noncommutative geometry@n, volume 1 : the tools
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`http://win.ua.ac.be/~lebruyn/LeBruyn2005d.pdf`
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Errata and addenda by Darij Grinberg

The following is a haphazard list of errors I found in "noncommutative geometry@n, volume 1 : the tools" by Lieven Le Bruyn.

All page numbers given below are to be understood as page numbers in the printed book (thus, page 1 is the first page of the Introduction, and not the first page of the PDF file).

11. Errata

- **Page 7:** Add a whitespace before "This" in "relations.This".
- **Page 11:** In the formula "$\sigma_l(\lambda_1, \ldots, \lambda_l) = \sum_{i_1<i_2<\ldots<i_l} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_l}$", replace "$\lambda_l$" by "$\lambda_n$".

  Also, it makes sense to define $\sigma_l(A) := \sigma_l(\lambda_1, \ldots, \lambda_n)$; this notation will be used just a few lines below.
- **Page 18, Theorem 1.3:** Replace "is a regular" by "be a regular".
- **Page 18, proof of Theorem 1.3:** Replace "functions on $\mathbb{C}^m$" by "functions on $\mathbb{C}^n$".
- **Page 18, proof of Theorem 1.3:** Replace "$f' \in \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)]$" by "$f' \in \mathbb{C}[\sigma_1, \ldots, \sigma_n]$, and thus $f' \circ \pi \in \mathbb{C}[\sigma_1(X), \ldots, \sigma_n(X)]$".
- **Page 18, proof of Theorem 1.3:** Replace "Moreover, f and f'" by "Moreover, f and $f' \circ \pi$".
- **Page 18:** Replace "$A \in A$" by "$A \in M_n$".
- **Page 19:** "defined by mapping $\lambda_{j+1}, \ldots, \lambda_j$ to zero" $\to$ "defined by mapping $\lambda_{j+1}, \ldots, \lambda_n$ to zero".
- **Page 19:** In the last displayed formula on this page, replace "$n-i$" by "$j-i$" (in "$\sum_{i=1}^{j} (-1)^i \phi(\sigma_i) \phi(s_{n-i})$").
- **Page 20:** "with diagonal entries $(\lambda_1, \ldots, \lambda_1)$" $\to$ "with diagonal entries $(\lambda_1, \ldots, \lambda_n)$".

- **Page 20, Theorem 1.4:** In the upper right corner of the matrix, replace "x_{nn}" by "x_{1n}".
- **Page 26:** The displayed formula "$\sigma.(v_1 \otimes \cdots \otimes v_m) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$" does **not** define a left action of S_m on $V_n^{\otimes m}$. It should be replaced either by the formula "$(v_1 \otimes \cdots \otimes v_m).\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$" defining a **right** action of S_m on $V_n^{\otimes m}$, or by the formula "$\sigma.(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$" defining a left action of S_m on $V_n^{\otimes m}$. (In either case, I'm afraid it might be necessary to make some more replacements further on.)
- **Page 27, proof of Theorem 1.5:** In the proof of part (2), replace "$End\left(V_n^{\otimes m}\right) \simeq M_{nm}$" by "$End\left(V_n^{\otimes m}\right) \simeq M_{n^m}$".
- **Page 27, proof of Theorem 1.5:** In the proof of part (2), replace "$End_{S_m}\left(V_m^{\otimes m}\right)$" by "$End_{S_m}\left(V_n^{\otimes m}\right)$".
- **Page 27:** In the displayed formula preceding Proposition 1.1, you write: "$\prod_i f_i\left(v_{\sigma'(i)}\right)$". This should be "$\prod_i f_i\left(v_{\sigma(i)}\right)$".
- **Page 27, Proposition 1.1:** Add a tensor sign between "$\cdots$" and "f_m" in "$f_1 \otimes \cdots f_m \otimes v_1 \otimes \cdots \otimes v_m$".

  The same mistake appears in the displayed formula preceding Proposition 1.1. It also appears one line above this formula.

  Furthermore, the displayed formula preceding Proposition 1.1 has another similar mistake: a tensor sign is missing between "$\cdots$" and "$v_{\sigma(m)}$" in "$v_{\sigma(1)} \otimes \cdots v_{\sigma(m)}$".
- **Page 30:** In the first displayed formula on this page, replace "t_{d_1}" by "$t_{d_1}(1)$".
- **Page 30:** In the long displayed formula that follows the words "then we get an expression", replace "$t_{d_1}^{s_{d_1}(1)}$" by "$t_{d_1}(1)^{s_{d_1}(1)}$".
- **Page 31:** Replace "to obtain an $n \times n$ matrix with coefficients in $M_n\left(\mathbb{C}\left[M_n^m\right]\right)$" by "to obtain an $n \times n$ matrix in $M_n\left(\mathbb{C}\left[M_n^m\right]\right)$" (or by "to obtain an $n \times n$ matrix with entries in $\mathbb{C}\left[M_n^m\right]$").
- **Page 32:** Replace "$X_i : M_n^m = M_n \oplus \ldots \oplus M_n^m \to M_n$" by "$X_i : M_n^m = M_n \oplus \ldots \oplus M_n \to M_n$" (there was one "$m$" exponent too much).
- **Page 33, proof of Proposition 1.3:** You write: "By theorem 1.6 we can therefore write

$$tr\left(fX_{m+1}\right) = \sum \underbrace{g_{i_1 \ldots i_l}}_{\in \mathbb{N}_n^m} tr\left(X_{i_1} \ldots X_{i_l} X_{m+1}\right) \qquad (1)$$

". I would suggest being a bit more precise here: First, replace the "$\sum$" sign by "$\sum\limits_{i_1,\ldots,i_l\in\{1,2,\ldots,m\}}$". Secondly, it is worth explaining why $tr\,(fX_{m+1})$ can be represented in the form (1): Namely, theorem 1.6 shows that $tr\,(fX_{m+1})$ can be represented as a polynomial in the invariants $tr\,\left(X_{i_1}\ldots X_{i_l}\right)$ (where $i_1,\ldots,i_l \in \{1,2,\ldots,m+1\}$). In other words, $tr\,(fX_{m+1})$ is a linear combination of products of the form $tr\,(X_{a_1}\ldots X_{a_l})\,tr\,\left(X_{b_1}\ldots X_{b_m}\right)\ldots tr\,\left(X_{h_1}\ldots X_{h_s}\right)$. Furthermore, in each such product, the matrix X_{m+1} occurs exactly once (since $tr\,(fX_{m+1})$ is linear in X_{m+1}); thus, we can push the unique factor in which X_{m+1} occurs to the end of the product and also push the X_{m+1} to the end of this factor. As a result, the product rewrites in the form $\underbrace{g_{i_1\ldots i_l}}_{\in\mathbb{N}_n^m} tr\,\left(X_{i_1}\ldots X_{i_l}X_{m+1}\right)$ for some $i_1,\ldots,i_l \in \{1,2,\ldots,m\}$. Altogether, we obtain a representation in the form (1).

- **Page 34, Proposition 1.4:** Replace "$l \leq 2^n-1$" by "$l \leq 2^n-2$". (I am not saying that "$l \leq 2^n-1$" is false; but your proof yields the better bound "$l \leq 2^n-2$".)

- **Page 34, proof of Proposition 1.4:** Replace "$R = \mathbb{T}_+/\mathbb{N}_+.\mathbb{T}_+$" by "$R = \mathbb{T}_+/\mathbb{N}_+.\mathbb{T}$". (With your definition "$R = \mathbb{T}_+/\mathbb{N}_+.\mathbb{T}_+$", your claim that "for any $x\in R$ we have that $x^n = 0$" would be unfounded, because the last term c_n on the left hand side of the equation "$t^n + c_1t^{n-1} + \ldots + c_n = 0$" does not lie in $\mathbb{N}_+.\mathbb{T}_+$.)

- **Page 34, proof of Proposition 1.4:** Replace "degree at most 2^n-1" by "degree at most 2^n-2".

- **Page 35, Theorem 1.8:** Replace "of length $l \leq 2^n$" by "of length $l \leq 2^n-1$".

- **Page 35, proof of Theorem 1.8:** Replace "of degree at most 2^n-1" by "of degree at most 2^n-2".

- **Page 35, proof of Theorem 1.8:** Replace "of degree at most 2^n." by "of degree at most 2^n-1.".

- **Page 35, proof of Theorem 1.8:** Replace "by all $tr\,(S')$" by "by $tr\,(\mathbb{S}')$".

- **Page 35, proof of Theorem 1.8:** Replace "$\mathbb{N}_+ = \mathbb{N}_n^m tr\,(\mathbb{T}_+)$" by "$\mathbb{N}_+ = \mathbb{N}_n^m tr\,(\mathbb{T}'_+)$".

- **Page 36, Example 1.4:** Replace "$det\,(X_i) = \frac{1}{2}\left(tr\,(X_i)^2 - tr\,(X_i)^2\right)$" by "$det\,(X_i) = \frac{1}{2}\left((tr\,(X_i))^2 - tr\,\left(X_i^2\right)\right)$".

- **Page 37:** On the first line of this page, replace "tableaux" by "tableau".

- **Page 37:** Replace "define different subgroups" by "sometimes define different subgroups". (It is easy to find two distinct tableaux of shape $(2,2)$ which define the same P_λ and the same Q_λ.)
- **Page 37:** In the definition of c_λ, replace "$c_\lambda = a_\lambda.b_\sigma$" by "$c_\lambda = a_\lambda.b_\lambda$".
- **Page 37, Theorem 1.9:** "representations" $\to$ "representation".
- **Page 37, proof of Theorem 1.9:** "consider the tableaux" $\to$ "consider the tableau".
- **Page 37, proof of Theorem 1.9:** "there is an element $p_1 \in T_\lambda$" should be "there is an element $p_1 \in P_\lambda$".
- **Page 37, proof of Theorem 1.9:** "$q' \in \sigma.Q_\lambda, \sigma^{-1}$" should be "$q' \in \sigma.Q_\lambda.\sigma^{-1}$".
- **Page 39:** Do you count $t(1)$ among the generators of $\mathbb{N}^\infty$ or not? (I suspect you don't, but I am not fully sure.)

  Also, I think your map $t : \mathbb{T}^\infty \to \mathbb{N}^\infty$ is not surjective, in spite of you using the $\twoheadrightarrow$ arrow when defining it. In fact, its image is $\mathbb{N}^\infty_+$.
- **Page 41:** You again define an action of S_d on $V^{\otimes d}$ by "$\sigma.(v_1 \otimes \cdots \otimes v_d) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$" (as on page 26). Again, this does not define a valid left action of S_d on $V^{\otimes d}$.
- **Page 41:** Replace "endomorphism $\sum a_\sigma \lambda_\sigma \in End(V^{\otimes m})$" by "endomorphism $\sum a_\sigma \lambda_\sigma \in End\left(V^{\otimes d}\right)$".
- **Page 42:** Replace "Then, the subgroup Q_λ of S_d which preserves each row of λ (or equivalently, each column of λ^*) is" by "Then, the subgroup Q_λ of S_d which preserves each row of λ^* (or equivalently, each column of λ) is".
- **Page 42:** You say that "we have proved" Theorem 1.10. But is this so? I see that you have proved the "if" direction of Theorem 1.10 (since you have shown that any Young symmetrizer c_λ corresponding to a partition λ with at least $n+1$ rows acts as zero on $V^{\otimes d}$). But the "only if" direction does not seem to follow from what you have done.

  Also, have you shown that the Young symmetrizers c_λ relative to partitions λ with at least $n+1$ rows actually span an ideal of $\mathbb{C}S_d$?

  (The "only if" direction can be derived from Corollary 6.6 in William Fulton, Joe Harris, *Representation Theory: A First Course*, Springer 2004. However, this requires some nontrivial representation theory. The claim that the Young symmetrizers c_λ relative to partitions λ with at least $n+1$ rows actually span an ideal of $\mathbb{C}S_d$ can be derived from the "if" and "only if" directions.)
- **Page 43, Theorem 1.10:** Replace "with a least" by "with at least".