**Noncommutative Schur functions and their applications**

*Sergey Fomin and Curtis Greene*

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**Errata and addenda by Darij Grinberg**

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The following is a list of errors and comments to the paper "Noncommutative Schur functions and their applications" by Sergey Fomin and Curtis Greene in the version of 12 December 1996. (This is not the version that was published in Discrete Mathematics 306 (2006), pp. 1080–1096; but almost all of the errors listed here exist in the published version as well. Of course, the numbering of the pages differs from that in the published version.)

# 8. Errata

- **Page 2, Section 1:** After "the sum ranges over all semi-standard tableaux $T$ of shape $\lambda$", add "with entries in $\{1, 2, \ldots, n\}$".
- **Page 2, Section 1:** After "For example, if $\lambda = (3, 2)$", add "and $n = 2$".
- **Page 2, Section 1:** After "the sum is over all semi-standard skew tableaux $T$ of shape $\lambda/\mu$", add "with entries in $\{1, 2, \ldots, n\}$".
- **Page 4, Section 1:** Replace "$u_i$ adds a box in row $i$" by "$u_i$ adds a box in column $i$". (This typo has been corrected in the published version of the paper.)
- **Page 4, Theorem 2:** It should be said that here and in the following, the conjugate of a partition $\lambda$ is denoted by $\lambda'$.

- **Page 5, just before Theorem 1.3:** I would add the following sentence right before Theorem 1.3: "An *ascent* of a word $w = a_1 a_2 \cdots a_m$ means an index $i \in \{1, 2, \ldots, m-1\}$ such that $a_i \leq a_{i+1}$."

- **Page 6, the paragraph above Example 2.2:** You write: "In the case of the plactic algebra, $F_h$ is just the Schur function $s_{\lambda^*}$". I think $\lambda^*$ means the transpose of the partition $\lambda$ here; but this should be explained. (It is not a common notation for the transpose. You seem to yourself use a different notation – namely, $\lambda'$ – later in the paper.)

- **Page 6, the paragraph above Example 2.2:** Replace "all standard tableaux T" by "all standard tableaux $T$" (the "$T$" should be in mathmode).

  This said, I don't understand the whole sentence around this: How do you manage to restrict attention to the standard tableaux? Many of the $h$'s have repeated letters. I would rather prove the fact that $F_h = s_{\lambda^*}$ using a variation on the RSK algorithm.

- **Page 7, Example 2.4:** Replace "are the permutations $a_1 \cdots a_n$" by "are the permutations $a_1 \cdots a_{n+1} \in S_{n+1}$ (written here in one-line notation, i.e., such that $a_i$ is the image of $i$)".

- **Page 10, (3.3):** After the equality (3.3), I suggest explaining that $\det \left( e_{\lambda'_i - \mu'_j + j - i} (\mathbf{u}) \right)$ means the determinant of the $N \times N$-matrix $\left( e_{\lambda'_i - \mu'_j + j - i} (\mathbf{u}) \right)_{1 \leq i \leq N,\ 1 \leq j \leq N}$, where $N$ is a nonnegative integer large enough to satisfy $N \geq \lambda_1$ and $N \geq \mu_1$.

- **Page 11, proof of Lemma 3.2:** "with families of lattice paths $\left(\pi_{1j_1}, \pi_{2j_2}, \ldots\right)$" $\to$ "with families of lattice paths $\left(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}\right)$, where $N$ is the size of the matrix on the right hand side of (3.3)".

  Likewise, replace "$\left(\pi_{1j_1}, \pi_{2j_2}, \ldots\right)$" by "$\left(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}\right)$" throughout the rest of this proof.

- **Page 11, proof of Lemma 3.2:** It would be good to explain what a "lattice path" is. Probably the simplest way to do so is the following: Let $G$ be the directed graph whose vertex set is $\mathbb{Z}^2$ and which has arcs from every lattice point $(\alpha, \beta) \in \mathbb{Z}^2$ to $(\alpha + 1, \beta)$ and to $(\alpha, \beta + 1)$ (and no further arcs). Then, a *lattice path* means a (directed) path on $G$.

- **Page 11, proof of Lemma 3.2:** In the caption of Figure 1, replace "$m(\pi_{21}) = 5432,\ m(\pi_{12}) = 5,\ m(\pi_{33}) = 51$" by "$m(\pi_{21}) = u_5 u_4 u_3 u_2,\ m(\pi_{12}) = u_5,\ m(\pi_{33}) = u_5 u_1$".

- **Page 11, proof of Lemma 3.2:** Replace "the expression $e_{\lambda'_i - \mu'_j + j - i} (\mathbf{u})$" by "the expression $e_{\lambda'_j - \mu'_i + i - j} (\mathbf{u})$".

- **Page 12, proof of Lemma 3.2:** Here you write: "The corresponding two paths must have consecutive terminal points, say $Q_k$ and $Q_{k+1}$". This claim (and also the implicit claim that only two paths intersect in the lexicographically maximal intersection) is not completely obvious and, in my opinion, warrants some justification. I give detailed proofs of these claims in Subsection 9.2 below. (Namely, these two claims are parts **(a)** and **(b)** of Lemma 9.1.)

- **Page 12, proof of Lemma 3.2:** After "Then $m(\pi_{i,k})\, m(\pi_{j,k+1})$ may be factored as", I'd add a footnote saying that we are abusing notation to equate any string $i_1 i_2 \cdots i_k$ of elements of $\{1, 2, \ldots, n\}$ with the corresponding product $u_{i_1} u_{i_2} \cdots u_{i_k}$.

- **Page 12, proof of Lemma 3.2:** You write: "Here $p$ labels the vertical segment in $\pi_{i,k}$ that lies just above $(\alpha, \beta)$". This relies on the fact that there is a vertical segment in $\pi_{i,k}$ that lies just above $(\alpha, \beta)$ (in other words, the path $\pi_{i,k}$ exits $(\alpha, \beta)$ in the northern direction). Again, this fact is not completely obvious; I shall prove it in Subsection 9.2 below. (Namely, this fact is Lemma 9.1 **(c)**.)

- **Page 12, proof of Lemma 3.2:** I think "$A = a_1 a_2 \ldots$ and $B = b_1 b_2 \ldots$" should be replaced by "$A = \ldots a_2 a_1$ and $B = \ldots b_2 b_1$". (It would also be good to mention that the "$\ldots$" symbols here do not signify infinite strings, but merely finite strings whose lengths are immaterial.) You should also mention that $\ell(A) \geq \ell(B)$, where $\ell(C)$ denotes the length of a string $C$.

- **Page 12, proof of Lemma 3.3:** I don't understand what you mean by "jeu-de-taquin transformations" here. I give a completely elementary proof of Lemma 3.3 in Subsection 9.3 below.

- **Page 13, proof of Lemma 3.3:** "by all configurations" $\to$ "by all families of paths".

- **Page 13, proof of Lemma 3.3:** "not involving $P_j, P_k, Q_k, Q_{k+1}$" $\to$ "not involving $P_i, P_j, Q_k, Q_{k+1}$".

- **Page 13, proof of Lemma 3.3:** You write: "Hence we have constructed a sign-reversing involution which leaves only terms of (3.3) corresponding to non-intersecting families of paths". This is not correct as stated. What you have constructed is not an involution on the set of all families $(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N})$ of paths. The problem is that each term

$$W_1 A pe_r(u_1, \ldots, u_{\alpha+\beta})\, Be_s(u_1, \ldots, u_{\alpha+\beta})\, W_2$$

is not a single monomial corresponding to a single family of paths, but rather a sum of such monomials for a certain set of families of paths (namely, for the set of all families that agree with the original family everywhere except on the initial segments of $\pi_{i,k}$ and $\pi_{j,k+1}$). By showing that it equals

$$W_1 Ape_s \left(u_1, \ldots, u_{\alpha+\beta}\right) Be_r \left(u_1, \ldots, u_{\alpha+\beta}\right) W_2,$$

you can conclude that the contributions of all these families to the determinant in (3.3) cancel against the contributions of another such set of families. This way, all terms in (3.3) contributed by intersecting families of paths are cancelled, whereas the terms contributed by non-intersecting families of paths are preserved. This shows that the determinant in (3.3) reduces to the sum of the contributions of non-intersecting families; however, this does not construct a sign-reversing involution on the single monomials in (3.3). (Fortunately, you don't need such an involution.)

- **Page 13, proof of Lemma 3.3:** After "Now the argument continues exactly as in the commutative case.", add "Every family $\left(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}\right)$ of non-intersecting paths $\pi_{ij_i}$ must have $(j_1, j_2, \ldots, j_N) = (1, 2, \ldots, N)$." See Subsection 9.4 below for the proof of this statement. (Namely, this statement is Lemma 9.5.)

- **Page 13, proof of Lemma 3.3:** After "and hence of Theorem 1.1" (in the last sentence of the proof), I would add "(since the (unique) algebra homomorphism from $\Lambda_n$ to $A$ that sends $e_1, e_2, e_3, \ldots$ to $e_1(\mathbf{u}), e_2(\mathbf{u}), e_3(\mathbf{u}), \ldots$ will send each $e_i$ (for $i \in \mathbb{Z}$) to the corresponding $e_i(\mathbf{u})$ (because $e_0(\mathbf{u}) = 1$ and $e_i(\mathbf{u}) = 0$ for all $i < 0$), and therefore will send each $s_{\lambda/\mu} = \det\left(e_{\lambda'_i - \mu'_j + j - i}\right)$ to $\det\left(e_{\lambda'_i - \mu'_j + j - i}(\mathbf{u})\right) = \mathfrak{J}_{\lambda/\mu}(\mathbf{u}) = s_{\lambda/\mu}(\mathbf{u})$ (by Lemma 3.2))".

- **Page 15, proof of Theorem 1.3:** After "the complete noncommutative analog of the Frobenius formula (5.2)", I would add ", namely the formula

$$p_\alpha(\mathbf{u}) = \operatorname{sgn}(\alpha) \sum_\lambda \chi_\lambda(\alpha) s_{\lambda'}(\mathbf{u})$$

" (just for the sake of clarity).

- **Page 18, Section 6:** "a linear operator in $V \otimes V$" → "a linear operator from $V \otimes V$ to $V \otimes V$".

- **Page 18, Section 6:** "Now let $u_i$ act in" → "Now let $u_i$ act on".

- **Page 18, Section 6:** "by $u_i = I^{(i-1)} \otimes u \otimes I^{(n-i)}$" → "by $u_i = I^{\otimes(i-1)} \otimes u \otimes I^{\otimes(n-i)}$".

- **Page 18, Section 6:** "an identity operator" → "the identity operator".

- **Page 18, Example 6.1:** After "and $\epsilon$ is a *right* unit of $\mathcal{M}$", add "(that is, $\epsilon \in \mathcal{M}$ satisfies $\alpha\epsilon = \alpha$ for each $\alpha \in \mathcal{M}$)". (The notion of a right unit used here is not very well-known.)

- **Page 18, Example 6.1:** After "If $\epsilon$ is a *left* unit of $\mathcal{M}$", add "(that is, $\epsilon \in \mathcal{M}$ satisfies $\epsilon\alpha = \alpha$ for each $\alpha \in \mathcal{M}$)".

- **Page 19, Lemma 6.6:** I think you want to assume that the operations $\wedge$ and $\vee$ are associative here. (At least, I need this assumption to prove the sufficiency.)

- **Page 20, Example 6.7:** I don't think $u$ is well-defined as stated: the right-hand side $p \cdot \alpha \otimes \beta + (1-p) \cdot \alpha \otimes \alpha$ is not bilinear in $(\alpha, \beta)$, so the equation does not yield a well-defined linear map from the tensor product $V \otimes V$. (But you can salvage this definition by requiring the equation to hold only in the case when $\alpha$ and $\beta$ belong to a fixed basis of $V$.)

# 9. Remarks and addenda

## 9.1. Another proof of the symmetry of $F_{h/g}$

On page 4, you say that "Note that $F_{h/g}$ is a function in the commuting variables $x_i$ alone; we shall later demonstrate that this is indeed a symmetric function, in the ordinary sense". It is, in fact, easy to derive the symmetry of $F_{h/g}$ from Theorem 1.1 (or, even better, from Lemma 3.1):

*Proof of the fact that $F_{h/g}$ is symmetric in the $x_i$:* The relations (1.2) and (1.3) hold. Therefore, the conditions of Lemma 3.1 are satisfied (in fact, the relations (3.2) follow immediately from (1.2), whereas the relations (3.2) follow from (1.2) when $j - i > 1$ and from (1.3) when $j - i = 1$). Hence, Lemma 3.1 yields that the elements $e_k(u_1, \ldots, u_n)$ for $k \in \mathbb{N}$ commute. These elements clearly commute with all of the $x_1, x_2, \ldots, x_m$. Thus, all of the elements $e_k(u_1, \ldots, u_n)$ and the elements $x_1, x_2, \ldots, x_m$ commute with each other. Therefore, the subalgebra of $A$ generated by the elements $e_k(u_1, \ldots, u_n)$ and the elements $x_1, x_2, \ldots, x_m$ is commutative[1]. Let $B$ denote this subalgebra.

Let $Z(A)$ denote the center of the ring $A$. For every $t \in Z(A)$, we have

$$\prod_{j=n}^{1} (1 + tu_j) = (1 + tu_n)(1 + tu_{n-1}) \cdots (1 + tu_1) = \sum_{k=0}^{n} e_k(u_1, \ldots, u_n) t^n. \qquad (1)$$

[1]Here, $A$ is supposed to be an algebra that contains $u_1, u_2, \ldots, u_n$ and $x_1, x_2, \ldots, x_m$.

(In fact, this is a noncommutative analogue of the classical formula that $\prod\limits_{j=1}^{n}\left(1+t\alpha_j\right) = \sum\limits_{k=0}^{n} e_k\left(\alpha_1,\ldots,\alpha_n\right)t^n$ for any elements $\alpha_1,\alpha_2,\ldots,\alpha_n,t$ of a commutative algebra; and it can be proven by induction over $n$ in the same way.) Now, for every $i\in\{1,2,\ldots,m\}$, we have

$$\prod_{j=n}^{1}\left(1+x_iu_j\right) = \sum_{k=0}^{n}\underbrace{e_k\left(u_1,\ldots,u_n\right)}_{\in B}\underbrace{x_i^n}_{\substack{\in B\\ (\text{since } x_i\in B)}} \qquad \text{(by (1), applied to } t=x_i\text{)}$$
$$\in \sum_{k=0}^{n} BB \subseteq B.$$

Thus, $\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)$ is a product of elements of $B$. Since $B$ is commutative, this shows that the order of these elements is immaterial – that is, we can reorder the entries of the (outer) product $\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)$ without changing its value. But reordering the entries of the (outer) product $\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)$ has the same effect as permuting the variables $x_1,x_2,\ldots,x_m$. Therefore, we conclude that permuting the variables $x_1,x_2,\ldots,x_m$ does not change the value of $\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)$. Therefore, permuting the variables $x_1,x_2,\ldots,x_m$ does not change the value of $\left\langle\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)g,h\right\rangle$. In other words, $\left\langle\prod\limits_{i=1}^{m}\prod\limits_{j=n}^{1}\left(1+x_iu_j\right)g,h\right\rangle$ is symmetric in the $x_1,x_2,\ldots,x_m$. Qed. □

## 9.2. Some details for the proof of Lemma 3.2

In this subsection, we shall give detailed proofs of three statements left unproved in the proof of Lemma 3.2.

The notations introduced during the proof of Lemma 3.2 shall be used throughout Subsection 9.2. We shall prove the claim "The corresponding two paths must have consecutive terminal points, say $Q_k$ and $Q_{k+1}$" made on page 12, as well as the implicit claim that only two paths intersect in the lexicographically maximal intersection. We shall also prove that the path that ends in $Q_k$ has a vertical segment that lies just above $(\alpha,\beta)$.

In the following, an "intersection point" will mean a point in which (at least) two of the paths $\pi_{1j_1},\pi_{2j_2},\ldots,\pi_{Nj_N}$ intersect. We shall now prove the following:

**Lemma 9.1.** Let $X$ denote the intersection point $(\alpha, \beta)$ with the pair $(\alpha + \beta, \alpha - \beta)$ lexicographically maximal. Then:

**(a)** Exactly two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$.

**(b)** These two paths must end at two points of the form $Q_k$ and $Q_{k+1}$ for some $k \in \{1, 2, \ldots, N-1\}$.

**(c)** The path (among these two paths) that ends in $Q_k$ contains the vertical segment $(\alpha, \beta) \mapsto (\alpha, \beta + 1)$, where $(\alpha, \beta) = X$.

Before we prove Lemma 9.1, let us introduce a notation. The *depth* of a lattice point $(\gamma, \delta) \in \mathbb{Z}^2$ will mean the sum $\gamma + \delta$; it will be denoted by $\mathbf{d}((\gamma, \delta))$. The *tilt* of a lattice point $(\gamma, \delta) \in \mathbb{Z}^2$ will mean the difference $\gamma - \delta$; it will be denoted by $\mathbf{t}((\gamma, \delta))$. Notice that the pair $(\mathbf{d}(T), \mathbf{t}(T))$ uniquely determines a point $T \in \mathbb{Z}^2$. In other words, if $T_1$ and $T_2$ are two points in $\mathbb{Z}^2$ such that $(\mathbf{d}(T_1), \mathbf{t}(T_1)) = (\mathbf{d}(T_2), \mathbf{t}(T_2))$, then

$$T_1 = T_2. \tag{2}$$

Notice also that every $T \in \mathbb{Z}^2$ satisfies

$$\mathbf{t}(T) \equiv \mathbf{d}(T) \bmod 2. \tag{3}$$

[2]

We have

$$\mathbf{d}(P_i) = 0 \qquad \text{for every } i \in \{1, 2, \ldots, N\}. \tag{4}$$

[3] Also,

$$\mathbf{d}(Q_j) = n \qquad \text{for every } j \in \{1, 2, \ldots, N\}. \tag{5}$$

[4] Now, let us consider the directed graph $G$ (which was defined above in my first comment on "page 11, proof of Lemma 3.2"). If $(U, V)$ is an arc of the directed graph $G$, then

$$\mathbf{d}(V) = \mathbf{d}(U) + 1 \tag{6}$$

[2] *Proof of (3):* Let $T \in \mathbb{Z}^2$. Let us write $T$ in the form $(\gamma, \delta)$ for some integers $\gamma$ and $\delta$. Then, $\mathbf{t}\left(\underbrace{T}_{=(\gamma,\delta)}\right) = \mathbf{t}((\gamma, \delta)) = \gamma - \delta$ (by the definition of $\mathbf{t}((\gamma, \delta))$) and $\mathbf{d}\left(\underbrace{T}_{=(\gamma,\delta)}\right) = \mathbf{d}((\gamma, \delta)) = \gamma + \delta$ (by the definition of $\mathbf{d}((\gamma, \delta))$). Now, $\mathbf{t}(T) = \gamma - \delta = \gamma + \underbrace{(-1)}_{\equiv 1 \bmod 2} \delta \equiv \gamma + \delta = \mathbf{d}(T) \bmod 2$. This proves (3).

[3] *Proof of (4):* Let $i \in \{1, 2, \ldots, N\}$. Then, the definition of $P_i$ yields $P_i = \left((i-1) - \mu_i', -(i-1) + \mu_i'\right)$, so that

$$\begin{aligned} \mathbf{d}(P_i) &= \mathbf{d}\left(\left((i-1) - \mu_i', -(i-1) + \mu_i'\right)\right) = \left((i-1) - \mu_i'\right) + \left(-(i-1) + \mu_i'\right) \\ &\qquad \left(\text{by the definition of } \mathbf{d}\left(\left((i-1) - \mu_i', -(i-1) + \mu_i'\right)\right)\right) \\ &= 0. \end{aligned}$$

This proves (4).

[4] *Proof of (5):* Let $j \in \{1, 2, \ldots, N\}$. Then, the definition of $Q_j$ yields $Q_j =$

[5] and

$$\mathbf{t}(U) - 1 \leq \mathbf{t}(V) \leq \mathbf{t}(U) + 1 \tag{7}$$

[6]. For every $i \in \{1, 2, \ldots, N\}$ and every $k \in \{0, 1, \ldots, n\}$,

$$\text{there exists exactly one point } F \text{ on the path } \pi_{ij_i} \text{ such that } \mathbf{d}(F) = k. \tag{8}$$

[7] We shall denote this point by $F_{i,k}$. Thus, for every $i \in \{1, 2, \ldots, N\}$ and every

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$\left(n + (j-1) - \lambda'_j, -(j-1) + \lambda'_j\right)$, so that

$$\begin{aligned}\mathbf{d}(Q_j) &= \mathbf{d}\left(\left(n + (j-1) - \lambda'_j, -(j-1) + \lambda'_j\right)\right) = \left(n + (j-1) - \lambda'_j\right) + \left(-(j-1) + \lambda'_j\right) \\ &\qquad \left(\text{by the definition of } \mathbf{d}\left(\left(n + (j-1) - \lambda'_j, -(j-1) + \lambda'_j\right)\right)\right) \\ &= n.\end{aligned}$$

This proves (5).

[5] *Proof of (6):* Let $(U, V)$ be an arc of the directed graph $G$. Write the point $U$ in the form $U = (\gamma, \delta)$. Then, $V$ is either $(\gamma + 1, \delta)$ or $(\gamma, \delta + 1)$ (since $(U, V)$ is an arc of $G$). In either of these two cases, we have $\mathbf{d}(V) = \mathbf{d}(U) + 1$. Hence, (6) is proven.

[6] *Proof of (7):* Let $(U, V)$ be an arc of the directed graph $G$. Write the point $U$ in the form $U = (\gamma, \delta)$. Then, $V$ is either $(\gamma + 1, \delta)$ or $(\gamma, \delta + 1)$ (since $(U, V)$ is an arc of $G$). We have $\mathbf{t}(V) = \mathbf{t}(U) + 1$ in the first of these two cases, and we have $\mathbf{t}(V) = \mathbf{t}(U) - 1$ in the second of these two cases. Thus, we have $\mathbf{t}(U) - 1 \leq \mathbf{t}(V) \leq \mathbf{t}(U) + 1$ in either case. Hence, (7) is proven.

[7] *Proof of (8):* Let $i \in \{1, 2, \ldots, N\}$. Write the path $\pi_{ij_i}$ in the form $(F_0, F_1, \ldots, F_\ell)$. Since the path $\pi_{ij_i}$ begins at $P_i$ and ends at $Q_{j_i}$, we must have $F_0 = P_i$ and $F_\ell = Q_{j_i}$. Moreover, for every $p \in \{0, 1, \ldots, \ell - 1\}$, the pair $\left(F_p, F_{p+1}\right)$ is an arc of the directed graph $G$ (since $(F_0, F_1, \ldots, F_\ell)$ is a path in $G$) and thus satisfies $\mathbf{d}\left(F_{p+1}\right) = \mathbf{d}\left(F_p\right) + 1$ (by (6), applied to $U = F_p$ and $V = F_{p+1}$).

Thus, we know that $\mathbf{d}\left(F_{p+1}\right) = \mathbf{d}\left(F_p\right) + 1$ for every $p \in \{0, 1, \ldots, \ell - 1\}$. In other words, $(\mathbf{d}(F_0), \mathbf{d}(F_1), \ldots, \mathbf{d}(F_\ell))$ is an arithmetic sequence with difference 1 (that is, $\mathbf{d}\left(F_p\right)$ grows by 1 every time we increase $p$ by 1). Thus,

$$\mathbf{d}\left(F_p\right) = \mathbf{d}(F_0) + p \qquad \text{for every } p \in \{0, 1, \ldots, \ell\}. \tag{9}$$

Since $\mathbf{d}\left(\underbrace{F_0}_{=P_i}\right) = \mathbf{d}(P_i) = 0$ (by (4)), this shows that

$$\mathbf{d}\left(F_p\right) = \underbrace{\mathbf{d}(F_0)}_{=0} + p = p \qquad \text{for every } p \in \{0, 1, \ldots, \ell\}. \tag{10}$$

Applying this to $p = \ell$, we obtain $\mathbf{d}(F_\ell) = \ell$. Thus, $\ell = \mathbf{d}\left(\underbrace{F_\ell}_{=Q_{j_i}}\right) = \mathbf{d}\left(Q_{j_i}\right) = n$ (by (5), applied to $j = j_i$).

Now, let $k \in \{0, 1, \ldots, n\}$. Thus, $k \in \{0, 1, \ldots, \ell\}$ (since $\ell = n$). Applying (10) to $p = k$, we obtain $\mathbf{d}(F_k) = k$. Hence, $F_k$ is a point on the path $\pi_{ij_i}$ (since $\pi_{ij_i} = (F_0, F_1, \ldots, F_\ell)$) satisfying $\mathbf{d}(F_k) = k$. In other words, $F_k$ is a point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$.

$k \in \{0, 1, \ldots, n\}$, we have

$$\mathbf{d}(F_{i,k}) = k. \qquad (11)$$

[8] Notice that for every $i \in \{1, 2, \ldots, N\}$, every point $U$ on the path $\pi_{ij_i}$ satisfies

$$\mathbf{d}(U) \in \{0, 1, \ldots, n\}. \qquad (12)$$

[9].

We have

$$F_{i,n} = Q_{j_i} \qquad \text{for every } i \in \{1, 2, \ldots, N\}. \qquad (14)$$

[10] Also, for every $i \in \{1, 2, \ldots, N\}$ and every $k \in \{0, 1, \ldots, n-1\}$,

$$\text{the pair } (F_{i,k}, F_{i,k+1}) \text{ is an arc of the path } \pi_{ij_i}. \qquad (15)$$

[11] Therefore, for every $i \in \{1, 2, \ldots, N\}$ and every $k \in \{0, 1, \ldots, n-1\}$,

$$\text{the pair } (F_{i,k}, F_{i,k+1}) \text{ is an arc of the directed graph } G. \qquad (16)$$

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On the other hand, let $F$ be any point on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$. Then, $F = F_p$ for some $p \in \{0, 1, \ldots, \ell\}$ (since $F$ is a point on the path $\pi_{ij_i} = (F_0, F_1, \ldots, F_\ell)$). Consider this $p$. Then, $k = \mathbf{d}\left(\underbrace{F}_{=F_p}\right) = \mathbf{d}(F_p) = p$ (by (10)), so that $F_k = F_p$ and thus $F = F_p = F_k$. Now, let us forget that we fixed $F$. We thus have shown that if $F$ is any point on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$, then $F = F_k$. Hence, there exists at most one point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$. Since there exists at least one such point (namely, $F_k$), we can thus conclude that there exists exactly one point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$. This proves (8).

[8] *Proof of (11):* Let $i \in \{1, 2, \ldots, N\}$ and $k \in \{0, 1, \ldots, n\}$. Then, $F_{i,k}$ is defined as the unique point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$. Hence, $F_{i,k}$ is a point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$. In other words, $F_{i,k}$ belongs to the path $\pi_{ij_i}$ and satisfies $\mathbf{d}(F_{i,k}) = k$. This proves (11).

[9] *Proof of (12):* Let $i \in \{1, 2, \ldots, N\}$. Let $U$ be a point on the path $\pi_{ij_i}$. Write the path $\pi_{ij_i}$ in the form $(F_0, F_1, \ldots, F_\ell)$. We can prove that

$$\mathbf{d}(F_p) = p \qquad \text{for every } p \in \{0, 1, \ldots, \ell\}. \qquad (13)$$

(This can be proven just as we proved (10) in our proof of (8).) Also, $\ell = n$. (Again, this can be proven just as in our proof of (8).)

But recall that $\pi_{ij_i} = (F_0, F_1, \ldots, F_\ell)$. Hence, $U$ is a point on the path $(F_0, F_1, \ldots, F_\ell)$ (since $U$ is a point on the path $\pi_{ij_i}$). In other words, there exists a $p \in \{0, 1, \ldots, \ell\}$ such that $U = F_p$. Consider this $p$. We have $p \in \{0, 1, \ldots, \ell\} = \{0, 1, \ldots, n\}$ (since $\ell = n$). Now, (13) yields $\mathbf{d}(F_p) = p$. So $\mathbf{d}\left(\underbrace{U}_{=F_p}\right) = \mathbf{d}(F_p) = p \in \{0, 1, \ldots, n\}$. This proves (12).

[10] *Proof of (14):* Let $i \in \{1, 2, \ldots, N\}$. Then, $Q_{j_i}$ is a point on the path $\pi_{ij_i}$ (since $\pi_{ij_i}$ is a path from $P_i$ to $Q_{j_i}$) and satisfies $\mathbf{d}(Q_{j_i}) = n$ (by (5), applied to $j = j_i$). Hence, $Q_{j_i}$ is a point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = n$. But we know (from (8), applied to $k = n$) that there exists exactly one such point; and we have denoted this point by $F_{i,n}$. Therefore, $Q_{j_i} = F_{i,n}$. This proves (14).

[11] *Proof of (15):* Let $i \in \{1, 2, \ldots, N\}$. Let $k \in \{0, 1, \ldots, n-1\}$. Then, both $F_{i,k}$ and $F_{i,k+1}$ are well-defined points on the path $\pi_{ij_i}$ (by the definition of $F_{i,k}$ and $F_{i,k+1}$). From $k \in \{0, 1, \ldots, n-1\}$, we obtain $k + 1 \in \{1, 2, \ldots, n\} \subseteq \{0, 1, \ldots, n\}$.

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We have

$$\mathbf{t}(Q_1) < \mathbf{t}(Q_2) < \cdots < \mathbf{t}(Q_N). \tag{17}$$

[13] Thus, the numbers $\mathbf{t}(Q_1), \mathbf{t}(Q_2), \ldots, \mathbf{t}(Q_N)$ are pairwise distinct. Hence, the points $Q_1, Q_2, \ldots, Q_N$ are pairwise distinct.

Let us now come to the proof of Lemma 9.1.

*Proof of Lemma 9.1.* Let us first observe that

$$\text{every intersection point } Y \text{ satisfies } \mathbf{d}(Y) \leq \mathbf{d}(X). \tag{18}$$

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We know that $F_{i,k}$ is the unique point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k$ (according to the definition of $F_{i,k}$). Thus, $F_{i,k}$ is a point on the path $\pi_{ij_i}$ and satisfies $\mathbf{d}(F_{i,k}) = k$. We have $\mathbf{d}(F_{i,k}) = k \neq n = \mathbf{d}(Q_{j_i})$ (because $\mathbf{d}(Q_{j_i}) = n$ (by (5), applied to $j = j_i$)), and thus $F_{i,k} \neq Q_{j_i}$. Therefore, $F_{i,k}$ is not the terminus of the path $\pi_{ij_i}$ (since the terminus of the path $\pi_{ij_i}$ is $Q_{j_i}$). Hence, the next point after $F_{i,k}$ on the path $\pi_{ij_i}$ is well-defined. Let $F'$ denote this next point.

Clearly, $(F_{i,k}, F')$ is an arc of the path $\pi_{ij_i}$ (since $F'$ is the next point after $F_{i,k}$ on the path $\pi_{ij_i}$). Thus, $(F_{i,k}, F')$ is an arc of the directed graph $G$ (since $\pi_{ij_i}$ is a path on this directed graph). Hence, (6) (applied to $U = F_{i,k}$ and $V = F'$) yields $\mathbf{d}(F') = \underbrace{\mathbf{d}(F_{i,k})}_{=k} + 1 = k + 1$. Also, $F'$ lies on $\pi_{ij_i}$. Hence, $F'$ is a point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = k + 1$. But we know that there exists exactly one such point $F$ (according to (8), applied to $k+1$ instead of $k$), and we have denoted this point $F$ by $F_{i,k+1}$. Hence, $F'$ is $F_{i,k+1}$. Thus, $(F_{i,k}, F_{i,k+1})$ is an arc of the path $\pi_{ij_i}$ (since $(F_{i,k}, F')$ is an arc of the path $\pi_{ij_i}$). This proves (15).

[12] *Proof of (16):* Let $i \in \{1, 2, \ldots, N\}$. Let $k \in \{0, 1, \ldots, n-1\}$. Then, (15) shows that the pair $(F_{i,k}, F_{i,k+1})$ is an arc of the path $\pi_{ij_i}$. Thus, the pair $(F_{i,k}, F_{i,k+1})$ is an arc of the directed graph $G$ (since $\pi_{ij_i}$ is a path on this directed graph). This proves (16).

[13] *Proof of (17):* Let $j \in \{1, 2, \ldots, N-1\}$. We will now show that $\mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$.

The definition of $Q_j$ yields $Q_j = \left(n + (j-1) - \lambda'_j, -(j-1) + \lambda'_j\right)$, so that $\mathbf{t}(Q_j) = \mathbf{t}\left(\left(n + (j-1) - \lambda'_j, -(j-1) + \lambda'_j\right)\right) = \left(n + (j-1) - \lambda'_j\right) - \left(-(j-1) + \lambda'_j\right) = n + 2(j-1) - 2\lambda'_j$. The same argument (applied to $j+1$ instead of $j$) yields $\mathbf{t}(Q_{j+1}) = n + 2((j+1)-1) - 2\lambda'_{j+1}$. But $\lambda'$ is a partition, and thus we have $\lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq \cdots$. Hence, $\lambda'_j \geq \lambda'_{j+1}$. Now,

$$\mathbf{t}(Q_j) = n + 2\left(\underbrace{j}_{<j+1} - 1\right) - 2\underbrace{\lambda'_j}_{\geq \lambda'_{j+1}} < n + 2((j+1)-1) - 2\lambda'_{j+1} = \mathbf{t}(Q_{j+1}).$$

Let us now forget that we fixed $j$. We thus have proven that $\mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$ for every $j \in \{1, 2, \ldots, N-1\}$. In other words, $\mathbf{t}(Q_1) < \mathbf{t}(Q_2) < \cdots < \mathbf{t}(Q_N)$. This proves (17).

[14] *Proof of (18):* We know (from the definition of $X$) that $X$ is the intersection point $(\alpha, \beta)$ with the pair $(\alpha + \beta, \alpha - \beta)$ lexicographically maximal. In other words, $X$ is the intersection point $(\alpha, \beta)$ with the pair $(\mathbf{d}((\alpha, \beta)), \mathbf{t}((\alpha, \beta)))$ lexicographically maximal (because every point $(\alpha, \beta) \in \mathbb{Z}^2$ satisfies $\mathbf{d}((\alpha, \beta)) = \alpha + \beta$ and $\mathbf{t}((\alpha, \beta)) = \alpha - \beta$). Hence, every intersection point $Y$ satisfies $(\mathbf{d}(X), \mathbf{t}(X)) \geq (\mathbf{d}(Y), \mathbf{t}(Y))$ in lexicographic order. Thus, every intersection point $Y$ satisfies $\mathbf{d}(X) \geq \mathbf{d}(Y)$. This proves (18).

If $k \in \{0, 1, \ldots, n\}$ is such that $k > \mathbf{d}(X)$, then any two distinct elements $\mathfrak{u}$ and $\mathfrak{v}$ of $\{1, 2, \ldots, N\}$ satisfy

$$\mathbf{t}(F_{\mathfrak{u},k}) \neq \mathbf{t}(F_{\mathfrak{v},k}). \qquad (19)$$

[15] Moreover, if $k \in \{0, 1, \ldots, n\}$ and if $\mathfrak{u}$ and $\mathfrak{v}$ are two elements of $\{1, 2, \ldots, N\}$, then

$$\mathbf{t}(F_{\mathfrak{v},k}) \neq \mathbf{t}(F_{\mathfrak{u},k}) + 1. \qquad (20)$$

[16]

But recall that $(j_1, j_2, \ldots, j_N)$ is a permutation of $\{1, 2, \ldots, N\}$. Denote this permutation by $\mathbf{j}$. Thus,

$$\mathbf{j}(i) = j_i \qquad \text{for every } i \in \{1, 2, \ldots, N\}. \qquad (21)$$

Now, for every $j \in \{1, 2, \ldots, N-1\}$, we have

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right) \qquad \text{for every } k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}. \quad (22)$$

[17] Consequently, if $k \in \{0, 1, \ldots, n\}$ is such that $k > \mathbf{d}(X)$, then

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),k}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),k}\right). \qquad (23)$$

[15] *Proof of (19):* Let $k \in \{0, 1, \ldots, n\}$ be such that $k > \mathbf{d}(X)$. Let $\mathfrak{u}$ and $\mathfrak{v}$ be two distinct elements of $\{1, 2, \ldots, N\}$. We need to prove (19).

Assume the contrary. Thus, (19) does not hold. In other words, $\mathbf{t}(F_{\mathfrak{u},k}) = \mathbf{t}(F_{\mathfrak{v},k})$. But $\mathbf{d}(F_{\mathfrak{u},k}) = k$ (by (11), applied to $i = \mathfrak{u}$) and $\mathbf{d}(F_{\mathfrak{v},k}) = k$ (by (11), applied to $i = \mathfrak{v}$). Hence, $\mathbf{d}(F_{\mathfrak{u},k}) = k = \mathbf{d}(F_{\mathfrak{v},k})$. Altogether, $\left(\underbrace{\mathbf{d}(F_{\mathfrak{u},k})}_{=\mathbf{d}(F_{\mathfrak{v},k})}, \underbrace{\mathbf{t}(F_{\mathfrak{u},k})}_{=\mathbf{t}(F_{\mathfrak{v},k})}\right) = (\mathbf{d}(F_{\mathfrak{v},k}), \mathbf{t}(F_{\mathfrak{v},k}))$. Thus, (2) (applied to $T_1 = F_{\mathfrak{u},k}$ and $T_2 = F_{\mathfrak{v},k}$) yields $F_{\mathfrak{u},k} = F_{\mathfrak{v},k}$.

But $F_{\mathfrak{u},k}$ is the unique point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = k$ (according to the definition of $F_{\mathfrak{u},k}$). Hence, $F_{\mathfrak{u},k}$ is a point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = k$. In particular, $F_{\mathfrak{u},k}$ is a point on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$. Similarly, $F_{\mathfrak{v},k}$ is a point on the path $\pi_{\mathfrak{v}j_\mathfrak{v}}$. In other words, $F_{\mathfrak{u},k}$ is a point on the path $\pi_{\mathfrak{v}j_\mathfrak{v}}$ (since $F_{\mathfrak{u},k} = F_{\mathfrak{v},k}$). The point $F_{\mathfrak{u},k}$ thus lies on both paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$. Hence, $F_{\mathfrak{u},k}$ is a point in which (at least) two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect (namely, the paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$). In other words, $F_{\mathfrak{u},k}$ is an intersection (since $\mathfrak{u}$ and $\mathfrak{v}$ are distinct). Hence, (18) (applied to $Y = F_{\mathfrak{u},k}$) yields $\mathbf{d}(F_{\mathfrak{u},k}) \leq \mathbf{d}(X)$. But this contradicts $\mathbf{d}(F_{\mathfrak{u},k}) = k > \mathbf{d}(X)$. This contradiction proves that our assumption was wrong. Thus, (19) is proven.

[16] *Proof of (20):* Let $k \in \{0, 1, \ldots, n\}$, and let $\mathfrak{u}$ and $\mathfrak{v}$ be two elements of $\{1, 2, \ldots, N\}$. We need to prove that (20) holds.

Assume the contrary. Thus, (20) does not hold. Thus, we have $\mathbf{t}(F_{\mathfrak{v},k}) = \mathbf{t}(F_{\mathfrak{u},k}) + 1$. But (3) (applied to $T = F_{\mathfrak{v},k}$) yields $\mathbf{t}(F_{\mathfrak{v},k}) \equiv \mathbf{d}(F_{\mathfrak{v},k}) \bmod 2$. Hence, $\mathbf{d}(F_{\mathfrak{v},k}) \equiv \mathbf{t}(F_{\mathfrak{v},k}) = \mathbf{t}(F_{\mathfrak{u},k}) + 1 \bmod 2$. But $\mathbf{d}(F_{\mathfrak{v},k}) = k$ (by (11), applied to $i = \mathfrak{v}$) and thus $k = \mathbf{d}(F_{\mathfrak{v},k}) \equiv \mathbf{t}(F_{\mathfrak{u},k}) + 1 \bmod 2$. However, (3) (applied to $T = F_{\mathfrak{u},k}$) yields $\mathbf{t}(F_{\mathfrak{u},k}) \equiv \mathbf{d}(F_{\mathfrak{u},k}) \bmod 2$. Also, $\mathbf{d}(F_{\mathfrak{u},k}) = k$ (by (11), applied to $i = \mathfrak{u}$), so that $\mathbf{t}(F_{\mathfrak{u},k}) \equiv \mathbf{d}(F_{\mathfrak{u},k}) = k \bmod 2$. Thus, $k \equiv \underbrace{\mathbf{t}(F_{\mathfrak{u},k})}_{\equiv k \bmod 2} + 1 \bmod 2 \equiv k + 1 \bmod 2$. Subtracting $k$ from this congruence, we obtain $0 \equiv 1 \bmod 2$, which is absurd. Hence, we have obtained a contradiction. Thus, our assumption was wrong, and (20) is proven.

[17] *Proof of (22):* Let $j \in \{1, 2, \ldots, N-1\}$. Set $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$. We need to prove

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Next, we notice that

$$\mathbf{d}(X) \in \{0,1,\ldots,n-1\}. \tag{24}$$

(22).

Let us (for the sake of contradiction) assume the contrary. Thus, not every $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ satisfies $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. In other words, not every $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ satisfies $\mathbf{t}(F_{\mathfrak{u},k}) \leq \mathbf{t}(F_{\mathfrak{v},k})$ (because $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$). In other words, there exists a $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Let $\ell$ be the **highest** such $k$. Thus, $\ell$ is a $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Hence, $\ell$ belongs to $\{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ and satisfies $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}(F_{\mathfrak{v},\ell})$. We have $\mathbf{t}(F_{\mathfrak{u},\ell}) \geq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1$ (since $\mathbf{t}(F_{\mathfrak{u},\ell})$ and $\mathbf{t}(F_{\mathfrak{v},\ell})$ are integers and satisfy $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}(F_{\mathfrak{v},\ell})$). In other words, $\mathbf{t}(F_{\mathfrak{v},\ell}) + 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell})$.

Let us first assume (for the sake of contradiction) that $\ell = n$. Then, $F_{\mathfrak{u},\ell} = F_{\mathfrak{u},n} = Q_{j_\mathfrak{u}}$ (by (14), applied to $i = \mathfrak{u}$). But the definition of $\mathbf{j}(\mathfrak{u})$ yields $\mathbf{j}(\mathfrak{u}) = j_\mathfrak{u}$, so that $j_\mathfrak{u} = \mathbf{j}(\mathfrak{u}) = j$ (since $\mathfrak{u} = \mathbf{j}^{-1}(j)$) and thus $Q_{j_\mathfrak{u}} = Q_j$. Also, from $\ell = n$, we obtain $F_{\mathfrak{v},\ell} = F_{\mathfrak{v},n} = Q_{j_\mathfrak{v}}$ (by (14), applied to $i = \mathfrak{v}$). But the definition of $\mathbf{j}(\mathfrak{v})$ yields $\mathbf{j}(\mathfrak{v}) = j_\mathfrak{v}$, so that $j_\mathfrak{v} = \mathbf{j}(\mathfrak{v}) = j+1$ (since $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$) and thus $Q_{j_\mathfrak{v}} = Q_{j+1}$. But $\mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$ (by (17)). Now, $\mathbf{t}\left(\underbrace{F_{\mathfrak{u},\ell}}_{=Q_{j_\mathfrak{u}}=Q_j}\right) = \mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$ contradicts $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}\left(\underbrace{F_{\mathfrak{v},\ell}}_{=Q_{j_\mathfrak{v}}=Q_{j+1}}\right) = \mathbf{t}(Q_{j+1})$. This contradiction proves that our assumption (that $\ell = n$) was wrong. Hence, we cannot have $\ell = n$.

We have $\mathbf{d}(X) \in \{0,1,\ldots,n\}$ (by (12), applied to $U = X$), thus $0 \leq \mathbf{d}(X)$. We have $\ell \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$, thus $\mathbf{d}(X) \leq \ell \leq n$. Thus, $\ell < n$ (since $\ell \leq n$ but not $\ell = n$) and $0 \leq \mathbf{d}(X) \leq \ell$. Hence, $0 \leq \ell < n$, so that $\ell \in \{0,1,\ldots,n-1\}$ and thus $\ell + 1 \in \{1,2,\ldots,n\} \subseteq \{0,1,\ldots,n\}$, so that $\ell + 1 \leq n$. Moreover, $\ell + 1 \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ (since $\mathbf{d}(X) \leq \ell \leq \ell + 1$ and $\ell + 1 \leq n$).

But $\ell \in \{0,1,\ldots,n-1\}$. Thus, (16) (applied to $k = \mathfrak{u}$ and $i = \ell$) shows that the pair $(F_{\mathfrak{u},\ell}, F_{\mathfrak{u},\ell+1})$ is an arc of the directed graph $G$. Therefore, (7) (applied to $U = F_{\mathfrak{u},\ell}$ and $V = F_{\mathfrak{u},\ell+1}$) shows that $\mathbf{t}(F_{\mathfrak{u},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{u},\ell}) + 1$. The same argument (but with every $\mathfrak{u}$ replaced by $\mathfrak{v}$) yields $\mathbf{t}(F_{\mathfrak{v},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1$. Thus, $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell}) \leq \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ (since $\mathbf{t}(F_{\mathfrak{u},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell+1})$). Combined with $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ (which is a consequence of (20), applied to $\ell + 1$ instead of $k$), this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) < \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$. Since $\mathbf{t}(F_{\mathfrak{v},\ell+1})$ and $\mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ are integers, this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq (\mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1) - 1 = \mathbf{t}(F_{\mathfrak{u},\ell+1})$.

We have $\mathbf{d}(X) \leq \ell < \ell + 1$ and thus $\ell + 1 > \mathbf{d}(X)$. Hence, (19) (applied to $k = \ell + 1$) yields $\mathbf{t}(F_{\mathfrak{u},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{v},\ell+1})$. In other words, $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{u},\ell+1})$. Combined with $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{u},\ell+1})$, this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) < \mathbf{t}(F_{\mathfrak{u},\ell+1})$. In other words, $\mathbf{t}(F_{\mathfrak{u},\ell+1}) > \mathbf{t}(F_{\mathfrak{v},\ell+1})$. Hence, $\ell + 1$ is a $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Since the highest such $k$ is $\ell$ (by the definition of $\ell$), this yields that $\ell + 1 \leq \ell$. But this contradicts $\ell + 1 > \ell$. This contradiction shows that our assumption was wrong. Hence, (22) is proven.

[18]*Proof of (23):* Let $k \in \{0,1,\ldots,n\}$ be such that $k > \mathbf{d}(X)$. Let $j \in \{1,2,\ldots,N-1\}$. Since $k \geq \mathbf{d}(X)$ (because $k > \mathbf{d}(X)$) and $k \leq n$ (since $k \in \{0,1,\ldots,n\}$), we have $k \in \{\mathbf{d}(X), \mathbf{d}(X)+1, \ldots, n\}$. Hence, (22) yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. But $\mathbf{j}(\mathbf{j}^{-1}(j)) = j \neq j+1 = \mathbf{j}(\mathbf{j}^{-1}(j+1))$, so that $\mathbf{j}^{-1}(j) \neq \mathbf{j}^{-1}(j+1)$. In other words, the positive integers $\mathbf{j}^{-1}(j)$ and $\mathbf{j}^{-1}(j+1)$ are distinct. Thus, (19) (applied to $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$)

[19] Now, if $\mathfrak{u} \in \{1, 2, \ldots, N\}$ and $\mathfrak{v} \in \{1, 2, \ldots, N\}$ are such that $F_{\mathfrak{u}, \mathbf{d}(X)} = F_{\mathfrak{v}, \mathbf{d}(X)}$, then

$$\mathbf{j}(\mathfrak{v}) \leq \mathbf{j}(\mathfrak{u}) + 1. \qquad (25)$$

[20]

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yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \neq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. Combined with $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$, this yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$.

Now, let us forget that we fixed $j \in \{1, 2, \ldots, N-1\}$. Thus, we have shown that $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$ for every $j \in \{1, 2, \ldots, N-1\}$. In other words, $\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),k}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),k}\right)$. This proves (23).

[19]*Proof of (24):* The point $X$ is an intersection point (in fact, $X$ is the intersection point $(\alpha, \beta)$ with the pair $(\alpha + \beta, \alpha - \beta)$ lexicographically maximal). In other words, $X$ is a point in which (at least) two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect. In other words, at least two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. In other words, there exist two distinct elements $\mathfrak{u}$ and $\mathfrak{v}$ of $\{1, 2, \ldots, N\}$ such that the paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$ intersect at $X$. Consider these $\mathfrak{u}$ and $\mathfrak{v}$. We can WLOG assume that $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v})$ (since otherwise, we can just switch $\mathfrak{u}$ with $\mathfrak{v}$). Assume this. We have $\mathfrak{u} \neq \mathfrak{v}$ (since $\mathfrak{u}$ and $\mathfrak{v}$ are distinct), and thus $\mathbf{j}(\mathfrak{u}) \neq \mathbf{j}(\mathfrak{v})$ (since $\mathbf{j}$ is injective (since $\mathbf{j}$ is a permutation)). Thus, $\mathbf{j}(\mathfrak{u}) < \mathbf{j}(\mathfrak{v})$ (since $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v})$).

The point $X$ lies on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ (since the paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$ intersect at $X$). Hence, (12) (applied to $i = \mathfrak{u}$ and $U = X$) yields $\mathbf{d}(X) \in \{0, 1, \ldots, n\}$.

We now assume (for the sake of contradiction) that $\mathbf{d}(X) = n$.

Recall that there exists exactly one point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = n$ (according to (8), applied to $i = \mathfrak{u}$ and $k = n$); this point is denoted by $F_{\mathfrak{u},n}$. Thus, $F_{\mathfrak{u},n}$ is the unique point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = n$. Since $X$ is a point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = n$ (because $X$ lies on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and satisfies $\mathbf{d}(X) = n$), this yields that $X$ is $F_{\mathfrak{u},n}$. In other words, $X = F_{\mathfrak{u},n}$. But (14) (applied to $i = \mathfrak{u}$) yields $F_{\mathfrak{u},n} = Q_{j_\mathfrak{u}}$. But the definition of $\mathbf{j}$ yields $\mathbf{j}(\mathfrak{u}) = j_\mathfrak{u}$. Hence, $Q_{\mathbf{j}(\mathfrak{u})} = Q_{j_\mathfrak{u}}$, so that $X = F_{\mathfrak{u},n} = Q_{j_\mathfrak{u}} = Q_{\mathbf{j}(\mathfrak{u})}$. The same argument (but with $\mathfrak{u}$ replaced by $\mathfrak{v}$) yields $X = Q_{\mathbf{j}(\mathfrak{v})}$.

But (17) yields $\mathbf{t}(Q_1) < \mathbf{t}(Q_2) < \cdots < \mathbf{t}(Q_N)$. In other words, if $\mathfrak{a} \in \{1, 2, \ldots, N\}$ and $\mathfrak{b} \in \{1, 2, \ldots, N\}$ satisfy $\mathfrak{a} < \mathfrak{b}$, then $\mathbf{t}(Q_\mathfrak{a}) < \mathbf{t}(Q_\mathfrak{b})$. Applying this to $\mathfrak{a} = \mathbf{j}(\mathfrak{u})$ and $\mathfrak{b} = \mathbf{j}(\mathfrak{v})$, we obtain $\mathbf{t}\left(Q_{\mathbf{j}(\mathfrak{u})}\right) < \mathbf{t}\left(Q_{\mathbf{j}(\mathfrak{v})}\right)$ (since $\mathbf{j}(\mathfrak{u}) < \mathbf{j}(\mathfrak{v})$). But this contradicts $\mathbf{t}\left(\underbrace{Q_{\mathbf{j}(\mathfrak{u})}}_{=X=Q_{\mathbf{j}(\mathfrak{v})}}\right) = \mathbf{t}\left(Q_{\mathbf{j}(\mathfrak{v})}\right)$. This contradiction proves that our assumption (that $\mathbf{d}(X) = n$) was wrong. Hence, we cannot have $\mathbf{d}(X) = n$. In other words, we have $\mathbf{d}(X) \neq n$. Combined with $\mathbf{d}(X) \in \{0, 1, \ldots, n\}$, this yields $\mathbf{d}(X) \in \{0, 1, \ldots, n\} \setminus \{n\} = \{0, 1, \ldots, n-1\}$. This proves (24).

[20]*Proof of (25):* Let $\mathfrak{u} \in \{1, 2, \ldots, N\}$ and $\mathfrak{v} \in \{1, 2, \ldots, N\}$ be such that $F_{\mathfrak{u}, \mathbf{d}(X)} = F_{\mathfrak{v}, \mathbf{d}(X)}$. We need to prove that (25) holds.

In fact, let us assume the contrary. Then, (25) does not hold. In other words, $\mathbf{j}(\mathfrak{v}) > \mathbf{j}(\mathfrak{u}) + 1$. Hence, $\mathbf{j}(\mathfrak{v}) > \mathbf{j}(\mathfrak{u}) + 1 > \mathbf{j}(\mathfrak{u})$, so that $\mathbf{j}(\mathfrak{v}) \neq \mathbf{j}(\mathfrak{u})$ and thus $\mathfrak{v} \neq \mathfrak{u}$. Hence, $\mathfrak{u} \neq \mathfrak{v}$.

We have $\mathbf{j}(\mathfrak{u}) \in \{1, 2, \ldots, N\}$, thus $\mathbf{j}(\mathfrak{u}) \geq 1$. We have $\mathbf{j}(\mathfrak{v}) \in \{1, 2, \ldots, N\}$, thus $\mathbf{j}(\mathfrak{v}) \leq N$. Hence, $N \geq \mathbf{j}(\mathfrak{v}) > \mathbf{j}(\mathfrak{u}) + 1 > \mathbf{j}(\mathfrak{u})$, and thus $\mathbf{j}(\mathfrak{u}) < N$. Therefore, $\mathbf{j}(\mathfrak{u}) \in \{1, 2, \ldots, N-1\}$ (since $\mathbf{j}(\mathfrak{u}) \geq 1$).

$> \mathbf{j}(\mathfrak{u})$, we conclude that $\mathbf{j}(\mathfrak{u}) + 1 \in \{1, 2, \ldots, N\}$ (since both $\mathbf{j}(\mathfrak{v})$ and $\mathbf{j}(\mathfrak{u})$ belong to $\{1, 2, \ldots, N\}$). Hence, $\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u}) + 1)$ is a well-defined element of $\{1, 2, \ldots, N\}$. Denote this element by $\mathfrak{w}$. Thus, $\mathfrak{w} = \mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u}) + 1)$.

From (24), we obtain $\mathbf{d}(X) \in \{0, 1, \ldots, n-1\}$, thus $\mathbf{d}(X) + 1 \in \{1, 2, \ldots, n\} \subseteq \{0, 1, \ldots, n\}$. Let $k = \mathbf{d}(X) + 1$. Then, $k = \mathbf{d}(X) + 1 \in \{0, 1, \ldots, n\}$ and $k = \mathbf{d}(X) + 1 > \mathbf{d}(X)$. Thus, (23) We now proceed to the proofs of the three parts of Lemma 9.1:

**(a)** We know that $X$ is an intersection point. Hence, (at least) two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. Therefore, in order to prove Lemma 9.1 **(a)**, we only need to show that no more than two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. Let us prove this.

Indeed, let us assume the contrary (for the sake of contradiction). Thus, more than two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. Hence, at least three of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. In other words, at least three of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ pass through $X$. In other words, there exist three pairwise distinct elements $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ of $\{1, 2, \ldots, N\}$ such that the paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$, $\pi_{\mathfrak{v}j_\mathfrak{v}}$ and $\pi_{\mathfrak{w}j_\mathfrak{w}}$ pass through $X$. Consider these $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$. We can assume WLOG that $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v}) \leq \mathbf{j}(\mathfrak{w})$ (since otherwise, we can just permute $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$). Assume this. The map $\mathbf{j}$ is a permutation, and thus injective. Hence, $\mathbf{j}(\mathfrak{u}) \neq \mathbf{j}(\mathfrak{v})$

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yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),k}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),k}\right)$. In other words, any $\mathfrak{a} \in \{1, 2, \ldots, N\}$ and $\mathfrak{b} \in \{1, 2, \ldots, N\}$ satisfying $\mathfrak{a} < \mathfrak{b}$ must satisfy

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathfrak{a}),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathfrak{b}),k}\right). \tag{26}$$

We know (from (16), applied to $\mathfrak{u}$ and $\mathbf{d}(X)$ instead of $i$ and $k$) that the pair $\left(F_{\mathfrak{u},\mathbf{d}(X)}, F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$ is an arc of the directed graph $G$. Hence, (7) (applied to $(U, V) = \left(F_{\mathfrak{u},\mathbf{d}(X)}, F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$) yields $\mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) - 1 \leq \mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)+1}\right) \leq \mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) + 1$. In other words,

$$\mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) - 1 \leq \mathbf{t}\left(F_{\mathfrak{u},k}\right) \leq \mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) + 1$$

(since $k = \mathbf{d}(X) + 1$). The same argument (but with $\mathfrak{u}$ replaced by $\mathfrak{v}$) yields

$$\mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)}\right) - 1 \leq \mathbf{t}\left(F_{\mathfrak{v},k}\right) \leq \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)}\right) + 1.$$

Hence, $\mathbf{t}\left(F_{\mathfrak{v},k}\right) \leq \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)}\right) + 1$, so that $\mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)}\right) \geq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$, so that $\mathbf{t}\left(\underbrace{F_{\mathfrak{u},\mathbf{d}(X)}}_{=F_{\mathfrak{v},\mathbf{d}(X)}}\right) = \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)}\right) \geq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$. But from $\mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) - 1 \leq \mathbf{t}\left(F_{\mathfrak{u},k}\right)$, we obtain $\mathbf{t}\left(F_{\mathfrak{u},k}\right) \geq \mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) - 1$, so that $\mathbf{t}\left(F_{\mathfrak{u},k}\right) + 1 \geq \mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)}\right) \geq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$.

On the other hand, $\mathbf{j}(\mathfrak{u}) < \mathbf{j}(u) + 1$. Hence, (26) (applied to $\mathfrak{a} = \mathbf{j}(\mathfrak{u})$ and $\mathfrak{b} = \mathbf{j}(\mathfrak{u}) + 1$) yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u})),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u})+1),k}\right)$. In other words, $\mathbf{t}\left(F_{\mathfrak{u},k}\right) < \mathbf{t}\left(F_{\mathfrak{w},k}\right)$ (since $\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u})) = \mathfrak{u}$ and $\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u}) + 1) = \mathfrak{w}$). Thus, $\mathbf{t}\left(F_{\mathfrak{u},k}\right) \leq \mathbf{t}\left(F_{\mathfrak{w},k}\right) - 1$ (since $\mathbf{t}\left(F_{\mathfrak{u},k}\right)$ and $\mathbf{t}\left(F_{\mathfrak{w},k}\right)$ are integers). Thus, $\mathbf{t}\left(F_{\mathfrak{u},k}\right) + 1 \leq \mathbf{t}\left(F_{\mathfrak{w},k}\right)$, so that $\mathbf{t}\left(F_{\mathfrak{w},k}\right) \geq \mathbf{t}\left(F_{\mathfrak{u},k}\right) + 1 \geq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$.

But we know that $\mathbf{j}(\mathfrak{v}) > \mathbf{j}(\mathfrak{u}) + 1$, so that $\mathbf{j}(\mathfrak{u}) + 1 < \mathbf{j}(\mathfrak{v})$. Hence, (26) (applied to $\mathfrak{a} = \mathbf{j}(\mathfrak{u}) + 1$ and $\mathfrak{b} = \mathbf{j}(\mathfrak{v})$) yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u})+1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{v})),k}\right)$. In other words, $\mathbf{t}\left(F_{\mathfrak{w},k}\right) < \mathbf{t}\left(F_{\mathfrak{v},k}\right)$ (since $\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{u}) + 1) = \mathfrak{w}$ and $\mathbf{j}^{-1}(\mathbf{j}(\mathfrak{v})) = \mathfrak{v}$). Thus, $\mathbf{t}\left(F_{\mathfrak{w},k}\right) \leq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$ (since $\mathbf{t}\left(F_{\mathfrak{w},k}\right)$ and $\mathbf{t}\left(F_{\mathfrak{v},k}\right)$ are integers). Combined with $\mathbf{t}\left(F_{\mathfrak{w},k}\right) \geq \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$, this yields $\mathbf{t}\left(F_{\mathfrak{w},k}\right) = \mathbf{t}\left(F_{\mathfrak{v},k}\right) - 1$. That is, $\mathbf{t}\left(F_{\mathfrak{v},k}\right) = \mathbf{t}\left(F_{\mathfrak{w},k}\right) + 1$.

But (20) (applied to $\mathfrak{w}$ instead of $\mathfrak{u}$) yields $\mathbf{t}\left(F_{\mathfrak{v},k}\right) \neq \mathbf{t}\left(F_{\mathfrak{w},k}\right) + 1$. This contradicts $\mathbf{t}\left(F_{\mathfrak{v},k}\right) = \mathbf{t}\left(F_{\mathfrak{w},k}\right) + 1$. This contradiction shows that our assumption was wrong. Hence, (25) is proven.

(since $\mathfrak{u} \neq \mathfrak{v}$ (since $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ are pairwise distinct)), so that $\mathbf{j}(\mathfrak{u}) < \mathbf{j}(\mathfrak{v})$ (since $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v})$). Thus, $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v}) - 1$ (since $\mathbf{j}(\mathfrak{u})$ and $\mathbf{j}(\mathfrak{v})$ are integers). In other words, $\mathbf{j}(\mathfrak{u}) + 1 \leq \mathbf{j}(\mathfrak{v})$.

From (24), we obtain $\mathbf{d}(X) \in \{0, 1, \ldots, n-1\} \subseteq \{0, 1, \ldots, n\}$. Thus, there exists exactly one point $F$ on the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ such that $\mathbf{d}(F) = \mathbf{d}(X)$ (according to (8), applied to $i = \mathfrak{u}$ and $k = \mathbf{d}(X)$); this point is denoted by $F_{\mathfrak{u},\mathbf{d}(X)}$. Thus, $F_{\mathfrak{u},\mathbf{d}(X)}$ is the unique point $F$ on the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ such that $\mathbf{d}(F) = \mathbf{d}(X)$. Since $X$ is a point $F$ on the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ such that $\mathbf{d}(F) = \mathbf{d}(X)$ (because $X$ lies on the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and satisfies $\mathbf{d}(X) = \mathbf{d}(X)$), this yields that $X$ is $F_{\mathfrak{u},\mathbf{d}(X)}$. In other words, $X = F_{\mathfrak{u},\mathbf{d}(X)}$. The same reasoning (applied to $\mathfrak{v}$ instead of $\mathfrak{u}$) yields $X = F_{\mathfrak{v},\mathbf{d}(X)}$. Thus, $F_{\mathfrak{u},\mathbf{d}(X)} = X = F_{\mathfrak{v},\mathbf{d}(X)}$. Hence, (25) yields $\mathbf{j}(\mathfrak{v}) \leq \mathbf{j}(\mathfrak{u}) + 1$. Combined with $\mathbf{j}(\mathfrak{u}) + 1 \leq \mathbf{j}(\mathfrak{v})$, this yields $\mathbf{j}(\mathfrak{v}) = \mathbf{j}(\mathfrak{u}) + 1$. The same argument (but applied to $\mathfrak{w}$ instead of $\mathfrak{v}$) shows that $\mathbf{j}(\mathfrak{w}) = \mathbf{j}(\mathfrak{u}) + 1$. Thus, $\mathbf{j}(\mathfrak{v}) = \mathbf{j}(\mathfrak{u}) + 1 = \mathbf{j}(\mathfrak{w})$, so that $\mathfrak{v} = \mathfrak{w}$ (since the map $\mathbf{j}$ is injective). But the integers $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ are pairwise distinct. Thus, $\mathfrak{v} \neq \mathfrak{w}$. This contradicts $\mathfrak{v} = \mathfrak{w}$. This contradiction proves that our assumption was wrong. Hence, we have shown that no more than two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. This completes the proof of Lemma 9.1 **(a)**.

**(b)** We know from Lemma 9.1 **(a)** that exactly two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. We now must show that these two paths must end at two points of the form $Q_k$ and $Q_{k+1}$ for some $k \in \{1, 2, \ldots, N-1\}$.

Let $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ be these two paths, with $\mathfrak{u}$ and $\mathfrak{v}$ being distinct elements of $\{1, 2, \ldots, N\}$. Thus, the two paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ intersect at $X$. In other words, the two paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ pass through $X$. We can assume WLOG that $\mathbf{j}(\mathfrak{u}) \leq \mathbf{j}(\mathfrak{v})$ (since otherwise, we can just permute $\mathfrak{u}$ and $\mathfrak{v}$). Assume this. The map $\mathbf{j}$ is a permutation, and thus injective. Hence, $\mathbf{j}(\mathfrak{u}) \neq \mathbf{j}(\mathfrak{v})$ (since $\mathfrak{u} \neq \mathfrak{v}$ (since $\mathfrak{u}$ and $\mathfrak{v}$ are distinct)). Now, we have $\mathbf{j}(\mathfrak{v}) = \mathbf{j}(\mathfrak{u}) + 1$. (This can be proven in exactly the same manner as we proved it in the proof of Lemma 9.1 **(a)**.) The definition of $\mathbf{j}$ yields $\mathbf{j}(\mathfrak{u}) = j_{\mathfrak{u}}$ and $\mathbf{j}(\mathfrak{v}) = j_{\mathfrak{v}}$. Thus, $j_{\mathfrak{v}} = \mathbf{j}(\mathfrak{v}) = \underbrace{\mathbf{j}(\mathfrak{u})}_{= j_{\mathfrak{u}}} + 1 = j_{\mathfrak{u}} + 1$.

But the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ ends at the point $Q_{j_{\mathfrak{u}}}$ (according to the definition of this path). The path $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at the point $Q_{j_{\mathfrak{v}}}$ (according to the definition of this path). In other words, the path $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at the point $Q_{j_{\mathfrak{u}}+1}$ (since $j_{\mathfrak{v}} = j_{\mathfrak{u}} + 1$). Hence, the paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ end at the points $Q_{j_{\mathfrak{u}}}$ and $Q_{j_{\mathfrak{u}}+1}$. Hence, the paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ end at two points of the form $Q_k$ and $Q_{k+1}$ for some $k \in \{1, 2, \ldots, N-1\}$ (namely, for $k = j_{\mathfrak{u}}$).

Now, recall that exactly two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$ – namely, the two paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$. We have shown that these two paths $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ end at two points of the form $Q_k$ and $Q_{k+1}$ for some $k \in \{1, 2, \ldots, N-1\}$. In other words, Lemma 9.1 **(b)** is proven.

**(c)** Let $(\alpha, \beta) = X$.

We know from Lemma 9.1 **(a)** that exactly two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$. Lemma 9.1 **(b)** furthermore shows that these two paths must end at two points of the form $Q_k$ and $Q_{k+1}$ for some $k \in \{1,2,\ldots,N-1\}$. Let $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ be these two paths, labelled in such a way that $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ ends at $Q_k$ whereas $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at $Q_{k+1}$ (where $k$ is as in the previous sentence). Furthermore, consider the $k \in \{1,2,\ldots,N-1\}$ for which this holds. From $k \in \{1,2,\ldots,N-1\}$, we obtain $k \in \{1,2,\ldots,N\}$ and $k+1 \in \{1,2,\ldots,N\}$.

The path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ ends at the point $Q_{j_{\mathfrak{u}}}$ (according to the definition of this path). In other words,

$$\left(\text{the point at which the path } \pi_{\mathfrak{u}j_{\mathfrak{u}}} \text{ ends}\right) = Q_{j_{\mathfrak{u}}}.$$

On the other hand,

$$\left(\text{the point at which the path } \pi_{\mathfrak{u}j_{\mathfrak{u}}} \text{ ends}\right) = Q_k$$

(since the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ ends at $Q_k$). Comparing these two equalities, we obtain $Q_{j_{\mathfrak{u}}} = Q_k$. From this, we obtain $j_{\mathfrak{u}} = k$ (since the points $Q_1, Q_2, \ldots, Q_N$ are pairwise distinct). The definition of $\mathbf{j}$ yields $\mathbf{j}(\mathfrak{u}) = j_{\mathfrak{u}} = k$. Hence, $\mathfrak{u} = \mathbf{j}^{-1}(k)$.

Furthermore, the path $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at the point $Q_{j_{\mathfrak{v}}}$ (according to the definition of this path). In other words,

$$\left(\text{the point at which the path } \pi_{\mathfrak{v}j_{\mathfrak{v}}} \text{ ends}\right) = Q_{j_{\mathfrak{v}}}.$$

On the other hand,

$$\left(\text{the point at which the path } \pi_{\mathfrak{v}j_{\mathfrak{v}}} \text{ ends}\right) = Q_{k+1}$$

(since the path $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at $Q_{k+1}$). Comparing these two equalities, we obtain $Q_{j_{\mathfrak{v}}} = Q_{k+1}$. From this, we obtain $j_{\mathfrak{v}} = k+1$ (since the points $Q_1, Q_2, \ldots, Q_N$ are pairwise distinct). The definition of $\mathbf{j}$ yields $\mathbf{j}(\mathfrak{v}) = j_{\mathfrak{v}} = k+1$. Hence, $\mathfrak{v} = \mathbf{j}^{-1}(k+1)$.

Now, (24) yields $\mathbf{d}(X) \in \{0,1,\ldots,n-1\}$, whence $\mathbf{d}(X)+1 \in \{1,2,\ldots,n\} \subseteq \{0,1,\ldots,n\}$. Hence, (23) (applied to $\mathbf{d}(X)+1$ instead of $k$) shows that

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),\mathbf{d}(X)+1}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),\mathbf{d}(X)+1}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),\mathbf{d}(X)+1}\right)$$

(since $\mathbf{d}(X)+1 > \mathbf{d}(X)$). In other words,

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(r),\mathbf{d}(X)+1}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(r+1),\mathbf{d}(X)+1}\right)$$

for each $r \in \{1,2,\ldots,N-1\}$. Applying this to $r = k$, we obtain

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(k),\mathbf{d}(X)+1}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(k+1),\mathbf{d}(X)+1}\right).$$

In view of $\mathfrak{u} = \mathbf{j}^{-1}(k)$ and $\mathfrak{v} = \mathbf{j}^{-1}(k+1)$, we can rewrite this as

$$\mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)+1}\right) < \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)+1}\right). \tag{27}$$

On the other hand, we have $X = F_{\mathfrak{u},\mathbf{d}(X)}$. (This can be shown in the same way as we proved it during the proof of Lemma 9.1 **(a)**.) Hence, $F_{\mathfrak{u},\mathbf{d}(X)} = X = (\alpha, \beta)$ (since we have set $(\alpha, \beta) = X$).

Recall that $\mathbf{d}(X) \in \{0, 1, \ldots, n-1\}$. Hence, (16) (applied to $\mathfrak{u}$ and $\mathbf{d}(X)$ instead of $i$ and $k$) yields that the pair $\left(F_{\mathfrak{u},\mathbf{d}(X)}, F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$ is an arc of the directed graph $G$. In other words, the pair $\left((\alpha, \beta), F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$ is an arc of the directed graph $G$ (since $F_{\mathfrak{u},\mathbf{d}(X)} = (\alpha, \beta)$). In other words, the directed graph $G$ has an arc from $(\alpha, \beta)$ to $F_{\mathfrak{u},\mathbf{d}(X)+1}$.

But the only arcs of $G$ that start in $(\alpha, \beta)$ are $((\alpha, \beta), (\alpha+1, \beta))$ and $((\alpha, \beta), (\alpha, \beta+1))$ (by the definition of $G$). Hence, the only points $Y$ such that the directed graph $G$ has an arc from $(\alpha, \beta)$ to $Y$ are $(\alpha+1, \beta)$ and $(\alpha, \beta+1)$. Thus, the point $F_{\mathfrak{u},\mathbf{d}(X)+1}$ must be one of these two points $(\alpha+1, \beta)$ and $(\alpha, \beta+1)$ (since the directed graph $G$ has an arc from $(\alpha, \beta)$ to $F_{\mathfrak{u},\mathbf{d}(X)+1}$). In other words, we have either $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha+1, \beta)$ or $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha, \beta+1)$.

Recall that the pair $\left((\alpha, \beta), F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$ is an arc of the directed graph $G$. The same argument (applied to $\mathfrak{v}$ instead of $\mathfrak{u}$) shows that the pair $\left((\alpha, \beta), F_{\mathfrak{v},\mathbf{d}(X)+1}\right)$ is an arc of the directed graph $G$. Hence, (7) (applied to $(\alpha, \beta)$ and $F_{\mathfrak{v},\mathbf{d}(X)+1}$ instead of $U$ and $V$) shows that

$$\mathbf{t}((\alpha, \beta)) - 1 \leq \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)+1}\right) \leq \mathbf{t}((\alpha, \beta)) + 1.$$

Hence, (27) becomes

$$\mathbf{t}\left(F_{\mathfrak{u},\mathbf{d}(X)+1}\right) < \mathbf{t}\left(F_{\mathfrak{v},\mathbf{d}(X)+1}\right) \leq \underbrace{\mathbf{t}((\alpha, \beta))}_{\substack{=\alpha-\beta \\ \text{(by the definition of } \mathbf{t}((\alpha,\beta)))}} + 1 = \alpha - \beta + 1 = (\alpha+1) - \beta. \tag{28}$$

However, if we had $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha+1, \beta)$, then we would have

$$\mathbf{t}\left(\underbrace{F_{\mathfrak{u},\mathbf{d}(X)+1}}_{=(\alpha+1,\beta)}\right) = \mathbf{t}((\alpha+1, \beta)) = (\alpha+1) - \beta$$

(by the definition of $\mathbf{t}((\alpha+1, \beta))$), which would contradict (28). Hence, we cannot have $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha+1, \beta)$. Thus, we have $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha, \beta+1)$ (since we have either $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha+1, \beta)$ or $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha, \beta+1)$).

But (15) (applied to $\mathfrak{u}$ and $\mathbf{d}(X)$ instead of $i$ and $k$) shows that the pair $\left(F_{\mathfrak{u},\mathbf{d}(X)}, F_{\mathfrak{u},\mathbf{d}(X)+1}\right)$ is an arc of the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$. In view of $F_{\mathfrak{u},\mathbf{d}(X)} = (\alpha, \beta)$ and $F_{\mathfrak{u},\mathbf{d}(X)+1} = (\alpha, \beta+1)$, we can rewrite this as follows: The pair $((\alpha, \beta), (\alpha, \beta+1))$

is an arc of the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$. In other words, the path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ contains the vertical segment $(\alpha, \beta) \mapsto (\alpha, \beta + 1)$.

Now, let us recall that exactly two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect at $X$, and these two paths are $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$. Among these two paths, the one that ends in $Q_k$ is $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ (since we have labelled them by $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ and $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ in such a way that $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ ends at $Q_k$ whereas $\pi_{\mathfrak{v}j_{\mathfrak{v}}}$ ends at $Q_{k+1}$). In other words, the path (among the two paths discussed in Lemma 9.1 **(b)**) that ends in $Q_k$ is $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$. As we have just shown, this path $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$ contains the vertical segment $(\alpha, \beta) \mapsto (\alpha, \beta + 1)$. In other words, the path (among the two paths discussed in Lemma 9.1 **(b)**) that ends in $Q_k$ contains the vertical segment $(\alpha, \beta) \mapsto (\alpha, \beta + 1)$ (because this path is $\pi_{\mathfrak{u}j_{\mathfrak{u}}}$). This proves Lemma 9.1 **(c)**. □

## 9.3. Proof of Lemma 3.3

As promised above, let me give a self-contained proof of Lemma 3.3. First, I will rewrite this lemma in a more self-contained fashion:

**Lemma 9.2.** Let $u_1, u_2, \ldots, u_n$ be $n$ elements of an associative algebra $\mathfrak{A}$ which satisfy the relations (1.2). In this lemma, "letters" will always mean elements of $\{1, 2, \ldots, n\}$, and "strings" will always mean finite words composed of these letters. Let $A = a_\alpha \ldots a_2 a_1$ and $B = b_\beta \ldots b_2 b_1$ be two strings such that $\alpha \geq \beta$. Let $X = x_1 x_2 \ldots x_\varphi$ and $Y = y_1 y_2 \cdots y_\psi$ be two further strings. Let $p$ be a letter. Assume that

$$\begin{aligned} & a_\alpha > a_{\alpha-1} > \cdots > a_1 > p > x_1 > x_2 > \cdots > x_\varphi, \\ & b_\beta > b_{\beta-1} > \cdots > b_1 > p, \qquad \text{and} \\ & (a_i \leq b_i \text{ for every } i \in \{1, 2, \ldots, \beta\}). \end{aligned}$$

Let $\mathbf{A} = u_{a_\alpha} \cdots u_{a_2} u_{a_1}$, $\mathbf{B} = u_{b_\beta} \cdots u_{b_2} u_{b_1}$, $\mathbf{p} = u_p$, $\mathbf{X} = u_{x_1} u_{x_2} \cdots u_{x_\varphi}$ and $\mathbf{Y} = u_{y_1} u_{y_2} \cdots u_{y_\psi}$. Then, $\mathbf{ApXBY} = \mathbf{ApBXY}$ in $\mathfrak{A}$.

Note that you denote the elements **A**, **B**, **p**, **X** and **Y** of $\mathfrak{A}$ (defined in Lemma 9.2) by $A$, $B$, $p$, $X$ and $Y$, but I want to avoid this abuse of notation. I have also renamed your algebra $A$ as $\mathfrak{A}$, since the letter $A$ is used for one of the strings. Thus, Lemma 9.2 yields the claim $ApXBY = ApBXY$ of your Lemma 3.3.

It thus remains to prove Lemma 9.2. Before we do so, we shall show two lemmas:

**Lemma 9.3.** Let $u_1, u_2, \ldots, u_n$ be $n$ elements of an associative algebra $\mathfrak{A}$ which satisfy the relations

$$u_j u_i u_k = u_j u_k u_i \qquad \text{for all } i < j < k. \tag{29}$$

In this lemma, "letters" will always mean elements of $\{1, 2, \ldots, n\}$, and "strings" will always mean finite words composed of these letters. Let $a$ and

$b$ be two letters such that $b > a$. Let $V = v_1 v_2 \dots v_\gamma$ be a string such that

$$a > v_1 > v_2 > \cdots > v_\gamma .$$

Let $\mathbf{a} = u_a$, $\mathbf{b} = u_b$ and $\mathbf{V} = u_{v_1} u_{v_2} \cdots u_{v_\gamma}$. Then, $\mathbf{aVb} = \mathbf{abV}$ in $\mathfrak{A}$.

*Proof of Lemma 9.3.* If $\gamma = 0$, then

$$\begin{aligned} \mathbf{V} &= u_{v_1} u_{v_2} \cdots u_{v_\gamma} = u_{v_1} u_{v_2} \cdots u_{v_0} \qquad (\text{since } \gamma = 0) \\ &= (\text{empty product}) = 1. \end{aligned}$$

Hence, if $\gamma = 0$, then $\mathbf{aVb} = \mathbf{abV}$ is obviously true (because $\mathbf{a} \underbrace{\mathbf{V}}_{=1} \mathbf{b} = \mathbf{ab}$ and $\mathbf{ab} \underbrace{\mathbf{V}}_{=1} = \mathbf{ab}$). Thus, for the rest of this proof, we can WLOG assume that $\gamma \neq 0$. Assume this. Thus, $\gamma \geq 1$ (since $\gamma \neq 0$ and $\gamma \in \mathbb{N}$). Therefore, $1 \in \{1, 2, \dots, \gamma\}$. Hence, $v_1$ is well-defined.

From $a > v_1 > v_2 > \cdots > v_\gamma$, we obtain $a > v_1$ (since $1 \in \{1, 2, \dots, \gamma\}$). Hence, $v_1 < a < b$ (since $b > a$). Thus, (29) (applied to $i = v_1$, $j = a$ and $k = b$) yields $u_a u_{v_1} u_b = u_a u_b u_{v_1}$.

We shall prove that every $\rho \in \{1, 2, \dots, \gamma\}$ satisfies

$$\mathbf{a} \left( u_{v_1} u_{v_2} \cdots u_{v_\rho} \right) \mathbf{b} = \mathbf{ab} \left( u_{v_1} u_{v_2} \cdots u_{v_\rho} \right). \tag{30}$$

[*Proof of (30):* We will prove (30) by induction over $\rho$:

*Induction base:* Comparing

$$\underbrace{\mathbf{a}}_{=u_a} \underbrace{\left( u_{v_1} u_{v_2} \cdots u_{v_1} \right)}_{=u_{v_1}} \underbrace{\mathbf{b}}_{=u_b} = u_a u_{v_1} u_b = u_a u_b u_{v_1}$$

with

$$\underbrace{\mathbf{a}}_{=u_a} \underbrace{\mathbf{b}}_{=u_b} \underbrace{\left( u_{v_1} u_{v_2} \cdots u_{v_1} \right)}_{=u_{v_1}} = u_a u_b u_{v_1},$$

we obtain $\mathbf{a} \left( u_{v_1} u_{v_2} \cdots u_{v_1} \right) \mathbf{b} = \mathbf{ab} \left( u_{v_1} u_{v_2} \cdots u_{v_1} \right)$. In other words, (30) holds for $\rho = 1$. This completes the induction base.

*Induction step:* Let $r \in \{1, 2, \dots, \gamma - 1\}$. Assume (as the induction hypothesis) that (30) holds for $\rho = r$. We must prove that (30) holds for $\rho = r + 1$.

Recall that $a > v_1 > v_2 > \cdots > v_\gamma$. Hence, $a > v_h$ for every $h \in \{1, 2, \dots, \gamma\}$. Applying this to $h = r$, we obtain $a > v_r$. Hence, $b > a > v_r$. On the other hand, $v_r > v_{r+1}$ (since $v_1 > v_2 > \cdots > v_\gamma$), so that $v_{r+1} < v_r < b$ (since $b > v_r$). Thus, (29) (applied to $i = v_{r+1}$, $j = v_r$ and $k = b$) yields $u_{v_r} u_{v_{r+1}} u_b = u_{v_r} u_b u_{v_{r+1}}$.

Recall that we assumed that (30) holds for $\rho = r$. In other words, we have

$$\mathbf{a} \left( u_{v_1} u_{v_2} \cdots u_{v_r} \right) \mathbf{b} = \mathbf{ab} \left( u_{v_1} u_{v_2} \cdots u_{v_r} \right). \tag{31}$$

Now, comparing

$$\mathbf{a} \underbrace{\left(u_{v_1}u_{v_2}\cdots u_{v_{r+1}}\right)}_{=\left(u_{v_1}u_{v_2}\cdots u_{v_{r-1}}\right)u_{v_r}u_{v_{r+1}}} \underbrace{\mathbf{b}}_{=u_b} = \mathbf{a}\left(u_{v_1}u_{v_2}\cdots u_{v_{r-1}}\right) \underbrace{u_{v_r}u_{v_{r+1}}u_b}_{=u_{v_r}u_b u_{v_{r+1}}}$$
$$= \mathbf{a} \underbrace{\left(u_{v_1}u_{v_2}\cdots u_{v_{r-1}}\right)u_{v_r}}_{=u_{v_1}u_{v_2}\cdots u_{v_r}} \underbrace{u_b}_{\substack{=\mathbf{b} \\ (\text{since } \mathbf{b}=u_b)}} u_{v_{r+1}}$$
$$= \underbrace{\mathbf{a}\left(u_{v_1}u_{v_2}\cdots u_{v_r}\right)\mathbf{b}}_{\substack{=\mathbf{ab}\left(u_{v_1}u_{v_2}\cdots u_{v_r}\right) \\ (\text{by (31)})}} u_{v_{r+1}}$$
$$= \mathbf{ab} \underbrace{\left(u_{v_1}u_{v_2}\cdots u_{v_r}\right)u_{v_{r+1}}}_{=u_{v_1}u_{v_2}\cdots u_{v_{r+1}}} = \mathbf{ab}\left(u_{v_1}u_{v_2}\cdots u_{v_{r+1}}\right).$$

In other words, (30) holds for $\rho = r+1$. This completes the induction step. Thus, (30) is proved by induction.]

We have $\gamma \in \{1, 2, \ldots, \gamma\}$ (since $\gamma \geq 1$). Hence, (30) (applied to $\rho = \gamma$) yields

$$\mathbf{a}\left(u_{v_1}u_{v_2}\cdots u_{v_\gamma}\right)\mathbf{b} = \mathbf{ab}\left(u_{v_1}u_{v_2}\cdots u_{v_\gamma}\right).$$

In view of $\mathbf{V} = u_{v_1}u_{v_2}\cdots u_{v_\gamma}$, this rewrites as $\mathbf{aVb} = \mathbf{abV}$. This proves Lemma 9.3. □

Our second lemma is a stronger version of Lemma 9.2:

**Lemma 9.4.** Let $u_1, u_2, \ldots, u_n$ be $n$ elements of an associative algebra $\mathfrak{A}$ which satisfy the relations (29). In this lemma, "letters" will always mean elements of $\{1, 2, \ldots, n\}$, and "strings" will always mean finite words composed of these letters. Let $A = a_\alpha \ldots a_2 a_1$ and $B = b_\beta \ldots b_2 b_1$ be two strings such that $\alpha \geq \beta - 1$. Let $X = x_1 x_2 \ldots x_\varphi$ be a further string. Let $p$ be a letter. Assume that

$$a_\alpha > a_{\alpha-1} > \cdots > a_1 > p > x_1 > x_2 > \cdots > x_\varphi,$$
$$b_\beta > b_{\beta-1} > \cdots > b_1 > p, \qquad \text{and}$$
$$\left(a_i \leq b_i \text{ for every } i \in \{1, 2, \ldots, \beta - 1\}\right).$$

Let $\mathbf{A} = u_{a_\alpha}\cdots u_{a_2}u_{a_1}$, $\mathbf{B} = u_{b_\beta}\cdots u_{b_2}u_{b_1}$, $\mathbf{p} = u_p$, and $\mathbf{X} = u_{x_1}u_{x_2}\cdots u_{x_\varphi}$. Then, $\mathbf{ApXB} = \mathbf{ApBX}$ in $\mathfrak{A}$.

*Proof of Lemma 9.4.* Let us extend the $\alpha$-tuple $(a_\alpha, a_{\alpha-1}, \ldots, a_1)$ to an $(\alpha+1)$-tuple $(a_\alpha, a_{\alpha-1}, \ldots, a_1, a_0)$ by setting $a_0 = p$. Thus, $a_0, a_1, \ldots, a_\alpha$ are well-defined letters. In other words, $a_j$ is a well-defined letter for each $j \in \{0, 1, \ldots, \alpha\}$.

But recall that we have the chain of inequalities

$$a_\alpha > a_{\alpha-1} > \cdots > a_1 > p > x_1 > x_2 > \cdots > x_\varphi.$$

In view of $a_0 = p$, this rewrites as

$$a_\alpha > a_{\alpha-1} > \cdots > a_1 > a_0 > x_1 > x_2 > \cdots > x_\varphi. \tag{32}$$

Also, $\mathbf{p} = u_p = u_{a_0}$ (since $p = a_0$).

From $\alpha \geq \beta - 1$, we obtain $\beta - 1 \leq \alpha$, thus $\{0, 1, \ldots, \beta - 1\} \subseteq \{0, 1, \ldots, \alpha\}$.

We have

$$\begin{aligned} &\underbrace{\mathbf{A}}_{=u_{a_\alpha} \cdots u_{a_2} u_{a_1}} \underbrace{\mathbf{p}}_{=u_{a_0}} \\ &= \left(u_{a_\alpha} \cdots u_{a_2} u_{a_1}\right) u_{a_0} = u_{a_\alpha} \cdots u_{a_1} u_{a_0} \\ &= \left(u_{a_\alpha} \cdots u_{a_{\beta+1}} u_{a_\beta}\right) \left(u_{a_{\beta-1}} u_{a_{\beta-2}} \cdots u_{a_0}\right) \end{aligned} \tag{33}$$

(since $\alpha \geq \beta - 1$).

If $i \in \{0, 1, \ldots, \beta\}$, then the numbers $i-1, i-2, \ldots, 0$ are elements of $\{0, 1, \ldots, \beta - 1\}$ and thus are elements of $\{0, 1, \ldots, \alpha\}$ (since $\{0, 1, \ldots, \beta - 1\} \subseteq \{0, 1, \ldots, \alpha\}$). Hence, if $i \in \{0, 1, \ldots, \beta\}$, then $a_{i-1}, a_{i-2}, \ldots, a_0$ are well-defined letters (since $a_j$ is a well-defined letter for each $j \in \{0, 1, \ldots, \alpha\}$). Thus, for each $i \in \{0, 1, \ldots, \beta\}$, we can define an element $\mathbf{A}_i$ of $\mathfrak{A}$ by

$$\mathbf{A}_i = u_{a_{i-1}} u_{a_{i-2}} \cdots u_{a_0}.$$

Consider this $\mathbf{A}_i$.

The definition of $\mathbf{A}_\beta$ yields $\mathbf{A}_\beta = u_{a_{\beta-1}} u_{a_{\beta-2}} \cdots u_{a_0}$. Hence,

$$\begin{aligned} &\left(u_{a_\alpha} \cdots u_{a_{\beta+1}} u_{a_\beta}\right) \underbrace{\mathbf{A}_\beta}_{=u_{a_{\beta-1}} u_{a_{\beta-2}} \cdots u_{a_0}} \\ &= \left(u_{a_\alpha} \cdots u_{a_{\beta+1}} u_{a_\beta}\right) \left(u_{a_{\beta-1}} u_{a_{\beta-2}} \cdots u_{a_0}\right) \\ &= \mathbf{A}\mathbf{p} \qquad \text{(by (33))}. \end{aligned} \tag{34}$$

For each $i \in \{0, 1, \ldots, \beta\}$, we set

$$\mathbf{B}_i = u_{b_i} u_{b_{i-1}} \cdots u_{b_1}.$$

Thus,

$$\mathbf{B}_0 = u_{b_0} u_{b_{0-1}} \cdots u_{b_1} = (\text{empty product}) = 1$$

and

$$\mathbf{B}_\beta = u_{b_\beta} u_{b_{\beta-1}} \cdots u_{b_1} = u_{b_\beta} \cdots u_{b_2} u_{b_1} = \mathbf{B}$$

(since $\mathbf{B} = u_{b_\beta} \cdots u_{b_2} u_{b_1}$).

We now claim that

$$\mathbf{A}_i \mathbf{X} \mathbf{B}_i = \mathbf{A}_i \mathbf{B}_i \mathbf{X} \tag{35}$$

for each $i \in \{0, 1, \ldots, \beta\}$.

[*Proof of (35):* We shall prove (35) by induction on $i$:

*Induction base:* Comparing $\mathbf{A}_0\mathbf{X}\underbrace{\mathbf{B}_0}_{=1} = \mathbf{A}_0\mathbf{X}$ with $\mathbf{A}_0\underbrace{\mathbf{B}_0}_{=1}\mathbf{X} = \mathbf{A}_0\mathbf{X}$, we obtain $\mathbf{A}_0\mathbf{X}\mathbf{B}_0 = \mathbf{A}_0\mathbf{B}_0\mathbf{X}$. In other words, (35) holds for $i = 0$. This completes the induction base.

*Induction step:* Let $s \in \{0, 1, \ldots, \beta - 1\}$. Assume (as the induction hypothesis) that (35) holds for $i = s$. We must prove that (35) holds for $i = s + 1$.

We have assumed that (35) holds for $i = s$. In other words, we have

$$\mathbf{A}_s\mathbf{X}\mathbf{B}_s = \mathbf{A}_s\mathbf{B}_s\mathbf{X}. \tag{36}$$

It is easy to see that $a_s < b_{s+1}$ [21]. Thus, $b_{s+1} > a_s$. Also, $s \in \{0, 1, \ldots, \beta - 1\}$ shows that $s \leq \beta - 1 \leq \alpha$.

The definition of $\mathbf{A}_s$ yields $\mathbf{A}_s = u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}$. The definition of $\mathbf{A}_{s+1}$ yields

$$\begin{aligned}\mathbf{A}_{s+1} &= u_{a_{(s+1)-1}}u_{a_{(s+1)-2}}\cdots u_{a_0} = u_{a_s}u_{a_{s-1}}\cdots u_{a_0} \qquad (\text{since } (s+1)-1 = s) \\ &= u_{a_s}\underbrace{\left(u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}\right)}_{\substack{=\mathbf{A}_s \\ (\text{since } \mathbf{A}_s = u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0})}} = u_{a_s}\mathbf{A}_s. \end{aligned} \tag{37}$$

The definition of $\mathbf{B}_s$ yields $\mathbf{B}_s = u_{b_s}u_{b_{s-1}}\cdots u_{b_1}$. The definition of $\mathbf{B}_{s+1}$ yields

$$\begin{aligned}\mathbf{B}_{s+1} &= u_{b_{s+1}}u_{b_{(s+1)-1}}\cdots u_{b_1} = u_{b_{s+1}}u_{b_s}\cdots u_{b_1} = u_{b_{s+1}}\underbrace{\left(u_{b_s}u_{b_{s-1}}\cdots u_{b_1}\right)}_{\substack{=\mathbf{B}_s \\ (\text{since } \mathbf{B}_s = u_{b_s}u_{b_{s-1}}\cdots u_{b_1})}} \\ &= u_{b_{s+1}}\mathbf{B}_s. \end{aligned} \tag{38}$$

From (32), we obtain $a_\alpha > a_{\alpha-1} > \cdots > a_1 > a_0$. Since $s \leq \alpha$, this yields $a_s > a_{s-1} > a_{s-2} > \cdots > a_0$.

[21] *Proof.* We are in one of the following two cases:

*Case 1:* We have $s = 0$.

*Case 2:* We have $s \neq 0$.

Let us first consider Case 1. In this case, we have $s = 0$. Hence, $a_s = a_0 = p$. Also, $s = 0$, so that $0 = s \in \{0, 1, \ldots, \beta - 1\}$, so that $0 \leq \beta - 1$ and thus $\beta \geq 1$. Hence, $1 \in \{1, 2, \ldots, \beta\}$, so that $b_1$ is well-defined. From $b_\beta > b_{\beta-1} > \cdots > b_1 > p$, we thus obtain $b_1 > p$. But $s = 0$, so that $s + 1 = 1$ and therefore $b_{s+1} = b_1 > p = a_s$ (since $a_s = p$). Thus, $a_s < b_{s+1}$. Hence, $a_s < b_{s+1}$ is proved in Case 1.

Let us now consider Case 2. In this case, we have $s \neq 0$. Combining this with $s \in \{0, 1, \ldots, \beta - 1\}$, we obtain $s \in \{0, 1, \ldots, \beta - 1\} \setminus \{0\} = \{1, 2, \ldots, \beta - 1\}$.

We assumed that ($a_i \leq b_i$ for every $i \in \{1, 2, \ldots, \beta - 1\}$). Applying this to $i = s$, we obtain $a_s \leq b_s$ (since $s \in \{1, 2, \ldots, \beta - 1\}$). But from $b_\beta > b_{\beta-1} > \cdots > b_1 > p$, we obtain $b_\beta > b_{\beta-1} > \cdots > b_1$. In other words, $b_{i+1} > b_i$ for each $i \in \{1, 2, \ldots, \beta - 1\}$. Applying this to $i = s$, we obtain $b_{s+1} > b_s$ (since $s \in \{1, 2, \ldots, \beta - 1\}$). Hence, $b_s < b_{s+1}$, so that $a_s \leq b_s < b_{s+1}$. Thus, $a_s < b_{s+1}$ is proved in Case 2.

We have now proved $a_s < b_{s+1}$ in both Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that $a_s < b_{s+1}$ always holds.

Now, let $V_1$ be the string defined by $V_1 = a_{s-1}a_{s-2}\cdots a_0$. This string satisfies $a_s > a_{s-1} > a_{s-2} > \cdots > a_0$ and $\mathbf{A}_s = u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}$. Hence, Lemma 9.3 (applied to $\gamma = s$ and $V = V_1$ and $(v_1, v_2, \ldots, v_\gamma) = (a_{s-1}, a_{s-2}, \ldots, a_0)$ and $a = a_s$ and $b = b_{s+1}$ and $\mathbf{V} = \mathbf{A}_s$ and $\mathbf{a} = u_{a_s}$ and $\mathbf{b} = u_{b_{s+1}}$) yields

$$u_{a_s}\mathbf{A}_s u_{b_{s+1}} = u_{a_s}u_{b_{s+1}}\mathbf{A}_s \tag{39}$$

(since $u_{a_s} = u_{a_s}$ and $u_{b_{s+1}} = u_{b_{s+1}}$).

Furthermore, let us define a $(s+\varphi)$-tuple $(v_1, v_2, \ldots, v_{s+\varphi})$ of letters by

$$(v_1, v_2, \ldots, v_{s+\varphi}) = (a_{s-1}, a_{s-2}, \ldots, a_0, x_1, x_2, \ldots, x_\varphi).$$

Let $V$ be the string defined by $V = v_1v_2\cdots v_{s+\varphi}$. Combining $a_s > a_{s-1} > a_{s-2} > \cdots > a_0$ with $a_0 > x_1 > x_2 > \cdots > x_\varphi$ (which follows from (32)), we see that

$$a_s > a_{s-1} > a_{s-2} > \cdots > a_0 > x_1 > x_2 > \cdots > x_\varphi.$$

In view of $(v_1, v_2, \ldots, v_{s+\varphi}) = (a_{s-1}, a_{s-2}, \ldots, a_0, x_1, x_2, \ldots, x_\varphi)$, we can rewrite this as

$$a_s > v_1 > v_2 > \cdots > v_s > v_{s+1} > v_{s+2} > \cdots > v_{s+\varphi}.$$

In other words,

$$a_s > v_1 > v_2 > \cdots > v_{s+\varphi}.$$

Moreover, from $(v_1, v_2, \ldots, v_{s+\varphi}) = (a_{s-1}, a_{s-2}, \ldots, a_0, x_1, x_2, \ldots, x_\varphi)$, we obtain

$$u_{v_1}u_{v_2}\cdots u_{v_{s+\varphi}} = u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}u_{x_1}u_{x_2}\cdots u_{x_\varphi}.$$

Comparing this with

$$\underbrace{\mathbf{A}_s}_{=u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}}\ \underbrace{\mathbf{X}}_{=u_{x_1}u_{x_2}\cdots u_{x_\varphi}} = u_{a_{s-1}}u_{a_{s-2}}\cdots u_{a_0}u_{x_1}u_{x_2}\cdots u_{x_\varphi},$$

we obtain

$$\mathbf{A}_s\mathbf{X} = u_{v_1}u_{v_2}\cdots u_{v_{s+\varphi}}.$$

Hence, Lemma 9.3 (applied to $\gamma = s+\varphi$ and $a = a_s$ and $b = b_{s+1}$ and $\mathbf{V} = \mathbf{A}_s\mathbf{X}$ and $\mathbf{a} = u_{a_s}$ and $\mathbf{b} = u_{b_{s+1}}$) yields

$$u_{a_s}\mathbf{A}_s\mathbf{X}u_{b_{s+1}} = u_{a_s}u_{b_{s+1}}\mathbf{A}_s\mathbf{X} \tag{40}$$

(since $u_{a_s} = u_{a_s}$ and $u_{b_{s+1}} = u_{b_{s+1}}$). Now,

$$\underbrace{\mathbf{A}_{s+1}}_{\substack{=u_{a_s}\mathbf{A}_s \\ \text{(by (37))}}}\mathbf{X}\underbrace{\mathbf{B}_{s+1}}_{\substack{=u_{b_{s+1}}\mathbf{B}_s \\ \text{(by (38))}}} = \underbrace{u_{a_s}\mathbf{A}_s\mathbf{X}u_{b_{s+1}}}_{\substack{=u_{a_s}u_{b_{s+1}}\mathbf{A}_s\mathbf{X} \\ \text{(by (40))}}}\mathbf{B}_s = u_{a_s}u_{b_{s+1}}\underbrace{\mathbf{A}_s\mathbf{X}\mathbf{B}_s}_{\substack{=\mathbf{A}_s\mathbf{B}_s\mathbf{X} \\ \text{(by (36))}}} = u_{a_s}u_{b_{s+1}}\mathbf{A}_s\mathbf{B}_s\mathbf{X}.$$

Comparing this with

$$\underbrace{\mathbf{A}_{s+1}}_{\substack{=u_{a_s}\mathbf{A}_s \\ \text{(by (37))}}} \underbrace{\mathbf{B}_{s+1}}_{\substack{=u_{b_{s+1}}\mathbf{B}_s \\ \text{(by (38))}}} \mathbf{X} = \underbrace{u_{a_s}\mathbf{A}_s u_{b_{s+1}}}_{\substack{=u_{a_s}u_{b_{s+1}}\mathbf{A}_s \\ \text{(by (39))}}} \mathbf{B}_s\mathbf{X} = u_{a_s}u_{b_{s+1}}\mathbf{A}_s\mathbf{B}_s\mathbf{X},$$

we obtain $\mathbf{A}_{s+1}\mathbf{X}\mathbf{B}_{s+1} = \mathbf{A}_{s+1}\mathbf{B}_{s+1}\mathbf{X}$. In other words, (35) holds for $i = s+1$. This completes the induction step. Hence, the induction proof of (35) is complete.]

Now, (35) (applied to $i = \beta$) yields

$$\mathbf{A}_\beta\mathbf{X}\mathbf{B}_\beta = \mathbf{A}_\beta\mathbf{B}_\beta\mathbf{X}.$$

Hence,

$$\left(u_{a_\alpha}\cdots u_{a_{\beta+1}}u_{a_\beta}\right)\underbrace{\mathbf{A}_\beta\mathbf{X}\mathbf{B}_\beta}_{=\mathbf{A}_\beta\mathbf{B}_\beta\mathbf{X}} = \underbrace{\left(u_{a_\alpha}\cdots u_{a_{\beta+1}}u_{a_\beta}\right)\mathbf{A}_\beta}_{\substack{=\mathbf{Ap} \\ \text{(by (34))}}}\underbrace{\mathbf{B}_\beta}_{=\mathbf{B}}\mathbf{X} = \mathbf{ApBX}.$$

Therefore,

$$\mathbf{ApBX} = \underbrace{\left(u_{a_\alpha}\cdots u_{a_{\beta+1}}u_{a_\beta}\right)\mathbf{A}_\beta}_{\substack{=\mathbf{Ap} \\ \text{(by (34))}}}\mathbf{X}\underbrace{\mathbf{B}_\beta}_{=\mathbf{B}} = \mathbf{ApXB}.$$

This proves Lemma 9.4. □

*Proof of Lemma 9.2.* We assumed that $u_1, u_2, \ldots, u_n$ satisfy the relations (1.2). Thus, in particular, the second equality of (1.2) is satisfied. In other words, we have

$$u_ju_iu_k = u_ju_ku_i \qquad \text{for all } i < j \leq k \text{ satisfying } |i-k| \geq 2.$$

Hence, in particular, we have

$$u_ju_iu_k = u_ju_ku_i \qquad \text{for all } i < j < k$$

(because every three letters $i, j, k$ satisfying $i < j < k$ satisfy $|i-k| \geq 2$ [22]). In other words, $u_1, u_2, \ldots, u_n$ satisfy the relations (29). From $\alpha \geq \beta$, we obtain $\alpha \geq \beta \geq \beta - 1$. Furthermore, we have assumed that ($a_i \leq b_i$ for every $i \in \{1, 2, \ldots, \beta\}$); thus, ($a_i \leq b_i$ for every $i \in \{1, 2, \ldots, \beta - 1\}$). Hence, Lemma 9.4 yields $\mathbf{ApXB} = \mathbf{ApBX}$ in $\mathfrak{A}$. Hence, $\underbrace{\mathbf{ApXB}}_{=\mathbf{ApBX}}\mathbf{Y} = \mathbf{ApBXY}$. This proves Lemma 9.2. □

[22] *Proof.* Let $i, j, k$ be three letters satisfying $i < j < k$. We must prove that $|i-k| \geq 2$.

We have $i < j$ and thus $i \leq j - 1$ (since $i$ and $j$ are integers). We have $j < k$ and thus $j \leq k - 1$ (since $j$ and $k$ are integers). Now, $i \leq \underbrace{j}_{\leq k-1} - 1 \leq (k-1) - 1 = k - 2$, so that $i - k \leq -2 < 0$ and therefore $|i-k| = -\underbrace{(i-k)}_{\leq -2} \geq -(-2) = 2$. Qed.

## 9.4. Proof of $(j_1, j_2, \ldots, j_N) = (1, 2, \ldots, N)$

The notations introduced during the proof of Lemma 3.2 shall be used throughout Subsection 9.4. We want to show the following lemma:

**Lemma 9.5.** Let $(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N})$ be a non-intersecting family of lattice paths, where each $\pi_{ij_i}$ is a path from $P_i$ to $Q_j$. Then, $(j_1, j_2, \ldots, j_N) = (1, 2, \ldots, N)$.

This lemma is used to link the left hand side of (3.3) with the terms on the right hand side of (3.3) that survive the cancellation.

Lemma 9.5 is a particular case of Proposition 2 in math.stackexchange question #2870640 ( `https://math.stackexchange.com/questions/2870640` ), but let us give a self-contained proof here.

*Proof of Lemma 9.5.* We shall use the notations introduced in Subsection 9.2. (In particular, we will use the points $F_{i,k}$ defined in Subsection 9.2.)

The following arguments mirror some of the arguments made at the beginning of the proof of Lemma 9.1 (but we are having it somewhat easier now, since the family $(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N})$ is non-intersecting).

If $k \in \{0, 1, \ldots, n\}$, then any two distinct elements $\mathfrak{u}$ and $\mathfrak{v}$ of $\{1, 2, \ldots, N\}$ satisfy

$$\mathbf{t}(F_{\mathfrak{u},k}) \neq \mathbf{t}(F_{\mathfrak{v},k}). \tag{41}$$

[23] Moreover, if $k \in \{0, 1, \ldots, n\}$ and if $\mathfrak{u}$ and $\mathfrak{v}$ are two elements of $\{1, 2, \ldots, N\}$, then

$$\mathbf{t}(F_{\mathfrak{v},k}) \neq \mathbf{t}(F_{\mathfrak{u},k}) + 1. \tag{42}$$

[24]

[23] *Proof of (41):* Let $k \in \{0, 1, \ldots, n\}$. Let $\mathfrak{u}$ and $\mathfrak{v}$ be two distinct elements of $\{1, 2, \ldots, N\}$. We need to prove (41).

Assume the contrary. Thus, (41) does not hold. In other words, $\mathbf{t}(F_{\mathfrak{u},k}) = \mathbf{t}(F_{\mathfrak{v},k})$. But $\mathbf{d}(F_{\mathfrak{u},k}) = k$ (by (11), applied to $i = \mathfrak{u}$) and $\mathbf{d}(F_{\mathfrak{v},k}) = k$ (by (11), applied to $i = \mathfrak{v}$). Hence, $\mathbf{d}(F_{\mathfrak{u},k}) = k = \mathbf{d}(F_{\mathfrak{v},k})$. Altogether, $\left(\underbrace{\mathbf{d}(F_{\mathfrak{u},k})}_{=\mathbf{d}(F_{\mathfrak{v},k})}, \underbrace{\mathbf{t}(F_{\mathfrak{u},k})}_{=\mathbf{t}(F_{\mathfrak{v},k})}\right) = (\mathbf{d}(F_{\mathfrak{v},k}), \mathbf{t}(F_{\mathfrak{v},k}))$. Thus, (2) (applied to $T_1 = F_{\mathfrak{u},k}$ and $T_2 = F_{\mathfrak{v},k}$) yields $F_{\mathfrak{u},k} = F_{\mathfrak{v},k}$.

But $F_{\mathfrak{u},k}$ is the unique point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = k$ (according to the definition of $F_{\mathfrak{u},k}$). Hence, $F_{\mathfrak{u},k}$ is a point $F$ on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$ such that $\mathbf{d}(F) = k$. In particular, $F_{\mathfrak{u},k}$ is a point on the path $\pi_{\mathfrak{u}j_\mathfrak{u}}$. Similarly, $F_{\mathfrak{v},k}$ is a point on the path $\pi_{\mathfrak{v}j_\mathfrak{v}}$. In other words, $F_{\mathfrak{u},k}$ is a point on the path $\pi_{\mathfrak{v}j_\mathfrak{v}}$ (since $F_{\mathfrak{u},k} = F_{\mathfrak{v},k}$). The point $F_{\mathfrak{u},k}$ thus lies on both paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$. Hence, $F_{\mathfrak{u},k}$ is a point in which (at least) two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect (namely, the paths $\pi_{\mathfrak{u}j_\mathfrak{u}}$ and $\pi_{\mathfrak{v}j_\mathfrak{v}}$). Thus, (at least) two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect. But no two of the paths $\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N}$ intersect (since the family $(\pi_{1j_1}, \pi_{2j_2}, \ldots, \pi_{Nj_N})$ is non-intersecting). The previous two sentences contradict each other. This contradiction proves that our assumption was wrong. Thus, (41) is proven.

[24] *Proof of (42):* This statement is precisely (20), and can be proved in the exact same way as (20) was proved.

But recall that $(j_1, j_2, \ldots, j_N)$ is a permutation of $\{1, 2, \ldots, N\}$. Denote this permutation by $\mathbf{j}$. Thus,

$$\mathbf{j}(i) = j_i \qquad \text{for every } i \in \{1, 2, \ldots, N\}. \tag{43}$$

Now, for every $j \in \{1, 2, \ldots, N-1\}$, we have

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right) \qquad \text{for every } k \in \{0, 1, \ldots, n\}. \tag{44}$$

[25] Consequently, if $k \in \{0, 1, \ldots, n\}$, then

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),k}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),k}\right). \tag{45}$$

[25] *Proof of (44):* Let $j \in \{1, 2, \ldots, N-1\}$. Set $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$. We need to prove (44).

Let us (for the sake of contradiction) assume the contrary. Thus, not every $k \in \{0, 1, \ldots, n\}$ satisfies $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. In other words, not every $k \in \{0, 1, \ldots, n\}$ satisfies $\mathbf{t}(F_{\mathfrak{u},k}) \leq \mathbf{t}(F_{\mathfrak{v},k})$ (because $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$). In other words, there exists a $k \in \{0, 1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Let $\ell$ be the **highest** such $k$. Thus, $\ell$ is a $k \in \{0, 1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Hence, $\ell$ belongs to $\{0, 1, \ldots, n\}$ and satisfies $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}(F_{\mathfrak{v},\ell})$. We have $\mathbf{t}(F_{\mathfrak{u},\ell}) \geq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1$ (since $\mathbf{t}(F_{\mathfrak{u},\ell})$ and $\mathbf{t}(F_{\mathfrak{v},\ell})$ are integers and satisfy $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}(F_{\mathfrak{v},\ell})$). In other words, $\mathbf{t}(F_{\mathfrak{v},\ell}) + 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell})$.

Let us first assume (for the sake of contradiction) that $\ell = n$. Then, $F_{\mathfrak{u},\ell} = F_{\mathfrak{u},n} = Q_{j_\mathfrak{u}}$ (by (14), applied to $i = \mathfrak{u}$). But the definition of $\mathbf{j}(\mathfrak{u})$ yields $\mathbf{j}(\mathfrak{u}) = j_\mathfrak{u}$, so that $j_\mathfrak{u} = \mathbf{j}(\mathfrak{u}) = j$ (since $\mathfrak{u} = \mathbf{j}^{-1}(j)$) and thus $Q_{j_\mathfrak{u}} = Q_j$. Also, from $\ell = n$, we obtain $F_{\mathfrak{v},\ell} = F_{\mathfrak{v},n} = Q_{j_\mathfrak{v}}$ (by (14), applied to $i = \mathfrak{v}$). But the definition of $\mathbf{j}(\mathfrak{v})$ yields $\mathbf{j}(\mathfrak{v}) = j_\mathfrak{v}$, so that $j_\mathfrak{v} = \mathbf{j}(\mathfrak{v}) = j+1$ (since $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$) and thus $Q_{j_\mathfrak{v}} = Q_{j+1}$. But $\mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$ (by (17)). Now, $\mathbf{t}\left(\underbrace{F_{\mathfrak{u},\ell}}_{=Q_{j_\mathfrak{u}}=Q_j}\right) = \mathbf{t}(Q_j) < \mathbf{t}(Q_{j+1})$ contradicts $\mathbf{t}(F_{\mathfrak{u},\ell}) > \mathbf{t}\left(\underbrace{F_{\mathfrak{v},\ell}}_{=Q_{j_\mathfrak{v}}=Q_{j+1}}\right) = \mathbf{t}(Q_{j+1})$. This contradiction proves that our assumption (that $\ell = n$) was wrong. Hence, we cannot have $\ell = n$.

Thus, $\ell < n$ (since $\ell \leq n$ but not $\ell = n$) and $0 \leq \ell$ (since $\ell \in \{0, 1, \ldots, n\}$). Hence, $0 \leq \ell < n$, so that $\ell \in \{0, 1, \ldots, n-1\}$ and thus $\ell + 1 \in \{1, 2, \ldots, n\} \subseteq \{0, 1, \ldots, n\}$. Therefore, (41) (applied to $k = \ell + 1$) yields $\mathbf{t}(F_{\mathfrak{u},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{v},\ell+1})$. In other words, $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{u},\ell+1})$.

But $\ell \in \{0, 1, \ldots, n-1\}$. Thus, (16) (applied to $k = \mathfrak{u}$ and $i = \ell$) shows that the pair $(F_{\mathfrak{u},\ell}, F_{\mathfrak{u},\ell+1})$ is an arc of the directed graph $G$. Therefore, (7) (applied to $U = F_{\mathfrak{u},\ell}$ and $V = F_{\mathfrak{u},\ell+1}$) shows that $\mathbf{t}(F_{\mathfrak{u},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{u},\ell}) + 1$. The same argument (but with every $\mathfrak{u}$ replaced by $\mathfrak{v}$) yields $\mathbf{t}(F_{\mathfrak{v},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1$. Thus, $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq \mathbf{t}(F_{\mathfrak{v},\ell}) + 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell}) \leq \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ (since $\mathbf{t}(F_{\mathfrak{u},\ell}) - 1 \leq \mathbf{t}(F_{\mathfrak{u},\ell+1})$). Combined with $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ (which is a consequence of (42), applied to $\ell + 1$ instead of $k$), this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) < \mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$. Since $\mathbf{t}(F_{\mathfrak{v},\ell+1})$ and $\mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1$ are integers, this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \leq (\mathbf{t}(F_{\mathfrak{u},\ell+1}) + 1) - 1 = \mathbf{t}(F_{\mathfrak{u},\ell+1})$.

Combined with $\mathbf{t}(F_{\mathfrak{v},\ell+1}) \neq \mathbf{t}(F_{\mathfrak{u},\ell+1})$, this yields $\mathbf{t}(F_{\mathfrak{v},\ell+1}) < \mathbf{t}(F_{\mathfrak{u},\ell+1})$. In other words, $\mathbf{t}(F_{\mathfrak{u},\ell+1}) > \mathbf{t}(F_{\mathfrak{v},\ell+1})$. Hence, $\ell + 1$ is a $k \in \{0, 1, \ldots, n\}$ such that $\mathbf{t}(F_{\mathfrak{u},k}) > \mathbf{t}(F_{\mathfrak{v},k})$. Since the highest such $k$ is $\ell$ (by the definition of $\ell$), this yields that $\ell + 1 \leq \ell$. But this contradicts $\ell + 1 > \ell$. This contradiction shows that our assumption was wrong. Hence, (44) is proven.

[26]

Now, let us show a chain of inequalities analogous to (17). Namely, we have

$$\mathbf{t}\left(P_1\right) < \mathbf{t}\left(P_2\right) < \cdots < \mathbf{t}\left(P_N\right). \tag{46}$$

[27]

On the other hand, we have

$$F_{i,0} = P_i \qquad \text{for every } i \in \{1,2,\ldots,N\}. \tag{47}$$

[28] Hence, using (45), we can easily see that

$$\mathbf{t}\left(P_{\mathbf{j}^{-1}(1)}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(2)}\right) < \cdots < \mathbf{t}\left(P_{\mathbf{j}^{-1}(N)}\right). \tag{48}$$

[29]

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[26] *Proof of (45):* Let $k \in \{0,1,\ldots,n\}$. Let $j \in \{1,2,\ldots,N-1\}$. Now, (44) yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. But $\mathbf{j}\left(\mathbf{j}^{-1}(j)\right) = j \neq j+1 = \mathbf{j}\left(\mathbf{j}^{-1}(j+1)\right)$, so that $\mathbf{j}^{-1}(j) \neq \mathbf{j}^{-1}(j+1)$. In other words, the positive integers $\mathbf{j}^{-1}(j)$ and $\mathbf{j}^{-1}(j+1)$ are distinct. Thus, (41) (applied to $\mathfrak{u} = \mathbf{j}^{-1}(j)$ and $\mathfrak{v} = \mathbf{j}^{-1}(j+1)$) yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \neq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$. Combined with $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) \leq \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$, this yields $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$.

Now, let us forget that we fixed $j \in \{1,2,\ldots,N-1\}$. Thus, we have shown that $\mathbf{t}\left(F_{\mathbf{j}^{-1}(j),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(j+1),k}\right)$ for every $j \in \{1,2,\ldots,N-1\}$. In other words, $\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),k}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),k}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),k}\right)$. This proves (45).

[27] *Proof of (46):* Let $i \in \{1,2,\ldots,N-1\}$. We will now show that $\mathbf{t}(P_i) < \mathbf{t}(P_{i+1})$.

The definition of $P_i$ yields $P_i = \left((i-1) - \mu'_i, -(i-1) + \mu'_i\right)$, so that $\mathbf{t}(P_i) = \mathbf{t}\left(\left((i-1) - \mu'_i, -(i-1) + \mu'_i\right)\right) = \left((i-1) - \mu'_i\right) - \left(-(i-1) + \mu'_i\right) = 2(i-1) - 2\mu'_i$. The same argument (applied to $i+1$ instead of $i$) yields $\mathbf{t}(P_{i+1}) = 2((i+1)-1) - 2\mu'_{i+1}$. But $\mu'$ is a partition, and thus we have $\mu'_1 \geq \mu'_2 \geq \mu'_3 \geq \cdots$. Hence, $\mu'_i \geq \mu'_{i+1}$. Now,

$$\mathbf{t}(P_i) = 2\left(\underbrace{i}_{<i+1} - 1\right) - 2\underbrace{\mu'_i}_{\geq \mu'_{i+1}} < 2((i+1)-1) - 2\mu'_{i+1} = \mathbf{t}(P_{i+1}).$$

Let us now forget that we fixed $i$. We thus have proven that $\mathbf{t}(P_i) < \mathbf{t}(P_{i+1})$ for every $i \in \{1,2,\ldots,N-1\}$. In other words, $\mathbf{t}(P_1) < \mathbf{t}(P_2) < \cdots < \mathbf{t}(P_N)$. This proves (46).

[28] *Proof of (47):* Let $i \in \{1,2,\ldots,N\}$. Then, $P_i$ is a point on the path $\pi_{ij_i}$ (since $\pi_{ij_i}$ is a path from $P_i$ to $Q_{j_i}$) and satisfies $\mathbf{d}(P_i) = 0$ (by (4)). Hence, $P_i$ is a point $F$ on the path $\pi_{ij_i}$ such that $\mathbf{d}(F) = 0$. But we know (from (8), applied to $k = 0$) that there exists exactly one such point; and we have denoted this point by $F_{i,0}$. Therefore, $P_i = F_{i,0}$. This proves (47).

[29] *Proof of (48):* Applying (45) to $k = 0$, we obtain

$$\mathbf{t}\left(F_{\mathbf{j}^{-1}(1),0}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(2),0}\right) < \cdots < \mathbf{t}\left(F_{\mathbf{j}^{-1}(N),0}\right) \tag{49}$$

(since $0 \in \{0,1,\ldots,n\}$).

Now, fix $\ell \in \{1,2,\ldots,N-1\}$. Then, $\mathbf{t}\left(F_{\mathbf{j}^{-1}(\ell),0}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(\ell+1),0}\right)$ (because of (49)). But $\ell \in \{1,2,\ldots,N-1\} \subseteq \{1,2,\ldots,N\}$, so that $\mathbf{j}^{-1}(\ell) \in \{1,2,\ldots,N\}$; hence, (47) (applied to

Now, let us fix some $k \in \{1, 2, \ldots, N-1\}$. We shall show that $\mathbf{j}(k) \leq \mathbf{j}(k+1)$.

Indeed, assume the contrary. Thus, $\mathbf{j}(k) > \mathbf{j}(k+1)$. Set $\mathfrak{u} = \mathbf{j}(k)$ and $\mathfrak{v} = \mathbf{j}(k+1)$. Thus, $\mathfrak{u} = \mathbf{j}(k) > \mathbf{j}(k+1) = \mathfrak{v}$ (since $\mathfrak{v} = \mathbf{j}(k+1)$), so that $\mathfrak{v} < \mathfrak{u}$. Furthermore, $\mathbf{j}^{-1}(\mathfrak{v}) = k+1$ (since $\mathfrak{v} = \mathbf{j}(k+1)$) and $\mathbf{j}^{-1}(\mathfrak{u}) = k$ (since $\mathfrak{u} = \mathbf{j}(k)$).

But (48) yields $\mathbf{t}\left(P_{\mathbf{j}^{-1}(\mathfrak{p})}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(\mathfrak{q})}\right)$ whenever $\mathfrak{p}$ and $\mathfrak{q}$ are two elements of $\{1, 2, \ldots, N\}$ satisfying $\mathfrak{p} < \mathfrak{q}$. Applying this to $\mathfrak{p} = \mathfrak{v}$ and $\mathfrak{q} = \mathfrak{u}$, we obtain $\mathbf{t}\left(P_{\mathbf{j}^{-1}(\mathfrak{v})}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(\mathfrak{u})}\right)$ (since $\mathfrak{v} < \mathfrak{u}$). This rewrites as $\mathbf{t}(P_{k+1}) < \mathbf{t}(P_k)$ (since $\mathbf{j}^{-1}(\mathfrak{v}) = k+1$ and $\mathbf{j}^{-1}(\mathfrak{u}) = k$).

But (46) yields $\mathbf{t}(P_k) < \mathbf{t}(P_{k+1})$. This contradicts $\mathbf{t}(P_{k+1}) < \mathbf{t}(P_k)$. This contradiction shows that our assumption was false. Hence, $\mathbf{j}(k) \leq \mathbf{j}(k+1)$ is proven.

Now, forget that we fixed $k$. We thus have shown that $\mathbf{j}(k) \leq \mathbf{j}(k+1)$ for each $k \in \{1, 2, \ldots, N-1\}$. In other words, $\mathbf{j}(1) \leq \mathbf{j}(2) \leq \cdots \leq \mathbf{j}(N)$.

But it is well-known (and easy to see) that the only permutation $\sigma$ of $\{1, 2, \ldots, N\}$ that satisfies $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(N)$ is the identity permutation id. In other words, if $\sigma$ is a permutation of $\{1, 2, \ldots, N\}$ that satisfies $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(N)$, then $\sigma = \mathrm{id}$. Applying this to $\sigma = \mathbf{j}$, we obtain $\mathbf{j} = \mathrm{id}$ (since $\mathbf{j}$ is a permutation of $\{1, 2, \ldots, N\}$ that satisfies $\mathbf{j}(1) \leq \mathbf{j}(2) \leq \cdots \leq \mathbf{j}(N)$). Hence, each $i \in \{1, 2, \ldots, N\}$ satisfies

$$\underbrace{\mathbf{j}}_{=\mathrm{id}}(i) = \mathrm{id}(i) = i$$

and therefore $i = \mathbf{j}(i) = j_i$ (by the definition of $\mathbf{j}$). In other words, $(1, 2, \ldots, N) = (j_1, j_2, \ldots, j_N)$. This proves Lemma 9.5. □

$i = \mathbf{j}^{-1}(\ell)$) yields $F_{\mathbf{j}^{-1}(\ell),0} = P_{\mathbf{j}^{-1}(\ell)}$. Also, from $\ell \in \{1, 2, \ldots, N-1\}$, we obtain $\ell + 1 \in \{1, 2, \ldots, N\}$, so that $\mathbf{j}^{-1}(\ell+1) \in \{1, 2, \ldots, N\}$; hence, (47) (applied to $i = \mathbf{j}^{-1}(\ell+1)$) yields $F_{\mathbf{j}^{-1}(\ell+1),0} = P_{\mathbf{j}^{-1}(\ell+1)}$.

Now, recall that $\mathbf{t}\left(F_{\mathbf{j}^{-1}(\ell),0}\right) < \mathbf{t}\left(F_{\mathbf{j}^{-1}(\ell+1),0}\right)$. This rewrites as $\mathbf{t}\left(P_{\mathbf{j}^{-1}(\ell)}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(\ell+1)}\right)$ (since $F_{\mathbf{j}^{-1}(\ell),0} = P_{\mathbf{j}^{-1}(\ell)}$ and $F_{\mathbf{j}^{-1}(\ell+1),0} = P_{\mathbf{j}^{-1}(\ell+1)}$).

Forget that we fixed $\ell$. We thus have proved that $\mathbf{t}\left(P_{\mathbf{j}^{-1}(\ell)}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(\ell+1)}\right)$ for each $\ell \in \{1, 2, \ldots, N-1\}$. In other words,

$$\mathbf{t}\left(P_{\mathbf{j}^{-1}(1)}\right) < \mathbf{t}\left(P_{\mathbf{j}^{-1}(2)}\right) < \cdots < \mathbf{t}\left(P_{\mathbf{j}^{-1}(N)}\right).$$

This proves (48).