Noncommutative determinants, Cauchy–Binet formulae, and Capelli–type identities, I. Generalizations of the Capelli and Turnbull identities

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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the paper by the numbers under which they appear in this paper (specifically, in its version arXiv:0809.3516v2, which is identical to its published version).

8. Errata

- **Proposition 1.2:** The notations "i_α" and "j_β" are undefined. While it isn't hard to guess what they mean, it would be good to explicitly define them: "Write the set I in the form $I = \{i_1 < i_2 < \cdots < i_r\}$, and write the set J in the form $J = \{j_1 < j_2 < \cdots < j_r\}$.".

  The same comment applies to the statements of Proposition 1.2′, Proposition 1.4, Proposition 1.5, Proposition 3.8, Proposition A.1 and Corollary A.3.

- **page 4:** You write: "presuppose that $1 \leq r \leq n$ (otherwise I and J would be nonexistent or empty)". It is not clear to me why you want to avoid the case of I and J being empty, unless your notion of a ring does not assume the existence of a 1 (but in this case, you should probably say this explicitly, and explicitly require I and J to be nonempty in the statements of your main results).

- **page 6, (1.21):** The equation (1.21) does not define a left action of $GL(m) \times GL(n)$ on $K^{m\times n}$. You probably want to replace it by "$X(M, N) = M^T XN$", which defines a **right** action of $GL(m) \times GL(n)$ on $K^{m\times n}$.

- **page 6:** In "faithful representation", remove the word "faithful". (Indeed, the representation of $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ on $K^{m\times n}$ you define is not faithful unless $m = n = 0$, because the elements $\sum_{i=1}^{m} L_{i,i}$ and $\sum_{j=1}^{n} R_{j,j}$ act identically.)

- **page 8, (1.26a):** Replace the "col-det A_{LI}" on the left hand side by a "col-det A_{IL}" (or, equivalently, by a "row-det A_{LI}").

  Let me also show a counterexample for the (non-corrected) version of (1.26a) that you stated:

  First of all, let me set $h = 0$ and $B = I_n$ (the $n \times n$ identity matrix). Then, $Q_{\text{col}} = 0$, $A^T B = A^T = A$ and col-det $B_{LI} = \delta_{LI}$ for all L. Hence, your (non-corrected) version of (1.26a) simplifies to col-det A_{JI} = col-det A_{IJ} in this case. I want to show that this is not (generally) correct. Indeed, let R be the $\mathbb{F}_2$-algebra with generators $a, a', b, b', c, c', d, d'$ and relations

$$[a,d] = 1, \qquad [b,c] = 1, \qquad [b',c'] = 1,$$
$$(\text{all other commutators}) = 0.$$

(This includes $[a',d'] = 0$.) Notice that $1 \neq 0$ in R (indeed, R can be viewed as the Clifford algebra of a symmetric bilinear form in 8 variables over $\mathbb{F}_2$; thus, R is an $\mathbb{F}_2$-vector space of dimension 2^8). Now, let $n = 4$, and let A be the $n \times n$-matrix $\begin{pmatrix} a' & b' & a & c \\ b' & d' & b & d \\ a & b & d' & c' \\ c & d & c' & a' \end{pmatrix}$. It is easy to see that A is column-pseudo-commutative and symmetric. The equalities (1.25) are clearly satisfied (since b_{kl} is always either 0 or 1). Take $I = \{3,4\}$ and $J = \{1,2\}$. Then, col-det A_{JI} = col-det $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$ and col-det A_{IJ} = col-det $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb$ are **not** equal (since $[b,c] = 1 \neq 0$ in R). Thus, (1.26a) cannot be true in the form you stated.

- **page 8, (1.28):** I suspect that this needs a correction similar to my above correction for (1.26a). (I have not checked yet.)
- **page 10, (1.38):** Why is per $\left(A^T B\right)_{IJ}$ well-defined on the right hand side of (1.38) (and on the line below)? The entries of $A^T B$ don't necessarily commute, or do they?
- **page 17:** In the definition of "*column-pseudo-commutative*", replace "$[M_{ij}, M_{kl}] = [M_{il}, M_{jk}]$" by "$[M_{ij}, M_{kl}] = [M_{il}, M_{kj}]$".
- **page 22, Remark:** It is worth explaining that here (and in the following), the letter "h" (without subscripts) denotes the matrix $\left(h_{jl}\right)_{j,l=1}^{n}$.
- **page 23, Corollary 3.5:** The notation used in "$F\left(\{\sigma(j)\}_{j\neq\alpha,\beta}\right)$" and "$G\left(\{\sigma(j)\}_{j\neq\alpha,\beta}\right)$" is a bit nonstandard; I would suggest defining it:

  "For any $\sigma \in \mathcal{S}_r$, we let $\{\sigma(j)\}_{j\neq\alpha,\beta}$ denote the $(r-2)$-tuple obtained from the r-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(r))$ by removing its α-th and β-th entries."
- **page 23, (3.12):** It would be better to rename the index "j" as "k" in (3.12) (just to make the notations more similar to those in (3.11)).

- **page 24:** The equality signs between (3.16a), (3.16b) and (3.16c) are somewhat nontrivial to justify, and I have spent some time trying to understand why they hold. I believe some explanations are warranted here. Let me provide them:

  First, we shall need an analogue of Corollary 3.5:

  Corollary 3.5b. Fix distinct elements $\alpha, \beta \in [r]$ and fix a set I of cardinality $|I| = r$. Let $A = \left(a_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq n}$ be a column-pseudo-commutative $m \times n$-matrix with coefficients in R. Then,

  $$\sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma)\, F\left(\{\sigma(j)\}_{j\neq\alpha,\beta}\right) \left[a_{li_{\sigma(\alpha)}}, a_{ki_{\sigma(\beta)}}\right] G\left(\{\sigma(j)\}_{j\neq\alpha,\beta}\right) = 0 \qquad (1)$$

  for arbitrary functions $F, G : [r]^{r-2} \to R$ and arbitrary indices $l, k \in [m]$.

  Proof of Corollary 3.5b. The column-pseudo-commutativity of A shows that

  $$\left[a_{li_{\sigma(\alpha)}}, a_{ki_{\sigma(\beta)}}\right] = \left[a_{li_{\sigma(\beta)}}, a_{ki_{\sigma(\alpha)}}\right]$$

  for any $\sigma \in \mathcal{S}_r$ and any $l, k \in [m]$. This means that the summand in (1) [excluding the factor of $\operatorname{sgn}(\sigma)$] is invariant under $\sigma \mapsto \sigma \circ (\alpha\beta)$. The claim then follows immediately from the Involution Lemma. □

  Now, let me explain the two equality signs:

Proof of the equality sign between (3.16a) and (3.16b). We have

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_1,\ldots,l_r\in[m]} a_{l_1 i_{\sigma(1)}}\Bigg[b_{l_1 j_1} a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}} \\
&\qquad - \sum_{s=2}^{r} h_{i_{\sigma(s)} j_1}\delta_{l_1 l_s} a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}}\Bigg] b_{l_2 j_2}\cdots b_{l_r j_r} \\
&= \sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\underbrace{\sum_{l_1,\ldots,l_r\in[m]} a_{l_1 i_{\sigma(1)}} b_{l_1 j_1} a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}}_{=\left(\sum_{l_1\in[m]} a_{l_1 i_{\sigma(1)}} b_{l_1 j_1}\right)\left(\sum_{l_2,\ldots,l_r\in[m]} a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}\right)} \\
&\qquad - \sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\underbrace{\sum_{l_1,\ldots,l_r\in[m]} a_{l_1 i_{\sigma(1)}}\sum_{s=2}^{r} h_{i_{\sigma(s)} j_1}\delta_{l_1 l_s} a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}}_{=\sum_{l_2,\ldots,l_r\in[m]}\sum_{s=2}^{r}\sum_{l_1\in[m]} a_{l_1 i_{\sigma(1)}} h_{i_{\sigma(s)} j_1}\delta_{l_1 l_s} a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}} \\
&= \sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\underbrace{\left(\sum_{l_1\in[m]} a_{l_1 i_{\sigma(1)}} b_{l_1 j_1}\right)}_{=(A^T B)_{i_{\sigma(1)} j_1}}\left(\sum_{l_2,\ldots,l_r\in[m]} a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}\right) \\
&\qquad - \sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\sum_{s=2}^{r}\underbrace{\sum_{l_1\in[m]} a_{l_1 i_{\sigma(1)}} h_{i_{\sigma(s)} j_1}\delta_{l_1 l_s}}_{=a_{l_s i_{\sigma(1)}} h_{i_{\sigma(s)} j_1}} a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2}\cdots b_{l_r j_r}
\end{aligned}$$

$$= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \left(A^T B\right)_{i_{\sigma(1)} j_1} \left(\sum_{l_2,\ldots,l_r \in [m]} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \right)$$

$$- \underbrace{\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \sum_{l_2,\ldots,l_r \in [m]} \sum_{s=2}^{r} a_{l_s i_{\sigma(1)}} h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}}_{= \sum_{l_2,\ldots,l_r \in [m]} \sum_{s=2}^{r} \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) a_{l_s i_{\sigma(1)}} h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}}$$

$$= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \left(A^T B\right)_{i_{\sigma(1)} j_1} \left(\sum_{l_2,\ldots,l_r \in [m]} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \right)$$

$$- \sum_{l_2,\ldots,l_r \in [m]} \sum_{s=2}^{r} \underbrace{\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) a_{l_s i_{\sigma(1)}} h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}}_{\substack{= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) h_{i_{\sigma(s)} j_1} a_{l_s i_{\sigma(1)}} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \\ \text{(here, we have used Corollary 3.5 to push the factor } h_{i_{\sigma(s)} j_1} \\ \text{to the left of the factor } a_{l_s i_{\sigma(1)}})}}$$

$$= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \left(A^T B\right)_{i_{\sigma(1)} j_1} \left(\sum_{l_2,\ldots,l_r \in [m]} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \right)$$

$$- \sum_{l_2,\ldots,l_r \in [m]} \sum_{s=2}^{r} \underbrace{\sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) h_{i_{\sigma(s)} j_1} a_{l_s i_{\sigma(1)}} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}}_{\substack{= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_s i_{\sigma(1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \\ \text{(here, we have repeatedly used Corollary 3.5b to push the factor } a_{l_s i_{\sigma(1)}} \\ \text{to the right until it settles between the factors } a_{l_{s-1} i_{\sigma(s-1)}} \text{ and } a_{l_{s+1} i_{\sigma(s+1)}})}}$$

$$\begin{aligned}
&= \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) \left(A^T B\right)_{i_{\sigma(1)} j_1} \left(\sum_{l_2,\ldots,l_r\in[m]} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \right) \\
&\qquad - \sum_{l_2,\ldots,l_r\in[m]} \sum_{s=2}^{r} \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) \, h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_s i_{\sigma(1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r} \\
&= \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma) \sum_{l_2,\ldots,l_r\in[m]} \left[\left(A^T B\right)_{i_{\sigma(1)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} \right. \\
&\qquad \left. - \sum_{s=2}^{r} h_{i_{\sigma(s)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_{s-1} i_{\sigma(s-1)}} a_{l_s i_{\sigma(1)}} a_{l_{s+1} i_{\sigma(s+1)}} \cdots a_{l_r i_{\sigma(r)}} \right] b_{l_2 j_2} \cdots b_{l_r j_r}.
\end{aligned}$$

This proves the equality sign between (3.16a) and (3.16b). □

Proof of the equality sign between (3.16b) and (3.16c). We have

$$
\begin{aligned}
&\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\Bigg[\left(A^TB\right)_{i_{\sigma(1)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}}\\
&\qquad -\sum_{s=2}^{r}h_{i_{\sigma(s)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1}i_{\sigma(s-1)}}a_{l_s i_{\sigma(1)}}a_{l_{s+1}i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}}\Bigg]b_{l_2 j_2}\cdots b_{l_r j_r}\\
&=\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\left(A^TB\right)_{i_{\sigma(1)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}}b_{l_2 j_2}\cdots b_{l_r j_r}\\
&\qquad \underbrace{-\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\sum_{s=2}^{r}h_{i_{\sigma(s)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1}i_{\sigma(s-1)}}a_{l_s i_{\sigma(1)}}a_{l_{s+1}i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}}b_{l_2 j_2}\cdots b_{l_r j_r}}_{\substack{=-\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\sum_{s=2}^{r}h_{i_{\sigma(1)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1}i_{\sigma(s-1)}}a_{l_s i_{\sigma(s)}}a_{l_{s+1}i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}}b_{l_2 j_2}\cdots b_{l_r j_r}\\ \text{(here, we have interchanged }\sigma(1)\text{ with }\sigma(s)\text{ (that is, substituted }\sigma\circ(1s)\text{ for }\sigma\text{), which}\\ \text{multiplies }\operatorname{sgn}(\sigma)\text{ by }-1\text{)}}}\\
&=\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\left(A^TB\right)_{i_{\sigma(1)}j_1}a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}}b_{l_2 j_2}\cdots b_{l_r j_r}\\
&\qquad +\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma)\sum_{l_2,\ldots,l_r\in[m]}\sum_{s=2}^{r}h_{i_{\sigma(1)}j_1}\underbrace{a_{l_2 i_{\sigma(2)}}\cdots a_{l_{s-1}i_{\sigma(s-1)}}a_{l_s i_{\sigma(s)}}a_{l_{s+1}i_{\sigma(s+1)}}\cdots a_{l_r i_{\sigma(r)}}}_{=a_{l_2 i_{\sigma(2)}}\cdots a_{l_r i_{\sigma(r)}}}b_{l_2 j_2}\cdots b_{l_r j_r}
\end{aligned}
$$

$$= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \sum_{l_2,\ldots,l_r \in [m]} \left(A^T B\right)_{i_{\sigma(1)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}$$
$$+ \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \sum_{l_2,\ldots,l_r \in [m]} \sum_{s=2}^{r} h_{i_{\sigma(1)} j_1} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}$$
$$= \sum_{\sigma \in \mathcal{S}_r} \operatorname{sgn}(\sigma) \left[\left(A^T B\right)_{i_{\sigma(1)} j_1} + \sum_{s=2}^{r} h_{i_{\sigma(1)} j_1} \right] \sum_{l_2,\ldots,l_r \in [m]} a_{l_2 i_{\sigma(2)}} \cdots a_{l_r i_{\sigma(r)}} b_{l_2 j_2} \cdots b_{l_r j_r}.$$

This proves the equality sign between (3.16b) and (3.16c). □

- **page 24, (3.17a):** Replace "det $\left(A^T\right)_{IL}$" by "col-det $\left(A^T\right)_{IL}$".
- **page 24, (3.18c):** I think the equality sign between (3.18b) and (3.18c) needs a more detailed proof. More precisely, I think that (3.18b) is a distraction, as it is not a logical stepping stone between (3.18a) and (3.18c); instead, the equality between (3.18a) and (3.18c) should be proven as follows:

  We begin with a lemma:

  Lemma 2.6′. If the square matrix M has weakly column-symmetric commutators, then:

  (a) The row-determinant is antisymmetric under permutation of rows, i.e.,

  $$\operatorname{row-det}\left({}^{\tau}M\right) = \operatorname{sgn}(\tau) \operatorname{row-det} M$$

  for any permutation τ.

  (b) If M has two equal rows, then $2 \operatorname{row-det} M = 0$.

  (c) If M has two equal rows and the elements in those rows commute among themselves, then $\operatorname{row-det} M = 0$.

  Proof of Lemma 2.6′. Lemma 2.6′ follows from Lemma 2.6 (applied to M^T instead of M). □

  Proof of the equality between (3.18a) and (3.18c): The matrix B is column-pseudo-commutative, and thus has column-symmetric commutators, and therefore has weakly column-symmetric commutators. For every $\tau \in \mathcal{S}_r$ and every $L \subseteq [m]$ satisfying $|L| = r$, the matrix B_{LJ} has weakly column-symmetric commutators (since B has weakly column-symmetric commutators); hence, Lemma 2.6′ **(a)** (applied to $M = B_{LJ}$) yields

  $$\operatorname{row-det}\left({}^{\tau}\left(B_{LJ}\right)\right) = \operatorname{sgn}(\tau) \operatorname{row-det} B_{LJ}. \tag{2}$$

Now,

$$\begin{aligned}
&\sum_L \underbrace{\left(\operatorname{row-det}\left(A^T\right)_{IL}\right)}_{=\sum_{\tau\in\mathcal{S}_r}\operatorname{sgn}(\tau) a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r}} (\operatorname{row-det} B_{LJ}) \\
&= \sum_L \sum_{\tau\in\mathcal{S}_r} \operatorname{sgn}(\tau)\, a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r} (\operatorname{row-det} B_{LJ}) \\
&= \sum_L \sum_{\tau\in\mathcal{S}_r} a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r} \underbrace{\operatorname{sgn}(\tau)\,(\operatorname{row-det} B_{LJ})}_{\substack{=\operatorname{row-det}\left({}^{\tau}\left(B_{LJ}\right)\right)\\ \text{(by (2))}}} \\
&= \sum_L \sum_{\tau\in\mathcal{S}_r} a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r} \underbrace{\operatorname{row-det}\left({}^{\tau}\left(B_{LJ}\right)\right)}_{=\sum_{\sigma\in\mathcal{S}_r}\operatorname{sgn}(\sigma) b_{l_{\tau(1)}j_{\sigma(1)}}\cdots b_{l_{\tau(r)}j_{\sigma(r)}}} \\
&= \sum_L \sum_{\tau\in\mathcal{S}_r} a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r} \sum_{\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma)\, b_{l_{\tau(1)}j_{\sigma(1)}}\cdots b_{l_{\tau(r)}j_{\sigma(r)}} \\
&= \sum_L \sum_{\tau,\sigma\in\mathcal{S}_r} \operatorname{sgn}(\sigma)\, a_{l_{\tau(1)}i_1}\cdots a_{l_{\tau(r)}i_r} b_{l_{\tau(1)}j_{\sigma(1)}}\cdots b_{l_{\tau(r)}j_{\sigma(r)}}.
\end{aligned}$$

This proves the equality between (3.18a) and (3.18c). □

- **page 25, (3.22a):** Replace "det B_{LJ}" by "row-det B_{LJ}".
- **page 26, Example 3.7:** Replace "the left-hand side of the identity" by "the left-hand side of the identity (1.9)" (otherwise it isn't clear what identity you mean).
- **page 27, Remarks:** In Remark 2, you write: "the replacements $A \to PAQ$ and $B \to RAS$". I am not sure, but I suspect you mean "$B \to RBS$" instead of "$B \to RAS$".

  (I have to admit I generally don't understand Remark 2.)
- **page 29, Lemma 4.1:** In (4.1b), replace "h_{jl}" by "δ_{jl}".
- **page 29, proof of Proposition 1.4:** In (4.4), replace "col-det A_{LI}" by "col-det A_{IL}". (Also, I don't think you need to say that "The first two steps in the proof are identical to those in Proposition 3.1". In order to justify (4.4), it is sufficient to observe that (4.4) follows from (3.13) because of $A^T = A$.)
- **page 30:** In (4.5a), replace "col-det A_{LI}" by "col-det A_{IL}".
- **page 30:** In (4.6a), replace "$(\det A_{LI})\,(\det B_{LJ})$" by "$(\operatorname{col-det} A_{IL})\,(\operatorname{col-det} B_{LJ})$" (or, equivalently, by "$(\operatorname{row-det} A_{LI})\,(\operatorname{col-det} B_{LJ})$").
- **pages 35–36:** Replace "Konstant" by "Kostant" several times on these pages.

- **page 39, Remark:** In Remark 2, you claim that "$\left[a_{ij}, a_{kl}\right] = 0$ and $\left[a_{ij}, b_{kl}\right] = -\delta_{ik} h_{jl}$ for all i, j, k, l *implies* $\left[a_{ij}, h_{kl}\right] = 0$ for all i, j, k, l, *provided that* $n \geq 2$". Let me give a quick proof of this claim:

  We have assumed that
  $$\left[a_{ij}, a_{kl}\right] = 0 \qquad \text{for all } i, j, k, l \tag{3}$$
  and that
  $$\left[a_{ij}, b_{kl}\right] = -\delta_{ik} h_{jl} \qquad \text{for all } i, j, k, l. \tag{4}$$

  Now, fix i, j, k, l, and assume that $n \geq 2$. We must prove that $\left[a_{ij}, h_{kl}\right] = 0$.

  There exists some i' such that $i' \neq i$ (since $n \geq 2$). Consider such an i'. From $i' \neq i$, we obtain $\delta_{ii'} = 0$. Now, (4) (applied to i' instead of k) yields $\left[a_{ij}, b_{i'l}\right] = -\underbrace{\delta_{ii'}}_{=0} h_{jl} = 0$. Also, (3) (applied to i', k, i and j instead of i, j, k and l) yields $\left[a_{i'k}, a_{ij}\right] = 0$.

  But (4) (applied to i', k and i' instead of i, j and k) yields $\left[a_{i'k}, b_{i'l}\right] = -\underbrace{\delta_{i'i'}}_{=1} h_{kl} = -h_{kl}$, so that $h_{kl} = -\left[a_{i'k}, b_{i'l}\right] = \left[b_{i'l}, a_{i'k}\right]$. Now, the Jacobi identity yields
  $$\left[a_{ij}, \left[b_{i'l}, a_{i'k}\right]\right] + \left[b_{i'l}, \left[a_{i'k}, a_{ij}\right]\right] + \left[a_{i'k}, \left[a_{ij}, b_{i'l}\right]\right] = 0.$$
  Hence,
  $$\begin{aligned} 0 &= \left[a_{ij}, \underbrace{\left[b_{i'l}, a_{i'k}\right]}_{=h_{kl}}\right] + \left[b_{i'l}, \underbrace{\left[a_{i'k}, a_{ij}\right]}_{=0}\right] + \left[a_{i'k}, \underbrace{\left[a_{ij}, b_{i'l}\right]}_{=0}\right] \\ &= \left[a_{ij}, h_{kl}\right] + \underbrace{\left[b_{i'l}, 0\right]}_{=0} + \underbrace{\left[a_{i'k}, 0\right]}_{=0} = \left[a_{ij}, h_{kl}\right]. \end{aligned}$$
  Hence, $\left[a_{ij}, h_{kl}\right] = 0$ is proven. □

9. Addenda

- **page 27:** Let me add an alternative proof of Proposition 3.8; it will derive this proposition from Proposition 1.2:

  Second proof of Proposition 3.8. Let K be the subset $\{x \in R \mid 2x = 0\}$ of R. Then, K is an ideal of R (this is straightforward to check). Let π be the canonical projection $R \to R/K$; this projection π is a ring homomorphism. For any $u \in \mathbb{N}$ and $v \in \mathbb{N}$, the ring homomorphism $\pi : R \to R/K$ induces a map $\pi^{u \times v} : R^{u \times v} \to (R/K)^{u \times v}$ that sends every matrix $\left(c_{i,j}\right)_{1 \leq i \leq u,\ 1 \leq j \leq v} \in R^{u \times v}$ to the matrix $\left(\pi\left(c_{i,j}\right)\right)_{1 \leq i \leq u,\ 1 \leq j \leq v} \in (R/K)^{u \times v}$.

Now, let us prove Proposition 3.8 (a). So we assume that A has column-symmetric commutators. Thus, the matrix $\pi^{m\times n}(A)$ is column-pseudo-commutative[1]. Applying the map π to the equality (3.25), we can easily obtain $\left[\pi\left(a_{i,j}\right), \pi\left(b_{k,l}\right)\right] = -\delta_{i,k}\pi\left(h_{j,l}\right)$ for all i, j, k, l. Hence, Proposition 1.2 **(a)** (applied to R/K, $\pi^{m\times n}(A)$, $\pi^{m\times n}(B)$ and $\left(\pi\left(h_{j,l}\right)\right)_{j,l=1}^{n}$ instead of R, A, B and $\left(h_{j,l}\right)_{j,l=1}^{n}$) shows that

$$\sum_{\substack{L\subseteq[m];\\ |L|=r}} \left(\operatorname{col-det}\left(\left(\pi^{m\times n}(A)\right)^{T}\right)_{IL}\right)\left(\operatorname{col-det}\left(\pi^{m\times n}(B)\right)_{LJ}\right)$$
$$= \operatorname{col-det}\left[\left(\left(\pi^{m\times n}(A)\right)^{T}\pi^{m\times n}(B)\right)_{IJ} + \widetilde{Q}_{\text{col}}\right], \qquad (5)$$

where
$$\left(\widetilde{Q}_{\text{col}}\right)_{\alpha\beta} = (r-\beta)\,\pi\left(h_{i_\alpha,j_\beta}\right) \qquad \text{for } 1\leq\alpha,\beta\leq r.$$

[1] *Proof.* In order to see this, we must prove the following two statements:

Statement 1: We have $\left[\pi\left(a_{i,j}\right), \pi\left(a_{k,l}\right)\right] = \left[\pi\left(a_{i,l}\right), \pi\left(a_{k,j}\right)\right]$ for all i, j, k, l.

Statement 2: We have $\left[\pi\left(a_{i,j}\right), \pi\left(a_{i,l}\right)\right] = 0$ for all i, j, l.

Proof of Statement 1: Let i, j, k, l be arbitrary. Then, $\left[a_{i,j}, a_{k,l}\right] = \left[a_{i,l}, a_{k,j}\right]$ (since the matrix A has column-symmetric commutators). Now, π is a ring homomorphism; thus,

$$\left[\pi\left(a_{i,j}\right), \pi\left(a_{k,l}\right)\right] = \pi\left(\underbrace{\left[a_{i,j}, a_{k,l}\right]}_{=\left[a_{i,l}, a_{k,j}\right]}\right) = \pi\left(\left[a_{i,l}, a_{k,j}\right]\right) = \left[\pi\left(a_{i,l}\right), \pi\left(a_{k,j}\right)\right]$$

(again since π is a ring homomorphism). This proves Statement 1.

Proof of Statement 2: Let i, j, l be arbitrary. The matrix A has column-symmetric commutators; thus, $\left[a_{i,j}, a_{k,l}\right] = \left[a_{i,l}, a_{k,j}\right]$ for every k. Applying this to $k = i$, we obtain $\left[a_{i,j}, a_{i,l}\right] = \left[a_{i,l}, a_{i,j}\right] = -\left[a_{i,j}, a_{i,l}\right]$. In other words, $2\left[a_{i,j}, a_{i,l}\right] = 0$. In other words, $\left[a_{i,j}, a_{i,l}\right] \in K$ (by the definition of K). Hence, $\pi\left(\left[a_{i,j}, a_{i,l}\right]\right) = 0$ (since π is the projection $R \to R/K$). But π is a ring homomorphism; thus, $\left[\pi\left(a_{i,j}\right), \pi\left(a_{i,l}\right)\right] = \pi\left(\left[a_{i,j}, a_{i,l}\right]\right) = 0$. This proves Statement 2.

Since π is a ring homomorphism, we have

$$\sum_{\substack{L\subseteq[m];\\ |L|=r}} \underbrace{\left(\operatorname{col-det}\left(\left(\pi^{m\times n}\left(A\right)\right)^{T}\right)_{IL}\right)}_{=\pi\left(\operatorname{col-det}\left(A^{T}\right)_{IL}\right)} \underbrace{\left(\operatorname{col-det}\left(\pi^{m\times n}\left(B\right)\right)_{LJ}\right)}_{=\pi\left(\operatorname{col-det}B_{LJ}\right)}$$
$$= \sum_{\substack{L\subseteq[m];\\ |L|=r}} \pi\left(\operatorname{col-det}\left(A^{T}\right)_{IL}\right)\pi\left(\operatorname{col-det}B_{LJ}\right)$$
$$= \pi\left(\sum_{\substack{L\subseteq[m];\\ |L|=r}} \left(\operatorname{col-det}\left(A^{T}\right)_{IL}\right)\left(\operatorname{col-det}B_{LJ}\right)\right)$$

and

$$\operatorname{col-det}\left[\underbrace{\left(\left(\pi^{m\times n}\left(A\right)\right)^{T}\pi^{m\times n}\left(B\right)\right)_{IJ}}_{=\pi^{r\times r}\left(A^{T}B\right)} + \underbrace{\widetilde{Q}_{\mathrm{col}}}_{=\pi^{r\times r}\left(Q_{\mathrm{col}}\right)}\right]$$
$$= \operatorname{col-det}\left[\underbrace{\pi^{r\times r}\left(\left(A^{T}B\right)_{IJ}\right) + \pi^{r\times r}\left(Q_{\mathrm{col}}\right)}_{=\pi^{r\times r}\left(\left(A^{T}B\right)_{IJ}+Q_{\mathrm{col}}\right)}\right]$$
$$= \operatorname{col-det}\left[\pi^{r\times r}\left(\left(A^{T}B\right)_{IJ} + Q_{\mathrm{col}}\right)\right] = \pi\left(\operatorname{col-det}\left[\left(A^{T}B\right)_{IJ} + Q_{\mathrm{col}}\right]\right).$$

Thus, (5) rewrites as

$$\pi\left(\sum_{\substack{L\subseteq[m];\\ |L|=r}} \left(\operatorname{col-det}\left(A^{T}\right)_{IL}\right)\left(\operatorname{col-det}B_{LJ}\right)\right)$$
$$= \pi\left(\operatorname{col-det}\left[\left(A^{T}B\right)_{IJ} + Q_{\mathrm{col}}\right]\right).$$

In other words,

$$\sum_{\substack{L\subseteq[m];\\ |L|=r}} \left(\operatorname{col-det}\left(A^{T}\right)_{IL}\right)\left(\operatorname{col-det}B_{LJ}\right)$$
$$\equiv \operatorname{col-det}\left[\left(A^{T}B\right)_{IJ} + Q_{\mathrm{col}}\right] \bmod K.$$

In other words,

$$2 \sum_{\substack{L \subseteq [m]; \\ |L| = r}} \left(\operatorname{col-det} \left(A^T \right)_{IL} \right) \left(\operatorname{col-det} B_{LJ} \right)$$
$$= 2 \operatorname{col-det} \left[\left(A^T B \right)_{IJ} + Q_{\mathrm{col}} \right]$$

(because two elements x and y of R satisfy $x \equiv y \bmod K$ if and only if they satisfy $2x = 2y$). This proves Proposition 3.8 (a). The proof of Proposition 3.8 (b) is similar. □

- **page 40, (A.17):** You are asking how to derive (A.17) from the Capelli identity. Let me sketch such a derivation. Before I do so, let me state a few lemmas:

  Lemma A.5. Let R be a (not-necessarily-commutative) ring. Let A be an $n \times n$-matrix with elements in R. Suppose that (A.1a) holds for all i, j, k, l. For any subset I of $[n]$, we let $\sum I$ denote the sum of all elements of I, and we let I^c denote the complement $[n] \setminus I$ of I. Let $r \in \mathbb{N}$. Let K and L be two r-element subsets of $[n]$. Then,

  $$\sum_{\substack{I \subseteq [n]; \\ |I| = r}} (-1)^{\sum K + \sum I} \left(\det A_{LI} \right) \left(\det A_{K^c I^c} \right) = \delta_{K,L} \det A.$$

  Proof of Lemma A.5. The equalities (A.1a) show that the entries $a_{i,j}$ of the matrix A mutually commute. Hence, the $\mathbb{Z}$-subalgebra of R generated by these entries $a_{i,j}$ is commutative. We can therefore WLOG assume that the ring R is commutative (because we can replace the ring R by this commutative $\mathbb{Z}$-subalgebra). Assume this.

  Now that R is commutative, Lemma A.5 becomes a well-known theorem (known as Laplace expansion in multiple rows, or multi-row Laplace expansion).[2] □

[2] In more details:

- In the case when $K = L$, the claim of Lemma A.5 says that

  $$\sum_{\substack{I \subseteq [n]; \\ |I| = r}} (-1)^{\sum L + \sum I} \left(\det A_{LI} \right) \left(\det A_{L^c I^c} \right) = \det A.$$

  This is [Grinbe16, Theorem 6.156 **(a)**] (applied to $P = L$).

- In the case when $K \neq L$, the claim of Lemma A.5 says that

  $$\sum_{\substack{I \subseteq [n]; \\ |I| = r}} (-1)^{\sum K + \sum I} \left(\det A_{LI} \right) \left(\det A_{K^c I^c} \right) = 0.$$

[*Remark:* Lemma A.5 remains valid even if we loosen its assumptions somewhat: Namely, we only need to require (A.1a) to hold for all i, j, k, l satisfying $j \neq l$ (as opposed to for all i, j, k, l). Proving this necessitates a more complicated argument, though.]

Lemma A.6. Let R, n, A, B and H be as in Proposition A.1. Let $r \in \mathbb{N}$. Let K and J be two subsets of $[n]$ such that $|K| = |J| = r$. Let s be a nonnegative integer. For every r-element subset I of $[n]$, define an $r \times r$-matrix $Q_{\operatorname{col},I,J}$ by

$$\left(Q_{\operatorname{col},I,J}\right)_{\alpha,\beta} = (r-\beta)\, h_{i_\alpha, j_\beta} \qquad \text{for all } 1 \leq \alpha \leq r \text{ and } 1 \leq \beta \leq r.$$

(In other words, $Q_{\operatorname{col},I,J}$ is the matrix that was denoted by Q_{col} in (A.2).) Then,

$$\begin{aligned}&(\det A)\left(\operatorname{col-det} B_{KJ}\right)(\det A)^s \\ &= (\det A)^s \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K + \sum I} \left(\det A_{K^c I^c}\right) \operatorname{col-det}\left[\left(A^T B + sH\right)_{IJ} + Q_{\operatorname{col},I,J}\right].\end{aligned}$$

Proof of Lemma A.6. The equalities (A.1a) show that the entries $a_{i,j}$ of the matrix A mutually commute. Hence, the $\mathbb{Z}$-subalgebra of R generated by these entries $a_{i,j}$ is commutative. Let R' denote this commutative $\mathbb{Z}$-subalgebra. Then, A is an $n \times n$-matrix over R'. Hence, all minors of A are elements of R', and therefore commute with each other (since R' is commutative). Moreover, any r-element subset I of $[n]$ satisfies

$$\det \underbrace{\left(A^T\right)_{IL}}_{=(A_{LI})^T} = \det\left(\left(A_{LI}\right)^T\right) = \det A_{LI} \tag{6}$$

(again because all entries of A lie in the commutative $\mathbb{Z}$-algebra R', and therefore the standard rules for determinants apply to A).

If I is an r-element subset of $[n]$, then

$$\begin{aligned}&\sum_{\substack{L\subseteq[n];\\ |L|=r}} \underbrace{\left(\det A_{LI}\right)}_{\substack{=\det\left(A^T\right)_{IL} \\ \text{(by (6))}}} \left(\operatorname{col-det} B_{LJ}\right)(\det A)^s \\ &= \sum_{\substack{L\subseteq[n];\\ |L|=r}} \left(\det\left(A^T\right)_{IL}\right)\left(\operatorname{col-det} B_{LJ}\right)(\det A)^s \\ &= (\det A)^s \operatorname{col-det}\left[\left(A^T B + sH\right)_{IJ} + Q_{\operatorname{col},I,J}\right]. \end{aligned} \tag{7}$$

This is [Grinbe16, Exercise 6.45 **(a)**] (applied to $P = K$ and $R = L$).

Thus, Lemma A.5 is proven in both cases.

(Indeed, this is simply the claim of Proposition A.1, because our $Q_{\operatorname{col},I,J}$ is the matrix Q_{col} from Proposition A.1.)

Now,

$$\begin{aligned}
&(\det A)^s \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} (\det A_{K^c I^c}) \operatorname{col-det}\left[\left(A^T B + sH\right)_{IJ} + Q_{\operatorname{col},I,J}\right] \\
&= \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} \underbrace{(\det A)^s (\det A_{K^c I^c})}_{\substack{=(\det A_{K^c I^c})(\det A)^s \\ \text{(since all minors of } A \\ \text{commute with each other)}}} \operatorname{col-det}\left[\left(A^T B + sH\right)_{IJ} + Q_{\operatorname{col},I,J}\right] \\
&= \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} (\det A_{K^c I^c}) \underbrace{(\det A)^s \operatorname{col-det}\left[\left(A^T B + sH\right)_{IJ} + Q_{\operatorname{col},I,J}\right]}_{\substack{= \sum_{\substack{L\subseteq[n];\\ |L|=r}} (\det A_{LI})\left(\operatorname{col-det} B_{LJ}\right)(\det A)^s \\ \text{(by (7))}}} \\
&= \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} (\det A_{K^c I^c}) \sum_{\substack{L\subseteq[n];\\ |L|=r}} (\det A_{LI}) (\operatorname{col-det} B_{LJ}) (\det A)^s \\
&= \sum_{\substack{L\subseteq[n];\\ |L|=r}} \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} \underbrace{(\det A_{K^c I^c}) (\det A_{LI})}_{\substack{=(\det A_{LI})(\det A_{K^c I^c}) \\ \text{(since all minors of } A \\ \text{commute with each other)}}} (\operatorname{col-det} B_{LJ}) (\det A)^s \\
&= \sum_{\substack{L\subseteq[n];\\ |L|=r}} \underbrace{\sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I} (\det A_{LI}) (\det A_{K^c I^c})}_{\substack{=\delta_{K,L} \det A \\ \text{(by Lemma A.5)}}} (\operatorname{col-det} B_{LJ}) (\det A)^s \\
&= \sum_{\substack{L\subseteq[n];\\ |L|=r}} \delta_{K,L} (\det A) (\operatorname{col-det} B_{LJ}) (\det A)^s = (\det A) (\operatorname{col-det} B_{KJ}) (\det A)^s
\end{aligned}$$

(because the $\delta_{K,L}$ factor in the sum has the effect of annihilating all addends except for the addend for $L = K$). This proves Lemma A.6. □

We can slightly simplify the statement of Lemma A.6 when H is the identity matrix:

Lemma A.7. Let R, n, A, B and H be as in Proposition A.1. Assume that $H = I_n$ (so that $h_{i,j} = \delta_{i,j}$ for all i and j). Let $r \in \mathbb{N}$. Let K and J be two subsets of $[n]$ such that $|K| = |J| = r$. Let s be a nonnegative integer. For every r-element subset I of $[n]$, define an $r \times r$-matrix $Q'_{\operatorname{col},I,J}$ by

$$\left(Q'_{\operatorname{col},I,J}\right)_{\alpha,\beta} = (r + s - \beta)\, \delta_{i_\alpha, j_\beta} \qquad \text{for all } 1 \le \alpha \le r \text{ and } 1 \le \beta \le r.$$

Then,

$$\begin{aligned}
&(\det A)\,(\operatorname{col-det} B_{KJ})\,(\det A)^s \\
&= (\det A)^s \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I}\,(\det A_{K^c I^c})\operatorname{col-det}\left[\left(A^T B\right)_{IJ} + Q'_{\mathrm{col},I,J}\right].
\end{aligned}$$

Proof of Lemma A.7. For every r-element subset I of $[n]$, define an $r\times r$-matrix $Q_{\mathrm{col},I,J}$ as in Lemma A.6. Then, for every r-element subset I of $[n]$, we have

$$\begin{aligned}
Q_{\mathrm{col},I,J} &= \left((r-\beta)\quad \underbrace{h_{i_\alpha,j_\beta}}_{\substack{=\delta_{i_\alpha,j_\beta}\\ (\text{since } H=I_n)}} \right)_{\alpha,\beta=1}^{r} \qquad \left(\text{by the definition of } Q_{\mathrm{col},I,J}\right) \\
&= \left((r-\beta)\,\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r}.
\end{aligned}$$

Now, for every r-element subset I of $[n]$, we have

$$\begin{aligned}
&s\underbrace{H_{IJ}}_{\substack{=\left(\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r}\\ (\text{since } H=I_n)}} + \underbrace{Q_{\mathrm{col},I,J}}_{=\left((r-\beta)\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r}} \\
&= s\left(\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r} + \left((r-\beta)\,\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r} = \left(\underbrace{s\delta_{i_\alpha,j_\beta} + (r-\beta)\,\delta_{i_\alpha,j_\beta}}_{=(r+s-\beta)\delta_{i_\alpha,j_\beta}}\right)_{\alpha,\beta=1}^{r} \\
&= \left((r+s-\beta)\,\delta_{i_\alpha,j_\beta}\right)_{\alpha,\beta=1}^{r} = Q'_{\mathrm{col},I,J} \qquad \left(\text{by the definition of } Q'_{\mathrm{col},I,J}\right)
\end{aligned} \tag{8}$$

and therefore

$$\left(A^T B + sH\right)_{IJ} + Q_{\mathrm{col},I,J} = \left(A^T B\right)_{IJ} + \underbrace{sH_{IJ} + Q_{\mathrm{col},I,J}}_{\substack{=Q'_{\mathrm{col},I,J}\\ (\text{by (8)})}} = \left(A^T B\right)_{IJ} + Q'_{\mathrm{col},I,J}. \tag{9}$$

Now, Lemma A.6 yields

$$\begin{aligned}
&(\det A)\left(\operatorname{col-det} B_{KJ}\right)(\det A)^{s} \\
&= (\det A)^{s} \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I}\left(\det A_{K^c I^c}\right) \operatorname{col-det}\left[\underbrace{\left(A^T B + sH\right)_{IJ} + Q_{\mathrm{col},I,J}}_{=(A^T B)_{IJ} + Q'_{\mathrm{col},I,J}}\right] \\
&= (\det A)^{s} \sum_{\substack{I\subseteq[n];\\ |I|=r}} (-1)^{\sum K+\sum I}\left(\det A_{K^c I^c}\right) \operatorname{col-det}\left[\left(A^T B\right)_{IJ} + Q'_{\mathrm{col},I,J}\right].
\end{aligned}$$

This proves Lemma A.7. □

Lemma A.8. Let K be a commutative ring. Let R be a K-algebra. Let M be a left R-module. Let $v \in M$. Let $k \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_k$ be k elements of R such that, for each $i \in [k]$, we have $a_i v \in Kv$. Let $b_1, b_2, \ldots, b_k$ be k elements of R such that, for each $i \in [k]$, we have $b_i v = 0$. Then,

$$(a_1 + b_1)(a_2 + b_2)\cdots(a_k + b_k)\, v = a_1 a_2 \cdots a_k v.$$

First proof of Lemma A.8 (sketched). The elements $b_1, b_2, \ldots, b_k$ annihilate v (because for each $i \in [k]$, we have $b_i v = 0$). The elements $a_1, a_2, \ldots, a_k$ each multiply v by a scalar factor (since for each $i \in [k]$, the element $a_i v$ of M is a scalar multiple of v). Hence, if we expand the sum $(a_1 + b_1)(a_2 + b_2)\cdots(a_k + b_k)\, v$, then all addends except for $a_1 a_2 \cdots a_k v$ vanish. Consequently, the sum equals $a_1 a_2 \cdots a_k v$. This proves Lemma A.8. □

Second proof of Lemma A.8. Let us give a more rigorous proof of Lemma A.8. We shall show that

$$(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)\, v = a_1 a_2 \cdots a_n v \tag{10}$$

for every $n \in \{0, 1, \ldots, k\}$.

[*Proof of (10):* We shall prove (10) by induction over n:

Induction base: If $n = 0$, then both products $(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)$ and $a_1 a_2 \cdots a_n$ are empty and thus equal 1. Hence, if $n = 0$, then both sides of (10) equal v. Hence, (10) is proven in the case when $n = 0$. This completes the induction base.

Induction step: Let $i \in \{0, 1, \ldots, k\}$ be positive. Assume that (10) holds for $n = i - 1$. We must prove that (10) holds for $n = i$.

We have assumed that (10) holds for $n = i - 1$. In other words, we have

$$(a_1 + b_1)(a_2 + b_2)\cdots(a_{i-1} + b_{i-1})\, v = a_1 a_2 \cdots a_{i-1} v.$$

Now, recall that $a_i v \in Kv$ (by one of the assumptions of Lemma A.8). In other words, $a_i v = \lambda v$ for some $\lambda \in K$. Consider this λ. Also, recall that $b_i v = 0$ (by one of the assumptions of Lemma A.8). Now, $(a_i + b_i) v = \underbrace{a_i v}_{=\lambda v} + \underbrace{b_i v}_{=0} = \lambda v$. Now,

$$\begin{aligned}
&(a_1 + b_1)(a_2 + b_2) \cdots (a_i + b_i) v \\
&= (a_1 + b_1)(a_2 + b_2) \cdots (a_{i-1} + b_{i-1}) \underbrace{(a_i + b_i) v}_{=\lambda v} \\
&= (a_1 + b_1)(a_2 + b_2) \cdots (a_{i-1} + b_{i-1}) \lambda v \\
&= \lambda \underbrace{(a_1 + b_1)(a_2 + b_2) \cdots (a_{i-1} + b_{i-1}) v}_{=a_1 a_2 \cdots a_{i-1} v} = a_1 a_2 \cdots a_{i-1} \underbrace{\lambda v}_{=a_i v} \\
&= a_1 a_2 \cdots a_{i-1} a_i v = a_1 a_2 \cdots a_i v.
\end{aligned}$$

In other words, (10) holds for $n = i$. This completes the induction step. Hence, (10) is proven by induction.]

Now, we can apply (10) to $n = k$. We thus obtain $(a_1 + b_1)(a_2 + b_2) \cdots (a_k + b_k) v = a_1 a_2 \cdots a_k v$. This proves Lemma A.8. □

Lemma A.9. Let K be a commutative ring. Let R be a K-algebra. Let M be a left R-module. Let $v \in M$. Let $k \in \mathbb{N}$. Let $A = (a_{i,j})_{i,j=1}^k$ and $B = (b_{i,j})_{i,j=1}^k$ be two $k \times k$-matrices over R. Assume that

$$a_{i,j} v \in Kv \tag{11}$$

for each $(i, j) \in [k]^2$. Assume that

$$b_{i,j} v = 0 \tag{12}$$

for each $(i, j) \in [k]^2$. Then,

$$(\operatorname{col-det}(A + B))(v) = (\operatorname{col-det} A)(v).$$

Proof of Lemma A.9. We have $A + B = (a_{i,j} + b_{i,j})_{i,j=1}^k$ (since $A = (a_{i,j})_{i,j=1}^k$ and $B = (b_{i,j})_{i,j=1}^k$). Thus, the definition of col-det $(A + B)$ yields

$$\begin{aligned}
&\operatorname{col-det}(A + B) \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left(a_{\sigma(1),1} + b_{\sigma(1),1}\right) \left(a_{\sigma(2),2} + b_{\sigma(2),2}\right) \cdots \left(a_{\sigma(k),k} + b_{\sigma(k),k}\right).
\end{aligned}$$

If we let both sides of this equality act on $v \in M$, then we obtain

$$\begin{aligned}
&(\operatorname{col-det}(A+B))(v) \\
&= \sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma) \underbrace{\left(a_{\sigma(1),1} + b_{\sigma(1),1}\right)\left(a_{\sigma(2),2} + b_{\sigma(2),2}\right) \cdots \left(a_{\sigma(k),k} + b_{\sigma(k),k}\right) v}_{\substack{= a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(k),k} v \\ \text{(by Lemma A.8, applied to } a_{\sigma(i),i} \text{ and } b_{\sigma(i),i} \text{ instead of } a_i \text{ and } b_i \\ \text{(because for each } i \in [k] \text{, we know that } a_{\sigma(i),i} v \in Kv \\ \text{(by (11)) and } b_{\sigma(i),i} v = 0 \text{ (by (12))))}}} \\
&= \underbrace{\sum_{\sigma \in \mathcal{S}_k} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(k),k}}_{\substack{= \operatorname{col-det} A \\ \text{(since this is how } \operatorname{col-det} A \text{ is defined)}}} v = (\operatorname{col-det} A)(v).
\end{aligned}$$

This proves Lemma A.9. □

We can finally prove (A.17) itself (with I renamed as K):

Theorem A.10. Let X, x_{ij}, ∂ and ∂_{ij} be as in Corollary A.3. Let s be a nonnegative integer. Let $k \in \mathbb{N}$. Let K and J be two subsets of $[n]$ of cardinality $|I| = |J| = k$. Then,

$$\begin{aligned}
&(\det \partial_{KJ}) (\det X)^s \\
&= s(s+1) \cdots (s+k-1) (\det X)^{s-1} \epsilon(K, J) (\det X_{K^c J^c}).
\end{aligned}$$

Sketch of a proof of Theorem A.10. WLOG assume that $s > 0$ (else, the claim is trivial).

Let R be the Weyl algebra $A_{n \times n}(K)$. Thus, X and ∂ are $n \times n$-matrices over R. It is easy to see that Lemma A.7 can be applied to $A = X$, $B = \partial$ and $H = I_n$. For every $r \in \mathbb{N}$ and every two r-element subsets I and J of $[n]$, define an $r \times r$-matrix $Q'_{\mathrm{col},I,J}$ by

$$\left(Q'_{\mathrm{col},I,J}\right)_{\alpha,\beta} = (r + s - \beta) \delta_{i_\alpha, j_\beta} \qquad \text{for all } 1 \leq \alpha \leq r \text{ and } 1 \leq \beta \leq r.$$

Then, every $r \in \mathbb{N}$ and every two r-element subsets I and J of $[n]$ satisfy

$$\operatorname{col-det} Q'_{\mathrm{col},I,J} = \delta_{I,J} s (s+1) \cdots (s+r-1) \tag{13}$$

[3].

[3]*Proof of (13):* Let $r \in \mathbb{N}$. Let I and J be two r-element subsets of $[n]$. We must prove (13).

We are in one of the following two cases:

Case 1: We have $I = J$.

Case 2: We have $I \neq J$.

Let us consider Case 1 first. In this case, we have $I = J$. Thus, every $\alpha \in [r]$ and $\beta \in [r]$

Now, Lemma A.7 (applied to $A = X$, $B = \partial$, $H = I_n$ and $r = k$) yields that for every two subsets K and J of $[n]$ satisfying $|K| = |J| = k$, we have

$$(\det X)\left(\operatorname{col-det} \partial_{KJ}\right)(\det X)^s$$
$$= (\det X)^s \sum_{\substack{I \subseteq [n]; \\ |I| = k}} (-1)^{\sum K + \sum I} \left(\det X_{K^c I^c}\right) \operatorname{col-det}\left[\left(X^T \partial\right)_{IJ} + Q'_{\mathrm{col},I,J}\right].$$

Applying both sides of this equality to the polynomial $1 \in K[X]$, we obtain

$$(\det X)\left(\operatorname{col-det} \partial_{KJ}\right)(\det X)^s$$
$$= (\det X)^s \sum_{\substack{I \subseteq [n]; \\ |I| = k}} (-1)^{\sum K + \sum I} \left(\det X_{K^c I^c}\right) \left(\operatorname{col-det}\left[\left(X^T \partial\right)_{IJ} + Q'_{\mathrm{col},I,J}\right]\right)(1). \tag{14}$$

---

satisfy

$$\begin{aligned}
\left(Q'_{\mathrm{col},I,J}\right)_{\alpha,\beta} &= (r+s-\beta)\,\delta_{i_\alpha,j_\beta} && \left(\text{by the definition of } Q'_{\mathrm{col},I,J}\right) \\
&= (r+s-\beta)\,\underbrace{\delta_{j_\alpha,j_\beta}}_{=\delta_{\alpha,\beta}} && \left(\text{since } i_\alpha = j_\alpha \text{ (since } I = J)\right) \\
&= (r+s-\beta)\,\delta_{\alpha,\beta}.
\end{aligned}$$

Hence, $Q'_{\mathrm{col},I,J}$ is a diagonal matrix with diagonal entries $r+s-1, r+s-2, \ldots, r+s-r$. Consequently, its column-determinant is

$$\begin{aligned}
\operatorname{col-det} Q'_{\mathrm{col},I,J} &= (r+s-1)(r+s-2)\cdots(r+s-r) \\
&= (r+s-1)(r+s-2)\cdots s = s(s+1)\cdots(s+r-1) \\
&= \delta_{I,J}\, s(s+1)\cdots(s+r-1)
\end{aligned}$$

(since $\underbrace{\delta_{I,J}}_{=1} s(s+1)\cdots(s+r-1) = s(s+1)\cdots(s+r-1)$). Thus, (13) is proven in Case 1.

Let us now consider Case 2. In this case, we have $I \neq J$. Hence, there exists some $g \in I$ such that $g \notin J$ (since $|I| = r = |J|$). Consider this g. Clearly, $g = i_\alpha$ for some $\alpha \in [r]$ (since $g \in I$). Consider this α. We have $i_\alpha = g \notin J$. Hence, there exists no $\beta \in [r]$ such that $i_\alpha = j_\beta$. In other words, $\delta_{i_\alpha,j_\beta} = 0$ for each $\beta \in [r]$. Thus, $(r+s-\beta)\underbrace{\delta_{i_\alpha,j_\beta}}_{=0} = 0$ for each $\beta \in [r]$. Hence, the whole α-th row of the matrix $Q'_{\mathrm{col},I,J}$ consists of zeroes (since the β-th entry of this row is $\left(Q'_{\mathrm{col},I,J}\right)_{\alpha,\beta} = (r+s-\beta)\,\delta_{i_\alpha,j_\beta} = 0$ for each $\beta \in [r]$). Thus, the column-determinant of this matrix is

$$\operatorname{col-det} Q'_{\mathrm{col},I,J} = 0 = \delta_{I,J}\, s(s+1)\cdots(s+r-1)$$

(since $\underbrace{\delta_{I,J}}_{=0} s(s+1)\cdots(s+r-1) = 0$). Thus, (13) is proven in Case 2.

We have now shown that (13) holds in each of the Cases 1 and 2. Thus, (13) always holds.

Now, fix a k-element subset I of $[n]$, and consider the polynomial

$$\left(\operatorname{col-det}\left[\left(X^T\partial\right)_{IJ}+Q'_{\mathrm{col},I,J}\right]\right)(1)\in K[X].$$

The $k\times k$-matrix $Q'_{\mathrm{col},I,J}$ has the property that, for each $(\alpha,\beta)\in[k]^2$, we have $\left(Q'_{\mathrm{col},I,J}\right)_{\alpha,\beta}(1)\in K1$ [4]. The $k\times k$-matrix $\left(X^T\partial\right)_{IJ}$ has the property that, for each $(\alpha,\beta)\in[k]^2$, we have $\left(\left(X^T\partial\right)_{IJ}\right)_{\alpha,\beta}(1)=0$ [5]. Hence, Lemma A.9 (applied to $M=K[X]$, $A=Q'_{\mathrm{col},I,J}$, $B=\left(X^T\partial\right)_{IJ}$ and $v=1$) shows that

$$\left(\operatorname{col-det}\left[Q'_{\mathrm{col},I,J}+\left(X^T\partial\right)_{IJ}\right]\right)(1)=\left(\operatorname{col-det}Q'_{\mathrm{col},I,J}\right)(1).$$

In other words,

$$\begin{aligned}\left(\operatorname{col-det}\left[\left(X^T\partial\right)_{IJ}+Q'_{\mathrm{col},I,J}\right]\right)(1)&=\underbrace{\left(\operatorname{col-det}Q'_{\mathrm{col},I,J}\right)}_{\substack{=\delta_{I,J}s(s+1)\cdots(s+k-1)\\ \text{(by (13), applied to }r=k)}}(1)\\&=\left(\delta_{I,J}s\left(s+1\right)\cdots\left(s+k-1\right)\right)(1)\\&=\delta_{I,J}s\left(s+1\right)\cdots\left(s+k-1\right).\qquad(15)\end{aligned}$$

Now, forget that we fixed I. We thus have proven (15) for each k-element

[4] since $\left(Q'_{\mathrm{col},I,J}\right)_{\alpha,\beta}=(r+s-\beta)\,\delta_{i_\alpha,j_\beta}\in K$

[5] since $\left(\left(X^T\partial\right)_{IJ}\right)_{\alpha,\beta}=\left(X^T\partial\right)_{i_\alpha,j_\beta}=\sum_{l=1}^{n}x_{l,i_\alpha}\partial_{l,j_\beta}$ and thus

$$\underbrace{\left(\left(X^T\partial\right)_{IJ}\right)_{\alpha,\beta}}_{=\sum_{l=1}^{n}x_{l,i_\alpha}\partial_{l,j_\beta}}(1)=\sum_{l=1}^{n}x_{l,i_\alpha}\underbrace{\partial_{l,j_\beta}(1)}_{\substack{=0\\ \text{(because each of the derivations }\partial_{u,v}\\ \text{annihilates the constant polynomial 1)}}}=\sum_{l=1}^{n}x_{l,i_\alpha}0=0$$

subset I of $[n]$. Now, (14) becomes

$$\begin{aligned}
&(\det X)\left(\operatorname{col-det}\partial_{KJ}\right)(\det X)^{s} \\
&= (\det X)^{s} \sum_{\substack{I\subseteq[n];\\ |I|=k}} (-1)^{\sum K+\sum I} \left(\det X_{K^c I^c}\right) \underbrace{\left(\operatorname{col-det}\left[\left(X^T\partial\right)_{IJ} + Q'_{\text{col},I,J}\right]\right)(1)}_{\substack{=\delta_{I,J}s(s+1)\cdots(s+k-1)\\ \text{(by (15))}}} \\
&= (\det X)^{s} \sum_{\substack{I\subseteq[n];\\ |I|=k}} (-1)^{\sum K+\sum I} \left(\det X_{K^c I^c}\right) \delta_{I,J} s\left(s+1\right)\cdots\left(s+k-1\right) \\
&= (\det X)^{s} \underbrace{(-1)^{\sum K+\sum J}}_{=\epsilon(K,J)} \left(\det X_{K^c J^c}\right) s\left(s+1\right)\cdots\left(s+k-1\right) \\
&\qquad \left(\begin{array}{c}\text{since the factor } \delta_{I,J} \text{ annihilates all addends in the sum,}\\ \text{except for the addend for } I=J\end{array}\right) \\
&= (\det X)^{s} \epsilon\left(K,J\right) \left(\det X_{K^c J^c}\right) s\left(s+1\right)\cdots\left(s+k-1\right).
\end{aligned}$$

We can cancel $\det X$ from this equality (since it is an equality inside $K[X]$); we thus obtain

$$\begin{aligned}
&\left(\operatorname{col-det}\partial_{KJ}\right)(\det X)^{s} \\
&= (\det X)^{s-1} \epsilon\left(K,J\right) \left(\det X_{K^c J^c}\right) s\left(s+1\right)\cdots\left(s+k-1\right) \\
&= s\left(s+1\right)\cdots\left(s+k-1\right)(\det X)^{s-1} \epsilon\left(K,J\right)\left(\det X_{K^c J^c}\right).
\end{aligned}$$

Since col-det $\partial_{KJ} = \det\partial_{KJ}$ (because all entries of the matrix ∂_{KJ} commute with each other), this rewrites as

$$\begin{aligned}
&\left(\det\partial_{KJ}\right)(\det X)^{s} \\
&= s\left(s+1\right)\cdots\left(s+k-1\right)(\det X)^{s-1} \epsilon\left(K,J\right)\left(\det X_{K^c J^c}\right).
\end{aligned}$$

This proves Theorem A.10. □

References

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