-1)^{*i}(\text{Prim } H) = 0$ and thus $(f-1)^{*i}(x) = 0$).

On the other hand, $x \in \text{Prim } H$, so that x is primitive, and thus $\epsilon(x) = 0$ (indeed, there is a well-known fact that every primitive element ξ of any bialgebra satisfies $\epsilon(\xi) = 0$).

Recall that (by the definition of $\log f$) we have

$$\log f = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(f-1)^{*n}}{n} = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(f-1)^{*i}}{i}$$

(here, we renamed the index n as i in the sum),

⁸This K -algebra $\text{Hom}_K(H, A)$ is the K -vector space of all K -linear maps $H \rightarrow A$, with convolution as multiplication.

⁹This follows from Lemma 3.12 if $A = H$, and is proven exactly in the same way in the general case.

so that

$$\begin{aligned}
(\log f)(x) &= \left(\sum_{i=1}^{\infty} (-1)^{i+1} \frac{(f-1)^{*i}}{i} \right) (x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(f-1)^{*i}(x)}{i} \\
&= \underbrace{(-1)^{1+1}}_{=1} \underbrace{\frac{(f-1)^{*1}(x)}{1}}_{=(f-1)^{*1}(x)=(f-1)(x)} + \sum_{i=2}^{\infty} (-1)^{i+1} \underbrace{\frac{(f-1)^{*i}(x)}{i}}_{=0} \\
&\hspace{15em} \text{(since } (f-1)^{*i}(x)=0 \text{ (by (25), since } i \geq 2)) \\
&= (f-1)(x) + \underbrace{\sum_{i=2}^{\infty} (-1)^{i+1} 0}_{=0} = (f-1)(x) = f(x) - \underbrace{1}_{=\eta \circ \epsilon}(x) = f(x) - \underbrace{(\eta \circ \epsilon)(x)}_{=\eta(\epsilon(x))} \\
&= f(x) - \eta \left(\underbrace{\epsilon(x)}_{=0} \right) = f(x) - \underbrace{\eta(0)}_{=0} = f(x).
\end{aligned}$$

Now forget that we fixed x . We thus have shown that every $x \in \text{Prim } H$ satisfies $(\log f)(x) = f(x)$. In other words, $(\log f)|_{\text{Prim } H} = f|_{\text{Prim } H}$. This proves (23).

d) Now to the proof of $\text{Prim } H \subset H^{(1)}$:

First, we notice that every $i \in \mathbb{N}$ satisfies $e^i(H) = H^{(i)}$. This is because, for every $i \in \mathbb{N}$, we have

$$\begin{aligned}
e^i(H) &= \bigoplus_{n \in \mathbb{N}} e_n^i(H_n) && \text{(because the } n\text{-th graded component of } e^i \text{ is } e_n^i \text{ (by Definition 3.3))} \\
&= \bigoplus_{n \geq i} \underbrace{e_n^i(H_n)}_{=H_n^{(i)}} && \left(\begin{array}{l} \text{here, we removed the addends with } n < i \text{ from our direct sum;} \\ \text{this did not change the sum (since } e_n^i = 0 \text{ when } n < i) \end{array} \right) \\
&= \bigoplus_{n \geq i} H_n^{(i)} = H^{(i)}.
\end{aligned}$$

Applied to $i = 1$, this yields $e^1(H) = H^{(1)}$.

On the other hand, by (13) (applied to $i = 1$), we have $e^1 = \frac{(\log I)^{*1}}{1!} = \frac{\log I}{1} = \log I$. Thus, $e^1|_{\text{Prim } H} = (\log I)|_{\text{Prim } H} = I|_{\text{Prim } H}$ (by (23), applied to $f = I$ and $A = H$). Hence, for every $x \in \text{Prim } H$, we have $e^1(x) = I(x)$. Thus, for every $x \in \text{Prim } H$, we have $x = I(x) = e^1(x) \in e^1(H) = H^{(1)}$. In other words, $\text{Prim } H \subset H^{(1)}$, qed.

Page 1079, line 7: Typo in this line: "sructure" should be "structure".

Page 1079, the equation after line 7: This equation is

$$\forall x \in A_n, \forall y \in A_m, [x, y] = x \otimes y - (-1)^{n \cdot m} y \otimes x.$$

This should instead be

$$\forall x \in A_n, \forall y \in A_m, [x, y] = xy - (-1)^{n \cdot m} yx.$$

Page 1079, Proposition 4.3: A proof of Proposition 4.3 is sketched in [C] (more precisely, in the proof of Theorem 3.8.3 in [C]).

Page 1080, Section 5: Here, Patras claims that

$$\Delta(x_{i_1} \otimes \dots \otimes x_{i_k}) = \sum_{p+q=k} \sum_{\beta} \operatorname{sgn}(\beta) \left(x_{i_{\beta(1)}} \otimes \dots \otimes x_{i_{\beta(p)}} \right) \otimes \left(x_{i_{\beta(p+1)}} \otimes \dots \otimes x_{i_{\beta(p+q)}} \right), \quad (26)$$

where the \sum_{β} sum is over all "opérateurs d'interclassement d'indices p et q "¹⁰.

He defines an "opérateur d'interclassement d'indices p et q " as an element σ of S_{p+q} satisfying

$$\sigma^{-1}(1) < \dots < \sigma^{-1}(p) \quad \text{and} \quad \sigma^{-1}(p+1) < \dots < \sigma^{-1}(p+q). \quad (27)$$

This is wrong. There are two ways to correct this mistake: Either replace the formula (26) by

$$\Delta(x_{i_1} \otimes \dots \otimes x_{i_k}) = \sum_{p+q=k} \sum_{\beta} \operatorname{sgn}(\beta) \left(x_{i_{\beta^{-1}(1)}} \otimes \dots \otimes x_{i_{\beta^{-1}(p)}} \right) \otimes \left(x_{i_{\beta^{-1}(p+1)}} \otimes \dots \otimes x_{i_{\beta^{-1}(p+q)}} \right), \quad (28)$$

where the \sum_{β} sum is still over all "opérateurs d'interclassement d'indices p et q ". Or

replace the formula (27) (in the definition of an "opérateur d'interclassement d'indices p et q ") by

$$\sigma(1) < \dots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \dots < \sigma(p+q).$$

In the following, I am going to assume that the mistake has been corrected in the first way (i. e., that the formula (26) has been replaced by (28)).

Page 1080, Section 5: Just a remark on $T(X)$. Patras gave the formula (26) for the comultiplication of $Th(X)$, and we corrected it to (28). There is an analogous formula for the comultiplication of $T(X)$, namely:

$$\Delta(x_{i_1} \otimes \dots \otimes x_{i_k}) = \sum_{p+q=k} \sum_{\beta} \left(x_{i_{\beta^{-1}(1)}} \otimes \dots \otimes x_{i_{\beta^{-1}(p)}} \right) \otimes \left(x_{i_{\beta^{-1}(p+1)}} \otimes \dots \otimes x_{i_{\beta^{-1}(p+q)}} \right). \quad (29)$$

Here, the \sum_{β} sum is still over all "opérateurs d'interclassement d'indices p et q ".

Page 1081, Proof of Proposition 5.1: Here, Patras writes:

"En explicitant les formules pour le produit et le coproduit dans $T(X)$ (resp. $Th(X)$) en termes d'opérateurs d'interclassement, on vérifie facilement que :

$$\Psi_n^k \in K[S_n] \leftrightarrow \operatorname{End}(T_n(X))$$

¹⁰I think that the proper English translation of the notion "opérateur d'interclassement d'indices p et q " is " (p, q) -shuffle"; but I am not sure about this. Anyway there seem to exist at least three non-equivalent definitions of " (p, q) -shuffle" in literature, so one should be careful when using this notion.

(resp. :

$$\Psi_n^k \in K[S_n] \hookrightarrow \text{End}(Th_n(X)) .”$$

Let me detail this argument:

First, let us work in $T(X)$. By repeated application of (29), we see that any $\ell \in \mathbb{N}$, any $k \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_\ell) \in \{1, 2, \dots, n\}^\ell$ satisfy

$$\begin{aligned} & \Delta^{[k]}(x_{i_1} \otimes \dots \otimes x_{i_\ell}) \\ &= \sum_{p_1+p_2+\dots+p_k=\ell} \sum_{\substack{\sigma \in S_\ell; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \\ & \quad \left(x_{i_{\sigma(1)}} \otimes \dots \otimes x_{i_{\sigma(p_1)}} \right) \otimes \left(x_{i_{\sigma(p_1+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2)}} \right) \otimes \left(x_{i_{\sigma(p_1+p_2+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2+p_3)}} \right) \\ & \quad \otimes \dots \otimes \left(x_{i_{\sigma(p_1+p_2+\dots+p_{k-1}+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \right). \end{aligned} \quad (30)$$

Hence, any $k \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in \{1, 2, \dots, n\}^n$ satisfy

$$\begin{aligned} & \Psi_n^k(x_{i_1} \otimes \dots \otimes x_{i_n}) \\ &= \underbrace{\Psi^k}_{=I^{*k}=\Pi^{[k]} \circ \Delta^{[k]}}(x_{i_1} \otimes \dots \otimes x_{i_n}) = (\Pi^{[k]} \circ \Delta^{[k]})(x_{i_1} \otimes \dots \otimes x_{i_n}) \\ &= \Pi^{[k]}(\Delta^{[k]}(x_{i_1} \otimes \dots \otimes x_{i_n})) \\ &= \sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \\ & \quad \left(x_{i_{\sigma(1)}} \otimes \dots \otimes x_{i_{\sigma(p_1)}} \right) \cdot \left(x_{i_{\sigma(p_1+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2)}} \right) \cdot \left(x_{i_{\sigma(p_1+p_2+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2+p_3)}} \right) \\ & \quad \cdot \dots \cdot \left(x_{i_{\sigma(p_1+p_2+\dots+p_{k-1}+1)}} \otimes \dots \otimes x_{i_{\sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \right) \\ & \quad \text{(by (30) and since } \Pi \text{ is the multiplication map)} \\ &= \sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \underbrace{x_{i_{\sigma(1)}} \otimes x_{i_{\sigma(2)}} \otimes \dots \otimes x_{i_{\sigma(n)}}}_{=\sigma^{-1}(x_{i_1} \otimes \dots \otimes x_{i_n})} \\ &= \sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \sigma^{-1}(x_{i_1} \otimes \dots \otimes x_{i_n}). \end{aligned}$$

Thus, for every $k \in \mathbb{N}$, the map $\Psi_n^k \in \text{End}(T_n(X))$ is the image of the element

$$\sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \sigma^{-1} \in K[S_n]$$

under the map $K[S_n] \hookrightarrow \text{End}(T_n(X))$. Similarly, we can show the analogous result for $Th_n(X)$ instead of $T_n(X)$: Namely, for every $k \in \mathbb{N}$, the map $\Psi_n^k \in \text{End}(Th_n(X))$ is the image of the element

$$\sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \text{sgn}(\sigma) \cdot \sigma^{-1} \in K[S_n]$$

under the map $K[S_n] \hookrightarrow \text{End}(Th_n(X))$.

Page 1082, Proof of Proposition 5.1: Here, Patras writes:

”Le reste de la proposition ne présente pas de difficultés.”

Let me try to make this part of the proof a bit more precise. Namely, let me show that the idempotents e_n^i and f_n^i of the algebra $K[S_n]$ are obtained from each other by the involution

$$\begin{aligned} K[S_n] &\rightarrow K[S_n] \\ \sigma &\mapsto \text{sgn}(\sigma) \cdot \sigma \end{aligned}$$

of the K -algebra $K[S_n]$.

In fact, let inv denote the K -vector space homomorphism

$$\begin{aligned} K[S_n] &\rightarrow K[S_n] \\ \sigma &\mapsto \text{sgn}(\sigma) \cdot \sigma \quad (\text{for every } \sigma \in S_n). \end{aligned}$$

It is easy to see that inv is a K -algebra homomorphism (this is more or less because $\text{sgn} : S_n \rightarrow \{-1, 1\}$ is a group homomorphism) and an involution (since $(\text{sgn}(\sigma))^2 = 1$ for every $\sigma \in S_n$). We must now prove that $\text{inv}(f_n^i) = e_n^i$. We will do this in two steps:

a) For every $k \in \mathbb{N}$, let $\Psi_{T,n}^k$ denote the element of $K[S_n]$ whose image under the map $K[S_n] \hookrightarrow \text{End}(T_n(X))$ is the Ψ_n^k of $T(X)$ (where by ” Ψ_n^k of $T(X)$ ”, we mean the map Ψ_n^k defined with respect to the graded bialgebra $T(X)$). Then, for every $k \in \mathbb{N}$, we have

$$\Psi_{T,n}^k = \sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \sigma^{-1}. \quad (31)$$

Similarly, for every $k \in \mathbb{N}$, let $\Psi_{Th,n}^k$ denote the element of $K[S_n]$ whose image under the map $K[S_n] \hookrightarrow \text{End}(Th_n(X))$ is the Ψ_n^k of $Th(X)$ (where by " Ψ_n^k of $Th(X)$ ", we mean the map Ψ_n^k defined with respect to the Hopf algebra $Th(X)$). Then, for every $k \in \mathbb{N}$, we have

$$\Psi_{Th,n}^k = \sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \text{sgn}(\sigma) \cdot \sigma^{-1}. \quad (32)$$

Comparing (31) with (32), we immediately see that $\text{inv}(\Psi_{T,n}^k) = \Psi_{Th,n}^k$.

b) Fix some $n \in \mathbb{N}$ and some $i \in \{0, 1, \dots, n\}$. Consider the N and the elements $\alpha_0, \alpha_1, \dots, \alpha_N$ whose existence is guaranteed by Lemma 3.10.

(Note that the equality (20) is a concretization of Patras' claim that "Ce morphisme peut, d'après 1.3 et 3.1, se réécrire comme une combinaison linéaire finie d'endomorphismes caractéristiques" on page 1081. Patras' assertion doesn't make it clear that the coefficients of this "combinaison linéaire" don't depend on H , but our Lemma 3.10 does, and we are going to use this now.)

Applying (20) to $H = T(X)$, we get $f_n^i = \sum_{k=0}^N \alpha_k \Psi_{T,n}^k$. Applying (20) to $H =$

¹¹This is because we showed above that for every $k \in \mathbb{N}$, the map $\Psi_n^k \in \text{End}(T_n(X))$ is the image of the element

$$\sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \sigma^{-1} \in K[S_n]$$

under the map $K[S_n] \hookrightarrow \text{End}(T_n(X))$.

¹²This is because we showed above that for every $k \in \mathbb{N}$, the map $\Psi_n^k \in \text{End}(Th_n(X))$ is the image of the element

$$\sum_{p_1+p_2+\dots+p_k=n} \sum_{\substack{\sigma \in S_n; \\ \sigma(1) < \sigma(2) < \dots < \sigma(p_1); \\ \sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2); \\ \sigma(p_1+p_2+1) < \sigma(p_1+p_2+2) < \dots < \sigma(p_1+p_2+p_3); \\ \dots; \\ \sigma(p_1+p_2+\dots+p_{k-1}+1) < \sigma(p_1+p_2+\dots+p_{k-1}+2) < \dots < \sigma(p_1+p_2+\dots+p_{k-1}+p_k)}} \text{sgn}(\sigma) \cdot \sigma^{-1} \in K[S_n]$$

under the map $K[S_n] \hookrightarrow \text{End}(Th_n(X))$.

$Th(X)$, we get $e_n^i = \sum_{k=0}^N \alpha_k \Psi_{Th,n}^k$. Thus,

$$\begin{aligned} \text{inv}(f_n^i) &= \text{inv}\left(\sum_{k=0}^N \alpha_k \Psi_{T,n}^k\right) && \left(\text{since } f_n^i = \sum_{k=0}^N \alpha_k \Psi_{T,n}^k\right) \\ &= \sum_{k=0}^N \alpha_k \underbrace{\text{inv}\left(\Psi_{T,n}^k\right)}_{=\Psi_{Th,n}^k} && (\text{since inv is } K\text{-linear}) \\ &= \sum_{k=0}^N \alpha_k \Psi_{Th,n}^k = e_n^i, \end{aligned}$$

qed.

Page 1082, three lines above Lemma 5.2: Patras writes: "Par définition, l'algèbre de Lie libre (resp. graduée) est la plus petite sous-algèbre de Lie (resp. graduée) de $T(X)$ (resp. de $Th(X)$) contenant X ." I don't think this really follows from the definition (but it follows from the Poincaré-Birkhoff-Witt theorem)¹³.

Page 1082: On this page, it should be said somewhere that $Th^*(X)$ is just an abbreviation for $Th^{*gr}(X)$ (and not the dual of $Th(X)$ as an ungraded K -vector space).

Page 1082: On the penultimate line of page 1082, Patras writes: "l'algèbre graduée \mathcal{T} ". This is inaccurate, since \mathcal{T} is not a graded algebra but the *completion* of a graded algebra (with respect to the canonical topology induced by the grading). Fortunately this does not prevent the conclusion (that the logarithm of S is well-defined) from being valid (it actually would not be valid if \mathcal{T} was just a graded algebra!).

Page 1083, proof of Proposition 5.3: On the second line of this page, there is a typo: $Tg(X)$ should be $Th(X)$.

Page 1083, proof of Proposition 5.3: Here, I don't understand why Patras claims that

$$Q_{i_1, \dots, i_k}^{(m)} = \Pi^{[m]} \circ (I - \eta \circ \epsilon)^{\otimes m} \circ \Delta^{[m]}(x_{i_1} \otimes \dots \otimes x_{i_k}).$$

But I think there is an alternative proof of Proposition 5.3 anyway. It is rather simple, but it uses a lot of notation and commonplace facts from linear algebra. So let us begin with some definitions that could just as well stand in a linear algebra text:

Definition 0.3. Whenever V and W are two K -vector spaces, then we denote by $\rho_{V,W}$ the K -linear map

$$\begin{aligned} V^* \otimes W &\rightarrow \text{Hom}_K(V, W), \\ f \otimes \xi &\mapsto (\text{the map } V \rightarrow W \text{ which sends every } x \in V \text{ to } f(x)\xi) \end{aligned}$$

¹³The paper [BF] gives a different proof of the fact that the free Lie algebra over X is the smallest Lie subalgebra of $T(X)$ containing X . This proof doesn't use the Poincaré-Birkhoff-Witt theorem but still is far from being trivial.

(where V^* denotes the dual of V). This map $\rho_{V,W}$ is injective (this can be proven by standard linear algebra, i. e., working with bases). In general, it is not surjective (but it is surjective if $\dim V < \infty$).

Next, a similarly elementary definition related to graded vector spaces:

Definition 0.4. (a) Whenever V and W are two graded K -vector spaces, we let $\text{dirsum}_{V,W}$ denote the canonical injection $\prod_{i \in \mathbb{N}} \text{Hom}_K(V_i, W_i) \rightarrow \text{Hom}_K(V, W)$ which takes every family $(f_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \text{Hom}_K(V_i, W_i)$ of maps to the direct sum $\bigoplus_{i \in \mathbb{N}} f_i : \bigoplus_{i \in \mathbb{N}} V_i \rightarrow \bigoplus_{i \in \mathbb{N}} W_i$ (this direct sum is, of course, a map $V \rightarrow W$, since $\bigoplus_{i \in \mathbb{N}} V_i = V$ and $\bigoplus_{i \in \mathbb{N}} W_i = W$).

(b) Whenever V is a graded K -vector space and W is a K -vector space, let us define a topology on the K -vector space $\text{Hom}_K(V, W)$ (this is the space of all K -linear maps (not only the graded ones) from V to W) as follows: Let \mathbb{N}_- denote the set $\{-1, 0, 1, 2, \dots\}$. For every $i \in \mathbb{N}_-$, let $V^{(i)}$ denote the subspace $V_0 + V_1 + \dots + V_i$ of V . For any $i \in \mathbb{N}_-$ and any $g \in \text{Hom}_K(V^{(i)}, W)$, let $\text{Hom}_{K,i,g}(V, W)$ denote the subset

$$\{f \in \text{Hom}_K(V, W) \mid f|_{V^{(i)}} = g\} \quad \text{of} \quad \text{Hom}_K(V, W).$$

Then, we define the topology on the K -vector space $\text{Hom}_K(V, W)$ to be the topology generated by the sets $\text{Hom}_{K,i,g}(V, W)$ with $i \in \mathbb{N}_-$ and $g \in \text{Hom}_K(V^{(i)}, W)$ (as open sets). (Note that $\text{Hom}_{K,-1,0}(V, W) = \text{Hom}_K(V, W)$.) This topology will be called the *right degree topology* on $\text{Hom}_K(V, W)$. (I am pretty sure that this topology has a different, more standard name, but I don't know it.)

Let us summarize a few (easy-to-prove) properties of this right degree topology:

Let V be a graded K -vector space and W a K -vector space. First of all, the right degree topology makes $\text{Hom}_K(V, W)$ into a Hausdorff topological space, so it makes sense to speak of "the limit" of a convergent sequence. Second, the right degree topology makes $\text{Hom}_K(V, W)$ into a complete set, as can be easily seen. Furthermore, it is easy to see that a sequence $(f_i)_{i \in \mathbb{N}}$ of K -linear maps $f_i : V \rightarrow W$ converges to a K -linear map $f : V \rightarrow W$ if and only if for every $i \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that every $n \geq N$ satisfies $f_n|_{V^{(i)}} = f|_{V^{(i)}}$. Thus, an infinite sum $\sum_{i \in \mathbb{N}} g_i$ of K -linear maps $g_i : V \rightarrow W$ converges to a K -linear map $g : V \rightarrow W$ if and only if for every $i \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that

$$g_N(V^{(i)}) = g_{N+1}(V^{(i)}) = g_{N+2}(V^{(i)}) = \dots = 0 \quad \text{and} \quad \left(\sum_{\substack{i \in \mathbb{N}; \\ i \leq N-1}} g_i \right) |_{V^{(i)}} = g|_{V^{(i)}}.$$

Consequently, it is easy to see that an infinite sum $\sum_{i \in \mathbb{N}} g_i$ of K -linear maps $g_i : V \rightarrow W$ converges if and only if for every $i \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that

$$g_N(V^{(i)}) = g_{N+1}(V^{(i)}) = g_{N+2}(V^{(i)}) = \dots = 0.$$

(In this case, the value of this infinite sum is the function g that sends every $x \in V$ to $\sum_{i \in \mathbb{N}} g_i(x)$; here, the infinite sum $\sum_{i \in \mathbb{N}} g_i(x)$ has a well-defined value since all but finitely many of its terms are zero.)

It is easy to see that if V and W are two graded K -vector spaces, then the map $\text{dirsum}_{V,W}$ is continuous, where the topology on the space $\prod_{i \in \mathbb{N}} \text{Hom}_K(V_i, W_i)$ is the canonical one obtained by seeing $\prod_{i \in \mathbb{N}} \text{Hom}_K(V_i, W_i)$ as the completion of the graded vector space $\bigoplus_{i \in \mathbb{N}} \text{Hom}_K(V_i, W_i)$, whereas the topology on $\text{Hom}_K(V, W)$ is the right degree topology.

Next, let us introduce a certain subspace of the tensor product of two graded spaces:

Definition 0.5. Let V and W be two graded K -vector spaces. Let $V \otimes W$ denote the K -vector subspace $\sum_{i \in \mathbb{N}} V_i \otimes W_i$ of $V \otimes W$. (Here, for every $i \in \mathbb{N}$, we consider $V_i \otimes W_i$ as a subspace of $V \otimes W$, because V_i is a subspace of V and W_i is a subspace of W .)

Here is an alternative definition of $V \otimes W$: Since $V = \bigoplus_{i \in \mathbb{N}} V_i$ and $W = \bigoplus_{j \in \mathbb{N}} W_j$, we have

$$V \otimes W = \left(\bigoplus_{i \in \mathbb{N}} V_i \right) \otimes \left(\bigoplus_{j \in \mathbb{N}} W_j \right) = \bigoplus_{(i,j) \in \mathbb{N}^2} (V_i \otimes W_j)$$

(since tensor products commute with direct sums). Now, let $V \otimes W$ denote the subspace $\bigoplus_{\substack{(i,j) \in \mathbb{N}^2; \\ i=j}} (V_i \otimes W_j)$ of this direct sum $\bigoplus_{(i,j) \in \mathbb{N}^2} (V_i \otimes W_j)$.

From this alternative description of $V \otimes W$ as the space $\bigoplus_{\substack{(i,j) \in \mathbb{N}^2; \\ i=j}} (V_i \otimes W_j)$,

it is clear that $V \otimes W \cong \bigoplus_{i \in \mathbb{N}} (V_i \otimes W_i)$. More precisely, the map

$$\begin{aligned} \bigoplus_{i \in \mathbb{N}} (V_i \otimes W_i) &\rightarrow V \otimes W, \\ (a_i)_{i \in \mathbb{N}} &\mapsto \sum_{i \in \mathbb{N}} a_i \end{aligned}$$

is a well-defined canonical isomorphism of graded K -vector spaces. We denote this isomorphism by $\text{combine}_{V,W}$.

The completion of the graded K -vector space $\bigoplus_{i \in \mathbb{N}} (V_i \otimes W_i)$ is $\prod_{i \in \mathbb{N}} (V_i \otimes W_i)$.

We denote the completion of the graded K -vector space $V \otimes W$ by $V \widehat{\otimes} W$.

Let also $\widehat{\text{combine}}_{V,W}$ denote the completion of the isomorphism $\text{combine}_{V,W} : \bigoplus_{i \in \mathbb{N}} (V_i \otimes W_i) \rightarrow V \otimes W$. Since $\text{combine}_{V,W}$ is an isomorphism of graded K -vector spaces, its completion $\widehat{\text{combine}}_{V,W}$ is an isomorphism $\prod_{i \in \mathbb{N}} (V_i \otimes W_i) \rightarrow V \widehat{\otimes} W$ of topological K -vector spaces. This isomorphism $\widehat{\text{combine}}_{V,W}$ is the $\widehat{\text{map}}$

$$\begin{aligned} \prod_{i \in \mathbb{N}} (V_i \otimes W_i) &\rightarrow V \widehat{\otimes} W, \\ (a_i)_{i \in \mathbb{N}} &\mapsto \sum_{i \in \mathbb{N}} a_i \end{aligned}$$

(where the sum $\sum_{i \in \mathbb{N}} a_i$ is automatically convergent by the completeness of $V \widehat{\otimes} W$).

The completion of the canonical inclusion $V \otimes W \rightarrow V \otimes W$ is an injective continuous map $V \widehat{\otimes} W \rightarrow V \widehat{\otimes} W$ of K -vector spaces. This injective map allows us to consider $V \widehat{\otimes} W$ as a K -vector subspace of $V \widehat{\otimes} W$. We are going to do so.

These three definitions were purely linear-algebraical (and topological to the extent they involved completions). It is time to tie in some algebras and coalgebras into this. First, the purely algebraic part:

Proposition 1.12. Let A and B be two graded K -algebras. Then, $A \otimes B$ is a graded K -subalgebra of $A \otimes B$, and $A \widehat{\otimes} B$ is a K -subalgebra of $A \widehat{\otimes} B$.

The proof of Proposition 1.12 is a completely straightforward check (it boils down to showing that $1_{A \otimes B} \in A_0 \otimes B_0$ and that $(A_i \otimes B_i) \cdot (A_j \otimes B_j) \subseteq A_{i+j} \otimes B_{i+j}$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, where we consider $A_u \otimes B_v$ as a K -vector subspace of $A \otimes B$ for all $u \in \mathbb{N}$ and $v \in \mathbb{N}$).

Now here is something more interesting:

Proposition 1.13. Let C be a locally-finite¹⁴ graded K -coalgebra, and let A be a graded K -algebra. Since C is a locally-finite graded K -coalgebra, the graded dual $C^{\star \text{gr}}$ becomes a graded K -algebra.

Consider the continuous K -vector space isomorphism $\widehat{\text{combine}}_{C^{\star \text{gr}}, A} : \prod_{i \in \mathbb{N}} (C_i^{\star \text{gr}} \otimes A_i) \rightarrow C^{\star \text{gr}} \widehat{\otimes} A$.

¹⁴Here, a graded K -vector space V is said to be *locally-finite* if for every $n \in \mathbb{N}$, the n -th graded component of V is a finite-dimensional K -vector space. So our notion of "locally finite" is exactly what Patras calls "de type fini".

For every $i \in \mathbb{N}$, we have an injective map $\rho_{C_i, A_i} : (C_i)^* \otimes A_i \rightarrow \text{Hom}_K(C_i, A_i)$. Since $(C_i)^* = C_i^{\star \text{gr}}$ for every $i \in \mathbb{N}$, this rewrites as follows: For every $i \in \mathbb{N}$, we have an injective map $\rho_{C_i, A_i} : C_i^{\star \text{gr}} \otimes A_i \rightarrow \text{Hom}_K(C_i, A_i)$. Thus, the product $\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} : \prod_{i \in \mathbb{N}} (C_i^{\star \text{gr}} \otimes A_i) \rightarrow \prod_{i \in \mathbb{N}} \text{Hom}_K(C_i, A_i)$ of these maps is also injective.

Finally, consider the K -vector space injection $\text{dirsum}_{C,A} : \prod_{i \in \mathbb{N}} \text{Hom}_K(C_i, A_i) \rightarrow \text{Hom}_K(C, A)$.

Denote by $\phi_{C,A}$ the map

$$\text{dirsum}_{C,A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^{\star \text{gr}}, A}^{-1} : C^{\star \text{gr}} \widehat{\otimes} A \rightarrow \text{Hom}_K(C, A).$$

Clearly, this map $\phi_{C,A}$ makes the diagram

$$\begin{array}{ccc} \prod_{i \in \mathbb{N}} (C_i^{\star \text{gr}} \otimes A_i) & \xrightarrow{\prod_{i \in \mathbb{N}} \rho_{C_i, A_i}} & \prod_{i \in \mathbb{N}} \text{Hom}_K(C_i, A_i) \\ \widehat{\text{combine}}_{C^{\star \text{gr}}, A} \downarrow \cong & & \downarrow \text{dirsum}_{C,A} \\ C^{\star \text{gr}} \widehat{\otimes} A & \xrightarrow{\phi_{C,A}} & \text{Hom}_K(C, A) \end{array}$$

commute.

(a) This map $\phi_{C,A}$ is an injective and continuous K -algebra homomorphism. (Here, the K -algebra structure on $C^{\star \text{gr}} \widehat{\otimes} A$ is given by the fact that $C^{\star \text{gr}} \widehat{\otimes} A$ is a K -subalgebra of $C^{\star \text{gr}} \widehat{\otimes} A$, whereas the K -algebra structure on $\text{Hom}_K(C, A)$ is defined to be the convolution.)

(b) For every $n \in \mathbb{N}$, every $\gamma \in C_n^{\star \text{gr}}$ and every $a \in A_n$, we have

$$\phi_{C,A}(\gamma \otimes a) = (\text{the map } C \rightarrow A \text{ which sends every } x \in C \text{ to } \gamma(x)a). \quad (33)$$

In other words, for every $n \in \mathbb{N}$, every $\gamma \in C_n^{\star \text{gr}}$ and every $a \in A_n$, we have

$$(\phi_{C,A}(\gamma \otimes a))(x) = \gamma(x)a \quad \text{for every } x \in C. \quad (34)$$

(c) Let $f : C \rightarrow A$ be a graded K -linear map. For every $k \in \mathbb{N}$, let $(c_{k,\ell})_{\ell \in I_k}$ be a basis of the K -vector space C_k , and let $(c_{k,\ell}^*)_{\ell \in I_k}$ be the basis of $C_k^{\star \text{gr}}$ dual to this basis $(c_{k,\ell})_{\ell \in I_k}$.¹⁵ Then,

$$f = \phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right).$$

¹⁵At this place, we are using the condition that C is locally-finite. (In fact, since C is locally-finite, the space C_k is finite-dimensional, so that every basis of C_k has a dual basis of $(C_k)^* = C_k^{\star \text{gr}}$.)

Proof of Proposition 1.13. (b) Let $n \in \mathbb{N}$, $\gamma \in C_n^{\text{gr}}$ and $a \in A_n$ be arbitrary. Define a family $(t_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} (C_i^{\text{gr}} \otimes A_i)$ by

$$\left(t_i = \begin{cases} \gamma \otimes a, & \text{if } i = n; \\ 0, & \text{if } i \neq n \end{cases} \quad \text{for every } i \in \mathbb{N} \right).$$

Then, $t_n = \gamma \otimes a$, whereas every $i \neq n$ satisfies $t_i = 0$.

Since we know that $\widehat{\text{combine}}_{C^{\text{gr}}, A}$ is the map

$$\begin{aligned} \prod_{i \in \mathbb{N}} (C_i^{\text{gr}} \otimes A_i) &\rightarrow C^{\text{gr}} \widehat{\otimes} A, \\ (a_i)_{i \in \mathbb{N}} &\mapsto \sum_{i \in \mathbb{N}} a_i, \end{aligned}$$

we have

$$\widehat{\text{combine}}_{C^{\text{gr}}, A} ((a_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} a_i \quad \text{for every } (a_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} (C_i^{\text{gr}} \otimes A_i). \quad (35)$$

Applied to $(a_i)_{i \in \mathbb{N}} = (t_i)_{i \in \mathbb{N}}$, this yields

$$\begin{aligned} \widehat{\text{combine}}_{C^{\text{gr}}, A} ((t_i)_{i \in \mathbb{N}}) &= \sum_{i \in \mathbb{N}} t_i = \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i=n}} t_i}_{=t_n=\gamma \otimes a} + \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i \neq n}} t_i}_{=0} = \gamma \otimes a. \\ &\quad \text{(since every } i \neq n \text{ satisfies } t_i=0) \end{aligned}$$

Thus,

$$(t_i)_{i \in \mathbb{N}} = \widehat{\text{combine}}_{C^{\text{gr}}, A}^{-1} (\gamma \otimes a).$$

On the other hand, by the definition of $\prod_{i \in \mathbb{N}} \rho_{C_i, A_i}$, we have $\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) (t_i)_{i \in \mathbb{N}} = (\rho_{C_i, A_i} (t_i))_{i \in \mathbb{N}}$, so that

$$\begin{aligned} \text{dirsum}_{C, A} \left(\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) ((t_i)_{i \in \mathbb{N}}) \right) &= \text{dirsum}_{C, A} ((\rho_{C_i, A_i} (t_i))_{i \in \mathbb{N}}) = \bigoplus_{i \in \mathbb{N}} \rho_{C_i, A_i} (t_i) \\ &\quad \text{(by the definition of } \text{dirsum}_{C, A} \text{)}. \end{aligned}$$

In other words,

$$\begin{aligned} \bigoplus_{i \in \mathbb{N}} \rho_{C_i, A_i} (t_i) &= \text{dirsum}_{C, A} \left(\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \underbrace{((t_i)_{i \in \mathbb{N}})}_{=\widehat{\text{combine}}_{C^{\text{gr}}, A}^{-1} (\gamma \otimes a)} \right) \\ &= \text{dirsum}_{C, A} \left(\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \left(\widehat{\text{combine}}_{C^{\text{gr}}, A}^{-1} (\gamma \otimes a) \right) \right) \\ &= \underbrace{\left(\text{dirsum}_{C, A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^{\text{gr}}, A}^{-1} \right)}_{=\phi_{C, A}} (\gamma \otimes a) = \phi_{C, A} (\gamma \otimes a). \end{aligned}$$

Recall that the map ρ_{C_n, A_n} was defined as the K -linear map

$$(C_n)^* \otimes A_n \rightarrow \text{Hom}_K(C_n, A_n), \\ f \otimes \xi \mapsto (\text{the map } C_n \rightarrow A_n \text{ which sends every } x \in C_n \text{ to } f(x)\xi).$$

Hence, $\rho_{C_n, A_n}(\gamma \otimes a) = (\text{the map } C_n \rightarrow A_n \text{ which sends every } x \in C_n \text{ to } \gamma(x)a)$. Hence,

$$(\rho_{C_n, A_n}(\gamma \otimes a))(y) = \gamma(y)a \quad \text{for every } y \in C_n. \quad (36)$$

Now let $x \in C$ be arbitrary. For every $i \in \mathbb{N}$, let x_i be the i -th graded component of x . Since every nonnegative integer $i \neq n$ satisfies $t_i = 0$, it is clear that every nonnegative integer $i \neq n$ satisfies $(\rho_{C_i, A_i}(t_i))(x_i) = \underbrace{(\rho_{C_i, A_i}(0))}_{=0}(x_i) = 0(x_i) = 0$.

Thus, $\sum_{\substack{i \in \mathbb{N}; \\ i \neq n}} (\rho_{C_i, A_i}(t_i))(x_i) = \sum_{\substack{i \in \mathbb{N}; \\ i \neq n}} 0 = 0$. On the other hand,

$$\begin{aligned} \underbrace{(\phi_{C, A}(\gamma \otimes a))}_{= \bigoplus_{i \in \mathbb{N}} \rho_{C_i, A_i}(t_i)}(x) &= \left(\bigoplus_{i \in \mathbb{N}} \rho_{C_i, A_i}(t_i) \right)(x) = \sum_{i \in \mathbb{N}} (\rho_{C_i, A_i}(t_i))(x_i) \\ &= \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i = n}} (\rho_{C_i, A_i}(t_i))(x_i)}_{= (\rho_{C_n, A_n}(t_n))(x_n)} + \underbrace{\sum_{\substack{i \in \mathbb{N}; \\ i \neq n}} (\rho_{C_i, A_i}(t_i))(x_i)}_{= 0} \\ &= \left(\rho_{C_n, A_n} \underbrace{(t_n)}_{= \gamma \otimes a} \right)(x_n) = (\rho_{C_n, A_n}(\gamma \otimes a))(x_n) \\ &= \underbrace{\gamma(x_n)}_{= \gamma(x)} a \quad (\text{by (36), applied to } x_n \text{ instead of } y) \\ &= \gamma(x)a. \end{aligned}$$

Now forget that we fixed x . We have just proven that every $x \in C$ satisfies $(\phi_{C, A}(\gamma \otimes a))(x) = \gamma(x)a$. Hence,

$$\phi_{C, A}(\gamma \otimes a) = (\text{the map } C \rightarrow A \text{ which sends every } x \in C \text{ to } \gamma(x)a).$$

This proves Proposition 1.13 **(b)**.

(a) We know that the maps $\text{dirsum}_{C, A}$, $\prod_{i \in \mathbb{N}} \rho_{C_i, A_i}$ and $\widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ are injective¹⁶. Hence, their composition $\text{dirsum}_{C, A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ must also be injective. Since this composition $\text{dirsum}_{C, A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ is the map $\phi_{C, A}$, this yields that $\phi_{C, A}$ is injective.

¹⁶*Proof.* The map $\text{dirsum}_{C, A}$ is injective by Definition 0.4 **(a)**. The map $\prod_{i \in \mathbb{N}} \rho_{C_i, A_i}$ is injective because the product of injective maps is injective, and because the map ρ_{C_i, A_i} is injective for each $i \in \mathbb{N}$ (by Definition 0.3). The map $\widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ is injective because it is the inverse of an isomorphism.

We know that the maps $\text{dirsum}_{C,A}, \prod_{i \in \mathbb{N}} \rho_{C_i, A_i}$ and $\widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ are continuous¹⁷.

Hence, their composition $\text{dirsum}_{C,A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ must also be continuous. Since this composition $\text{dirsum}_{C,A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ is the map $\phi_{C,A}$, this yields that $\phi_{C,A}$ is continuous.

So we now know that $\phi_{C,A}$ is an injective and continuous K -vector space homomorphism. We still need to check that $\phi_{C,A}$ is a K -algebra homomorphism.

Let us first show that

$$\phi_{C,A}(\gamma \otimes a) * \phi_{C,A}(\delta \otimes b) = \phi_{C,A}((\gamma * \delta) \otimes ab) \quad (37)$$

for all $n \in \mathbb{N}, m \in \mathbb{N}, a \in A_n, b \in A_m, \gamma \in C_n^{* \text{ gr}}$ and $\delta \in C_m^{* \text{ gr}}$.

Proof of (37). Let $n \in \mathbb{N}, m \in \mathbb{N}, a \in A_n, b \in A_m, \gamma \in C_n^{* \text{ gr}}$ and $\delta \in C_m^{* \text{ gr}}$ be arbitrary.

Let $y \in C$. Since $\Delta(y) \in C \otimes C$, we can write the tensor $\Delta(y)$ in the form $\Delta(y) = \sum_{\ell=1}^L \lambda_\ell c_\ell \otimes d_\ell$ for some $L \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_L$ of K , some elements c_1, c_2, \dots, c_L of C , and some elements d_1, d_2, \dots, d_L of C . Consider this L , these $\lambda_1, \lambda_2, \dots, \lambda_L$, these c_1, c_2, \dots, c_L , and these d_1, d_2, \dots, d_L .

By the definition of the convolution,

$$\phi_{C,A}(\gamma \otimes a) * \phi_{C,A}(\delta \otimes b) = \mu_A \circ (\phi_{C,A}(\gamma \otimes a) \otimes \phi_{C,A}(\delta \otimes b)) \circ \Delta_C,$$

¹⁷*Proof.* The map $\text{dirsum}_{C,A}$ is continuous by Definition 0.4 (b). The map $\prod_{i \in \mathbb{N}} \rho_{C_i, A_i}$ is continuous because it is the completion of a graded map (namely, of the graded map $\bigoplus_{i \in \mathbb{N}} \rho_{C_i, A_i}$). The map $\widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$ is continuous because it is the inverse of an isomorphism of topological K -vector spaces.

Compared with $(\phi_{C,A}((\gamma * \delta) \otimes ab))(y) = (\gamma * \delta)(y) \cdot ab$ (by (34), applied to y , $\gamma * \delta$ and ab instead of x , γ and a), this yields

$$(\phi_{C,A}(\gamma \otimes a) * \phi_{C,A}(\delta \otimes b))(y) = (\phi_{C,A}((\gamma * \delta) \otimes ab))(y).$$

Now forget that we fixed y . We thus have shown that every $y \in C$ satisfies $(\phi_{C,A}(\gamma \otimes a) * \phi_{C,A}(\delta \otimes b))(y) = (\phi_{C,A}((\gamma * \delta) \otimes ab))(y)$. In other words, $\phi_{C,A}(\gamma \otimes a) * \phi_{C,A}(\delta \otimes b) = \phi_{C,A}((\gamma * \delta) \otimes ab)$. This proves (37).

Next let us show a slightly more general fact: Let us show that

$$\phi_{C,A}(t) * \phi_{C,A}(s) = \phi_{C,A}(ts) \quad (38)$$

for all $n \in \mathbb{N}$, $m \in \mathbb{N}$, $t \in C_n^{\star \text{gr}} \otimes A_n$ and $s \in C_m^{\star \text{gr}} \otimes A_m$. Here, the product ts is to be understood as a product in the K -algebra $C^{\star \text{gr}} \widehat{\otimes} A$ (which contains t and s because both $C_n^{\star \text{gr}} \otimes A_n$ and $C_m^{\star \text{gr}} \otimes A_m$ canonically inject into $C^{\star \text{gr}} \widehat{\otimes} A$).

Proof of (38). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, $t \in C_n^{\star \text{gr}} \otimes A_n$ and $s \in C_m^{\star \text{gr}} \otimes A_m$ be arbitrary.

Since t is a tensor in $C_n^{\star \text{gr}} \otimes A_n$, we can write t in the form $t = \sum_{i=1}^I \lambda_i \gamma_i \otimes a_i$ for some $I \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_I$ of K , some elements $\gamma_1, \gamma_2, \dots, \gamma_I$ of $C_n^{\star \text{gr}}$, and some elements a_1, a_2, \dots, a_I of A_n . Consider this I , these $\lambda_1, \lambda_2, \dots, \lambda_I$, these $\gamma_1, \gamma_2, \dots, \gamma_I$, and these a_1, a_2, \dots, a_I .

Since s is a tensor in $C_m^{\star \text{gr}} \otimes A_m$, we can write s in the form $s = \sum_{j=1}^J \nu_j \delta_j \otimes b_j$ for some $J \in \mathbb{N}$, some elements $\nu_1, \nu_2, \dots, \nu_J$ of K , some elements $\delta_1, \delta_2, \dots, \delta_J$ of $C_m^{\star \text{gr}}$, and some elements b_1, b_2, \dots, b_J of A_m . Consider this J , these $\nu_1, \nu_2, \dots, \nu_J$, these $\delta_1, \delta_2, \dots, \delta_J$, and these b_1, b_2, \dots, b_J .

Now,

$$\begin{aligned} \phi_{C,A} \underbrace{\left(\sum_{i=1}^I \lambda_i \gamma_i \otimes a_i \right)}_{(t)} * \phi_{C,A} \underbrace{\left(\sum_{j=1}^J \nu_j \delta_j \otimes b_j \right)}_{(s)} &= \underbrace{\phi_{C,A} \left(\sum_{i=1}^I \lambda_i \gamma_i \otimes a_i \right)}_{\substack{= \sum_{i=1}^I \lambda_i \phi_{C,A}(\gamma_i \otimes a_i) \\ \text{(since } \phi_{C,A} \text{ is } K\text{-linear)}}} * \underbrace{\phi_{C,A} \left(\sum_{j=1}^J \nu_j \delta_j \otimes b_j \right)}_{\substack{= \sum_{j=1}^J \nu_j \phi_{C,A}(\delta_j \otimes b_j) \\ \text{(since } \phi_{C,A} \text{ is } K\text{-linear)}}} \\ &= \left(\sum_{i=1}^I \lambda_i \phi_{C,A}(\gamma_i \otimes a_i) \right) * \left(\sum_{j=1}^J \nu_j \phi_{C,A}(\delta_j \otimes b_j) \right) \\ &= \sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j \underbrace{\phi_{C,A}(\gamma_i \otimes a_i) * \phi_{C,A}(\delta_j \otimes b_j)}_{\substack{= \phi_{C,A}((\gamma_i * \delta_j) \otimes a_i b_j) \\ \text{(by (37), applied to } \gamma_i, a_i, \delta_j \text{ and } b_j \\ \text{instead of } \gamma, a, \delta \text{ and } b)}} \\ &= \sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j \phi_{C,A}((\gamma_i * \delta_j) \otimes a_i b_j). \quad (39) \end{aligned}$$

Meanwhile, multiplying the equations $t = \sum_{i=1}^I \lambda_i \gamma_i \otimes a_i$ and $s = \sum_{j=1}^J \nu_j \delta_j \otimes b_j$, we obtain

$$\begin{aligned} ts &= \left(\sum_{i=1}^I \lambda_i \gamma_i \otimes a_i \right) \left(\sum_{j=1}^J \nu_j \delta_j \otimes b_j \right) = \sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j \underbrace{(\gamma_i \otimes a_i) (\delta_j \otimes b_j)}_{=(\gamma_i * \delta_j) \otimes a_i b_j} \\ &\quad \text{(by the definition of the product in } C^{*\text{gr}} \widehat{\otimes} A) \\ &= \sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j (\gamma_i * \delta_j) \otimes a_i b_j, \end{aligned}$$

so that

$$\phi_{C,A}(ts) = \phi_{C,A} \left(\sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j (\gamma_i * \delta_j) \otimes a_i b_j \right) = \sum_{i=1}^I \sum_{j=1}^J \lambda_i \nu_j \phi_{C,A}((\gamma_i * \delta_j) \otimes a_i b_j)$$

(since $\phi_{C,A}$ is K -linear). Compared to (39), this yields $\phi_{C,A}(t) * \phi_{C,A}(s) = \phi_{C,A}(ts)$. This proves (38).

Next we are going to show that

$$\phi_{C,A}(t) * \phi_{C,A}(s) = \phi_{C,A}(ts) \quad (40)$$

for any $t \in C^{*\text{gr}} \widehat{\otimes} A$ and $s \in C^{*\text{gr}} \widehat{\otimes} A$.

Proof of (40). Let $t \in C^{*\text{gr}} \widehat{\otimes} A$ and $s \in C^{*\text{gr}} \widehat{\otimes} A$ be arbitrary.

Write the family $\widehat{\text{combine}}_{C^{*\text{gr}}, A}^{-1}(t) \in \prod_{i \in \mathbb{N}} (C_i^{*\text{gr}} \otimes A_i)$ in the form $(t_i)_{i \in \mathbb{N}}$. Then,

$$\begin{aligned} t &= \widehat{\text{combine}}_{C^{*\text{gr}}, A}((t_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} t_i \quad \text{(by (35), applied to } (a_i)_{i \in \mathbb{N}} = (t_i)_{i \in \mathbb{N}}) \\ &= \sum_{n \in \mathbb{N}} t_n \quad \text{(here, we renamed the index } i \text{ as } n), \end{aligned}$$

so that $\phi_{C,A}(t) = \phi_{C,A} \left(\sum_{n \in \mathbb{N}} t_n \right) = \sum_{n \in \mathbb{N}} \phi_{C,A}(t_n)$ (since $\phi_{C,A}$ is a continuous K -vector space homomorphism).

Write the family $\widehat{\text{combine}}_{C^{*\text{gr}}, A}^{-1}(s) \in \prod_{i \in \mathbb{N}} (C_i^{*\text{gr}} \otimes A_i)$ in the form $(s_i)_{i \in \mathbb{N}}$. Then,

$$\begin{aligned} s &= \widehat{\text{combine}}_{C^{*\text{gr}}, A}((s_i)_{i \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} s_i \quad \text{(by (35), applied to } (a_i)_{i \in \mathbb{N}} = (s_i)_{i \in \mathbb{N}}) \\ &= \sum_{m \in \mathbb{N}} s_m \quad \text{(here, we renamed the index } i \text{ as } m), \end{aligned}$$

so that $\phi_{C,A}(s) = \phi_{C,A} \left(\sum_{m \in \mathbb{N}} s_m \right) = \sum_{m \in \mathbb{N}} \phi_{C,A}(s_m)$ (since $\phi_{C,A}$ is a continuous K -vector space homomorphism).

Now it is time to notice that $\text{Hom}_K(C, A)$ is a topological K -algebra, i. e., the convolution is a continuous map $\text{Hom}_K(C, A) \times \text{Hom}_K(C, A) \rightarrow \text{Hom}_K(C, A)$.¹⁸

Furthermore, $C^{\star \text{gr}} \widehat{\otimes} A$ is a topological K -algebra (in fact, for any two K -algebras \mathfrak{A} and \mathfrak{B} , the K -algebra $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is a topological K -algebra).

Multiplying the equalities $\phi_{C,A}(t) = \sum_{n \in \mathbb{N}} \phi_{C,A}(t_n)$ and $\phi_{C,A}(s) = \sum_{m \in \mathbb{N}} \phi_{C,A}(s_m)$, we obtain

$$\begin{aligned} \phi_{C,A}(t) * \phi_{C,A}(s) &= \left(\sum_{n \in \mathbb{N}} \phi_{C,A}(t_n) \right) * \left(\sum_{m \in \mathbb{N}} \phi_{C,A}(s_m) \right) = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \underbrace{(\phi_{C,A}(t_n)) * (\phi_{C,A}(s_m))}_{= \phi_{C,A}(t_n s_m)} \\ &\quad \left(\begin{array}{c} \text{since } \text{Hom}_K(C, A) \text{ is a topological } K\text{-algebra, so that} \\ \text{there is a "distributive law" for convergent infinite sums in } \text{Hom}_K(C, A) \end{array} \right) \\ &= \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \phi_{C,A}(t_n s_m). \end{aligned}$$

On the other hand, multiplying the equalities $t = \sum_{n \in \mathbb{N}} t_n$ and $s = \sum_{m \in \mathbb{N}} s_m$, we obtain

$$\begin{aligned} ts &= \left(\sum_{n \in \mathbb{N}} t_n \right) * \left(\sum_{m \in \mathbb{N}} s_m \right) = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} t_n * s_m \\ &\quad \left(\begin{array}{c} \text{since } C^{\star \text{gr}} \widehat{\otimes} A \text{ is a topological } K\text{-algebra, so that} \\ \text{there is a "distributive law" for convergent infinite sums in } C^{\star \text{gr}} \widehat{\otimes} A \end{array} \right), \end{aligned}$$

so that

$$\begin{aligned} \phi_{C,A}(ts) &= \phi_{C,A} \left(\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} t_n s_m \right) = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \phi_{C,A}(t_n s_m) \\ &\quad \text{(since } \phi_{C,A} \text{ is a continuous } K\text{-linear map)} \\ &= \phi_{C,A}(t) * \phi_{C,A}(s). \end{aligned}$$

Thus, (40) is proven.

Finally, let us recall that we denote the unity of a K -algebra U by 1_U . Then,

¹⁸This fact does not even require all conditions of Proposition 1.13. As long as C is a graded K -coalgebra (not necessarily locally-finite) and A is a K -algebra (not necessarily graded), the convolution algebra $\text{Hom}_K(C, A)$ (with the right degree topology) is a topological K -algebra. This follows from Lemma 1.14 further below.

$1_{C^* \text{gr} \widehat{\otimes} A} = 1_{C^* \text{gr}} \otimes 1_A$, so that

$$\begin{aligned}
\phi_{C,A} \left(1_{C^* \text{gr} \widehat{\otimes} A} \right) &= \phi_{C,A} (1_{C^* \text{gr}} \otimes 1_A) = \left(\text{the map } C \rightarrow A \text{ which sends every } x \in C \text{ to } \underbrace{1_{C^* \text{gr}}(x)}_{=\epsilon} 1_A \right) \\
&\quad \text{(by (33), applied to } n = 0, \gamma = 1_{C^* \text{gr}} \text{ and } a = 1_A) \\
&= \left(\text{the map } C \rightarrow A \text{ which sends every } x \in C \text{ to } \underbrace{\epsilon(x) 1_A}_{=\eta(\epsilon(x))=(\eta \circ \epsilon)(x)} \right) \\
&= (\text{the map } C \rightarrow A \text{ which sends every } x \in C \text{ to } (\eta \circ \epsilon)(x)) \\
&= \eta \circ \epsilon = 1_{\text{Hom}_K(C,A)}.
\end{aligned}$$

Combined with the fact that (40) holds for any $t \in C^* \text{gr} \widehat{\otimes} A$ and $s \in C^* \text{gr} \widehat{\otimes} A$, this yields that $\phi_{C,A}$ is a K -algebra homomorphism. This completes the proof of Proposition 1.13 **(b)**.

(c) First of all, applying (35) to $(a_i)_{i \in \mathbb{N}} = \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right)_{i \in \mathbb{N}}$, we obtain

$$\widehat{\text{combine}}_{C^* \text{gr}, A} \left(\left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right)_{i \in \mathbb{N}} \right) = \sum_{i \in \mathbb{N}} \sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) = \sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell})$$

(here, we renamed the index i as k in the first sum). Thus,

$$\left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right)_{i \in \mathbb{N}} = \widehat{\text{combine}}_{C^* \text{gr}, A}^{-1} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right).$$

Now let $x \in C$ be arbitrary. For every $i \in \mathbb{N}$, let x_i be the i -th graded component of x . Then, $x = \sum_{i \in \mathbb{N}} x_i$.

Since $\phi_{C,A} = \text{dirsum}_{C,A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1}$, we have

$$\begin{aligned}
& \phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \\
&= \left(\text{dirsum}_{C,A} \circ \left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \circ \widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1} \right) \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \\
&= \text{dirsum}_{C,A} \left(\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \underbrace{\left(\widehat{\text{combine}}_{C^* \text{ gr}, A}^{-1} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \right)}_{= \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right)_{i \in \mathbb{N}}} \right) \\
&= \text{dirsum}_{C,A} \left(\underbrace{\left(\prod_{i \in \mathbb{N}} \rho_{C_i, A_i} \right) \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right)_{i \in \mathbb{N}}}_{= \left(\rho_{C_i, A_i} \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right) \right)_{i \in \mathbb{N}} \text{ (by the definition of } \prod_{i \in \mathbb{N}} \rho_{C_i, A_i})} \right) \\
&= \text{dirsum}_{C,A} \left(\left(\rho_{C_i, A_i} \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right) \right)_{i \in \mathbb{N}} \right) = \bigoplus_{i \in \mathbb{N}} \underbrace{\left(\rho_{C_i, A_i} \left(\sum_{\ell \in I_i} c_{i,\ell}^* \otimes f(c_{i,\ell}) \right) \right)}_{= \sum_{\ell \in I_i} \rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell})) \text{ (since } \rho_{C_i, A_i} \text{ is } K\text{-linear})} \\
&\quad \text{(by the definition of } \text{dirsum}_{C,A}) \\
&= \bigoplus_{i \in \mathbb{N}} \left(\sum_{\ell \in I_i} \rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell})) \right),
\end{aligned}$$

so that

$$\begin{aligned}
\left(\phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \right) (x) &= \left(\bigoplus_{i \in \mathbb{N}} \left(\sum_{\ell \in I_i} \rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell})) \right) \right) (x) \\
&= \sum_{i \in \mathbb{N}} \left(\sum_{\ell \in I_i} \rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell})) \right) (x_i) \\
&= \sum_{i \in \mathbb{N}} \sum_{\ell \in I_i} (\rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell}))) (x_i). \quad (41)
\end{aligned}$$

Recall that, for every $i \in \mathbb{N}$, the map ρ_{C_i, A_i} was defined as the K -linear map

$$\begin{aligned}
(C_i)^* \otimes A_i &\rightarrow \text{Hom}_K(C_i, A_i), \\
f \otimes \xi &\mapsto (\text{the map } C_i \rightarrow A_i \text{ which sends every } x \in C_i \text{ to } f(x)\xi).
\end{aligned}$$

Hence, for every $i \in \mathbb{N}$ and every $\ell \in I_i$, we have

$$\rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell})) = (\text{the map } C_i \rightarrow A_i \text{ which sends every } x \in C_i \text{ to } c_{i,\ell}^*(x) f(c_{i,\ell})).$$

Hence, for every $i \in \mathbb{N}$ and every $\ell \in I_i$, we have

$$(\rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell}))) (x_i) = c_{i,\ell}^*(x_i) f(c_{i,\ell}) \quad (42)$$

(because $x_i \in C_i$). On the other hand, for every $i \in \mathbb{N}$, every $y \in C_i$ satisfies $\sum_{\ell \in I_i} c_{i,\ell}^*(y) c_{i,\ell} = y$ (because $(c_{i,\ell}^*)_{\ell \in I_i}$ is the basis of C_i^{gr} dual to the basis $(c_{i,\ell})_{\ell \in I_i}$ of C_i). Applied to $y = x_i$, this yields

$$\sum_{\ell \in I_i} c_{i,\ell}^*(x_i) c_{i,\ell} = x_i \quad (43)$$

for every $i \in \mathbb{N}$.

The equality (41) now becomes

$$\begin{aligned} \left(\phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \right) (x) &= \sum_{i \in \mathbb{N}} \sum_{\ell \in I_i} \underbrace{(\rho_{C_i, A_i} (c_{i,\ell}^* \otimes f(c_{i,\ell}))) (x_i)}_{=c_{i,\ell}^*(x_i) f(c_{i,\ell}) \text{ (by (42))}} = \sum_{i \in \mathbb{N}} \underbrace{\sum_{\ell \in I_i} c_{i,\ell}^*(x_i) f(c_{i,\ell})}_{=f\left(\sum_{\ell \in I_i} c_{i,\ell}^*(x_i) c_{i,\ell}\right) \text{ (since } f \text{ is } K\text{-linear)}} \\ &= \sum_{i \in \mathbb{N}} f \left(\underbrace{\sum_{\ell \in I_i} c_{i,\ell}^*(x_i) c_{i,\ell}}_{=x_i \text{ (by (43))}} \right) = \sum_{i \in \mathbb{N}} f(x_i) \\ &= f \left(\underbrace{\sum_{i \in \mathbb{N}} x_i}_{=x} \right) \quad (\text{since } f \text{ is } K\text{-linear}) \\ &= f(x). \end{aligned}$$

Now forget that we fixed x . We have thus proven that every $x \in C$ satisfies $\left(\phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) \right) (x) = f(x)$. In other words, $\phi_{C,A} \left(\sum_{k \in \mathbb{N}} \sum_{\ell \in I_k} c_{k,\ell}^* \otimes f(c_{k,\ell}) \right) = f$. This proves Proposition 1.13 (c).

This completes the proof of Proposition 1.13, but we still need to show the following fact that we used:

Lemma 1.14. Let C be a graded K -coalgebra, and let A be a K -algebra. Consider the convolution K -algebra $\text{Hom}_K(C, A)$, endowed with the right degree topology. Then, $\text{Hom}_K(C, A)$ is a topological K -algebra, i. e., the convolution is a continuous map $\text{Hom}_K(C, A) \times \text{Hom}_K(C, A) \rightarrow \text{Hom}_K(C, A)$.

Before we prove Lemma 1.14, two very trivial facts:

Lemma 1.15. Let C be a graded K -coalgebra. Let \mathbb{N}_- denote the set $\{-1, 0, 1, 2, \dots\}$. For every $i \in \mathbb{N}_-$, let $C^{(i)}$ denote the subspace $C_0 + C_1 + \dots + C_i$ of C . Then, for every $i \in \mathbb{N}_-$, the subset $C^{(i)}$ is a subcoalgebra of C .

Lemma 1.16. Let C be a K -coalgebra. Let D be a subcoalgebra of C . Let $f : C \rightarrow A$ and $g : C \rightarrow A$ be two K -linear maps. Then, $(f * g) |_{D=} = (f |_{D=}) * (g |_{D=})$.

Proof of Lemma 1.15. Let $i \in \mathbb{N}_-$. We have

$$C_k \subseteq C^{(i)} \text{ for every } k \in \{0, 1, \dots, i\}. \quad (44)$$

19

Since C is a graded coalgebra, we have $\Delta(C_n) \subseteq \sum_{\ell=0}^n C_\ell \otimes C_{n-\ell}$ for every $n \in \mathbb{N}$.

Now, $C^{(i)} = C_0 + C_1 + \dots + C_i = \sum_{n=0}^i C_n$, so that

$$\begin{aligned} \Delta(C^{(i)}) &= \Delta\left(\sum_{n=0}^i C_n\right) = \sum_{n=0}^i \underbrace{\Delta(C_n)}_{\subseteq \sum_{\ell=0}^n C_\ell \otimes C_{n-\ell}} \quad (\text{since } \Delta \text{ is } K\text{-linear}) \\ &\subseteq \sum_{n=0}^i \sum_{\ell=0}^n \underbrace{C_\ell}_{\subseteq C^{(i)}} \otimes \underbrace{C_{n-\ell}}_{\subseteq C^{(i)}} \\ &\quad \begin{array}{l} \text{(by (44) (applied to } k=\ell), \text{ because } \\ \ell \leq n \leq i \text{ and thus } \ell \in \{0, 1, \dots, i\}) \end{array} \quad \otimes \quad \begin{array}{l} \text{(by (44) (applied to } k=n-\ell), \text{ because } \\ n-\ell \leq n \leq i \text{ and thus } n-\ell \in \{0, 1, \dots, i\}) \end{array} \\ &\subseteq \sum_{n=0}^i \sum_{\ell=0}^n C^{(i)} \otimes C^{(i)} \subseteq C^{(i)} \otimes C^{(i)} \quad (\text{since } C^{(i)} \otimes C^{(i)} \text{ is a } K\text{-vector space}). \end{aligned}$$

Hence, $C^{(i)}$ is a subcoalgebra of C . This proves Lemma 1.15.

Proof of Lemma 1.16. Let ι denote the canonical inclusion map $D \rightarrow C$. Then, ι is a K -coalgebra homomorphism, so that $\Delta_C \circ \iota = (\iota \otimes \iota) \circ \Delta_D$. But by the definition of convolution, we have the two equalities $f * g = \mu_A \circ (f \otimes g) \circ \Delta_C$ and $(f |_{D=}) * (g |_{D=}) = \mu_A \circ ((f |_{D=}) \otimes (g |_{D=})) \circ \Delta_D$. Besides, $f |_{D=} = f \circ \iota$ (because ι is the inclusion map $D \rightarrow C$) and $g |_{D=} = g \circ \iota$ (for the same reason). Now,

$$\begin{aligned} (f * g) |_{D=} &= \underbrace{(f * g)}_{=\mu_A \circ (f \otimes g) \circ \Delta_C} \circ \iota \quad (\text{since } \iota \text{ is the inclusion map } D \rightarrow C) \\ &= \mu_A \circ (f \otimes g) \circ \underbrace{\Delta_C \circ \iota}_{=(\iota \otimes \iota) \circ \Delta_D} = \mu_A \circ \underbrace{(f \otimes g) \circ (\iota \otimes \iota)}_{=(f \circ \iota) \otimes (g \circ \iota)} \circ \Delta_D \\ &= \mu_A \circ \left(\underbrace{(f \circ \iota)}_{=f |_{D=}} \otimes \underbrace{(g \circ \iota)}_{=g |_{D=}} \right) \circ \Delta_D = \mu_A \circ ((f |_{D=}) \otimes (g |_{D=})) \circ \Delta_D \\ &= (f |_{D=}) * (g |_{D=}). \end{aligned}$$

¹⁹*Proof.* Let $k \in \{0, 1, \dots, i\}$. Then, C_k is an addend of the sum $C_0 + C_1 + \dots + C_i$. Hence, $C_k \subseteq C_0 + C_1 + \dots + C_i = C^{(i)}$, and thus (44) is proven.

This proves Lemma 1.16.

Proof of Lemma 1.14. Let \mathbb{N}_- denote the set $\{-1, 0, 1, 2, \dots\}$. For every $i \in \mathbb{N}_-$, let $C^{(i)}$ denote the subspace $C_0 + C_1 + \dots + C_i$ of C . For any $i \in \mathbb{N}_-$ and any $g \in \text{Hom}_K(C^{(i)}, A)$, let $\text{Hom}_{K,i,g}(C, A)$ denote the subset

$$\{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = g\} \quad \text{of} \quad \text{Hom}_K(C, A).$$

By the definition of the right degree topology (in Definition 0.4 **(b)**), the right degree topology on $\text{Hom}_K(C, A)$ is the topology generated by the sets $\text{Hom}_{K,i,g}(C, A)$ with $i \in \mathbb{N}_-$ and $g \in \text{Hom}_K(C^{(i)}, A)$ (as open sets). In other words, the set

$$\{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\}$$

is a basis of the right degree topology on $\text{Hom}_K(C, A)$.

We recall a very easy fact from general topology:

$$\left(\begin{array}{l} \text{If } \mathfrak{A} \text{ and } \mathfrak{B} \text{ are two topological spaces, and } \mathcal{S} \text{ is a basis of the topology on } \mathfrak{B}, \\ \text{and if } T : \mathfrak{A} \rightarrow \mathfrak{B} \text{ is a map such that (the set } T^{-1}(U) \text{ is open for every } U \in \mathcal{S}), \\ \text{then } T \text{ is continuous} \end{array} \right). \quad (45)$$

(In brief, this fact means that the continuity of a map needs not be checked on every open set - it is enough to check it on the open sets of a basis).

Let \mathfrak{A} be the topological space $\text{Hom}_K(C, A) \times \text{Hom}_K(C, A)$. Let \mathfrak{B} be the topological space $\text{Hom}_K(C, A)$. Let $T : \text{Hom}_K(C, A) \times \text{Hom}_K(C, A) \rightarrow \text{Hom}_K(C, A)$ be the product map of the K -algebra $\text{Hom}_K(C, A)$. (In other words, T is the map which takes any $(f, g) \in \text{Hom}_K(C, A) \times \text{Hom}_K(C, A)$ to the convolution $f * g$.) We are going to prove that T is continuous.

Note that $\mathfrak{B} = \text{Hom}_K(C, A)$, so that $\mathfrak{B} \times \mathfrak{B} = \text{Hom}_K(C, A) \times \text{Hom}_K(C, A) = \mathfrak{A}$.

Denote by \mathcal{S} the set $\{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\}$.

We know that $\{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\}$ is a basis of the topology on $\text{Hom}_K(C, A)$. Since $\{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\} = \mathcal{S}$ and $\text{Hom}_K(C, A) = \mathfrak{B}$, this rewrites as follows: The set \mathcal{S} is a basis of the topology on \mathfrak{B} .

Now we will prove that

$$\text{the set } T^{-1}(U) \text{ is open for every } U \in \mathcal{S}. \quad (46)$$

In order to prove this, we will show that every $i \in \mathbb{N}_-$ and every $g \in \text{Hom}_K(C^{(i)}, A)$ satisfy

$$\begin{aligned} & T^{-1}(\text{Hom}_{K,i,g}(C, A)) \\ &= \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A); \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A). \end{aligned} \quad (47)$$

Proof of (47). Let $i \in \mathbb{N}_-$ and $g \in \text{Hom}_K(C^{(i)}, A)$.

a) Let s be an element of $T^{-1}(\text{Hom}_{K,i,g}(C, A))$. Then, $s \in \text{Hom}_K(C, A) \times \text{Hom}_K(C, A)$, so we can write s in the form $s = (s_1, s_2)$ for some $s_1 \in \text{Hom}_K(C, A)$ and $s_2 \in \text{Hom}_K(C, A)$. Consider these s_1 and s_2 . Let $t_1 = s_1|_{C^{(i)}}$ and $t_2 = s_2|_{C^{(i)}}$.

By Lemma 1.15, we know that $C^{(i)}$ is a subcoalgebra of C .

Since $s = (s_1, s_2)$, we have $T(s) = T(s_1, s_2) = s_1 * s_2$ (because T is the product map of the K -algebra $\text{Hom}_K(C, A)$). Thus,

$$\begin{aligned} s_1 * s_2 &= T(s) \in \text{Hom}_{K,i,g}(C, A) && \text{(since } s \in T^{-1}(\text{Hom}_{K,i,g}(C, A))) \\ &= \{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = g\}, \end{aligned}$$

so that $(s_1 * s_2)|_{C^{(i)}} = g$. Compared with

$$\begin{aligned} (s_1 * s_2)|_{C^{(i)}} &= \underbrace{(s_1|_{C^{(i)}})}_{=t_1} * \underbrace{(s_2|_{C^{(i)}})}_{=t_2} \\ &\quad \left(\begin{array}{c} \text{by Lemma 1.16 (applied to } C^{(i)}, s_1 \text{ and } s_2 \text{ instead of } D, f \text{ and } g), \\ \text{since } C^{(i)} \text{ is a subcoalgebra of } C \end{array} \right) \\ &= t_1 * t_2, \end{aligned}$$

this yields $t_1 * t_2 = g$. On the other hand, the definition of $\text{Hom}_{K,i,t_1}(C, A)$ says that

$$\text{Hom}_{K,i,t_1}(C, A) = \{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = t_1\}.$$

Hence, from $s_1|_{C^{(i)}} = t_1$, we get $s_1 \in \{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = t_1\} = \text{Hom}_{K,i,t_1}(C, A)$.

Similarly, $s_2 \in \text{Hom}_{K,i,t_2}(C, A)$.

Since $s_1 \in \text{Hom}_{K,i,g_1}(C, A)$ and $s_2 \in \text{Hom}_{K,i,g_2}(C, A)$, we have $(s_1, s_2) \in \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$. Hence,

$$\begin{aligned} s &= (s_1, s_2) \in \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A) \\ &\subseteq \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A); \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A) \\ &\quad \text{(since } t_1 * t_2 = g). \end{aligned}$$

Now forget that we fixed s . We have proven that every $s \in T^{-1}(\text{Hom}_{K,i,g}(C, A))$ satisfies $s \in \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A); \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$. In

other words,

$$\begin{aligned} &T^{-1}(\text{Hom}_{K,i,g}(C, A)) \\ &\subseteq \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A); \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A). \end{aligned} \quad (48)$$

b) Now let z be an element of $\bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A); \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times$

$\text{Hom}_{K,i,g_2}(C, A)$. Then, there exists some $(\gamma_1, \gamma_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A)$ such that $\gamma_1 * \gamma_2 = g$ and $z \in \text{Hom}_{K,i,\gamma_1}(C, A) \times \text{Hom}_{K,i,\gamma_2}(C, A)$. Consider this (γ_1, γ_2) .

Since $z \in \text{Hom}_{K,i,\gamma_1}(C, A) \times \text{Hom}_{K,i,\gamma_2}(C, A)$, we can write z in the form $z = (z_1, z_2)$ for some $z_1 \in \text{Hom}_{K,i,\gamma_1}(C, A)$ and $z_2 \in \text{Hom}_{K,i,\gamma_2}(C, A)$. Consider these z_1 and z_2 .

Since

$$z_1 \in \text{Hom}_{K,i,\gamma_1}(C, A) = \{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = \gamma_1\}$$

(by the definition of $\text{Hom}_{K,i,\gamma_1}(C, A)$), we have $z_1|_{C^{(i)}} = \gamma_1$. Similarly, $z_2|_{C^{(i)}} = \gamma_2$. Now, $z = (z_1, z_2)$ yields $T(z) = T(z_1, z_2) = z_1 * z_2$ (since T is the product map of the K -algebra $\text{Hom}_K(C, A)$), so that

$$\begin{aligned} (T(z))|_{C^{(i)}} &= (z_1 * z_2)|_{C^{(i)}} = \underbrace{(z_1|_{C^{(i)}})}_{=\gamma_1} * \underbrace{(z_2|_{C^{(i)}})}_{=\gamma_2} \\ &\left(\begin{array}{l} \text{by Lemma 1.16 (applied to } C^{(i)}, z_1 \text{ and } z_2 \text{ instead of } D, f \text{ and } g), \\ \text{since } C^{(i)} \text{ is a subcoalgebra of } C \end{array} \right) \\ &= \gamma_1 * \gamma_2 = g, \end{aligned}$$

so that $T(z) \in \{f \in \text{Hom}_K(C, A) \mid f|_{C^{(i)}} = g\} = \text{Hom}_{K,i,g}(C, A)$ and thus $z \in T^{-1}(\text{Hom}_{K,i,g}(C, A))$.

Now forget that we fixed z . We have thus shown that every $z \in \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A) \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$ satisfies $z \in T^{-1}(\text{Hom}_{K,i,g}(C, A))$. In other words,

$$\begin{aligned} &\bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A) \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A) \\ &\subseteq T^{-1}(\text{Hom}_{K,i,g}(C, A)). \end{aligned}$$

Combining this relation with (48), we obtain (47). This proves (47).

Proof of (46). We have $\mathcal{S} = \{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\}$. Hence,

$$\text{Hom}_{K,i,h}(C, A) \in \mathcal{S} \quad \text{for every } i \in \mathbb{N}_- \text{ and } h \in \text{Hom}_K(C^{(i)}, A). \quad (49)$$

Now let $U \in \mathcal{S}$. Then, $U \in \mathcal{S} = \{\text{Hom}_{K,i,g}(C, A) \mid i \in \mathbb{N}_-; g \in \text{Hom}_K(C^{(i)}, A)\}$. Hence, there exists some $i \in \mathbb{N}_-$ and some $g \in \text{Hom}_K(C^{(i)}, A)$ such that $U = \text{Hom}_{K,i,g}(C, A)$. Consider these i and g .

For every $(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A)$, the subset $\text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$ of \mathfrak{A} is open.²⁰ Thus, the set

$$\bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A) \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$$

is a union of open sets, and therefore open itself. Since

$$\begin{aligned} T^{-1}(U) &= T^{-1}(\text{Hom}_{K,i,g}(C, A)) \quad (\text{because } U = \text{Hom}_{K,i,g}(C, A)) \\ &= \bigcup_{\substack{(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A) \\ g_1 * g_2 = g}} \text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A) \end{aligned}$$

²⁰*Proof.* Let $(g_1, g_2) \in \text{Hom}_K(C^{(i)}, A) \times \text{Hom}_K(C^{(i)}, A)$. Then, $\text{Hom}_{K,i,g_1}(C, A) \in \mathcal{S}$ (by (49), applied to $h = g_1$) and $\text{Hom}_{K,i,g_2}(C, A) \in \mathcal{S}$ (by (49), applied to $h = g_2$).

Since \mathcal{S} is a basis of the topology on \mathfrak{B} , every element of \mathcal{S} is an open subset of \mathfrak{B} . Thus, both $\text{Hom}_{K,i,g_1}(C, A)$ and $\text{Hom}_{K,i,g_2}(C, A)$ are open subsets of \mathfrak{B} (since $\text{Hom}_{K,i,g_1}(C, A) \in \mathcal{S}$ and $\text{Hom}_{K,i,g_2}(C, A) \in \mathcal{S}$). Hence (by the definition of the product topology), their cartesian product $\text{Hom}_{K,i,g_1}(C, A) \times \text{Hom}_{K,i,g_2}(C, A)$ is an open subset of $\mathfrak{B} \times \mathfrak{B} = \mathfrak{A}$, qed.

(by (47)), this rewrites as follows: The set $T^{-1}(U)$ is open. This proves (46).

Now that (46) is proven, we can apply (45) and conclude that T is continuous. Since $T : \text{Hom}_K(C, A) \times \text{Hom}_K(C, A) \rightarrow \text{Hom}_K(C, A)$ is the product map of the K -algebra $\text{Hom}_K(C, A)$, this shows that the product map of the K -algebra $\text{Hom}_K(C, A)$ is continuous. Thus, $\text{Hom}_K(C, A)$ is a topological K -algebra (since we already know that $\text{Hom}_K(C, A)$ is a topological K -vector space). This proves Lemma 1.14.

Now, finally, the reason why were doing all of this:

Alternative proof of Proposition 5.3. First of all, let $C = Th(X)$ and $A = Th(X)$. For every graded K -linear map $f : C \rightarrow A$, let S_f denote the element

$$\sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k}) \in \mathcal{T}.$$

Consider the elements $T \in \mathcal{T}$ and $S \in \mathcal{T}$ defined on page 1082 of Patras's paper. These elements were defined by

$$T = \sum_{k \in \mathbb{N}^*} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (x_{i_1} \otimes \dots \otimes x_{i_k})$$

and $S = 1 \otimes 1 + T$.

Recall that I denotes the identity map $\text{id}_{Th(X)}$ of $Th(X)$. Thus, I is a graded K -linear map $C \rightarrow A$ (since $C = Th(X)$ and $A = Th(X)$), and we have

$$\begin{aligned} S_I &= \sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes \underbrace{I(x_{i_1} \otimes \dots \otimes x_{i_k})}_{=(x_{i_1} \otimes \dots \otimes x_{i_k})} && \text{(by the definition of } S_I) \\ &= \sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (x_{i_1} \otimes \dots \otimes x_{i_k}) \\ &= \underbrace{\sum_{(i_1, i_2, \dots, i_0) \in \{1, 2, \dots, n\}^0} (x_{i_1}^* \otimes \dots \otimes x_{i_0}^*) \otimes (x_{i_1} \otimes \dots \otimes x_{i_0})}_{\substack{=(\text{empty tensor product}) \otimes (\text{empty tensor product}) \\ \text{(since there exists only one } (i_1, i_2, \dots, i_0) \in \{1, 2, \dots, n\}^0, \text{ and this} \\ (i_1, i_2, \dots, i_0) \text{ is the empty 0-tuple)}}} \\ &\quad + \underbrace{\sum_{k \in \mathbb{N}^*} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (x_{i_1} \otimes \dots \otimes x_{i_k})}_{=T} \\ &= \underbrace{(\text{empty tensor product})}_{=1} \otimes \underbrace{(\text{empty tensor product})}_{=1} + T = 1 \otimes 1 + T = S. \end{aligned}$$

Now let us show that

$$\left(\begin{array}{l} \text{for every graded } K\text{-linear map } f : C \rightarrow A, \text{ the element} \\ S_f \text{ lies in } C^{\text{gr}} \widehat{\otimes} A \text{ and satisfies } \phi_{C,A}(S_f) = f. \end{array} \right) \quad (50)$$

Proof of (50). Let $f : C \rightarrow A$ be a graded K -linear map. For every $k \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} \underbrace{(x_{i_1}^* \otimes \dots \otimes x_{i_k}^*)}_{\in C_k^{\text{gr}}} \otimes \underbrace{f(x_{i_1} \otimes \dots \otimes x_{i_k})}_{\substack{\in A_k \\ \text{(since } x_{i_1} \otimes \dots \otimes x_{i_k} \in C_k \\ \text{and since } f \text{ is graded)}}} \\ & \in \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} C_k^{\text{gr}} \otimes A_k \subseteq C_k^{\text{gr}} \otimes A_k \end{aligned}$$

(since $C_k^{\text{gr}} \otimes A_k$ is a K -vector space). Hence,

$$\left(\sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k}) \right)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} (C_k^{\text{gr}} \otimes A_k).$$

Now, recall (from Definition 0.5) that $\widehat{\text{combine}}_{C^{\text{gr}}, A}$ is the map

$$\begin{aligned} \prod_{i \in \mathbb{N}} (C_i^{\text{gr}} \otimes A_i) & \rightarrow C^{\text{gr}} \widehat{\otimes} A, \\ (a_i)_{i \in \mathbb{N}} & \mapsto \sum_{i \in \mathbb{N}} a_i. \end{aligned}$$

Renaming the indices i as k in this formula, we see that $\widehat{\text{combine}}_{C^{\text{gr}}, A}$ is the map

$$\begin{aligned} \prod_{k \in \mathbb{N}} (C_k^{\text{gr}} \otimes A_k) & \rightarrow C^{\text{gr}} \widehat{\otimes} A, \\ (a_k)_{k \in \mathbb{N}} & \mapsto \sum_{k \in \mathbb{N}} a_k. \end{aligned}$$

Thus,

$$\begin{aligned} & \widehat{\text{combine}}_{C^{\text{gr}}, A} \left(\left(\sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k}) \right)_{k \in \mathbb{N}} \right) \\ & = \sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k}) = S_f, \end{aligned}$$

so that

$$\begin{aligned} S_f & \in \widehat{\text{combine}}_{C^{\text{gr}}, A} \left(\left(\sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k}) \right)_{k \in \mathbb{N}} \right) \\ & \subseteq \widehat{\text{combine}}_{C^{\text{gr}}, A} \left(\prod_{k \in \mathbb{N}} (C_k^{\text{gr}} \otimes A_k) \right) \subseteq C^{\text{gr}} \widehat{\otimes} A. \end{aligned}$$

Thus, we have proven the first part of the claim of (50).

Now to the second part: For every $k \in \mathbb{N}$, notice that $(x_{i_1} \otimes \dots \otimes x_{i_k})_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k}$ is a basis of the K -vector space C_k (since $C = Th(X)$), and $(x_{i_1}^* \otimes \dots \otimes x_{i_k}^*)_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k}$

is the basis of the K -vector space $C_k^{\star \text{gr}}$ dual to this basis $(x_{i_1} \otimes \dots \otimes x_{i_k})_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k}$. Hence, Proposition 1.13 **(c)** (applied to $(c_{k, \ell})_{\ell \in I_k} = (x_{i_1} \otimes \dots \otimes x_{i_k})_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k}$ and $(c_{k, \ell}^*)_{\ell \in I_k} = (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*)_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k}$) yields that

$$f = \phi_{C, A} \left(\underbrace{\sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes f(x_{i_1} \otimes \dots \otimes x_{i_k})}_{=S_f} \right) = \phi_{C, A}(S_f).$$

Thus we have shown that $S_f \in C^{\star \text{gr}} \widehat{\otimes} A$ and that $\phi_{C, A}(S_f) = f$. Hence, (50) is proven.

Applying (50) to $f = I$, we conclude that S_I lies in $C^{\star \text{gr}} \widehat{\otimes} A$ and satisfies $\phi_{C, A}(S_I) = I$.

Since $\phi_{C, A}$ commutes with taking the logarithm (because $\phi_{C, A}$ is a continuous K -algebra homomorphism, and because every continuous K -algebra homomorphism commutes with taking the logarithm), we have $\log(\phi_{C, A}(S_I)) = \phi_{C, A}(\log(S_I))$. Thus,

$$\log \underbrace{I}_{=\phi_{C, A}(S_I)} = \log(\phi_{C, A}(S_I)) = \phi_{C, A} \left(\log \underbrace{(S_I)}_{=S} \right) = \phi_{C, A}(\log S).$$

But applying (50) to $f = \log I$ (this is allowed since $\log I$ is graded), we conclude that $S_{\log I}$ lies in $C^{\star \text{gr}} \widehat{\otimes} A$ and satisfies $\phi_{C, A}(S_{\log I}) = \log I$. Thus, $\phi_{C, A}(S_{\log I}) = \log I = \phi_{C, A}(\log S)$. Since $\phi_{C, A}$ is injective (by Proposition 1.13 **(a)**), this yields $S_{\log I} = \log S$.

Hence,

$$\begin{aligned}
\log S &= S_{\log I} = \sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&\quad \text{(by the definition of } S_{\log I}\text{)} \\
&= \underbrace{\sum_{(i_1, i_2, \dots, i_0) \in \{1, 2, \dots, n\}^0} (x_{i_1}^* \otimes \dots \otimes x_{i_0}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_0})}_{\substack{=(\text{empty tensor product}) \otimes (\log I)(\text{empty tensor product}) \\ (\text{since there exists only one } (i_1, i_2, \dots, i_0) \in \{1, 2, \dots, n\}^0, \text{ and this} \\ (i_1, i_2, \dots, i_0) \text{ is the empty 0-tuple)}}} \\
&\quad + \sum_{k \in \mathbb{N}^*} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&= \underbrace{(\text{empty tensor product})}_{=1} \otimes (\log I) \underbrace{(\text{empty tensor product})}_{=1} \\
&\quad + \underbrace{\sum_{\substack{k \in \mathbb{N}^* \\ k \geq 1}} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k})}_{= \sum_{k \geq 1}} \\
&= \underbrace{1 \otimes (\log I)(1)}_{=0 \text{ (since } (\log I)(1)=0)} \\
&\quad + \sum_{k \geq 1} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&= \sum_{k \geq 1} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k}). \tag{51}
\end{aligned}$$

Now let us remember that Patras (on page 1082-1083) defines the elements Q_{i_1, \dots, i_k} (for all $k \in \mathbb{N}^*$ and all $(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k$) by decomposing $\log S$ into the form

$$\log S = \sum_{k \geq 1} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes Q_{i_1, \dots, i_k}.$$

Due to (51), this decomposition is clearly given by

$$\log S = \sum_{k \geq 1} \sum_{(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k} (x_{i_1}^* \otimes \dots \otimes x_{i_k}^*) \otimes (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k}).$$

Hence, $Q_{i_1, \dots, i_k} = (\log I)(x_{i_1} \otimes \dots \otimes x_{i_k})$ for every $k \in \mathbb{N}^*$ and every $(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k$.

Now, denote the connected cocommutative graded Hopf algebra $Th(X)$ by H . By (13) (applied to $i = 1$), we have $e^1 = \frac{(\log I)^{*1}}{1!} = \frac{\log I}{1} = \log I$. Now, every $k \in \mathbb{N}^*$ and

every $(i_1, i_2, \dots, i_k) \in \{1, 2, \dots, n\}^k$ satisfy

$$\begin{aligned}
Q_{i_1, \dots, i_k} &= \underbrace{(\log I)(x_{i_1} \otimes \dots \otimes x_{i_k})}_{=e^1} = e^1(x_{i_1} \otimes \dots \otimes x_{i_k}) = e_k^1(x_{i_1} \otimes \dots \otimes x_{i_k}) \\
&\quad (\text{since } x_{i_1} \otimes \dots \otimes x_{i_k} \in H_k) \\
&\in e_k^1(H) = H_k^{(1)} \quad \left(\text{since } H_k^{(1)} \text{ was defined as } e_k^1(H) \right) \\
&\subseteq H^{(1)} \quad \left(\begin{array}{l} \text{since } H^{(1)} \text{ was defined as } \bigoplus_{n \geq 1} H_n^{(1)}, \text{ and thus contains } H_k^{(1)} \\ \text{(since } H_k^{(1)} \text{ is an addend of the sum } \bigoplus_{n \geq 1} H_n^{(1)}) \end{array} \right) \\
&= \text{Prim } H \quad (\text{by Lemma 4.1, since } H \text{ is connected cocommutative graded}) \\
&= \text{Prim}(Th(X)) \quad (\text{since } H = Th(X)).
\end{aligned}$$

This proves both parts of Proposition 5.3.

Page 1085, proof of Lemma 6.3: This proof is an induction proof, but the induction base (i. e., the case $n = 1$) is missing. Fortunately, this is not much of a problem since the induction base can be done by a slight simplification of the argument from the induction step.

The proof itself is nice, but let me rewrite it for more clarity. We start with a lemma:

Lemma 6.5. Let K be a field of characteristic p , where p is a prime. Let C be a K -vector space, and let D be a K -vector subspace of C . Let $\alpha : C \rightarrow C$ be a K -linear map such that

$$(\alpha(x) = x \quad \text{for every } x \in D) \quad (52)$$

and

$$(\alpha(x) \equiv x \pmod{D} \quad \text{for every } x \in C). \quad (53)$$

Then, $\alpha^p = \text{id}_C$.

Proof of Lemma 6.5. We claim that for every $\ell \in \mathbb{N}$, we have

$$\alpha^\ell(x) = x + \ell(\alpha(x) - x) \quad \text{for every } x \in C. \quad (54)$$

Proof of (54). We are going to prove (54) by induction over ℓ :

Induction base: We have $\underbrace{\alpha^0}_{=\text{id}}(x) = \text{id}(x) = x = x + 0(\alpha(x) - x)$ for every $x \in C$.

In other words, (54) holds for $\ell = 0$. This completes the induction base.

Induction step: Let $L \in \mathbb{N}$. Assume that (54) holds for $\ell = L$. We now must prove (54) for $\ell = L + 1$.

Since (54) holds for $\ell = L$, we have $\alpha^L(x) = x + L(\alpha(x) - x)$ for every $x \in C$. On the other hand, every $x \in C$ satisfies $\alpha(x) \equiv x \pmod{D}$ (by (53)) and thus $\alpha(x) - x \in D$.

Thus, for every $x \in C$, we have

$$\begin{aligned}
\alpha^{L+1}(x) &= \alpha \left(\underbrace{\alpha^L(x)}_{=x+L(\alpha(x)-x)} \right) = \alpha(x + L(\alpha(x) - x)) \\
&= \underbrace{\alpha(x)}_{=x+(\alpha(x)-x)} + L \underbrace{\alpha(\alpha(x) - x)}_{= \alpha(x)-x} \quad (\text{since } \alpha \text{ is } K\text{-linear}) \\
&\quad \text{(by (52) (applied to } \alpha(x)-x \text{ instead of } x), \\
&\quad \text{because } \alpha(x)-x \in D) \\
&= x + \underbrace{(\alpha(x) - x) + L(\alpha(x) - x)}_{=(L+1)(\alpha(x)-x)} = x + (L+1)(\alpha(x) - x).
\end{aligned}$$

In other words, (54) is proven for $\ell = L + 1$. This completes the induction step. Thus, the induction proof of (54) is complete.

Now, applying (54) to $\ell = p$, we obtain $\alpha^p(x) = x + p(\alpha(x) - x)$ for every $x \in C$. Thus, every $x \in C$ satisfies

$$\alpha^p(x) = x + \underbrace{p(\alpha(x) - x)}_{=0 \text{ (since } K \text{ has characteristic } p)} = x = \text{id}_C(x).$$

In other words, $\alpha^p = \text{id}_C$. This proves Lemma 6.5.

Lemma 6.6. Let K be a field of characteristic p , where p is a prime. Let k be an integer not divisible by p . Let C be a K -vector space, and let D be a K -vector subspace of C . Let $\beta : C \rightarrow C$ be a K -linear map such that

$$(\beta^{p-1}(x) = x \quad \text{for every } x \in D) \quad (55)$$

and

$$(\beta(x) \equiv kx \pmod{D} \quad \text{for every } x \in C). \quad (56)$$

Then, $(\beta^{p-1})^p = \text{id}_C$.

Proof of Lemma 6.6. We claim that for every $\ell \in \mathbb{N}$, we have

$$\beta^\ell(x) \equiv k^\ell x \pmod{D} \quad \text{for every } x \in C. \quad (57)$$

Proof of (57). We are going to prove (57) by induction over ℓ :

Induction base: We have $\underbrace{\beta^0}_{=\text{id}}(x) = \text{id}(x) = x$ and $\underbrace{k^0}_{=1}x = x$ for every $x \in C$.

Thus, $\beta^0(x) = x = k^0x$ for every $x \in C$. In other words, (57) holds for $\ell = 0$. This completes the induction base.

Induction step: Let $L \in \mathbb{N}$. Assume that (57) holds for $\ell = L$. We now must prove (57) for $\ell = L + 1$.

Since (57) holds for $\ell = L$, we have

$$\beta^L(x) \equiv k^L x \pmod{D} \quad \text{for every } x \in C. \quad (58)$$

Thus, for every $x \in C$, we have

$$\begin{aligned} \beta^{L+1}(x) &= \beta^L(\beta(x)) \equiv k^L \underbrace{\beta(x)}_{\substack{\equiv kx \pmod{D} \\ \text{(by (56))}}} && \text{(by (58), applied to } \beta(x) \text{ instead of } x) \\ &\equiv \underbrace{k^L k}_{=k^{L+1}} x = k^{L+1}x \pmod{D}. \end{aligned}$$

In other words, (57) is proven for $\ell = L + 1$. This completes the induction step. Thus, the induction proof of (57) is complete.

Now notice that, by Fermat's Little Theorem, we have $k^{p-1} \equiv 1 \pmod{p}$ (since p is a prime, and k is an integer not divisible by p). Hence, $k^{p-1} - 1$ is divisible by p , so that $(k^{p-1} - 1)x = 0$ for every $x \in C$ (since K has characteristic p). On the other hand, applying (57) to $\ell = p - 1$, we obtain $\beta^{p-1}(x) \equiv k^{p-1}x \pmod{D}$ for every $x \in C$. Thus, every $x \in C$ satisfies

$$\beta^{p-1}(x) \equiv k^{p-1}x = x + \underbrace{(k^{p-1}x - x)}_{=(k^{p-1}-1)x=0} = x \pmod{D}.$$

Combined with (55), this yields that we can apply Lemma 6.5 to $\alpha = \beta^{p-1}$. This yields $(\beta^{p-1})^p = \text{id}_C$, and thus Lemma 6.6 is proven.

The next lemma we need is a basic property of connected graded bialgebras:

Lemma 1.17. Let H be a connected graded bialgebra over a field K (not necessarily of characteristic p). Let n be a positive integer. Let $x \in H_n$. Then,

$$\Delta(x) \in x \otimes 1 + 1 \otimes x + \sum_{k=1}^{n-1} H_k \otimes H_{n-k}.$$

Note that this Lemma 1.17 is a slight rewriting of Proposition II.1.1 of [M], but let us give a proof for the sake of completeness.

Proof of Lemma 1.17. In the following, id will always denote the identity map id_H of H .

We have $x \in H_n$ and thus

$$\begin{aligned} \Delta(x) \in \Delta(H_n) &\subseteq \sum_{k=0}^n H_k \otimes H_{n-k} && \text{(since } H \text{ is a graded coalgebra)} \\ &= H_0 \otimes \underbrace{H_{n-0}}_{=H_n} + \sum_{k=1}^{n-1} H_k \otimes H_{n-k} + H_n \otimes \underbrace{H_{n-n}}_{=H_0} \\ &= H_0 \otimes H_n + \sum_{k=1}^{n-1} H_k \otimes H_{n-k} + H_n \otimes H_0. \end{aligned}$$

Hence, there exist some $u \in H_0 \otimes H_n$, $v \in \sum_{k=1}^{n-1} H_k \otimes H_{n-k}$ and $w \in H_n \otimes H_0$ such that $\Delta(x) = u + v + w$. Consider these u , v and w .

We now will prove the two equalities $u = x \otimes 1$ and $w = 1 \otimes x$. The proofs of these equalities are analogous (they only differ in the order of the tensorands), so it will be enough to prove $w = 1 \otimes x$ only.

According to my definition of a connected graded bialgebra (see my remark about "Page 1070, fifth line of this page"), the map $\epsilon|_{H_0}: H_0 \rightarrow K$ is an isomorphism (since H is connected). Thus, the map $\text{id} \otimes (\epsilon|_{H_0}): H \otimes H_0 \rightarrow H \otimes K$ is an isomorphism.

Adding the equalities $u \in H_0 \otimes H_n$ and $v \in \sum_{k=1}^{n-1} H_k \otimes H_{n-k}$, we obtain

$$u + v \in H_0 \otimes H_n + \sum_{k=1}^{n-1} H_k \otimes H_{n-k} = \sum_{k=0}^{n-1} H_k \otimes H_{n-k},$$

so that

$$\begin{aligned} (\text{id} \otimes \epsilon)(u + v) &\in (\text{id} \otimes \epsilon) \left(\sum_{k=0}^{n-1} H_k \otimes H_{n-k} \right) = \sum_{k=0}^{n-1} \underbrace{(\text{id} \otimes \epsilon)(H_k \otimes H_{n-k})}_{=\text{id}(H_k) \otimes \epsilon(H_{n-k})} \\ &\quad (\text{since } \text{id} \otimes \epsilon \text{ is } K\text{-linear}) \\ &= \sum_{k=0}^{n-1} \text{id}(H_k) \otimes \underbrace{\epsilon(H_{n-k})}_{=0} = \sum_{k=0}^{n-1} \text{id}(H_k) \otimes 0 = 0. \\ &\quad (\text{since } n-k \geq 1 \text{ (because } k \leq n-1), \text{ and since } H \text{ is a graded coalgebra}) \end{aligned}$$

In other words, $(\text{id} \otimes \epsilon)(u + v) = 0$.

Let kan_1 be the canonical isomorphism $H \rightarrow H \otimes K$ which sends every $x \in H$ to $x \otimes 1$. By the axioms of a coalgebra, $(\text{id} \otimes \epsilon) \circ \Delta = \text{kan}_1$ (since H is a coalgebra). Hence,

$$\begin{aligned} ((\text{id} \otimes \epsilon) \circ \Delta)(x) &= \text{kan}_1(x) = \underbrace{x}_{=\text{id}(x)} \otimes \underbrace{1}_{=\epsilon(1) = (\epsilon|_{H_0})(1)} \quad (\text{by the definition of } \text{kan}_1) \\ &\quad (\text{since } 1 \in H_0) \\ &= \text{id}(x) \otimes (\epsilon|_{H_0})(1) = (\text{id} \otimes (\epsilon|_{H_0}))(x \otimes 1). \end{aligned}$$

Compared with

$$\begin{aligned} ((\text{id} \otimes \epsilon) \circ \Delta)(x) &= (\text{id} \otimes \epsilon) \left(\underbrace{\Delta(x)}_{=u+v+w} \right) = (\text{id} \otimes \epsilon)(u + v + w) = \underbrace{(\text{id} \otimes \epsilon)(u + v)}_{=0} + (\text{id} \otimes \epsilon)(w) \\ &= (\text{id} \otimes \epsilon)(w) = \underbrace{((\text{id} \otimes \epsilon)|_{H \otimes H_0})}_{=\text{id} \otimes (\epsilon|_{H_0})}(w) \quad \left(\text{since } w \in \underbrace{H_n \otimes H_0}_{\subseteq H} \subseteq H \otimes H_0 \right) \\ &= (\text{id} \otimes (\epsilon|_{H_0}))(w), \end{aligned}$$

this yields $(\text{id} \otimes (\epsilon|_{H_0}))(w) = (\text{id} \otimes (\epsilon|_{H_0}))(x \otimes 1)$. Since $\text{id} \otimes (\epsilon|_{H_0})$ is injective (because $\text{id} \otimes (\epsilon|_{H_0})$ is an isomorphism), this yields $w = x \otimes 1$. As we have said, the proof of $u = 1 \otimes x$ is similar to the proof of $w = x \otimes 1$ that we just did (except we have to switch the order of the tensorands). Thus, we now have

$$\Delta(x) = u + v + w = \underbrace{w}_{=x \otimes 1} + \underbrace{u}_{=1 \otimes x} + \underbrace{v}_{\in \sum_{k=1}^{n-1} H_k \otimes H_{n-k}} \in x \otimes 1 + 1 \otimes x + \sum_{k=1}^{n-1} H_k \otimes H_{n-k}.$$

This proves Lemma 1.17.

Corollary 1.18. Let H be a connected graded bialgebra over a field K (not necessarily of characteristic p). Let $\ell \in \mathbb{N}$.

(a) We have $\Psi^\ell(1) = 1$ and $(\Psi^\ell - I)(H_0) = 0$.

(b) Let n be a positive integer. Let $x \in H_n$. Then,

$$\Psi^\ell(x) \equiv \ell x \pmod{\left(\sum_{i=1}^{n-1} H_i H_{n-i}\right) \cap H_n}. \quad (59)$$

Proof of Corollary 1.18. (a) There is a fact that every K -linear map $f : H \rightarrow H$ satisfying $f(1) = 1$ must also satisfy $f^{*\nu}(1) = 1$ for every $\nu \in \mathbb{N}$.

This fact is very easy to prove by induction over ℓ (the details are left to the reader). Applying this fact to $f = I$ and $\nu = \ell$, we conclude that $I^{*\ell}(1) = 1$.

Since Ψ^ℓ was defined by $\Psi^\ell = I^{*\ell}$, we have $\Psi^\ell(1) = I^{*\ell}(1) = 1$.

On the other hand, since H is connected, the map $\epsilon|_{H_0} : H_0 \rightarrow K$ is an isomorphism.

Consider the family (1) of vectors in the K -vector space H_0 . The image of this family under the isomorphism $\epsilon|_{H_0}$ is $\left(\underbrace{(\epsilon|_{H_0})(1)}_{=\epsilon(1)=1}\right) = (1)$, and this is a basis of the K -vector space K . Hence, the family (1) is a basis of the K -vector space H_0 (because every family of vectors in a vector space whose image under some isomorphism is a basis must itself be a basis). Hence, $H_0 = K \cdot 1$ (where 1 denotes the unity of the K -algebra H). Thus,

$$\begin{aligned} (\Psi^\ell - I)(H_0) &= (\Psi^\ell - I)(K \cdot 1) = K \cdot \underbrace{(\Psi^\ell - I)(1)}_{=\Psi^\ell(1)-I(1)} \\ &\quad \text{(since } \Psi^\ell - I \text{ is } K\text{-linear)} \\ &= K \cdot \left(\underbrace{\Psi^\ell(1)}_{=1} - \underbrace{I(1)}_{=1}\right) = K \cdot \underbrace{(1-1)}_{=0} = 0. \end{aligned}$$

This proves Corollary 1.18 (a).

(b) Note first that $x \in H_n$ yields $\epsilon(x) \in \epsilon(H_n) = 0$ (since H is a graded coalgebra and $n > 0$), so that $\epsilon(x) = 0$.

We are going to prove (59) by induction over ℓ :

Induction base: We have

$$\underbrace{\Psi^0}_{=\eta \circ \epsilon}(x) = (\eta \circ \epsilon)(x) = \eta(\underbrace{\epsilon(x)}_{=0}) = \eta(0) = 0 \equiv 0x \pmod{\left(\sum_{i=1}^{n-1} H_i H_{n-i}\right) \cap H_n}.$$

Thus, (59) holds for $\ell = 0$. This completes the induction base.

Induction step: Let $L \in \mathbb{N}$ be arbitrary. Assume that (59) holds for $\ell = L$. We now must show that (59) holds for $\ell = L + 1$.

Since Ψ^L and Ψ^{L+1} are graded maps, we have $\Psi^L(H_n) \subseteq H_n$ and $\Psi^{L+1}(H_n) \subseteq H_n$.

By the definition of Ψ^L , we have $\Psi^L = I^{*L}$. Also, Corollary 1.18 **(a)** (applied to $\ell = L$) yields $\Psi^L(1) = 1$ and $(\Psi^L - I)(H_0) = 0$.

By the definition of Ψ^{L+1} , we have

$$\Psi^{L+1} = I^{*(L+1)} = I * \underbrace{I^{*L}}_{=\Psi^L} = I * \Psi^L = \mu \circ (I \otimes \Psi^L) \circ \Delta$$

(by the definition of convolution). Thus,

$$\begin{aligned} \Psi^{L+1}(x) &= (\mu \circ (I \otimes \Psi^L) \circ \Delta)(x) = \mu \left(\begin{array}{c} (I \otimes \Psi^L) \quad (\Delta(x)) \\ \in x \otimes 1 + 1 \otimes x + \sum_{k=1}^{n-1} H_k \otimes H_{n-k} \\ \text{(by Lemma 1.17)} \end{array} \right) \\ &= \mu \left(\underbrace{(I \otimes \Psi^L) \left(x \otimes 1 + 1 \otimes x + \sum_{k=1}^{n-1} H_k \otimes H_{n-k} \right)}_{=(I \otimes \Psi^L)(x \otimes 1) + (I \otimes \Psi^L)(1 \otimes x) + \sum_{k=1}^{n-1} (I \otimes \Psi^L)(H_k \otimes H_{n-k})} \right) \\ &\quad \text{(since } I \otimes \Psi^L \text{ is } K\text{-linear)} \\ &= \mu \left(\underbrace{(I \otimes \Psi^L)(x \otimes 1)}_{=I(x) \otimes \Psi^L(1)} + \underbrace{(I \otimes \Psi^L)(1 \otimes x)}_{=I(1) \otimes \Psi^L(x)} + \sum_{k=1}^{n-1} \underbrace{(I \otimes \Psi^L)(H_k \otimes H_{n-k})}_{=I(H_k) \otimes \Psi^L(H_{n-k})} \right) \\ &= \mu \left(I(x) \otimes \Psi^L(1) + I(1) \otimes \Psi^L(x) + \sum_{k=1}^{n-1} I(H_k) \otimes \Psi^L(H_{n-k}) \right) \\ &= \underbrace{I(x)}_{=x} \cdot \underbrace{\Psi^L(1)}_{=1} + \underbrace{I(1)}_{=1} \cdot \Psi^L(x) + \sum_{k=1}^{n-1} \underbrace{I(H_k)}_{=H_k} \cdot \underbrace{\Psi^L(H_{n-k})}_{\subseteq H_{n-k}} \\ &\quad \text{(since } \Psi^L \text{ is a graded map)} \\ &\quad \text{(since } I \text{ is the multiplication map)} \\ &\subseteq x + \Psi^L(x) + \sum_{k=1}^{n-1} H_k H_{n-k} = x + \Psi^L(x) + \sum_{i=1}^{n-1} H_i H_{n-i} \end{aligned}$$

(here, we renamed the index k as i in the sum), so that $\Psi^{L+1}(x) - x - \Psi^L(x) \in \sum_{i=1}^{n-1} H_i H_{n-i}$. Combined with $\Psi^{L+1}(x) - x - \Psi^L(x) \in H_n$ (this is because $x \in H_n$ and thus $\Psi^L(x) \in \Psi^L(H_n) \subseteq H_n$ and $\Psi^{L+1}(x) \in \Psi^{L+1}(H_n) \subseteq H_n$), this yields

$$\Psi^{L+1}(x) - x - \Psi^L(x) \in \left(\sum_{i=1}^{n-1} H_i H_{n-i} \right) \cap H_n.$$

In other words,

$$\begin{aligned} \Psi^{L+1} &\equiv x + \underbrace{\Psi^L(x)}_{\equiv Lx \text{ mod } \left(\sum_{i=1}^{n-1} H_i H_{n-i} \right) \cap H_n} \equiv x + Lx = (L+1)x \text{ mod } \left(\sum_{i=1}^{n-1} H_i H_{n-i} \right) \cap H_n. \\ &\quad \text{(because (59) holds for } \ell=L\text{)} \end{aligned}$$

In other words, (59) holds for $\ell = L + 1$. Thus, the induction step is done. The induction proof of (59) is therefore complete.

So now we know that (59) holds for every ℓ . In other words, Corollary 1.18 (b) is proven.

Finally, a consequence of Proposition 1.4:

Corollary 1.19. Let H be a bialgebra, a graded bialgebra or a Hopf algebra. Assume that H is commutative or cocommutative. Then the characteristic operations (defined in Definition 1.2) satisfy $(\Psi^k)^s = \Psi^{k^s}$ (where $(\Psi^k)^s$ means $\underbrace{\Psi^k \circ \Psi^k \circ \dots \circ \Psi^k}_{s \text{ times}}$) for all $k \in \mathbb{N}$ and $s \in \mathbb{N}$.

Proof of Corollary 1.19. Fix some $k \in \mathbb{N}$. We will prove $(\Psi^k)^s = \Psi^{k^s}$ by induction over s :

Induction base: We have $(\Psi^k)^0 = I = I^{*1} = \Psi^1$ (because Ψ^1 was defined as I^{*1}) and $\Psi^{k^0} = \Psi^1$. Thus, $(\Psi^k)^0 = \Psi^1 = \Psi^{k^0}$. In other words, $(\Psi^k)^s = \Psi^{k^s}$ holds for $s = 0$. This completes the induction base.

Induction step: Let $S \in \mathbb{N}$. Assume that $(\Psi^k)^s = \Psi^{k^s}$ holds for $s = S$. We must then prove that $(\Psi^k)^{s+1} = \Psi^{k^{s+1}}$ also holds for $s = S + 1$.

Since $(\Psi^k)^s = \Psi^{k^s}$ holds for $s = S$, we have $(\Psi^k)^S = \Psi^{k^S}$. Applying Proposition 1.4 to $l = k^S$, we get $\Psi^k \circ \Psi^{k^S} = \Psi^{k \cdot k^S} = \Psi^{k^{S+1}}$ (since $k \cdot k^S = k^{S+1}$). Hence, $(\Psi^k)^{S+1} = \Psi^k \circ \underbrace{(\Psi^k)^S}_{=\Psi^{k^S}} = \Psi^k \circ \Psi^{k^S} = \Psi^{k^{S+1}}$. In other words, $(\Psi^k)^s = \Psi^{k^s}$ is proven to

hold for $s = S + 1$. This completes the induction step. Thus, the induction proof of $(\Psi^k)^s = \Psi^{k^s}$ is complete. In other words, Corollary 1.19 is proven.

Now to the actual proof of Lemma 6.3:

Proof of Lemma 6.3: We WLOG assume that H is a graded bialgebra. (The case when H is a Hopf algebra is analogous.)

We WLOG assume that H is commutative. (The case when H is cocommutative can be obtained from the case when H is commutative by dualization using Proposition 3.9.)

Fix some $k \in \mathbb{Z}$ such that $k \not\equiv 0 \pmod{p}$. Thus, k is not divisible by p . Thus, by Fermat's Little Theorem, $p \mid k^{p-1} - 1$ (since p is prime), so that

$$(k^{p-1} - 1)(x) = 0 \quad \text{for every } x \in H \quad (60)$$

(since K has characteristic p).

We must prove that every positive integer n satisfies

$$\rho_n \left(\left(\Psi^{k^{p^{n-1}}} \right)^{p-1} \right) = \rho_n(I). \quad (61)$$

In fact, we will prove (61) by induction over n :

Induction base: We have $p^{1-1} = p^0 = 1$, so that $\Psi^{k^{p^{1-1}}} = \Psi^{k^1} = \Psi^k$, so that $\left(\Psi^{k^{p^{1-1}}} \right)^{p-1} = (\Psi^k)^{p-1} = \Psi^{k^{p-1}}$ (by Corollary 1.19, applied to $s = p - 1$).

Every $x \in H_1$ satisfies

$$\Psi^{k^{p-1}}(x) \equiv k^{p-1}x \bmod \left(\sum_{i=1}^{1-1} H_i H_{1-i} \right) \cap H_1$$

(by Corollary 1.18 (b), applied to $\ell = k^{p-1}$ and $n = 1$). Since $\underbrace{\left(\sum_{i=1}^{1-1} H_i H_{1-i} \right)}_{=(\text{empty sum})=0} \cap H_1 =$

$0 \cap H_1 = 0$, this becomes $\Psi^{k^{p-1}}(x) \equiv k^{p-1}x \bmod 0$. Hence, every $x \in H_1$ satisfies $\Psi^{k^{p-1}}(x) \equiv k^{p-1}x \bmod 0$, so that

$$\begin{aligned} \Psi^{k^{p-1}}(x) &= k^{p-1}x = x + \underbrace{(k^{p-1}x - x)}_{=(k^{p-1}-1)x=0} = x. \\ &\quad \text{(by (60))} \end{aligned}$$

Thus, every $x \in H_1$ satisfies $(\Psi^{k^{p-1}} - I)(x) = \underbrace{\Psi^{k^{p-1}}(x)}_{=x} - \underbrace{I(x)}_{=x} = x - x = 0$ and thus

$x \in \text{Ker}(\Psi^{k^{p-1}} - I)$. Hence, $H_1 \subseteq \text{Ker}(\Psi^{k^{p-1}} - I)$. In other words, $(\Psi^{k^{p-1}} - I)(H_1) = 0$.

On the other hand, Corollary 1.18 (a) (applied to $\ell = k^{p-1}$) yields $\Psi^{k^{p-1}}(1) = 1$ and $(\Psi^{k^{p-1}} - I)(H_0) = 0$.

Now, since $\rho_1(\Psi^{k^{p-1}} - I)$ is the restriction of the map $\Psi^{k^{p-1}} - I$ to $\bigoplus_{i=0}^1 H_i$, we have

$$\begin{aligned} \text{Im}(\rho_1(\Psi^{k^{p-1}} - I)) &= (\Psi^{k^{p-1}} - I) \underbrace{\left(\bigoplus_{i=0}^1 H_i \right)}_{=H_0 \oplus H_1 = H_0 + H_1} = (\Psi^{k^{p-1}} - I)(H_0 + H_1) \\ &\quad \text{(since direct sums are sums)} \\ &= \underbrace{(\Psi^{k^{p-1}} - I)(H_0)}_{=0} + \underbrace{(\Psi^{k^{p-1}} - I)(H_1)}_{=0} \quad \left(\text{since } \Psi^{k^{p-1}} - I \text{ is } K\text{-linear} \right) \\ &= 0 + 0 = 0, \end{aligned}$$

so that $\rho_1(\Psi^{k^{p-1}} - I) = 0$. Thus,

$$\begin{aligned} 0 &= \rho_1(\Psi^{k^{p-1}} - I) = \rho_1 \underbrace{(\Psi^{k^{p-1}})}_{=(\Psi^{k^{p-1}-1})^{p-1}} - \rho_1(I) \quad \text{(since } \rho_1 \text{ is } K\text{-linear)} \\ &= \rho_1 \left((\Psi^{k^{p-1}-1})^{p-1} \right) - \rho_1(I), \end{aligned}$$

so that $\rho_1 \left((\Psi^{k^{p-1}-1})^{p-1} \right) = \rho_1(I)$. In other words, (61) holds for $n = 1$. This completes the induction base.

Induction step: Let N be a positive integer. Assume that (61) holds for $n = N$. We must then prove that (61) also holds for $n = N + 1$.

Since (61) holds for $n = N$, we have

$$\rho_N \left(\left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \right) = \rho_N (I).$$

Every $x \in \bigoplus_{i=0}^N H_i$ satisfies $\left(\Psi^{k^{p^{N-1}}} \right)^{p-1} (x) = \left(\rho_N \left(\left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \right) \right) (x)$ (since $\rho_N \left(\left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \right)$ is the restriction of the map $\left(\Psi^{k^{p^{N-1}}} \right)^{p-1}$ to $\bigoplus_{i=0}^N H_i$) and $I(x) = (\rho_N(I))(x)$ (since $\rho_N(I)$ is the restriction of the map I to $\bigoplus_{i=0}^N H_i$). Thus,

$$\left(\begin{array}{c} \text{every } x \in \bigoplus_{i=0}^N H_i \text{ satisfies} \\ \left(\Psi^{k^{p^{N-1}}} \right)^{p-1} (x) = \underbrace{\left(\rho_N \left(\left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \right) \right)}_{=\rho_N(I)} (x) = (\rho_N(I))(x) = I(x) = x. \end{array} \right). \quad (62)$$

Let E be the K -vector subspace $\bigoplus_{i=0}^N H_i$ of H . Then, (62) rewrites as follows:

$$\text{Every } x \in E \text{ satisfies } \left(\Psi^{k^{p^{N-1}}} \right)^{p-1} (x) = x. \quad (63)$$

On the other hand,

$$\text{every } j \in \{0, 1, \dots, N\} \text{ satisfies } H_j \subseteq E \quad (64)$$

(because for every $j \in \{0, 1, \dots, N\}$, the space H_j is an addend of the direct sum $\bigoplus_{i=0}^N H_i$,

and thus is contained in $\bigoplus_{i=0}^N H_i = E$).

We are now going to show that

$$\left(\Psi^{k^{p^{N(p-1)}}} - I \right) (H_n) = 0 \quad \text{for every } n \in \{0, 1, \dots, N+1\}. \quad (65)$$

Proof of (65). Let $n \in \{0, 1, \dots, N+1\}$ be arbitrary. Then, $n \leq N+1$, so that $n-1 \leq N$.

Corollary 1.18 (a) (applied to $\ell = k^{p^{N(p-1)}}$) yields $\Psi^{k^{p^{N(p-1)}}} (1) = 1$ and $\left(\Psi^{k^{p^{N(p-1)}}} - I \right) (H_0) = 0$. Hence, (65) is already proven when $n = 0$. Thus, for the rest of the proof, we can WLOG assume that $n \neq 0$. Assume this. Then, n is a positive integer.

Since $\Psi^{k^{p^{N-1}}}$ is a graded map, it satisfies $\Psi^{k^{p^{N-1}}} (H_n) \subseteq H_n$. Hence, it restricts to a K -linear map $\beta : H_n \rightarrow H_n$ which satisfies $\left(\beta(x) = \Psi^{k^{p^{N-1}}} (x) \text{ for every } x \in H_n \right)$. Consider this β . Then,

$$\beta^\ell (x) = \left(\Psi^{k^{p^{N-1}}} \right)^\ell (x) \quad \text{for every } \ell \in \mathbb{N} \text{ and } x \in H_n. \quad (66)$$

Let C be the K -vector space H_n . Let D be the K -vector subspace $\left(\sum_{i=1}^{n-1} H_i H_{n-i}\right) \cap H_n$ of C . Since $C = H_n$, the map β is a map $C \rightarrow C$ (since β is a map $H_n \rightarrow H_n$).

Our goal is to apply Lemma 6.6. In order to do so, we must show that the conditions (55) and (56) of Lemma 6.6 are satisfied. Let us first prove that the condition (56) is satisfied:

Since k is not divisible by p , and since p is prime, we have $k^{p-1} \equiv 1 \pmod{p}$ (by Fermat's Little Theorem). Thus, $(k^{p-1})^{1+p+p^2+\dots+p^{N-2}} \equiv 1^{1+p+p^2+\dots+p^{N-2}} = 1 \pmod{p}$. But since $(k^{p-1})^{1+p+p^2+\dots+p^{N-2}} = k^{(p-1)(1+p+p^2+\dots+p^{N-2})} = k^{p^{N-1}-1}$ (because $(p-1)(1+p+p^2+\dots+p^{N-2}) = p^{N-1}-1$), this becomes $k^{p^{N-1}-1} \equiv 1 \pmod{p}$, so that $k^{p^N} = k \underbrace{k^{p^{N-1}-1}}_{\equiv 1 \pmod{p}} \equiv k \pmod{p}$. Hence,

$p \mid k^{p^N} - k$. Thus, $(k^{p^N} - k)(x) = 0$ for every $x \in C$ (since K has characteristic p).

Now, every $x \in C$ satisfies

$$\begin{aligned} \beta(x) &= \Psi^{k^{p^{N-1}}}(x) \equiv k^{p^{N-1}}x && \left(\text{by Corollary 1.18 (b) (applied to } \ell = k^{p^{N-1}}\text{), since } x \in C = H_n\right) \\ &= kx + \underbrace{(k^{p^{N-1}}x - kx)}_{=(k^{p^N}-k)(x)=0} = kx \pmod{\left(\sum_{i=1}^{n-1} H_i H_{n-i}\right) \cap H_n}. \end{aligned}$$

Since $\left(\sum_{i=1}^{n-1} H_i H_{n-i}\right) \cap H_n = D$, this rewrites as follows:

$$\text{Every } x \in C \text{ satisfies } \beta(x) \equiv kx \pmod{D}. \quad (67)$$

Now to checking the condition (55).

By Proposition 1.4, the characteristic operations of H are algebra homomorphisms (since H is commutative); in particular, $\Psi^{k^{p^{N-1}(p-1)}}$ is an algebra homomorphism. Since $(\Psi^{k^{p^{N-1}}})^{p-1} = \Psi^{k^{p^{N-1}(p-1)}}$ (by Corollary 1.19, applied to $k^{p^{N-1}}$ and $p-1$ instead of k and s), this yields that $(\Psi^{k^{p^{N-1}}})^{p-1}$ is a K -algebra homomorphism.

²¹*Proof of (66).* We will prove (66) by induction over ℓ :

Induction base: We have $\underbrace{\beta^0}_{=\text{id}_{H_n}}(x) = \text{id}_{H_n}(x) = x = \underbrace{\text{id}}_{=(\Psi^{k^{p^{N-1}}})^0}(x) = (\Psi^{k^{p^{N-1}}})^0(x)$. Thus, (66)

holds for $\ell = 0$. The induction base is now complete.

Induction step: Let $L \in \mathbb{N}$. Assume that (66) holds for $\ell = L$. Now we must show that (66) also holds for $\ell = L+1$.

Let $x \in H_n$. We can apply (66) to L and $\beta(x)$ instead of ℓ and x (since we assumed that (66) holds for $\ell = L$). This gives us $\beta^L(\beta(x)) = (\Psi^{k^{p^{N-1}}})^L(\beta(x))$. Thus,

$$\beta^{L+1}(x) = \beta^L(\beta(x)) = \left(\Psi^{k^{p^{N-1}}}\right)^L \underbrace{(\beta(x))}_{=\Psi^{k^{p^{N-1}}}(x)} = \left(\Psi^{k^{p^{N-1}}}\right)^L \left(\Psi^{k^{p^{N-1}}}(x)\right) = \left(\Psi^{k^{p^{N-1}}}\right)^{L+1}(x).$$

Thus, we have proven that $\beta^{L+1}(x) = (\Psi^{k^{p^{N-1}}})^{L+1}(x)$ for every $x \in H_n$. In other words, (66) holds for $\ell = L+1$. This completes the induction step. Thus, the induction proof of (66) is complete.

Now, let $x \in D$ be arbitrary. (Note that we are requiring $x \in D$ now, not only $x \in C$.) Then,

$$\begin{aligned}
x \in D &= \left(\sum_{i=1}^{n-1} H_i H_{n-i} \right) \cap H_n \subseteq \sum_{i=1}^{n-1} \underbrace{H_i}_{\subseteq E} \underbrace{H_{n-i}}_{\subseteq E} \\
&\quad \begin{array}{l} \text{(since } i \leq n-1 \leq N, \text{ so that } \\ i \in \{0, 1, \dots, N\}, \text{ so that } H_i \subseteq E \\ \text{(by (64), applied to } j=i)) \end{array} \quad \begin{array}{l} \text{(since } i \geq 1, \text{ thus } n-i \leq n-1 \leq N, \text{ so that } \\ n-i \in \{0, 1, \dots, N\}, \text{ so that } H_{n-i} \subseteq E \\ \text{(by (64), applied to } j=n-i)) \end{array} \\
&\subseteq \sum_{i=1}^{n-1} EE \subseteq EE \quad \text{(since } EE \text{ is a } K\text{-vector space)} \\
&= \text{(the set of all } K\text{-linear combinations of elements of the form } ee' \text{ with } e \in E \text{ and } e' \in E).
\end{aligned}$$

Hence, x is a K -linear combination of elements of the form ee' with $e \in E$ and $e' \in E$.

In other words, we can write x in the form $x = \sum_{i=1}^I \lambda_i e_i e'_i$ for some $I \in \mathbb{N}$, some elements $\lambda_1, \lambda_2, \dots, \lambda_I$ of K , some elements e_1, e_2, \dots, e_I of E , and some elements e'_1, e'_2, \dots, e'_I of E . Consider this I , these $\lambda_1, \lambda_2, \dots, \lambda_I$, these e_1, e_2, \dots, e_I , and these e'_1, e'_2, \dots, e'_I . Then,

$$\begin{aligned}
\beta^{p-1}(x) &= \left(\Psi^{k^{p^{N-1}}} \right)^{p-1}(x) \quad \text{(by (66) (applied to } \ell = p-1), \text{ since } x \in D \subseteq E = H_n) \\
&= \left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \left(\sum_{i=1}^I \lambda_i e_i e'_i \right) \quad \left(\text{since } x = \sum_{i=1}^I \lambda_i e_i e'_i \right) \\
&= \sum_{i=1}^I \lambda_i \underbrace{\left(\Psi^{k^{p^{N-1}}} \right)^{p-1}(e_i)}_{=e_i} \underbrace{\left(\Psi^{k^{p^{N-1}}} \right)^{p-1}(e'_i)}_{=e'_i} \\
&\quad \begin{array}{l} \text{(by (63), applied to } e_i \text{ instead of } x) \\ \text{(by (63), applied to } e'_i \text{ instead of } x) \end{array} \\
&\quad \left(\text{since } \left(\Psi^{k^{p^{N-1}}} \right)^{p-1} \text{ is a } K\text{-algebra homomorphism} \right) \\
&= \sum_{i=1}^I \lambda_i e_i e'_i = x.
\end{aligned}$$

Now forget that we fixed x . We have thus shown that every $x \in D$ satisfies $\beta^{p-1}(x) = x$. Combined with (67), this shows that all conditions of Lemma 6.6 are satisfied. Hence, we can apply Lemma 6.6, and obtain $(\beta^{p-1})^p = \text{id}_C$. Thus, $\text{id}_C = (\beta^{p-1})^p = \beta^{(p-1)p}$. Hence, every $x \in C$ satisfies

$$\begin{aligned}
x &= \underbrace{\text{id}_C}_{=\beta^{(p-1)p}}(x) = \beta^{(p-1)p}(x) = \left(\Psi^{k^{p^{N-1}}} \right)^{(p-1)p}(x) \quad \text{(by (66), applied to } \ell = (p-1)p) \\
&= \Psi^{k^{p^{N-1}(p-1)p}}(x) \quad \left(\begin{array}{l} \text{since Corollary 1.19} \\ \text{(applied to } k^{p^{N-1}} \text{ and } (p-1)p \text{ instead of } k \text{ and } s) \\ \text{yields } \left(\Psi^{k^{p^{N-1}}} \right)^{(p-1)p} = \Psi^{k^{p^{N-1}(p-1)p}} \end{array} \right) \\
&= \Psi^{k^{p^N(p-1)}}(x) \quad \left(\text{since } p^{N-1}(p-1)p = \underbrace{p^{N-1}p}_{=p^N}(p-1) = p^N(p-1) \right)
\end{aligned}$$

and thus

$$\left(\Psi^{k^{p^N(p-1)}} - I\right)(x) = \underbrace{\Psi^{k^{p^N(p-1)}}(x)}_{=x} - \underbrace{I(x)}_{=x} = x - x = 0,$$

so that $x \in \text{Ker}\left(\Psi^{k^{p^N(p-1)}} - I\right)$. In other words, $C \subseteq \text{Ker}\left(\Psi^{k^{p^N(p-1)}} - I\right)$. Hence, $\left(\Psi^{k^{p^N(p-1)}} - I\right)(C) = 0$. Since $C = H_n$, this becomes $\left(\Psi^{k^{p^N(p-1)}} - I\right)(H_n) = 0$. This proves (65).

Now, $\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}} - I\right)$ is the restriction of the map $\Psi^{k^{p^N(p-1)}} - I$ to $\bigoplus_{i=0}^{N+1} H_i$. Hence,

$$\begin{aligned} & \text{Im}\left(\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}} - I\right)\right) \\ &= \left(\Psi^{k^{p^N(p-1)}} - I\right) \underbrace{\left(\bigoplus_{i=0}^{N+1} H_i\right)}_{= \sum_{i=0}^{N+1} H_i} = \left(\Psi^{k^{p^N(p-1)}} - I\right) \sum_{i=0}^{N+1} H_i \\ & \hspace{10em} \text{(since direct sums are sums)} \\ &= \sum_{i=0}^{N+1} \underbrace{\left(\Psi^{k^{p^N(p-1)}} - I\right)(H_i)}_{=0 \text{ (by (65), applied to } n=i)} \quad \left(\text{since } \Psi^{k^{p^N(p-1)}} - I \text{ is } K\text{-linear}\right) \\ &= \sum_{i=0}^{N+1} 0 = 0, \end{aligned}$$

so that $\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}} - I\right) = 0$. Since $\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}} - I\right) = \rho_{N+1}\left(\Psi^{k^{p^N(p-1)}}\right) - \rho_{N+1}(I)$ (because ρ_{N+1} is K -linear), this becomes $\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}}\right) - \rho_{N+1}(I) = 0$, so that $\rho_{N+1}\left(\Psi^{k^{p^N(p-1)}}\right) = \rho_{N+1}(I)$.

Since $\left(\Psi^{k^{p^{N+1-1}}}\right)^{p-1} = \left(\Psi^{k^{p^N}}\right)^{p-1} = \Psi^{k^{p^N(p-1)}}$ (by Corollary 1.19, applied to k^{p^N} and $p-1$ instead of k and s), we have $\rho_{N+1}\left(\left(\Psi^{k^{p^{N+1-1}}}\right)^{p-1}\right) = \rho_{N+1}\left(\Psi^{k^{p^N(p-1)}}\right) = \rho_{N+1}(I)$. Thus, (61) holds for $n = N + 1$. This completes the induction step. Thus, the induction proof of (61) is completed, and with it the proof of Lemma 6.3.

Page 1086, Proposition 6.4: I think the condition that " $k \not\equiv 0 [p], k \not\equiv 1 [p]$ " has to be replaced by the (stronger) condition that k be a primitive root modulo p . Otherwise, the sum $H^{(1)} \oplus \dots \oplus H^{(p-1)}$ won't be a well-defined direct sum anymore (since some of the addends will be equal).

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