Birational rowmotion on a rectangle over a noncommutative ring

(detailed version)

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Abstract. We extend the periodicity of birational rowmotion for rectangular posets to the case when the base field is replaced by a noncommutative ring (under appropriate conditions). This resolves a conjecture from 2014. The proof uses a novel approach and is fully self-contained.

Consider labelings of a finite poset P by $|P| + 2$ elements of a ring $\mathbb{K}$: one label associated with each poset element and two constant labels for the added top and bottom elements in $\widehat{P}$. *Birational rowmotion* is a partial map on such labelings. It was originally defined by Einstein and Propp [EinPro13] for $\mathbb{K} = \mathbb{R}$ as a lifting (via detropicalization) of *piecewise-linear rowmotion*, a map on the order polytope $\mathcal{O}(P) := \{\text{order-preserving } f : P \to [0,1]\}$. The latter, in turn, extends the well-studied rowmotion map on the set of order ideals (or more properly, the set of order filters) of P, which correspond to the vertices of $\mathcal{O}(P)$. Dynamical properties of these combinatorial maps sometimes (but not always) extend to the birational level, while results proven at the birational level always imply their combinatorial counterparts. Allowing $\mathbb{K}$ to be noncommutative, we generalize the birational level even further, and some properties are in fact lost at this step.

In 2014, the authors gave the first proof of periodicity for birational rowmotion on rectangular posets (when P is a product of two chains) for $\mathbb{K}$ a field, and conjectured that it survives (in an appropriately twisted form) in the noncommutative case. In this paper, we prove this noncommutative periodicity and a concomitant antipodal reciprocity formula. We end with some conjectures about periodicity for other posets, and the question of whether our results can be extended to (noncommutative) semirings.

It has been observed by Glick and Grinberg that, in the commutative case, periodicity of birational rowmotion can be used to derive Zamolodchikov periodicity in the type AA case, and vice-versa. However, for noncommutative $\mathbb{K}$, Zamolodchikov periodicity fails even in small examples (no matter what order the factors are multiplied), while noncommutative birational rowmotion continues to exhibit periodicity. Thus, our result can be viewed as a lateral generalization of Zamolodchikov periodicity to the noncommutative setting.

Keywords: rowmotion; posets; noncommutative rings; semirings; Zamolodchikov periodicity; root systems; promotion; trees; graded posets; Grassmannian; tropicalization.

Contents

Introduction

The goal of this paper is to extend the periodicity of birational rowmotion for rectangular posets to the case when the base field is replaced by a noncommutative ring (under appropriate conditions). This resolves a conjecture from 2014. The proof uses a novel approach (even in the commutative case) and is fully self-contained.

Let P be a finite poset, and let $\widehat{P}$ be the same poset with two extra elements added: one global minimum and one global maximum. For the time being, let $\mathbb{K}$ be a field. A $\mathbb{K}$-*labeling of* P means a map from $\widehat{P}$ to $\mathbb{K}$; we view it as a way of labeling each element of $\widehat{P}$ by an element of $\mathbb{K}$. Birational rowmotion, as studied conventionally, is a rational map R on such labelings (i.e., a rational map $R : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$). It was introduced by Einstein and Propp [EinPro13] for $\mathbb{K} = \mathbb{R}$, generalizing (via the tropical limit[1]) the well-studied *combinatorial rowmotion* map on order ideals of P [BrSchr74, StWi11, ProRob13, ThoWil19].

Birational rowmotion can be defined as a composition of "toggles": For each $v \in P$, we define the v-*toggle* as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ that modifies a $\mathbb{K}$-labeling f by changing the label $f(v)$ to[2]

$$\left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot (f(v))^{-1} \cdot \left(\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} (f(u))^{-1} \right)^{-1},$$

[1] See [Kirill00, Section 4.1] for what we mean by the "tropical limit" here, and [KirBer95] for one of the earliest example of detropicalization (i.e., the generalization of a combinatorial map to a rational one).

[2] The notations $\lessdot$ and $\gtrdot$ mean "covered by" and "covers", respectively (see Sections 1 and 3 for details).

while leaving all the other labels of f unchanged. Now, birational rowmotion R is the composition of all the v-toggles, where v runs over the poset P from top to bottom. (That is, we pick a linear extension $(v_1, v_2, \ldots, v_n)$ of P, and set $R = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n}$.)

Dynamical properties at the combinatorial level sometimes extend to higher levels, while results proven at the birational level always imply their combinatorial counterparts. In particular, while combinatorial rowmotion always has finite order (since it is an invertible map on a finite set), there is no reason to expect periodicity at all at the higher levels. Indeed, for many nice posets, birational rowmotion has infinite order, including for the Boolean algebra of order 3 (or those in [Roby15, Fig. 6]), and there are only a few infinite classes where it appears to have finite order (mostly posets associated with representation theory, e.g., root or minuscule posets). In these cases the order of birational rowmotion is generally the *same* as for combinatorial rowmotion, e.g., $p+q$ for $P = [p] \times [q]$.

In 2014, the authors gave the first proof of periodicity of birational rowmotion for rectangular posets (i.e., when P is a product of two chains) and $\mathbb{K}$ a field [GriRob14]. The main idea of this proof was to embed the space of labelings into an appropriate Grassmannian (where in each "sufficiently generic" $\mathbb{K}$-labeling, the labels can be expressed as ratios of certain minors of a matrix) and use particular Plücker relations to derive the result. There were several serious technical hurdles to overcome.

The definition of birational rowmotion relies entirely on addition, multiplication and inverses in $\mathbb{K}$. Thus, it is natural to extend it to the case when $\mathbb{K}$ is a ring (not necessarily commutative), or even just a semiring. (At this level, birational rowmotion is no longer a rational map, just a partial map.) However, there is no guarantee that the properties of birational rowmotion survive at this level for every poset; and indeed, sometimes they do not (see, e.g., Example 13.9). However, in 2014, the authors experimentally observed that the periodicity for rectangular posets appears to hold even in this noncommutative setting, as long as it is appropriately modified: After $p+q$ iterations of birational rowmotion, the labels are not returned to their original states, but rather to certain "twisted variants" thereof (resembling, but not the same as, conjugates). See Example 3.19 to get the sense of this.

Strikingly, this noncommutative generalization has resisted all approaches that have previously succeeded in the commutative case. The determinantal computations involved in the proof in [GriRob14] can be extended to the noncommutative setting using the *quasideterminants* of Gelfand and Retakh, but it seems impossible to make a rigorous proof out of it (lacking, e.g., any useful notation of Zariski topology in this setting, it is not clear what it means for a $\mathbb{K}$-labeling to be "generic"). The alternative proof of commutative periodicity found by Musiker and Roby [MusRob17] (via a lattice-path formula for iterates of birational rowmotion) could not be generalized as well. Thus the noncommutative case remained an open problem.[3]

[3]This is not the first time that rational maps in algebraic combinatorics have been generalized to the noncommutative case; some other instances are [IyuShk14, BerRet15, Rupel17, GonKon21]. Each time, the generalizations have been much harder to prove, not least because very little of the commutative groundwork is (currently?) available at the noncommutative level. For instance, it is insufficient to work over the "free skew fields", since an identity between rational expressions can be true in all

At some point, Glick and Grinberg noticed that the Y-variables in the type-AA Zamolodchikov periodicity theorem of Volkov [Volk06] could be written as ratios of labels under iterated birational rowmotion [Roby15, § 4.4]; this allows the periodicity in one setting to be derived from that in the other (with some work). However, for noncommutative $\mathbb{K}$, Zamolodchikov periodicity fails even in small examples such as $r = r' = 2$ (no matter what order we multiply the factors), while noncommutative birational rowmotion continues to exhibit periodicity. This approach is therefore unavailable in the noncommutative case as well.

In this paper, we prove the periodicity of birational rowmotion and a concomitant antipodal reciprocity formula over an arbitrary noncommutative ring. The proof proceeds from first principles, by studying certain values V_ℓ^v and A_ℓ^v and their products along paths in the rectangle. At the core of the proof is a "conversion lemma" (Lemma 9.2), which provides an identity between a certain sum of V_ℓ^v products and a certain sum of A_ℓ^v products for the same ℓ; this equality does not actually depend on the concept of rowmotion and might be of interest on its own. Another important step is the reduction of the reciprocity claim to the labels on the "lower boundary" of the rectangle (i.e., to the labels at the elements of the form $(i, 1)$ and $(1, j)$). This reduction requires subtraction, which is why we are only addressing the case of a ring, not of a semiring; the latter remains open.

A few words are in order about the relation between our birational rowmotion and a parallel construction. Combinatorial rowmotion seems first to have been defined not on the set $J(P)$ of order ideals of P, but rather on the set $\mathcal{A}(P)$ of *antichains* of P [BrSchr74]. The standard bijection between $J(P)$ and $\mathcal{A}(P)$ (by taking maximal elements of $I \in J(P)$ or saturating down from an antichain) makes it easy to go between the two maps and to see that they have the same periodicity. However, some dynamic properties (e.g., homomesy) that depend on the sets themselves are not so easily translated. Just as Einstein and Propp lifted combinatorial rowmotion on $J(P)$ to a birational map and we continued to the noncommutative context, Joseph and Roby did a parallel lifting on the antichain side: from antichain rowmotion to piecewise-linear rowmotion on the *chain polytope*, $\mathcal{C}(P)$, to birational antichain rowmotion, and finally to noncommutative antichain rowmotion [JosRob20, JosRob21]. In particular they lifted "transfer maps" (originally defined by Stanley to go between $\mathcal{O}(P)$ and $\mathcal{C}(P)$ [Stan86]) from the piecewise-linear to the birational and noncommutative realms. These serve as equivariant bijections at each level, thus showing that periodicity at each level is equivalent for the order-ideal and antichain liftings. But they were unable to find a new proof of periodicity for the piecewise-linear and higher levels, relying instead on the periodicity results for birational order-ideal rowmotion to deduce it for birational antichain rowmotion. They also lifted a useful invariant, the *Stanley–Thomas word*, which cyclically rotates with antichain rowmotion at each level. At the combinatorial level, this gives an equivariant bijection that proves periodicity [ProRob14, § 3.3.2]; however, it is no longer a bijection at

skew fields yet fail in some noncommutative rings (such as the identity $x(yx)^{-1}y = 1$). For this reason, while natural from an algebraic point of view, the noncommutative setting is only recently and slowly getting explored.

higher levels. Our paper completes the story in the case of a ring: Via the transfer maps mentioned above, the periodicity of noncommutative birational order-ideal rowmotion entails the periodicity of noncommutative birational antichain rowmotion.

The paper is structured in a fairly straightforward way: In the first sections (Sections 1 to 3), we introduce our noncommutative setup and define birational rowmotion in it. These include technicalities about partial maps and the definition of noncommutative toggles. In Section 4, we state our main results. In the sections that follow, we build an arsenal of lemmas to prove these results; the proofs are completed in Section 11. (The structure of the proof is outlined at the end of Section 4.) In Sections 12 and 13, we discuss avenues for further work: a possible generalization to semirings and conjectured periodicity claims for other posets. In the final Section 14, we apply our techniques to arbitrary posets (not just rectangles), obtaining two identities.

A 12-page survey of the results of this paper (with the main steps of the proof outlined) can be found in the extended abstract [GriRob23].

0.1. Remark on the level of detail

This paper comes in two versions: a regular one and a more detailed one. The regular version is optimized for readability, leaving out the more straightforward parts and technical arguments. The more detailed version has many of them expanded.

This is the more detailed version of the paper. The two versions share the same .tex file, with the only difference that there are two lines in the preamble of the file which need to be modified in order to switch between the short and the detailed version. Namely, these lines are

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\excludecomment{verlong}
\includecomment{vershort}
```

for the short version and

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\includecomment{verlong}
\excludecomment{vershort}
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for the detailed one. It is also available on the arXiv page of this paper.

0.2. Acknowledgments

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Computations using the SageMath computer algebra system [S+09] provided essential data for us to conjecture some of the results.

1. Linear extensions of posets

This section collects a few standard notions concerning posets and their linear extensions, needed to define the main characters of our paper. Readers familiar with the subject may wish to skip forward to Section 2 or Section 3. We start by defining general notations identical with those in [GriRob14], to which we refer the reader for commentary and comparison to other references.

Convention 1.1. We let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

Definition 1.2. Let P be a poset, and $u, v \in P$.

(a) We will use the symbols $\leqslant$, $<$, $\geqslant$ and $>$ to denote the lesser-or-equal relation, the lesser relation, the greater-or-equal relation and the greater relation, respectively, of the poset P. (Thus, for example, "$u < v$" means "u is smaller than v with respect to the partial order on P".)

(b) The elements u and v of P are said to be *incomparable* if we have neither $u \leqslant v$ nor $u \geqslant v$.

(c) We write $u \lessdot v$ if we have $u < v$ and there is no $w \in P$ such that $u < w < v$. One often says that "u is covered by v" to signify that $u \lessdot v$.

(d) We write $u \gtrdot v$ if we have $u > v$ and there is no $w \in P$ such that $u > w > v$. (Thus, $u \gtrdot v$ holds if and only if $v \lessdot u$.) One often says that "u covers v" to signify that $u \gtrdot v$.

(e) An element u of P is called *maximal* if every $w \in P$ satisfying $w \geqslant u$ satisfies $w = u$. In other words, an element u of P is called *maximal* if there is no $w \in P$ such that $w > u$.

(f) An element u of P is called *minimal* if every $w \in P$ satisfying $w \leqslant u$ satisfies $w = u$. In other words, an element u of P is called *minimal* if there is no $w \in P$ such that $w < u$.

These notations may become ambiguous when an element belongs to several different posets simultaneously. In such cases, we will disambiguate them by adding the words "in P" (where P is the poset which we want to use).[4]

Convention 1.3. From now on, for the rest of the paper, **we fix a finite poset** P. Most of our results will concern the case when P has a rather specific form (viz., a rectangular poset, i.e., a Cartesian product of two finite chains), but we do not assume this straightaway.

Definition 1.4. A *linear extension* of P will mean a list $(v_1, v_2, \ldots, v_m)$ of the elements of P such that

- each element of P occurs exactly once in this list, and
- any $i, j \in \{1, 2, \ldots, m\}$ satisfying $v_i < v_j$ (in P) must satisfy $i < j$ (in $\mathbb{Z}$).

A linear extension of P is also known as a *topological sorting* of P.

We will use the following well-known fact:

Theorem 1.5. There exists a linear extension of P.

Definition 1.6. The set of all linear extensions of P will be called $\mathcal{L}(P)$. Thus, $\mathcal{L}(P) \neq \varnothing$ (by Theorem 1.5).

The reader can easily verify the following proposition:

Proposition 1.7. Let $(v_1, v_2, \ldots, v_m)$ be a linear extension of P. Let $i \in \{1, 2, \ldots, m-1\}$ be such that the elements v_i and v_{i+1} of P are incomparable. Then $(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m)$ (this is the tuple obtained from the tuple $(v_1, v_2, \ldots, v_m)$ by interchanging the adjacent entries v_i and v_{i+1}) is a linear extension of P as well.

We will also use the following folklore result:[5]

Proposition 1.8. Let $\sim$ denote the equivalence relation on $\mathcal{L}(P)$ generated by the following requirement: For any linear extension $(v_1, v_2, \ldots, v_m)$ of P and any $i \in \{1, 2, \ldots, m-1\}$ such that the elements v_i and v_{i+1} of P are incomparable, we set

$$(v_1, v_2, \ldots, v_m) \sim (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m).$$

Then any two elements of $\mathcal{L}(P)$ are equivalent under the relation $\sim$.

Proofs of Proposition 1.8 can be found in [GriRob14, Proposition 1.7], in [AyKlSc12, Proposition 4.1 (for the $\pi' = \pi\tau_j$ case)], in [Etienn84, Lemma 1] and in [Gyoja86, Lemma

[4] For instance, if R denotes the poset $\mathbb{Z}$ endowed with the **reverse** of its usual order, then we say (for instance) that "$0 > 3$ in R" rather than just "$0 > 3$" (to avoid mistaking our statement for an absurd claim about the usual order on $\mathbb{Z}$).

[5] Particular cases of Proposition 1.8 have a tendency to appear in various parts of combinatorics; see [DefKra21, Proposition 1.3] for a few such references.

4.2][6]. See also [Naatz00, Proposition 2.2] for a stronger claim (describing a shortest way to transform a given linear extension into another by successively swapping adjacent incomparable entries).

Another well-known fact says that any nonempty finite poset has a minimal element and a maximal element. In other words:

Proposition 1.9. Assume that $P \neq \varnothing$. Then:

(a) The poset P has a minimal element.

(b) The poset P has a maximal element.

Proof sketch. **(a)** For any $p \in P$, let n_p denote the number of all $q \in P$ satisfying $q < p$. Argue that if $a, b \in P$ are two elements satisfying $a < b$, then $n_a < n_b$ (since any $q \in P$ satisfying $q < a$ must also satisfy $q < b$, and furthermore the element a satisfies $a < b$ but not $a < a$). Use this to conclude that any element $p \in P$ with minimum n_p must be a minimal element of P.

(b) The proof is analogous to the proof of part **(a)**; just replace some (not all!) "$<$" signs by "$>$" signs. □

2. Inverses in rings

Convention 2.1. From now on, for the rest of this paper, **we fix a ring** $\mathbb{K}$. This ring is not required to be commutative, but must have a unity and be associative.

For example, $\mathbb{K}$ can be $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{C}$ or a polynomial ring or a matrix ring over any of these. In almost all previous work on birational rowmotion (with the exception of [JosRob20] and [JosRob21]), only commutative rings (and, occasionally, semirings) were considered; by removing the commutativity assumption, we are invalidating many of the methods used in prior research. We suspect that the level of generality can be increased even further, replacing our ring $\mathbb{K}$ by a semiring (i.e., a "ring without subtraction"); however, this poses new difficulties which we will not surmount in the present work. (See Section 12 for more about this.)

Even as we do not assume our ring $\mathbb{K}$ to be a division ring, we will nevertheless take multiplicative inverses of elements of $\mathbb{K}$ on many occasions. These inverses do not always exist, but when they do exist, they are unique; thus, we introduce a notation for them:

[6] Note that the sources [AyKlSc12], [Etienn84] and [Gyoja86] define linear extensions of P as bijections $\beta : \{1, 2, \ldots, n\} \to P$ (where $n = |P|$) whose inverse map β^{-1} is order-preserving. This is equivalent to our definition (indeed, if $\beta : \{1, 2, \ldots, n\} \to P$ is a linear extension of P in their sense, then the list $(\beta(1), \beta(2), \ldots, \beta(n))$ is a linear extension of P in our sense).

Definition 2.2. Let a be an element of $\mathbb{K}$.

(a) An *inverse* of a means an element $b \in \mathbb{K}$ such that $ab = ba = 1$. This inverse is unique when it exists, and will be denoted by $\overline{a}$. (A more standard notation for it is a^{-1}, but we prefer the notation $\overline{a}$ since it helps keep our formulas short.)

(b) We say that the element a of $\mathbb{K}$ is *invertible* if it has an inverse.

The following well-known properties of inverses will often be used without mention:

Proposition 2.3.

(a) If a is an invertible element of $\mathbb{K}$, then its inverse $\overline{a}$ is invertible as well, and its inverse is $\overline{\overline{a}} = a$.

(b) If a and b are two invertible elements of $\mathbb{K}$, then their product ab is invertible as well, and its inverse is $\overline{ab} = \overline{b} \cdot \overline{a}$.

(c) If $a_1, a_2, \ldots, a_m$ are several invertible elements of $\mathbb{K}$, then their product $a_1 a_2 \cdots a_m$ is invertible as well, and its inverse is $\overline{a_1 a_2 \cdots a_m} = \overline{a_m} \cdot \overline{a_{m-1}} \cdot \cdots \cdot \overline{a_1}$.

The converse of Proposition 2.3 **(b)** does not necessarily hold: A product ab of two elements a and b of $\mathbb{K}$ can be invertible even when neither a nor b is[7].

The next property of inverses is less well-known:[8]

Proposition 2.4. Let a and b be two elements of $\mathbb{K}$ such that $a+b$ is invertible. Then:

(a) We have $a \cdot \overline{a+b} \cdot b = b \cdot \overline{a+b} \cdot a$.

(b) If both a and b are invertible, then $\overline{a} + \overline{b}$ is invertible as well and its inverse is

$$\overline{\overline{a} + \overline{b}} = a \cdot \overline{a+b} \cdot b.$$

Proof. **(a)** Comparing

$$a \cdot \overline{a+b} \cdot a + a \cdot \overline{a+b} \cdot b = a \cdot \underbrace{\overline{a+b} \cdot (a+b)}_{=1} = a$$

with

$$a \cdot \overline{a+b} \cdot a + b \cdot \overline{a+b} \cdot a = \underbrace{(a+b) \cdot \overline{a+b}}_{=1} \cdot a = a,$$

[7]See https://math.stackexchange.com/questions/627562 for examples of such situations.

[8]Proposition 2.4 **(a)** will not be used in what follows, but its proof provides a good warm-up exercise in manipulating inverses in a noncommutative ring.

we obtain

$$a \cdot \overline{a+b} \cdot a + a \cdot \overline{a+b} \cdot b = a \cdot \overline{a+b} \cdot a + b \cdot \overline{a+b} \cdot a.$$

Subtracting $a \cdot \overline{a+b} \cdot a$ from both sides of this equality, we obtain $a \cdot \overline{a+b} \cdot b = b \cdot \overline{a+b} \cdot a$. This proves Proposition 2.4 **(a)**.

(b) Assume that both a and b are invertible. Set $x := \overline{a} + \overline{b}$ and $y := a \cdot \overline{a+b} \cdot b$. From $x = \overline{a} + \overline{b}$, we obtain $x \cdot a = \left(\overline{a} + \overline{b}\right) \cdot a = \underbrace{\overline{a}a}_{=1} + \overline{b}a = 1 + \overline{b}a$. Comparing this with

$$\overline{b} \cdot (a+b) = \overline{b}a + \underbrace{\overline{b}b}_{=1} = \overline{b}a + 1 = 1 + \overline{b}a,$$

we obtain $x \cdot a = \overline{b} \cdot (a+b)$. Now, from $y = a \cdot \overline{a+b} \cdot b$, we obtain

$$x \cdot y = \underbrace{x \cdot a}_{=\overline{b} \cdot (a+b)} \cdot \overline{a+b} \cdot b = \overline{b} \cdot \underbrace{(a+b) \cdot \overline{a+b}}_{=1} \cdot b = \overline{b} \cdot b = 1.$$

Furthermore, from $x = \overline{a} + \overline{b}$, we obtain $b \cdot x = b \cdot \left(\overline{a} + \overline{b}\right) = b\overline{a} + \underbrace{b\overline{b}}_{=1} = b\overline{a} + 1$. Comparing this with

$$(a+b) \cdot \overline{a} = \underbrace{a\overline{a}}_{=1} + b\overline{a} = 1 + b\overline{a} = b\overline{a} + 1,$$

we obtain $b \cdot x = (a+b) \cdot \overline{a}$. Now, from $y = a \cdot \overline{a+b} \cdot b$, we obtain

$$y \cdot x = a \cdot \overline{a+b} \cdot \underbrace{b \cdot x}_{=(a+b) \cdot \overline{a}} = a \cdot \underbrace{\overline{a+b} \cdot (a+b)}_{=1} \cdot \overline{a} = a \cdot \overline{a} = 1.$$

From $x \cdot y = 1$ and $y \cdot x = 1$, we conclude that y is an inverse of x. In other words, $a \cdot \overline{a+b} \cdot b$ is an inverse of $\overline{a} + \overline{b}$ (since $x = \overline{a} + \overline{b}$ and $y = a \cdot \overline{a+b} \cdot b$). Thus, $\overline{a} + \overline{b}$ is invertible and its inverse is $\overline{\overline{a} + \overline{b}} = a \cdot \overline{a+b} \cdot b$. This proves Proposition 2.4 **(b)**. □

3. Noncommutative birational rowmotion

In this section, we introduce the basic objects whose nature we will investigate: labelings of a finite poset P by elements of a ring, and a partial map between them called "birational rowmotion". These labelings generalize the field-valued labelings studied in [GriRob14], which in turn generalize the piecewise-linear labelings of [EinPro13], which in turn generalize the order ideals of P. Many of the definitions that follow will imitate analogous definitions made (in somewhat lesser generality) in [GriRob14].

3.1. The extended poset $\widehat{P}$

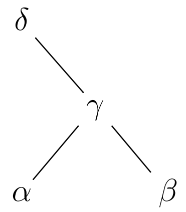
Definition 3.1. We define a poset $\widehat{P}$ as follows: As a set, let $\widehat{P}$ be the disjoint union of the set P with the two-element set $\{0, 1\}$. The smaller-or-equal relation $\leqslant$ on $\widehat{P}$ will be given by

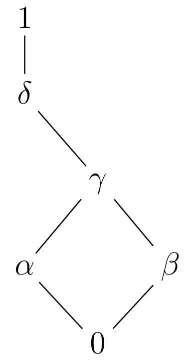
$$(a \leqslant b) \iff ((a \in P \text{ and } b \in P \text{ and } a \leqslant b \text{ in } P) \text{ or } a = 0 \text{ or } b = 1).$$

Here and in the following, we regard the canonical injection of the set P into the disjoint union $\widehat{P}$ as an inclusion; thus, P becomes a subposet of $\widehat{P}$.

In the terminology of Stanley's [Stan11, Section 3.2], this poset $\widehat{P}$ is the ordinal sum $\{0\} \oplus P \oplus \{1\}$.

Example 3.2. Let us represent posets by their Hasse diagrams. Then:

If $P =$ 

, then $\widehat{P} =$ 

.

Remark 3.3. The following observations are easy to check and will be used tacitly:

(a) An element $p \in P$ satisfies $0 \lessdot p$ in $\widehat{P}$ if and only if p is a minimal element of P.

(b) An element $p \in P$ satisfies $1 \gtrdot p$ in $\widehat{P}$ if and only if p is a maximal element of P.

(c) We have $0 \lessdot 1$ in $\widehat{P}$ if and only if $P = \varnothing$.

(d) Two elements $p, q \in P$ satisfy $p \lessdot q$ in $\widehat{P}$ if and only if they satisfy $p \lessdot q$ in P. An analogous statement holds with the symbol "$\lessdot$" replaced by "$\gtrdot$".

Convention 3.4. Let u and v be two elements of P. Then u and v are also elements of $\widehat{P}$ (since we are regarding P as a subposet of $\widehat{P}$). Thus, strictly speaking, statements like "$u < v$" or "$u \lessdot v$" are ambiguous because it is not clear whether they are referring to the poset P or to the poset $\widehat{P}$. However, this ambiguity is harmless, because it is easily seen that the truth of each of the statements "$u < v$", "$u \leqslant v$", "$u > v$", "$u \geqslant v$", "$u \lessdot v$", "$u \gtrdot v$" and "u and v are incomparable" is independent on whether it refers to the poset P or to the poset $\widehat{P}$. We are going to therefore omit mentioning the poset in these statements, unless there are other reasons for us to do so.

3.2. $\mathbb{K}$-labelings

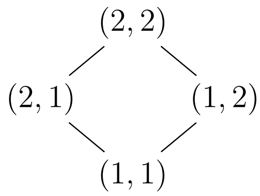
Let us now define the type of object on which our maps will act:

Definition 3.5. A $\mathbb{K}$-*labeling of* P will mean a map $f : \widehat{P} \to \mathbb{K}$. Thus, $\mathbb{K}^{\widehat{P}}$ is the set of all $\mathbb{K}$-labelings of P. If f is a $\mathbb{K}$-labeling of P and v is an element of $\widehat{P}$, then $f(v)$ will be called the *label of* f *at* v.

Example 3.6. Assume that P is the poset $\{1,2\} \times \{1,2\}$ with order relation defined by setting

$$(i,k) \leqslant (i',k') \qquad \text{if and only if } (i \leqslant i' \text{ and } k \leqslant k').$$

This poset will later be called the "2×2-rectangle" in Definition 4.2. It has Hasse diagram



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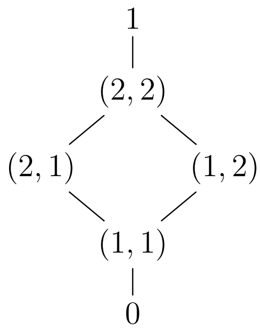
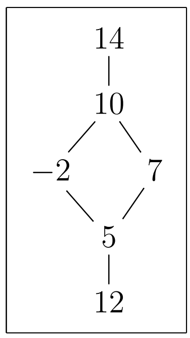
The extended poset $\widehat{P}$ has Hasse diagram



.

We recall that a $\mathbb{K}$-labeling of P is a map $f : \widehat{P} \to \mathbb{K}$. We can visualize such a $\mathbb{K}$-labeling by replacing, in the Hasse diagram of $\widehat{P}$, each element $v \in \widehat{P}$ by the label $f(v)$. For example, the $\mathbb{Z}$-labeling of P that sends 0, $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, and 1 to 12, 5, 7, -2, 10, and 14, respectively can be visualized as follows:



. (1)

For example, its label at $(1,2)$ is 7.

(The rectangular box around the drawing (1) is meant to signal that it shows a $\mathbb{K}$-labeling, not a poset. We will follow this convention throughout this paper.)

3.3. Partial maps

We will next define the notion of a partial map, to formalize the idea of an operation whose result may be undefined, such as division on $\mathbb{Q}$ (since division by zero is undefined). We will use $\bot$ as a symbol for such undefined values:

Convention 3.7. We fix an object called $\bot$. In the following, we tacitly assume that none of the sets we will consider contains this object $\bot$ (unless otherwise specified).

The reader can think of $\bot$ as a "division-by-zero error" (more precisely, a "division-by-a-non-invertible-element error", since 0 is often not the only non-invertible element of $\mathbb{K}$).

Definition 3.8. Let X and Y be two sets. A *partial map* from X to Y means a map from X to $Y \sqcup \{\bot\}$.

If f is a partial map from X to Y, then f can be canonically extended to a map from $X \sqcup \{\bot\}$ to $Y \sqcup \{\bot\}$ by setting $f(\bot) := \bot$. We always consider f to be extended in this way.

If f is a partial map from X to Y, then the set $\{x \in X \mid f(x) \neq \bot\}$ will be called the *domain of definition* of f.

We view the element $\bot$ as an "undefined output" – i.e., we think of a partial map f from X to Y as a "map" from X to Y that is defined only on some elements of X (namely, on those whose image under this map is not $\bot$). Thus, for example, in $\mathbb{Q}$, division is a partial map because division by 0 is undefined:

Example 3.9. The map

$$\mathbb{Q} \to \mathbb{Q} \sqcup \{\bot\},$$
$$x \mapsto \begin{cases} 1/x, & \text{if } x \neq 0; \\ \bot, & \text{if } x = 0 \end{cases}$$

is a partial map from $\mathbb{Q}$ to $\mathbb{Q}$.

Partial maps can be composed much like usual maps:

Definition 3.10.

(a) Let X, Y and Z be three sets. Let f be a partial map from Y to Z. Let g be a partial map from X to Y.

Then $f \circ g$ denotes the partial map from X to Z that sends

$$\text{each } x \in X \text{ to } \begin{cases} f(g(x)), & \text{if } g(x) \neq \bot\,; \\ \bot, & \text{if } g(x) = \bot\,. \end{cases}$$

(Following our convention that $f(\bot)$ is understood to be $\bot$, we could simplify the right hand side to just $f(g(x))$, but we nevertheless subdivided it into two cases just to stress the different branches in our "control flow".)

This partial map $f \circ g$ is called the *composition* of f and g.

(b) This notion of composition lets us define a category whose objects are sets and whose morphisms are partial maps. (The identity maps in this category are the obvious ones: i.e., the maps $\mathrm{id} : X \to X \sqcup \{\bot\}$ that send each $x \in X$ to $x \in X \subseteq X \sqcup \{\bot\}$.)

(c) Thus, if X is any set, and if f is any partial map from X to X, then we can define $f^k := \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ for any $k \in \mathbb{N}$.

Convention 3.11. Let X and Y be two sets. We will write "$f : X \dashrightarrow Y$" for "f is a partial map from X to Y" (just as maps from X to Y are denoted "$f : X \to Y$").

A warning is worth making: While we are using the symbol $\dashrightarrow$ for partial maps here, the same symbol has been used for rational maps in [GriRob14]. The two uses serve similar purposes (they both model "maps defined only on those inputs for which the relevant denominators are invertible"), but they have some technical differences. Rational maps are defined only when $\mathbb{K}$ is an infinite field[9], but are well-behaved in many ways that partial maps are not. (For example, a rational map is uniquely determined if its values on a Zariski-dense subset of its domain are known, but no such claims can be made for partial maps.) Thus, by working with partial maps instead of rational maps, we are freeing ourselves from technical assumptions on $\mathbb{K}$, but at the same time forcing ourselves to be explicit about the domains on which our partial maps are defined.

3.4. Toggles

Recall that $\mathbb{K}^{\widehat{P}}$ denotes the set of $\mathbb{K}$-labelings of a poset P (that is, the set of all maps $\widehat{P} \to \mathbb{K}$). Next, we define *(noncommutative) toggles*: certain (fairly simple) partial maps on this set.

Definition 3.12. Let $v \in P$. We define a partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ as follows: If

[9] It stands to reason that a notion of "rational map" should exist for a sufficiently wide class of infinite skew-fields as well, but we have not encountered a satisfactory theory of such maps in the literature. See https://mathoverflow.net/questions/362724/ for a discussion of how this theory might start. It appears unlikely, however, that such "noncommutative rational maps" exist in the generality that we are working in (viz., arbitrary rings).

$f \in \mathbb{K}^{\widehat{P}}$ is any $\mathbb{K}$-labeling of P, then the $\mathbb{K}$-labeling $T_v f \in \mathbb{K}^{\widehat{P}}$ is given by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases} \qquad (2)$$

for all $w \in \widehat{P}$.

Here, we agree that if any part of the expression $\left(\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}$ is not well-defined (i.e., if one of the values $f(u)$ and $f(v)$ appearing in it is undefined, or if $f(v)$ is not invertible, or if $f(u)$ is not invertible for some $u \in \widehat{P}$ satisfying $u \gtrdot v$, or if the sum $\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}$ is not invertible), then $T_v f$ is understood to be $\bot$.

This partial map T_v is called the *v-toggle* or the *toggle at v*.

Thus, the partial map T_v is a "local" transformation: it only changes the label at the element v (unless its result is $\bot$).

Remark 3.13. You are reading Definition 3.12 right: We set $T_v f = \bot$ if any of $\overline{f(v)}$ and $\overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}$ fails to be well-defined. Thus, in this case, none of the values $(T_v f)(w)$ exists. It may appear more natural to leave only the value $(T_v f)(v)$ undefined, while letting all other values $(T_v f)(w)$ equal the respective values $f(w)$. Our choice to "panic and crash", however, will be more convenient for some of our proofs.

The v-toggle T_v is called a "*noncommutative order toggle*" in [JosRob20, Definition 5.6]. When the ring $\mathbb{K}$ is commutative, this v-toggle T_v is an "involution" in the sense that each $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$ satisfying $T_v(T_v f) \neq \bot$ satisfies $T_v(T_v f) = f$. For noncommutative $\mathbb{K}$, this is usually not the case; an "inverse" partial map[10] can be obtained by flipping the order of the factors on the right hand side of (2). (This "inverse" appears in [JosRob20] under the name "noncommutative order elggot".)

The following proposition is trivially obtained by rewriting (2); we are merely stating it for easier reference in proofs:

Proposition 3.14. Let $v \in P$. For every $f \in \mathbb{K}^{\widehat{P}}$ satisfying $T_v f \neq \bot$, we have the following:

[10]We are putting the word "inverse" in scare quotes since we are talking about partial maps, but the two maps are as close to being mutually inverse as partial maps can be.

(a) Every $w \in \widehat{P}$ such that $w \neq v$ satisfies $(T_v f)(w) = f(w)$.

(b) We have

$$(T_v f)(v) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}.$$

Furthermore, the following "locality principle" (part of [JosRob20, Proposition 5.8]) is easy to check:[11]

Proposition 3.15. Let $v \in P$ and $w \in P$. Then $T_v \circ T_w = T_w \circ T_v$, unless we have either $v \lessdot w$ or $w \lessdot v$.

Proof of Proposition 3.15. In the case when $\mathbb{K}$ is commutative, this is essentially [GriRob14, Proposition 2.10], except that we are now working with partial maps instead of rational maps. The proof below is an adaptation of the proof given in [GriRob14] to the general (noncommutative) case, but it is structured more carefully in order to pay the requisite attention to cases when some values are $\perp$.

Let us first forget that we fixed v and w. We now introduce a convenient notation. Namely, if $f \in \mathbb{K}^{\widehat{P}}$ is a $\mathbb{K}$-labeling, and if $v \in P$, then we define the element $X_v(f) \in \mathbb{K} \sqcup \{\perp\}$ as follows: We set

$$X_v(f) := \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}$$

if the right hand side of this equation is well-defined; otherwise, we set $X_v(f) := \perp$.

The equality (2) from the definition of the v-toggle T_v can thus be rewritten as follows:

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ X_v(f), & \text{if } w = v \end{cases} \qquad \text{for all } w \in \widehat{P}. \tag{3}$$

This holds for any $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$ and any $v \in P$, provided that the expression $\left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}$ is well-defined (i.e., provided that $X_v(f) \neq \perp$).

[11] In the following, equalities between partial maps are understood in the strongest possible sense: Two partial maps $F : X \dashrightarrow Y$ and $G : X \dashrightarrow Y$ satisfy $F = G$ if and only if each $x \in X$ satisfies $F(x) = G(x)$. This entails, in particular, that $F(x) = \perp$ holds if and only if $G(x) = \perp$. Thus, $F = G$ is a stronger requirement than merely saying that "$F(x) = G(x)$ whenever neither $F(x)$ nor $G(x)$ is $\perp$".

Thus, the definition of the v-toggle T_v can be restated as follows: For any $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$ and any element $v \in P$, we let $T_v f$ be the $\mathbb{K}$-labeling defined by (3) if $X_v(f) \neq \bot$; otherwise, we set $T_v f := \bot$. In particular, for any $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$ and any element $v \in P$, we have

$$T_v f = \bot \text{ if and only if } X_v(f) = \bot. \tag{4}$$

Now, let $v \in P$ and $w \in P$ be two elements that satisfy neither $v \lessdot w$ nor $w \lessdot v$. We must show that $T_v \circ T_w = T_w \circ T_v$.

If $v = w$, then this is obvious. Thus, we WLOG assume that $v \neq w$.

We fix a $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$. We will show that $(T_v \circ T_w) f = (T_w \circ T_v) f$.

First we prove the following observation:

Observation 1: If $T_w f \neq \bot$, then $X_v(T_w f) = X_v(f)$.

Proof of Observation 1. Assume that $T_w f \neq \bot$.

We have $v \neq w$ and thus

$$(T_w f)(v) = f(v) \tag{5}$$

(by Proposition 3.14 **(a)**, applied to w and v instead of v and w).

For each $u \in \widehat{P}$ satisfying $u \gtrdot v$, we have $u \neq w$ (because if we had $u = w$, then we would have $w = u \gtrdot v$ and thus $v \lessdot w$, which would contradict the fact that we don't have $v \lessdot w$) and therefore

$$(T_w f)(u) = f(u)$$

(by Proposition 3.14 **(a)**, applied to w and u instead of v and w). Hence,

$$\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{(T_w f)(u)} = \sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{f(u)}. \tag{6}$$

For each $u \in \widehat{P}$ satisfying $u \lessdot v$, we have $u \neq w$ (because if we had $u = w$, then we would have $w = u \lessdot v$, which would contradict the fact that we don't have $w \lessdot v$) and therefore

$$(T_w f)(u) = f(u)$$

(by Proposition 3.14 **(a)**, applied to w and u instead of v and w). Hence,

$$\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} (T_w f)(u) = \sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u). \tag{7}$$

The definition of $X_v(f)$ yields

$$X_v(f) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{f(u)}}, \tag{8}$$

with the understanding that the right hand side is understood to be $\bot$ if any of its sub-expressions is not well-defined.

The definition of $X_v\left(T_w f\right)$ yields

$$X_v\left(T_w f\right) = \left(\sum_{\substack{u\in\widehat{P};\\ u<v}} \left(T_w f\right)(u)\right) \cdot \overline{\left(T_w f\right)(v)} \cdot \overline{\sum_{\substack{u\in\widehat{P};\\ u>v}} \overline{\left(T_w f\right)(u)}}$$

$$= \left(\sum_{\substack{u\in\widehat{P};\\ u<v}} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u\in\widehat{P};\\ u>v}} \overline{f(u)}} \qquad \text{(by (7), (6) and (5))}.$$

Comparing this with (8), we obtain $X_v\left(T_w f\right) = X_v(f)$. This proves Observation 1. $\square$

Now, we are in one of the following four cases:

Case 1: We have $X_v(f) \neq \bot$ and $X_w(f) \neq \bot$.

Case 2: We have $X_v(f) = \bot$ and $X_w(f) \neq \bot$.

Case 3: We have $X_v(f) \neq \bot$ and $X_w(f) = \bot$.

Case 4: We have $X_v(f) = \bot$ and $X_w(f) = \bot$.

Let us first consider Case 1. In this case, we have $X_v(f) \neq \bot$ and $X_w(f) \neq \bot$. From $X_v(f) \neq \bot$, we obtain $T_v f \neq \bot$ (by (4)). Similarly, $T_w f \neq \bot$. Thus, Observation 1 yields $X_v\left(T_w f\right) = X_v(f) \neq \bot$. From this, we obtain $T_v\left(T_w f\right) \neq \bot$ (since (4) (applied to $T_w f$ instead of f) shows that we have $T_v\left(T_w f\right) = \bot$ if and only if $X_v\left(T_w f\right) = \bot$). Similarly, $T_w\left(T_v f\right) \neq \bot$.

Now, let $x \in \widehat{P}$. Then

$$\left(T_w f\right)(x) = \begin{cases} f(x), & \text{if } x \neq w; \\ X_w(f), & \text{if } x = w \end{cases} \tag{9}$$

(by (3), applied to x and w instead of w and v). Furthermore, (3) (applied to $T_w f$ and x

instead of f and w) yields

$$\begin{aligned}
(T_v (T_w f)) (x) &= \begin{cases} (T_w f) (x), & \text{if } x \neq v; \\ X_v (T_w f), & \text{if } x = v \end{cases} \\
&= \begin{cases} (T_w f) (x), & \text{if } x \neq v; \\ X_v (f), & \text{if } x = v \end{cases} \qquad \text{(since } X_v (T_w f) = X_v (f)\text{)} \\
&= \begin{cases} \begin{cases} f (x), & \text{if } x \neq w; \\ X_w (f), & \text{if } x = w, \end{cases} & \text{if } x \neq v; \\ X_v (f), & \text{if } x = v \end{cases} \qquad \text{(by (9))} \\
&= \begin{cases} f (x), & \text{if } x \neq v \text{ and } x \neq w; \\ X_w (f), & \text{if } x \neq v \text{ and } x = w; \\ X_v (f), & \text{if } x = v \end{cases} \\
&= \begin{cases} f (x), & \text{if } x \neq v \text{ and } x \neq w; \\ X_w (f), & \text{if } x = w; \\ X_v (f), & \text{if } x = v \end{cases} \qquad (10)
\end{aligned}$$

(since the condition "$x \neq v$ and $x = w$" is equivalent to "$x = w$" (because $v \neq w$)). The same argument (but with the roles of v and w interchanged) shows that

$$\begin{aligned}
(T_w (T_v f)) (x) &= \begin{cases} f (x), & \text{if } x \neq w \text{ and } x \neq v; \\ X_v (f), & \text{if } x = v; \\ X_w (f), & \text{if } x = w \end{cases} \\
&= \begin{cases} f (x), & \text{if } x \neq w \text{ and } x \neq v; \\ X_w (f), & \text{if } x = w; \\ X_v (f), & \text{if } x = v \end{cases} \\
&= \begin{cases} f (x), & \text{if } x \neq v \text{ and } x \neq w; \\ X_w (f), & \text{if } x = w; \\ X_v (f), & \text{if } x = v. \end{cases}
\end{aligned}$$

Comparing this with (10), we obtain $(T_v (T_w f)) (x) = (T_w (T_v f)) (x)$.

Forget that we fixed x. We thus have proved that $(T_v (T_w f)) (x) = (T_w (T_v f)) (x)$ for each $x \in \widehat{P}$. In other words, $T_v (T_w f) = T_w (T_v f)$. Thus, $(T_v \circ T_w) f = T_v (T_w f) = T_w (T_v f) = (T_w \circ T_v) f$. We have therefore proved $(T_v \circ T_w) f = (T_w \circ T_v) f$ in Case 1.

Let us now consider Case 2. In this case, we have $X_v (f) = \perp$ and $X_w (f) \neq \perp$. From $X_v (f) = \perp$, we obtain $T_v f = \perp$ (by (4)) and therefore $T_w (T_v f) = T_w (\perp) = \perp$. On the other hand, (4) (applied to w instead of v) shows that $T_w f = \perp$ if and only if $X_w (f) = \perp$. Hence, we have $T_w f \neq \perp$ (since $X_w (f) \neq \perp$). Therefore, Observation 1 yields $X_v (T_w f) = X_v (f) = \perp$. However, (4) (applied to $T_w f$ instead of f) shows that $T_v(T_w f) = \bot$ if and only if $X_v(T_w f) = \bot$. Hence, we have $T_v(T_w f) = \bot$ (since $X_v(T_w f) = \bot$). Altogether, we now obtain

$$(T_v \circ T_w) f = T_v(T_w f) = \bot = T_w(T_v f) = (T_w \circ T_v) f.$$

Hence, we have proved $(T_v \circ T_w) f = (T_w \circ T_v) f$ in Case 2.

The proof of $(T_v \circ T_w) f = (T_w \circ T_v) f$ in Case 3 is analogous to the proof we just showed in Case 2; we only need to interchange v with w.

Finally, let us consider Case 4. In this case, we have $X_v(f) = \bot$ and $X_w(f) = \bot$. From $X_v(f) = \bot$, we obtain $T_v f = \bot$ (by (4)). Similarly, $T_w f = \bot$. Now,

$$(T_v \circ T_w) f = T_v \left(\underbrace{T_w f}_{=\bot} \right) = T_v(\bot) = \bot.$$

Similarly, $(T_w \circ T_v) f = \bot$. Comparing these two equalities, we obtain $(T_v \circ T_w) f = (T_w \circ T_v) f$. Hence, we have proved $(T_v \circ T_w) f = (T_w \circ T_v) f$ in Case 4.

We have now proved $(T_v \circ T_w) f = (T_w \circ T_v) f$ in all four Cases 1, 2, 3 and 4. Thus, $(T_v \circ T_w) f = (T_w \circ T_v) f$ always holds.

Forget that we fixed f. We thus have shown that $(T_v \circ T_w) f = (T_w \circ T_v) f$ for each $f \in \mathbb{K}^{\widehat{P}}$. In other words, $T_v \circ T_w = T_w \circ T_v$. This proves Proposition 3.15. □

As a particular case of Proposition 3.15, we have the following:

Corollary 3.16. Let v and w be two elements of P which are incomparable. Then $T_v \circ T_w = T_w \circ T_v$.

Proof. Since v and w are incomparable, we have neither $v \lessdot w$ nor $w \lessdot v$. Thus, Proposition 3.15 yields $T_v \circ T_w = T_w \circ T_v$. This proves Corollary 3.16. □

Corollary 3.17. Let $(v_1, v_2, \ldots, v_m)$ be a linear extension of P. Then the partial map $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ is independent of the choice of the linear extension $(v_1, v_2, \ldots, v_m)$.

Proof of Corollary 3.17. Forget that we fixed the linear extension $(v_1, v_2, \ldots, v_m)$.

If $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ is a linear extension of P, then we denote the partial map $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m}$ by $R_{\mathbf{v}}$. We must prove that this partial map $R_{\mathbf{v}}$ is independent of the choice of the linear extension $\mathbf{v}$. In other words, we must prove the following claim:

Claim 1: If $\mathbf{v}$ and $\mathbf{w}$ are any two linear extensions of P, then $R_{\mathbf{v}} = R_{\mathbf{w}}$.

Our proof of Claim 1 will rely on Proposition 1.8.

Consider the equivalence relation $\sim$ on $\mathcal{L}(P)$ introduced in Proposition 1.8. According to Proposition 1.8, any two elements of $\mathcal{L}(P)$ are equivalent under the relation $\sim$.

We say that two linear extensions $\mathbf{v}$ and $\mathbf{w}$ of P are *adjacent* if and only if they can be written in the forms

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, \ldots, v_m) \qquad \text{and} \\ \mathbf{w} &= (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m), \end{aligned}$$

where $i \in \{1, 2, \ldots, m-1\}$ is such that the elements v_i and v_{i+1} of P are incomparable. In other words, we say that two linear extensions $\mathbf{v}$ and $\mathbf{w}$ of P are *adjacent* if and only if $\mathbf{w}$ can be obtained from $\mathbf{v}$ by swapping two consecutive entries, provided that these two entries are incomparable. It is clear that the relation "adjacent" is symmetric: i.e., if two linear extensions $\mathbf{v}$ and $\mathbf{w}$ are adjacent, then $\mathbf{w}$ and $\mathbf{v}$ are adjacent as well (because if we swap two entries of $\mathbf{v}$ and then swap them again, then they end up back in their original positions).

Now, we notice the following fact:

Claim 2: If $\mathbf{v}$ and $\mathbf{w}$ are two adjacent linear extensions of P, then $R_{\mathbf{v}} = R_{\mathbf{w}}$.

Proof of Claim 2. Let $\mathbf{v}$ and $\mathbf{w}$ be two adjacent linear extensions of P. According to the definition of "adjacent", we can thus write $\mathbf{v}$ and $\mathbf{w}$ in the forms

$$\begin{aligned} \mathbf{v} &= (v_1, v_2, \ldots, v_m) \qquad \text{and} \\ \mathbf{w} &= (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m), \end{aligned}$$

where $i \in \{1, 2, \ldots, m-1\}$ is such that the elements v_i and v_{i+1} of P are incomparable. Write them in this form, and consider this i.

Since v_i and v_{i+1} are incomparable, we have $T_{v_i} \circ T_{v_{i+1}} = T_{v_{i+1}} \circ T_{v_i}$ (by Corollary 3.16).

The definition of $R_{\mathbf{v}}$ yields

$$\begin{aligned} R_{\mathbf{v}} &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} \qquad (\text{since } \mathbf{v} = (v_1, v_2, \ldots, v_m)) \\ &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{i-1}} \circ \underbrace{T_{v_i} \circ T_{v_{i+1}}}_{= T_{v_{i+1}} \circ T_{v_i}} \circ T_{v_{i+2}} \circ T_{v_{i+3}} \circ \cdots \circ T_{v_m} \\ &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{i-1}} \circ T_{v_{i+1}} \circ T_{v_i} \circ T_{v_{i+2}} \circ T_{v_{i+3}} \circ \cdots \circ T_{v_m}. \end{aligned}$$

On the other hand, the definition of $R_{\mathbf{w}}$ yields

$$\begin{aligned} R_{\mathbf{w}} &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{i-1}} \circ T_{v_{i+1}} \circ T_{v_i} \circ T_{v_{i+2}} \circ T_{v_{i+3}} \circ \cdots \circ T_{v_m} \\ &\qquad (\text{since } \mathbf{w} = (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m)). \end{aligned}$$

Comparing these two equalities, we obtain $R_{\mathbf{v}} = R_{\mathbf{w}}$. This proves Claim 2. □

Now, recall that the equivalence relation $\sim$ is generated by the elementary relations

$$(v_1, v_2, \ldots, v_m) \sim (v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, v_{i+3}, \ldots, v_m),$$

where $(v_1, v_2, \ldots, v_m)$ is a linear extension of P and where $i \in \{1, 2, \ldots, m-1\}$ is chosen such that the elements v_i and v_{i+1} of P are incomparable. In other words, the equivalence relation $\sim$ is generated by the elementary relations

$$\mathbf{v} \sim \mathbf{w}, \qquad \text{where } \mathbf{v} \text{ and } \mathbf{w} \text{ are adjacent linear extensions}$$

(by the definition of "adjacent"). In other words, the equivalence relation $\sim$ is the reflexive, transitive and symmetric closure of the relation "adjacent". In other words, the following holds:

> *Claim 3:* Let $\mathbf{v}$ and $\mathbf{w}$ be two linear extensions of P. Then we have $\mathbf{v} \sim \mathbf{w}$ if and only if there exists a tuple $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ of linear extensions of P such that $\mathbf{u}_0 = \mathbf{v}$ and $\mathbf{u}_k = \mathbf{w}$ and such that for each $i \in \{1, 2, \ldots, k\}$, we have
>
> $$(\mathbf{u}_{i-1} \text{ and } \mathbf{u}_i \text{ are adjacent}) \text{ or } (\mathbf{u}_i \text{ and } \mathbf{u}_{i-1} \text{ are adjacent}).$$

We are now ready to prove Claim 1:

Proof of Claim 1. Let $\mathbf{v}$ and $\mathbf{w}$ be any two linear extensions of P. Then $\mathbf{v}$ and $\mathbf{w}$ are two elements of $\mathcal{L}(P)$. Hence, $\mathbf{v} \sim \mathbf{w}$ (since any two elements of $\mathcal{L}(P)$ are equivalent under the relation $\sim$). Thus, Claim 3 shows that there exists a tuple $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ of linear extensions of P such that $\mathbf{u}_0 = \mathbf{v}$ and $\mathbf{u}_k = \mathbf{w}$ and such that for each $i \in \{1, 2, \ldots, k\}$, we have

$$(\mathbf{u}_{i-1} \text{ and } \mathbf{u}_i \text{ are adjacent}) \text{ or } (\mathbf{u}_i \text{ and } \mathbf{u}_{i-1} \text{ are adjacent}). \tag{11}$$

Consider this tuple $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_k)$.

Let $i \in \{1, 2, \ldots, k\}$. Then ($\mathbf{u}_{i-1}$ and $\mathbf{u}_i$ are adjacent) or ($\mathbf{u}_i$ and $\mathbf{u}_{i-1}$ are adjacent) (by (11)). In either of these two cases, we conclude that $\mathbf{u}_{i-1}$ and $\mathbf{u}_i$ are adjacent (since the relation "adjacent" is symmetric). Hence, Claim 2 (applied to $\mathbf{u}_{i-1}$ and $\mathbf{u}_i$ instead of $\mathbf{v}$ and $\mathbf{w}$) yields $R_{\mathbf{u}_{i-1}} = R_{\mathbf{u}_i}$.

Forget that we fixed i. We thus have proved the equality $R_{\mathbf{u}_{i-1}} = R_{\mathbf{u}_i}$ for each $i \in \{1, 2, \ldots, k\}$. Combining all these equalities, we obtain

$$R_{\mathbf{u}_0} = R_{\mathbf{u}_1} = R_{\mathbf{u}_2} = \cdots = R_{\mathbf{u}_k}.$$

Hence, $R_{\mathbf{u}_0} = R_{\mathbf{u}_k}$. In view of $\mathbf{u}_0 = \mathbf{v}$ and $\mathbf{u}_k = \mathbf{w}$, we can rewrite this as $R_{\mathbf{v}} = R_{\mathbf{w}}$. This proves Claim 1. □

Thus, the proof of Corollary 3.17 is complete (since Claim 1 is proved). □

3.5. Birational rowmotion

Recall that P is a finite poset. Corollary 3.17 lets us make the following definition.

Definition 3.18. *Birational rowmotion* (or, more precisely, the *birational rowmotion of* P) is defined as the partial map $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $(v_1, v_2, \ldots, v_m)$ is a linear extension of P. This partial map is well-defined, because

- Theorem 1.5 shows that a linear extension of P exists, and
- Corollary 3.17 shows that the partial map $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m}$ is independent of the choice of the linear extension $(v_1, v_2, \ldots, v_m)$.

This partial map will be denoted by R.

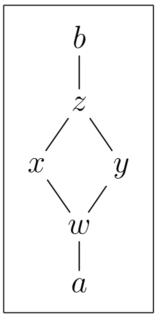
Birational rowmotion is called "birational NOR-motion" (and denoted NOR) in the paper [JosRob20, Definition 5.9][12]. When $\mathbb{K}$ is commutative, it agrees with the standard concept of birational rowmotion as studied in [EinPro13] and [GriRob14].

[12]To be more precise, [JosRob20, Definition 5.9] works in a slightly less general context, requiring $\mathbb{K}$ to be a skew field and that $f(0) = 1$ and $f(1) = C$ for some C in the center of $\mathbb{K}$.

Example 3.19. Let us demonstrate the effect of birational toggles and birational rowmotion. Namely, for this example, we let P be the poset $P = \{1,2\} \times \{1,2\}$ introduced in Example 3.6.

In order to disencumber our formulas, we agree to write $g(i,j)$ for $g((i,j))$ when g is a $\mathbb{K}$-labeling of P and (i,j) is an element of P.

As in Example 3.6, we visualize a $\mathbb{K}$-labeling f of P by replacing, in the Hasse diagram of $\widehat{P}$, each element $v \in \widehat{P}$ by the label $f(v)$. Let f be a $\mathbb{K}$-labeling sending 0, $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, and 1 to a, w, y, x, z, and b, respectively (for some elements a, b, x, y, z, w of $\mathbb{K}$); this f is then visualized as follows:

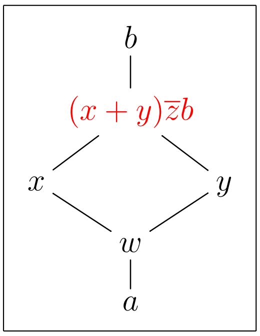
$f =$ 

.

(As before, we draw $(2,1)$ on the western corner and $(1,2)$ on the eastern corner.)

Now, recall the definition of birational rowmotion R on our poset P. Since the list $((1,1),(1,2),(2,1),(2,2))$ is a linear extension of P, we have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. Let us track how this transforms our labeling f:

We first apply $T_{(2,2)}$, obtaining the $\mathbb{K}$-labeling

$T_{(2,2)} f =$ 

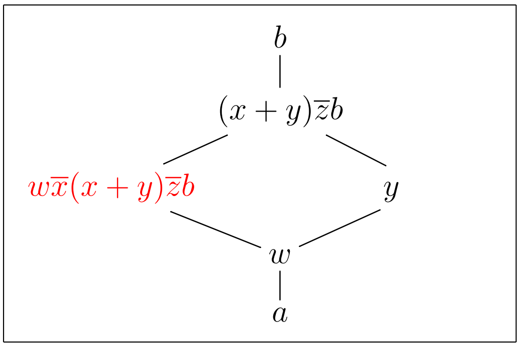
(where we colored the label at $(2,2)$ red to signify that it is the label at the element which got toggled). Indeed, the only label that changes under $T_{(2,2)}$ is the one at $(2,2)$, and this label becomes

$$\begin{aligned}\left(T_{(2,2)} f\right)(2,2) &= \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot (2,2)}} f(u)\right) \cdot \overline{f(2,2)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot (2,2)}} \overline{f(u)}} \\ &= (f(1,2) + f(2,1)) \cdot \overline{f(2,2)} \cdot \overline{\overline{f(1)}} \\ &= (y + x) \cdot \overline{z} \cdot \overline{\overline{b}} = (x + y) \cdot \overline{z} \cdot b.\end{aligned}$$

(We assume that z and b are indeed invertible; otherwise, $T_{(2,2)}f$ would be $\bot$ and would remain $\bot$ after any further toggles. Likewise, as we apply further toggles, we assume that everything else we need to invert is invertible.)

Having applied $T_{(2,2)}$, we next apply $T_{(2,1)}$, obtaining

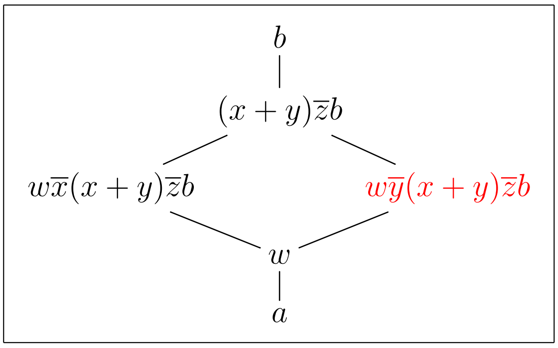
$T_{(2,1)}T_{(2,2)}f =$



.

Next, we apply $T_{(1,2)}$, obtaining

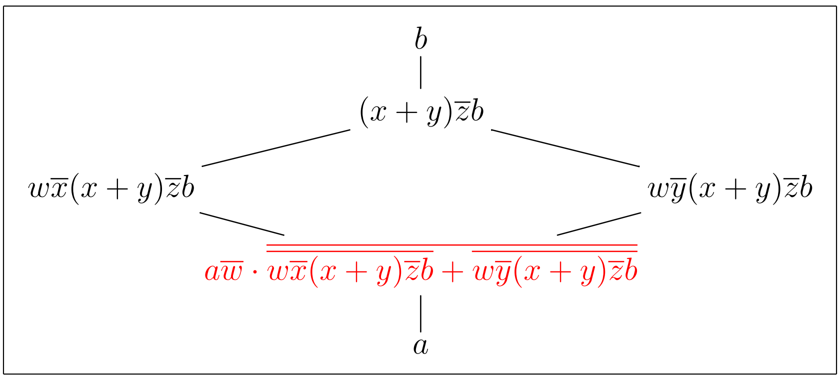
$T_{(1,2)}T_{(2,1)}T_{(2,2)}f =$



.

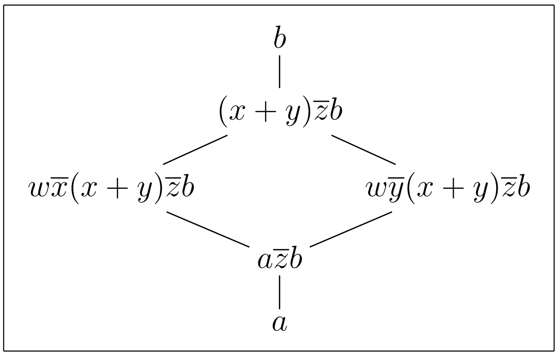
Finally, we apply $T_{(1,1)}$, resulting in

$T_{(1,1)}T_{(1,2)}T_{(2,1)}T_{(2,2)}f =$



.

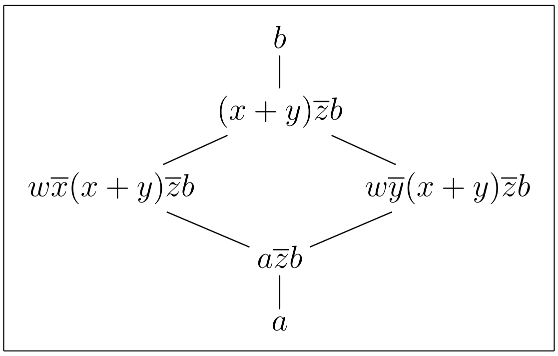
The unwieldy expression $\overline{w}\cdot\overline{\overline{w\overline{x}(x+y)\overline{z}b}+\overline{w\overline{y}(x+y)\overline{z}b}}$ in the label at $(1,1)$ can be simplified to $\overline{z}b$ (using standard laws such as $\overline{p}\cdot\overline{q}=\overline{qp}$ and distributivity), so this rewrites as

$T_{(1,1)}T_{(1,2)}T_{(2,1)}T_{(2,2)}f =$ 

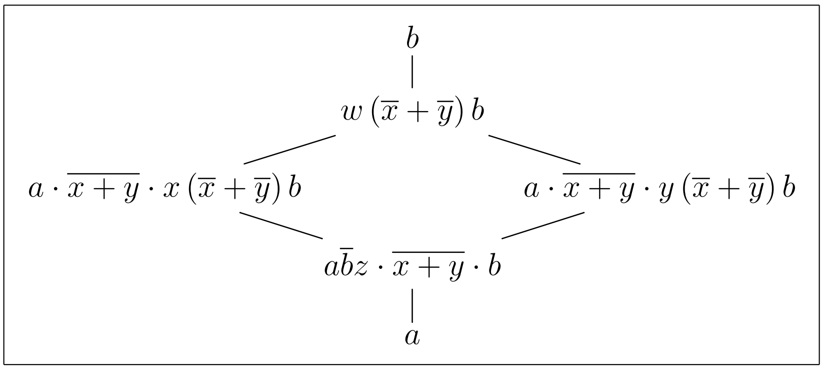
.

We thus have computed Rf (since $Rf = T_{(1,1)}T_{(1,2)}T_{(2,1)}T_{(2,2)}f$).

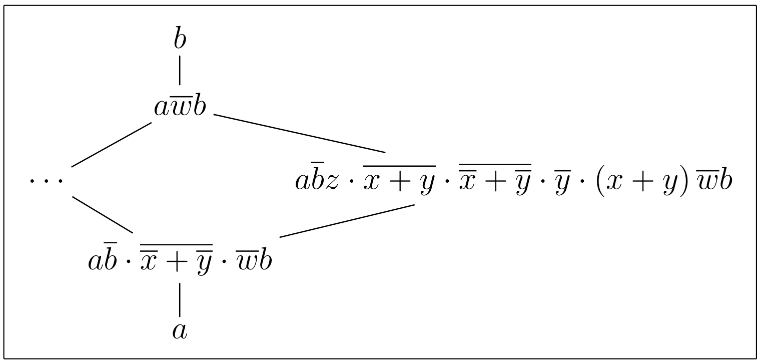
By repeating this procedure (or just substituting the labels of Rf obtained as variables), we can compute R^2f, R^3f etc., obtaining

$Rf =$ 

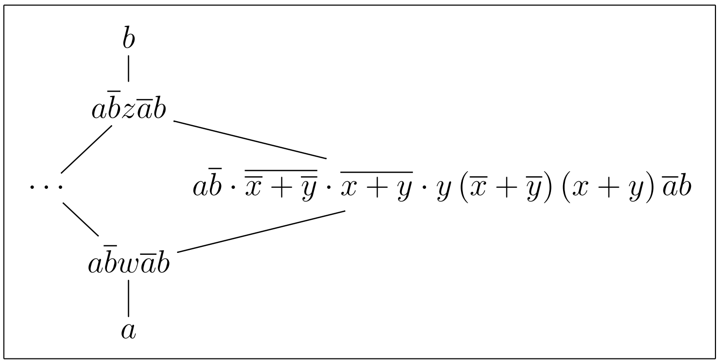
,

$R^2f =$ 

,

$R^3f =$ 

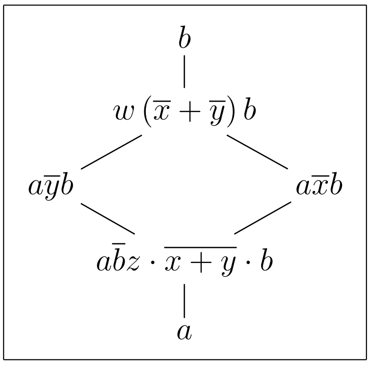
,

$R^4 f =$ 

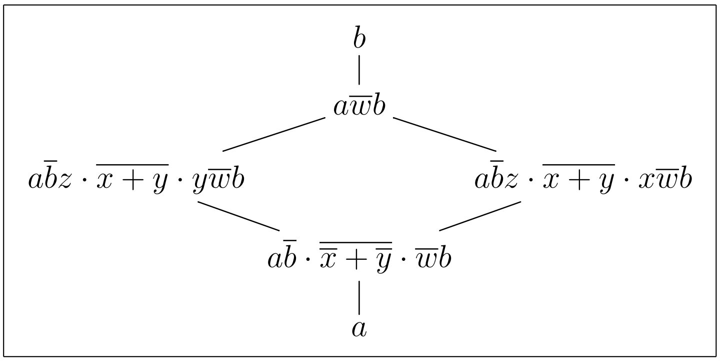
.

Here, we have omitted the label at $(2,1)$ for both $R^3 f$ and $R^4 f$, since it can be obtained from the respective label at $(1,2)$ by interchanging x with y (thanks to an obvious symmetry between $(1,2)$ and $(2,1)$).

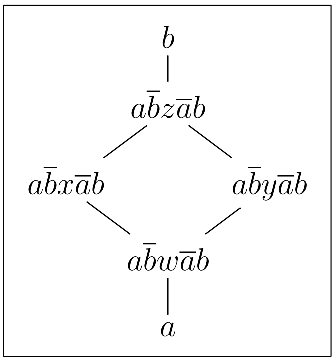
The above might suggest that the labels get progressively more complicated as we apply R over and over. For a general poset P, this is indeed the case. However, for our poset $P = \{1,2\} \times \{1,2\}$, a surprising periodicity-like pattern emerges. Indeed, our above expressions for $R^2 f,\ R^3 f,\ R^4 f$ can be simplified as follows:

$R^2 f =$ 

,

$R^3 f =$ 

,

$R^4 f =$ 

.

Thus, the labels of $R^4 f$ are closely related to those of f: For each $v \in P$, we have

$$\left(R^4 f\right)(v) = a\overline{b} \cdot f(v) \cdot \overline{a}b.$$

(This holds for $v = 0$ and $v = 1$ as well, as one can easily check.) Note that if $ab = ba$, then this entails that $(R^4 f)(v)$ is conjugate to v in $\mathbb{K}$.

In Theorem 4.7, we will generalize this phenomenon to arbitrary "rectangular" posets – i.e., posets of the form $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ with entrywise order. The "period" in this situation will be $p + q$.

Our $P = \{1, 2\} \times \{1, 2\}$ example also exhibits a reciprocity-like phenomenon. Indeed, our above expressions for $Rf,\ R^2 f,\ R^3 f$ reveal that

$$\begin{aligned}
(Rf)(1,1) &= a\overline{z}b = a \cdot \overline{f(2,2)} \cdot b; \\
\left(R^2 f\right)(1,2) &= a\overline{x}b = a \cdot \overline{f(2,1)} \cdot b; \\
\left(R^2 f\right)(2,1) &= a\overline{y}b = a \cdot \overline{f(1,2)} \cdot b; \\
\left(R^3 f\right)(2,2) &= a\overline{w}b = a \cdot \overline{f(1,1)} \cdot b.
\end{aligned}$$

These equalities relate the label of $R^{i+j-1} f$ at an element (i, j) with the label of f at the element $(3 - i,\ 3 - j)$ (which is, visually speaking, the "antipode" of the former element (i, j) on the Hasse diagram of P). To be specific, they say that

$$\left(R^{i+j-1} f\right)(i,j) = a \cdot \overline{f(3-i,\ 3-j)} \cdot b$$

for any $(i, j) \in P$. This too can be generalized to arbitrary rectangles (Theorem 4.8).

In the above calculation, we used the linear extension $((1,1), (1,2), (2,1), (2,2))$ of P to compute R as $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. We could have just as well used the linear extension $((1,1), (2,1), (1,2), (2,2))$, obtaining the same result. But we could not have used the list $((1,1), (1,2), (2,2), (2,1))$ (for example), since it is not a linear extension (and indeed, $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,2)} \circ T_{(2,1)}$ would not give rise to any similar phenomenon).

This example shows that birational rowmotion behaves unexpectedly well for some posets. There are also some more serious motivations to study it: Birational rowmotion for commutative $\mathbb{K}$ generalizes Schützenberger's classical "promotion" map on semistandard tableaux (see [GriRob14, Remark 11.6]), and is closely related to the Zamolodchikov periodicity conjecture in type AA (see [Roby15, §4.4]). The case of a noncommutative ring $\mathbb{K}$ appears more baroque, but we expect it to find a combinatorial meaning sooner or later.

Before we formalize and prove the above phenomena, we first consider some general properties of R. We begin with an implicit description of birational rowmotion that does not involve toggles (but is essentially a restatement of Definition 3.18):

Proposition 3.20. Let $v \in P$. Let $f \in \mathbb{K}^{\widehat{P}}$. Assume that $Rf \neq \bot$. Then

$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Rf)(u)}}.$$

Proof. In the case when $\mathbb{K}$ is commutative, this is [GriRob14, Proposition 2.16]. The proof given in [GriRob14] can be easily modified to apply to the general case as well. Here are the details:

Let $(v_1, v_2, \ldots, v_m)$ be a linear extension of P. (Such a linear extension exists, because of Theorem 1.5.)

Let $i \in \{1, 2, \ldots, m\}$ be the index satisfying $v_i = v$. Thus, $T_{v_i} = T_v$.

By the definition of birational rowmotion R, we have $R = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m}$.

Define two partial maps

$$A := T_{v_{i+1}} \circ T_{v_{i+2}} \circ \cdots \circ T_{v_m} \qquad \text{and} \qquad B := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{i-1}}$$

from $\mathbb{K}^{\widehat{P}}$ to $\mathbb{K}^{\widehat{P}}$. Then

$$\begin{aligned} R &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} = \underbrace{T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{i-1}}}_{=B} \circ \underbrace{T_{v_i}}_{=T_v} \circ \underbrace{T_{v_{i+1}} \circ T_{v_{i+2}} \circ \cdots \circ T_{v_m}}_{=A} \\ &= B \circ T_v \circ A. \end{aligned} \tag{12}$$

Define the $\mathbb{K}$-labeling $g := T_v(Af)$. Thus,

$$\underbrace{R}_{\substack{=B \circ T_v \circ A \\ \text{(by (12))}}} f = (B \circ T_v \circ A) f = B \left(\underbrace{T_v(Af)}_{=g} \right) = Bg.$$

Hence, $Bg = Rf \neq \bot = B(\bot)$, and thus $g \neq \bot$. Hence, $T_v(Af) = g \neq \bot = T_v(\bot)$, so that $Af \neq \bot$. Now:

- Each of the maps T_{v_j} with $j \neq i$ leaves the label at v unchanged when acting on a $\mathbb{K}$-labeling (since $j \neq i$ entails $v_j \neq v_i = v$). Hence, each of the maps B and A

leaves the label at v unchanged (since B and A are compositions of maps T_{v_j} with $j \neq i$). Thus, $(Bg)(v) = g(v)$ and $(Af)(v) = f(v)$. Now,

$$\begin{aligned}
\underbrace{(Rf)}_{=Bg}(v) = (Bg)(v) &= \underbrace{g}_{=T_v(Af)}(v) \\
&= (T_v(Af))(v) \\
&= \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} (Af)(u) \right) \cdot \overline{(Af)(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Af)(u)}} \\
&\qquad \left(\begin{array}{c} \text{by Proposition 3.14 } \textbf{(b)}, \\ \text{applied to } Af \text{ instead of } f \end{array} \right) \\
&= \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} (Af)(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Af)(u)}} \qquad (13)
\end{aligned}$$

(since $(Af)(v) = f(v)$).

- Let $u \in \widehat{P}$ be such that $u \lessdot v$. Then $u < v = v_i$ in $\widehat{P}$. Hence, u is none of the elements $v_{i+1}, v_{i+2}, \ldots, v_m$ (because $(v_1, v_2, \ldots, v_m)$ is a linear extension of P). Thus, each of the maps $T_{v_{i+1}}, T_{v_{i+2}}, \ldots, T_{v_m}$ leaves the label at u invariant when acting on a $\mathbb{K}$-labeling. Therefore, the map A also leaves the label at u invariant (since A is a composition of these maps $T_{v_{i+1}}, T_{v_{i+2}}, \ldots, T_{v_m}$). Hence, $(Af)(u) = f(u)$.

  Forget that we fixed u. We have thus shown that

$$(Af)(u) = f(u) \qquad \text{for every } u \in \widehat{P} \text{ such that } u \lessdot v. \qquad (14)$$

- Let $u \in \widehat{P}$ be such that $u \gtrdot v$. Then $u > v = v_i$ in $\widehat{P}$. Hence, u is none of the elements $v_1, v_2, \ldots, v_{i-1}$ (because $(v_1, v_2, \ldots, v_m)$ is a linear extension of P). Thus, each of the maps $T_{v_1}, T_{v_2}, \ldots, T_{v_{i-1}}$ leaves the label at u invariant when acting on a $\mathbb{K}$-labeling. Therefore, B also leaves the label at u invariant (since B is a composition of these maps $T_{v_1}, T_{v_2}, \ldots, T_{v_{i-1}}$). Since T_v also leaves the label at u invariant (because $u \neq v$ (since $u > v$)), this yields that the composition $B \circ T_v$ also leaves the label at u invariant. Hence, $((B \circ T_v)(Af))(u) = (Af)(u)$, so that

$$(Af)(u) = ((B \circ T_v)(Af))(u) = \left(\underbrace{(B \circ T_v \circ A)}_{\substack{=R \\ \text{(by (12))}}} f \right)(u) = (Rf)(u).$$

  Forget that we fixed u. We thus have proven that

$$(Af)(u) = (Rf)(u) \qquad \text{for every } u \in \widehat{P} \text{ such that } u \gtrdot v. \qquad (15)$$

Now, substituting (14) and (15) into (13), we obtain

$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P}; \\ u < v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u > v}} \overline{(Rf)(u)}}.$$

This proves Proposition 3.20. ☐

The following near-trivial fact completes the picture:

Proposition 3.21. Let $f \in \mathbb{K}^{\widehat{P}}$. Assume that $Rf \neq \bot$. Then $(Rf)(0) = f(0)$ and $(Rf)(1) = f(1)$.

Proof. Let $(v_1, v_2, \ldots, v_m)$ be a linear extension of P. (Such a linear extension exists, because of Theorem 1.5.)

By the definition of birational rowmotion R, we have $R = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m}$.

Each of the maps T_{v_j} with $j \in \{1, 2, \ldots, m\}$ leaves the label at 0 unchanged when acting on a $\mathbb{K}$-labeling (since $0 \neq v_j$ (because $0 \notin P$ whereas $v_j \in P$)). Therefore, the map R also leaves the label at 0 unchanged (since R is the composition $T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m}$ of these maps T_{v_j}). Hence, $(Rf)(0) = f(0)$. Similarly, $(Rf)(1) = f(1)$. This proves Proposition 3.21. ☐

A trivial corollary of Proposition 3.21 is:

Corollary 3.22. Let $f \in \mathbb{K}^{\widehat{P}}$ and $\ell \in \mathbb{N}$. Assume that $R^\ell f \neq \bot$. Then $\left(R^\ell f\right)(0) = f(0)$ and $\left(R^\ell f\right)(1) = f(1)$.

(Recall that $\mathbb{N}$ denotes the set $\{0, 1, 2, \ldots\}$ in this paper.)

Proof of Corollary 3.22. Let $i \in \{1, 2, \ldots, \ell\}$. Then $R^{\ell-i}(R^i f) = R^\ell f \neq \bot = R^{\ell-i}(\bot)$, so that $R^i f \neq \bot$. In other words, $R(R^{i-1} f) \neq \bot$ (since $R(R^{i-1} f) = R^i f$). Hence, we can apply Proposition 3.21 to $R^{i-1} f$ instead of f. As a result, we obtain

$$\left(R\left(R^{i-1} f\right)\right)(0) = \left(R^{i-1} f\right)(0) \qquad \text{and} \qquad \left(R\left(R^{i-1} f\right)\right)(1) = \left(R^{i-1} f\right)(1).$$

From $R^i f = R(R^{i-1} f)$, we now obtain $(R^i f)(0) = (R(R^{i-1} f))(0) = (R^{i-1} f)(0)$. Hence, $(R^{i-1} f)(0) = (R^i f)(0)$.

Forget that we fixed i. We thus have proved the equality $(R^{i-1} f)(0) = (R^i f)(0)$ for each $i \in \{1, 2, \ldots, \ell\}$. Combining all of these equalities, we obtain

$$\left(R^0 f\right)(0) = \left(R^1 f\right)(0) = \left(R^2 f\right)(0) = \cdots = \left(R^\ell f\right)(0).$$

Hence, $\left(R^\ell f\right)(0) = \underbrace{\left(R^0 f\right)}_{=f}(0) = f(0)$. A similar argument shows that $\left(R^\ell f\right)(1) = f(1)$. Thus, Corollary 3.22 is proven. ☐

3.6. Well-definedness lemmas

We next show some simple lemmas which say that certain inverses exist under the assumption that $R^\ell f$ is well-defined for some values of ℓ. These lemmas are easy and unexciting, but are necessary in order to rigorously prove the more substantial results that will follow. We recommend the reader skip the proofs, at least on a first reading.

Lemma 3.23. Let $f \in \mathbb{K}^{\widehat{P}}$ and $k, \ell \in \mathbb{N}$ satisfy $k \leqslant \ell$ and $R^\ell f \neq \bot$. Then, $R^k f \neq \bot$.

Proof. From $k \leqslant \ell$, we obtain $R^\ell f = R^{\ell-k}\left(R^k f\right)$. Thus, if we had $R^k f = \bot$, then we would obtain

$$R^\ell f = R^{\ell-k}\left(\underbrace{R^k f}_{=\bot}\right) = R^{\ell-k}(\bot) = \bot,$$

which would contradict $R^\ell f \neq \bot$. Hence, we must have $R^k f \neq \bot$. This proves Lemma 3.23. ☐

Lemma 3.24. Let $f \in \mathbb{K}^{\widehat{P}}$ satisfy $Rf \neq \bot$. Let $v \in P$. Then, $f(v)$ is invertible.

Proof. Proposition 3.20 yields

$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{(Rf)(u)}}.$$

Thus, in particular, $\overline{f(v)}$ is well-defined. In other words, $f(v)$ is invertible. This proves Lemma 3.24. ☐

Lemma 3.25. Assume that $P \neq \varnothing$. Let $f \in \mathbb{K}^{\widehat{P}}$ satisfy $Rf \neq \bot$. Then, $f(1)$ is invertible.

Proof. We have $P \neq \varnothing$. Thus, the poset P has a maximal element y (by Proposition 1.9 **(b)**). Consider this y. The element y of P is maximal. Thus, in $\widehat{P}$, we have $1 \gtrdot y$ (by Remark 3.3 **(b)**). In other words, 1 is a $u \in \widehat{P}$ satisfying $u \gtrdot y$.

We have $Rf \neq \bot$. Therefore, Proposition 3.20 (applied to $v = y$) yields

$$(Rf)(y) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} f(u)\right) \cdot \overline{f(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{(Rf)(u)}}.$$

Hence, in particular, $\overline{(Rf)(u)}$ is well-defined for each $u \in \widehat{P}$ satisfying $u \gtrdot y$. We can apply this to $u = 1$ (since 1 is a $u \in \widehat{P}$ satisfying $u \gtrdot y$), and thus conclude that $\overline{(Rf)(1)}$ is well-defined. In other words, $(Rf)(1)$ is invertible. However, Proposition 3.21 yields $(Rf)(1) = f(1)$. Thus, $f(1)$ is invertible (since $(Rf)(1)$ is invertible). This proves Lemma 3.25. ☐

Lemma 3.26. Assume that $P \neq \varnothing$. Let $f \in \mathbb{K}^{\widehat{P}}$ satisfy $R^2 f \neq \bot$. Then, $f(0)$ and $f(1)$ are invertible.

Proof. We have $P \neq \varnothing$. Thus, the poset P has a minimal element x (by Proposition 1.9 **(a)**). Consider this x.

From $R(Rf) = R^2 f \neq \bot = R(\bot)$, we obtain $Rf \neq \bot$; thus, $Rf \in \mathbb{K}^{\widehat{P}}$. Hence, Lemma 3.25 yields that $f(1)$ is invertible. Furthermore, Lemma 3.24 (applied to Rf and x instead of f and v) yields that $(Rf)(x)$ is invertible (since $R(Rf) \neq \bot$).

Recall again that $Rf \neq \bot$. Hence, Proposition 3.20 (applied to $v = x$) yields

$$(Rf)(x) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot x}} f(u) \right) \cdot \overline{f(x)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}}. \tag{16}$$

The only $u \in \widehat{P}$ satisfying $u \lessdot x$ is the element 0 of $\widehat{P}$ (since x is a minimal element of P). Thus, $\sum_{\substack{u \in \widehat{P}; \\ u \lessdot x}} f(u) = f(0)$. Hence, (16) rewrites as

$$(Rf)(x) = f(0) \cdot \overline{f(x)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}}. \tag{17}$$

This equality shows that $\overline{f(x)}$ and $\overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}}$ are well-defined. Thus, the elements $f(x)$ and $\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}$ are invertible. Multiplying both sides of the equality (17) by $\left(\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)} \right) \cdot f(x)$ on the right, we obtain

$$\begin{aligned} (Rf)(x) \cdot \left(\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)} \right) \cdot f(x) &= f(0) \cdot \overline{f(x)} \cdot \underbrace{\overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}} \cdot \left(\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)} \right)}_{=1} \cdot f(x) \\ &= f(0) \cdot \underbrace{\overline{f(x)} \cdot f(x)}_{=1} = f(0). \end{aligned}$$

The left hand side of this equality is invertible (since it is the product of the three invertible elements $(Rf)(x)$, $\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot x}} \overline{(Rf)(u)}$ and $f(x)$). Thus, its right hand side is invertible as well. In other words, $f(0)$ is invertible. This completes the proof of Lemma 3.26. ☐

Lemma 3.27. Let $v \in P$. Assume that v is not a minimal element of P. Then, there exists at least one element $w \in P$ satisfying $v \gtrdot w$.

Proof. The element v of P is not minimal. Thus, there exists some $u \in P$ satisfying $u < v$. In other words, the set $P_{<v} := \{u \in P \mid u < v\}$ is nonempty. Consider this set $P_{<v}$ as a subposet of P (with its partial order inherited from P). Then, $P_{<v}$ is a finite poset and thus has a maximal element (by Proposition 1.9 **(b)**, applied to $P_{<v}$ instead of P). Let m be this maximal element. Then, $m \in P_{<v} = \{u \in P \mid u < v\}$; in other words, $m \in P$ and $m < v$. There exists no element of $P_{<v}$ that is larger than m (since m is a **maximal** element of $P_{<v}$).

If there was some $w \in P$ satisfying $m < w < v$, then this w would belong to $P_{<v}$ (since $w < v$) but would be larger than m in the poset $P_{<v}$ (since $m < w$); this would contradict the fact that there exists no element of $P_{<v}$ that is larger than m. Hence, there is no $w \in P$ satisfying $m < w < v$. In other words, we have $m \lessdot v$ in P (since $m < v$). In other words, we have $v \gtrdot m$ in P. Hence, there exists at least one element $w \in P$ satisfying $v \gtrdot w$ (namely, $w = m$). This proves Lemma 3.27. $\square$

Lemma 3.28. Let $f \in \mathbb{K}^{\widehat{P}}$ satisfy $Rf \neq \perp$. Let $v \in P$. Assume that v is not a minimal element of P. Then, $(Rf)(v)$ is invertible.

Proof. Lemma 3.27 shows that there exists at least one element $w \in P$ satisfying $v \gtrdot w$. Consider this w. Proposition 3.20 (applied to w instead of v) yields

$$(Rf)(w) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot w}} f(u) \right) \cdot \overline{f(w)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot w}} \overline{(Rf)(u)}}.$$

In particular, $\overline{(Rf)(u)}$ is well-defined for each $u \in \widehat{P}$ satisfying $u \gtrdot w$. Applying this to $u = v$, we conclude that $\overline{(Rf)(v)}$ is well-defined (since $v \in P \subseteq \widehat{P}$ and $v \gtrdot w$). In other words, $(Rf)(v)$ is invertible. $\square$

Lemma 3.29. Assume that $P \neq \varnothing$. Let $f \in \mathbb{K}^{\widehat{P}}$ satisfy $Rf \neq \perp$. Let $v \in \widehat{P}$. Assume that $f(0)$ is invertible. Then, $(Rf)(v)$ is invertible.

Proof. We are in one of the following four cases:

Case 1: We have $v = 0$.

Case 2: We have $v = 1$.

Case 3: The element v is a minimal element of P.

Case 4: The element v is neither 0 nor 1 nor a minimal element of P.

Let us first consider Case 1. In this case, we have $v = 0$. Hence, $(Rf)(v) = (Rf)(0) = f(0)$ (by Proposition 3.21). Hence, $(Rf)(v)$ is invertible (since $f(0)$ is invertible). This proves Lemma 3.29 in Case 1.

Let us now consider Case 2. In this case, we have $v = 1$. Hence, $(Rf)(v) = (Rf)(1) = f(1)$ (by Proposition 3.21). However, $f(1)$ is invertible (by Lemma 3.25). In other words, $(Rf)(v)$ is invertible (since $(Rf)(v) = f(1)$). This proves Lemma 3.29 in Case 2.

Next, let us consider Case 3. In this case, the element v is a minimal element of P. Hence, the only $u \in \widehat{P}$ satisfying $u \lessdot v$ is the element 0. Therefore, $\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) = f(0)$. Now, Proposition 3.20 yields

$$(Rf)(v) = \underbrace{\left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right)}_{= f(0)} \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Rf)(u)}} = f(0) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Rf)(u)}}.$$

The right hand side of this equality is a product of three invertible elements (since $f(0)$ is invertible, and since $\overline{f(v)}$ and $\overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Rf)(u)}}$ are invertible[13]), and thus itself is invertible. Thus, the left hand side is invertible as well. In other words, $(Rf)(v)$ is invertible. This proves Lemma 3.29 in Case 3.

Finally, let us consider Case 4. In this case, the element v is neither 0 nor 1 nor a minimal element of P. However, $v \in P$ (since v is neither 0 nor 1). Thus, Lemma 3.28 yields that $(Rf)(v)$ is invertible. This proves Lemma 3.29 in Case 4.

We have now proved Lemma 3.29 in all four Cases 1, 2, 3 and 4. □

4. The rectangle: statements of the results

4.1. The $p \times q$-rectangle

As promised, we now state the phenomena observed in Example 3.19 in greater generality (and afterwards prove them). First we define the posets on which these phenomena manifest:

Definition 4.1. For $p \in \mathbb{Z}$, we let $[p]$ denote the totally ordered set $\{1, 2, \ldots, p\}$ (with its usual total order: $1 < 2 < \cdots < p$). This set is empty if $p \leqslant 0$.

We recall that the *Cartesian product* $P \times Q$ of two posets P and Q is defined to be the set $P \times Q$, equipped with the entrywise partial order (i.e., the partial order in which two pairs $(p_1, q_1) \in P \times Q$ and $(p_2, q_2) \in P \times Q$ satisfy $(p_1, q_1) \leqslant (p_2, q_2)$ if and only if $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$).

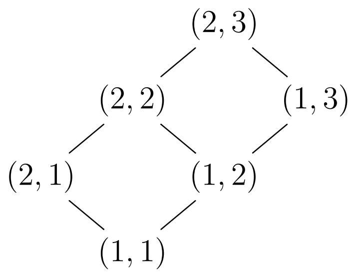
[13] because an inverse is always invertible

Definition 4.2. Let p and q be two positive integers. The $p \times q$*-rectangle* will mean the Cartesian product $[p] \times [q]$ of the two posets $[p]$ and $[q]$. Explicitly, this is the set $[p] \times [q] = \{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$, equipped with the partial order defined as follows: For two elements (i, j) and (i', j') of $[p] \times [q]$, we set $(i, j) \leqslant (i', j')$ if and only if ($i \leqslant i'$ and $j \leqslant j'$).

Henceforth, if we speak of $[p] \times [q]$, we implicitly assume that p and q are two positive integers.

The $p \times q$-rectangle has been denoted by $\operatorname{Rect}(p, q)$ in [GriRob14].

Example 4.3. Here is the Hasse diagram of the 2×3-rectangle $[2] \times [3]$:



(18)

Convention 4.4. In the following, the Hasse diagram of a $p \times q$-rectangle will always be drawn as in (18). That is, the elements (i, j) of $[p] \times [q]$ will be aligned in a rectangular grid, with the x-axis going southeast to northwest and the y-axis going southwest to northeast. Thus, for instance, the northwestern neighbor of an element (i, j) is always $(i + 1, j)$.

Two elements s and t of $\widehat{P}$ will be called *adjacent* if they satisfy $s \gtrdot t$ or $t \gtrdot s$.

The poset $[p] \times [q]$ has a unique minimal element, $(1, 1)$, and a unique maximal element, (p, q). Its covering relation can be characterized by the following easy remark (which will be used without explicit mention):

Remark 4.5. Let (i, j) and (i', j') be two elements of $[p] \times [q]$. Then, $(i, j) \lessdot (i', j')$ if and only if (i', j') is either $(i + 1, j)$ or $(i, j + 1)$.

Convention 4.6. Let $P = [p] \times [q]$. If f is a function defined on P or on $\widehat{P}$, and if (i, j) is any element of P, then we will write $f(i, j)$ for $f((i, j))$.

4.2. Periodicity

The following theorem (conjectured by the first author in 2014) generalizes the periodicity-like phenomenon seen in Example 3.19:

Theorem 4.7 (Periodicity theorem for the $\boldsymbol{p}\times\boldsymbol{q}$-rectangle)**.** Let $P=[p]\times[q]$, and let $f\in\mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^{p+q}f\neq\bot$. Set $a=f(0)$ and $b=f(1)$. Then, a and b are invertible, and for any $x\in\widehat{P}$ we have

$$\left(R^{p+q}f\right)(x)=a\overline{b}\cdot f(x)\cdot\overline{a}b. \tag{19}$$

If the ring $\mathbb{K}$ is commutative[14], then (19) simplifies to $\left(R^{p+q}f\right)(x)=f(x)$ (since commutativity of $\mathbb{K}$ yields $a\overline{b}\cdot f(x)\cdot\overline{a}b=\underbrace{a\overline{a}}_{=1}\cdot f(x)\cdot\underbrace{\overline{b}b}_{=1}=f(x)$). Thus, if $\mathbb{K}$ is commutative, then the claim of Theorem 4.7 can be rewritten as $R^{p+q}f=f$, generalizing the main part of [GriRob14, Theorem 11.5] (which itself generalizes similar properties of rowmotion operators on other levels). Unlike in [GriRob14, Theorem 11.5], we cannot honestly claim that $R^{p+q}=\mathrm{id}$ even when $\mathbb{K}$ is commutative, since the partial map R^{p+q} takes the value $\bot$ on some $\mathbb{K}$-labelings f (while id does not).

The parallel result for birational antichain rowmotion [JosRob20, Conjecture 5.10] follows from Theorem 4.7.

4.3. Reciprocity

Theorem 4.7 shows that the "periodicity phenomenon" we have observed on $[2]\times[2]$ in Example 3.19 was not a coincidence. The "reciprocity phenomenon" is similarly the $p=q=2$ case of a general fact:

Theorem 4.8 (Reciprocity theorem for $\boldsymbol{p}\times\boldsymbol{q}$-rectangle)**.** Let $P=[p]\times[q]$. Fix $\ell\in\mathbb{N}$, and let $f\in\mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^{\ell}f\neq\bot$. Set $a=f(0)$ and $b=f(1)$. Let $(i,j)\in P$ satisfy $\ell-i-j+1\geqslant 0$. Then,

$$\left(R^{\ell}f\right)(i,j)=a\cdot\overline{\left(R^{\ell-i-j+1}f\right)(p+1-i,\ q+1-j)}\cdot b. \tag{20}$$

Theorem 4.8 directly generalizes the analogous theorem [GriRob14, Theorem 11.7] in the commutative setting[15].

[14] or, more generally, if the elements a and b commute with each other and with $f(x)$

[15] In fact, let us assume that $\mathbb{K}$ is a (commutative) field. Keep using the notation of Theorem 4.8. Then, Theorem 4.8 (applied to $\ell=i+j-1$) yields

$$\begin{aligned}\left(R^{i+j-1}f\right)(i,j)&=a\cdot\overline{\left(R^{(i+j-1)-i-j+1}f\right)(p+1-i,\ q+1-j)}\cdot b\\&=a\cdot\overline{f(p+1-i,\ q+1-j)}\cdot b\qquad\left(\text{since }\underbrace{R^{(i+j-1)-i-j+1}}_{=R^0=\mathrm{id}}f=f\right)\\&=\frac{ab}{f(p+1-i,\ q+1-j)}.\end{aligned}$$

4.4. The structure of the proofs

Theorems 4.8 and 4.7 are the main results of this paper, and most of it will be devoted to their proofs. We first summarize the large-scale structure of these proofs:

1. In Section 5, we show that twisted periodicity (Theorem 4.7) follows from reciprocity (Theorem 4.8). Thus, proving the latter will suffice.

2. In Section 6, we introduce some notations. Some of these notations (a, b and x_ℓ) are mere abbreviations for the labels of $R^\ell f$, while others (A_ℓ^v, $\overline{V}_\ell^v$, $A_\ell^{\mathbf{p}}$, $\overline{V}_\ell^{\mathbf{p}}$, $A_\ell^{u\to v}$ and $\overline{V}_\ell^{u\to v}$) stand for certain derived quantities and will play a more active role. We also define "paths" on the poset P, and introduce a few of their basic features.

3. In Section 7, we prove a few simple results. The most important of these results are Proposition 7.3 (which reveals how birational rowmotion transforms $A_{\ell-1}^v$ into $\overline{V}_\ell^v$) and Theorem 7.6 (which allows us to recover the original labels x_ℓ from either A_ℓ^v or $\overline{V}_\ell^v$).

4. In Section 8, we prove Theorem 4.8 in the case when $(i,j) = (1,1)$. This proof warrants its own section both because it is conceptually easier than the general case, and because it requires some "well-definedness" technicalities that are (surprisingly) not needed in any other cases.

5. In Section 9, we saddle the main workhorse of our proof: a lemma (Lemma 9.2) that connects certain $A_\ell^{u\to v}$ quantities with certain $\overline{V}_\ell^{u\to v}$ quantities with the same ℓ. We prove this using a variant of paths, which we call "path-jump-paths" and which allow us to interpolate between $A_\ell^{u\to v}$ and $\overline{V}_\ell^{u\to v}$.

6. In Section 10, we combine the previous results with this lemma to prove Theorem 4.8 in the case when $j = 1$.

7. In Section 11, we finally complete the proof of Theorem 4.8 in the general case. This requires almost no new ideas, just an induction that extends Theorem 4.8 from four "adjacent" elements of P (labeled u, m, s, t in diagram (**??**)) to the fifth element v.

5. Twisted periodicity follows from reciprocity

Our first step towards the proofs of twisted periodicity (Theorem 4.7) and reciprocity (Theorem 4.8) is to show that the latter implies the former.[16]

Solving this equation for $f(p+1-i,\ q+1-j)$, we find

$$f(p+1-i,\ q+1-j) = \frac{ab}{(R^{i+j-1}f)(i,j)} = \frac{f(0)\cdot f(1)}{(R^{i+j-1}f)(i,j)} \qquad (\text{since } a = f(0) \text{ and } b = f(1)),$$

which is precisely the claim of [GriRob14, Theorem 11.7] (with k renamed as j).

[16] This reduction is not new; it appears already in [MusRob17, proof of Corollary 2.12] in the commutative case.

Proof of Theorem 4.7 using Theorem 4.8. Assume that Theorem 4.8 has been proved. Now, let p, q, P, f, a and b be as in Theorem 4.7. From $p \geqslant 1$ and $q \geqslant 1$, we obtain $p+q \geqslant 1+1=2$, so that $2 \leqslant p+q$. Hence, from $R^{p+q}f \neq \bot$, we obtain $R^2 f \neq \bot$ (by Lemma 3.23). Therefore, Lemma 3.26 yields that $f(0)$ and $f(1)$ are invertible. In other words, a and b are invertible (since $a = f(0)$ and $b = f(1)$).

Let $x \in \widehat{P}$. It thus remains to prove the equality (19). Let us first verify that this equality holds for $x = 0$. Indeed,

$$a\overline{b} \cdot \underbrace{f(0)}_{=a} \cdot \overline{a}b = a\overline{b} \cdot \underbrace{a \cdot \overline{a}}_{=1} b = a \underbrace{\overline{b} \cdot b}_{=1} = a = \left(R^{p+q}f\right)(0)$$

(because Corollary 3.22 (applied to $\ell = p+q$) yields $\left(R^{p+q}f\right)(0) = f(0) = a$). In other words, $\left(R^{p+q}f\right)(0) = a\overline{b} \cdot f(0) \cdot \overline{a}b$. In other words, the equality (19) holds for $x = 0$.

Similarly, this equality also holds for $x = 1$. Thus, for the rest of this proof, we WLOG assume that x is neither 0 nor 1. Hence, $x \in P = [p] \times [q]$. In other words, $x = (i,j)$ for some $i \in [p]$ and $j \in [q]$. Consider these i and j.

From $R^{p+q}f \neq \bot$, we obtain $R^1 f \neq \bot$ (by Lemma 3.23, since $1 \leqslant 2 \leqslant p+q$). In other words, $Rf \neq \bot$. Thus, Lemma 3.24 (applied to $v = (i,j)$) yields that $f(i,j)$ is invertible. Hence, $\overline{f(i,j)}$ is well-defined.

The element $\overline{f(i,j)}$ of $\mathbb{K}$ is invertible (since it has inverse $f(i,j)$), and so are the elements a and b (as we have proved above). Thus, Proposition 2.3 **(c)** yields

$$\overline{a \cdot \overline{f(i,j)} \cdot b} = \overline{b} \cdot \underbrace{\overline{\overline{f(i,j)}}}_{=f(i,j)} \cdot \overline{a} = \overline{b} \cdot f(i,j) \cdot \overline{a}. \tag{21}$$

Set $i' := p+1-i$ and $j' := q+1-j$. Thus,

$$p+1-\underbrace{i'}_{=p+1-i} = p+1-(p+1-i) = i \qquad \text{and}$$
$$q+1-\underbrace{j'}_{=q+1-j} = q+1-(q+1-j) = j$$

and

$$p+q-i-j+1 = \underbrace{p+1-i}_{=i'} + \underbrace{q+1-j}_{=j'} - 1 = i'+j'-1.$$

Therefore, $i'+j'-1 = p+q-\underbrace{i}_{\leqslant p}-\underbrace{j}_{\leqslant q}+1 \geqslant p+q-p-q+1 = 1$, so that $i'+j'-1 \in \mathbb{N}$. Moreover, $i' = p+1-i \in [p]$ (since $i \in [p]$) and $j' \in [q]$ (similarly). Hence, $(i',j') \in [p] \times [q] = P$.

We have $i' \leqslant p$ (since $i' \in [p]$) and similarly $j' \leqslant q$. Adding these two inequalities together, we find $i'+j' \leqslant p+q$. Hence, $i'+j'-1 \leqslant i'+j' \leqslant p+q$ and thus $R^{i'+j'-1}f \neq \bot$ (by Lemma 3.23, since $R^{p+q}f \neq \bot$). Also, $(i'+j'-1)-i'-j'+1 = 0 \geqslant 0$.

Thus, Theorem 4.8 (applied to $i'+j'-1$, i' and j' instead of ℓ, i and j) yields

$$\begin{aligned}\left(R^{i'+j'-1}f\right)(i',j') &= a\cdot\overline{\left(\underbrace{R^{(i'+j'-1)-i'-j'+1}}_{\substack{=R^0\\ (\text{since } (i'+j'-1)-i'-j'+1=0)}} f\right)\left(\underbrace{p+1-i'}_{=i},\ \underbrace{q+1-j'}_{=j}\right)}\cdot b\\ &= a\cdot\overline{\underbrace{\left(R^0 f\right)}_{\substack{=f\\ (\text{since } R^0=\mathrm{id})}}(i,j)}\cdot b\\ &= a\cdot\overline{f(i,j)}\cdot b. \end{aligned} \tag{22}$$

However, we also have $p+q-i-j+1=i'+j'-1\geqslant 1\geqslant 0$. Thus, Theorem 4.8 (applied to $\ell=p+q$) yields

$$\begin{aligned}\left(R^{p+q}f\right)(i,j) &= a\cdot\overline{\left(R^{p+q-i-j+1}f\right)(p+1-i,\ q+1-j)}\cdot b\\ &= a\cdot\overline{\left(R^{i'+j'-1}f\right)(i',j')}\cdot b\\ &\qquad \left(\begin{array}{c}\text{since } p+q-i-j+1=i'+j'-1\\ \text{and } p+1-i=i' \text{ and } q+1-j=j'\end{array}\right)\\ &= a\cdot\underbrace{\overline{a\cdot\overline{f(i,j)}\cdot b}}_{\substack{=\overline{b}\cdot f(i,j)\cdot\overline{a}\\ (\text{by (21)})}}\cdot b \qquad (\text{by (22)})\\ &= a\overline{b}\cdot f(i,j)\cdot\overline{a}b.\end{aligned}$$

Since $x=(i,j)$, we can rewrite this as

$$\left(R^{p+q}f\right)(x) = a\overline{b}\cdot f(x)\cdot\overline{a}b.$$

This proves the equality (19). Theorem 4.7 is thus proved, assuming that Theorem 4.8 holds. □

6. Proof of reciprocity: notations

It now suffices to prove Theorem 4.8, which will be the ultimate goal of the next few sections. First we introduce some notations that will be used throughout these sections.

Fix two positive integers p and q. Assume that $P=[p]\times[q]$. Let $f\in\mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling of P. Set

$$a := f(0) \qquad \text{and} \qquad b := f(1).$$

For any $x=(i,j)\in P$, we define an element $x^{\sim}\in P$ by

$$x^{\sim} := (p+1-i,\ q+1-j).$$

We call this element $x^{\sim}$ the *antipode* of x. Thus, the desired equality (20) can be rewritten as

$$\left(R^{\ell} f\right)(x) = a \cdot \overline{\left(R^{\ell-i-j+1} f\right)\left(x^{\sim}\right)} \cdot b \tag{23}$$

for $x = (i, j)$.

For any $x \in \widehat{P}$ and $\ell \in \mathbb{N}$, we write

$$x_{\ell} := \left(R^{\ell} f\right)(x), \tag{24}$$

which is well-defined whenever $R^{\ell} f \neq \bot$. This compact notation will make upcoming formulas more readable.

In particular, for each $x \in \widehat{P}$, we have

$$x_0 = \underbrace{\left(R^0 f\right)}_{=f}(x) = f(x).$$

Moreover, for each $\ell \in \mathbb{N}$ satisfying $R^{\ell} f \neq \bot$, we have

$$\begin{aligned} 0_{\ell} = \left(R^{\ell} f\right)(0) &= f(0) \qquad \text{(by Corollary 3.22)} \\ &= a \end{aligned} \tag{25}$$

and

$$\begin{aligned} 1_{\ell} = \left(R^{\ell} f\right)(1) &= f(1) \qquad \text{(by Corollary 3.22)} \\ &= b. \end{aligned} \tag{26}$$

We can further rewrite the equality (23) as $x_{\ell} = a \cdot \overline{x^{\sim}_{\ell-i-j+1}} \cdot b$ (since $x_{\ell} = \left(R^{\ell} f\right)(x)$ and $x^{\sim}_{\ell-i-j+1} = \left(R^{\ell-i-j+1} f\right)\left(x^{\sim}\right)$). Hence, our desired Theorem 4.8 takes the following form:

Theorem 4.8, restated. If $x = (i, j) \in P$ and $\ell \in \mathbb{N}$ satisfy $\ell - i - j + 1 \geqslant 0$ and $R^{\ell} f \neq \bot$, then

$$x_{\ell} = a \cdot \overline{x^{\sim}_{\ell-i-j+1}} \cdot b. \tag{27}$$

Proposition 3.20 yields that for each $v \in P$, we have[17]

$$(Rf)(v) = \left(\sum_{u \lessdot v} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum_{u \gtrdot v} \overline{(Rf)(u)}}. \tag{28}$$

Here, the summation index u under both sums is understood to range over $\widehat{P}$; from now on, this will always be understood if not otherwise specified.

[17]assuming that $Rf \neq \bot$

For each $v \in P$ and $\ell \in \mathbb{N}$ satisfying $R^{\ell+1} f \neq \bot$, we have

$$\begin{aligned}
\left(R^{\ell+1} f\right)(v) &= \left(R\left(R^{\ell} f\right)\right)(v) \qquad \left(\text{since } R^{\ell+1} f = R\left(R^{\ell} f\right)\right) \\
&= \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} \left(R^{\ell} f\right)(u)\right) \cdot \overline{\left(R^{\ell} f\right)(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{\left(R\left(R^{\ell} f\right)\right)(u)}} \\
&\qquad \left(\begin{array}{c}\text{by Proposition 3.20, applied to } R^{\ell} f \text{ instead of } f \\ \left(\text{since } R\left(R^{\ell} f\right) = R^{\ell+1} f \neq \bot\right)\end{array}\right) \\
&= \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} \left(R^{\ell} f\right)(u)\right) \cdot \overline{\left(R^{\ell} f\right)(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{\left(R^{\ell+1} f\right)(u)}} \\
&\qquad \left(\text{since } R\left(R^{\ell} f\right) = R^{\ell+1} f\right) \\
&= \left(\sum_{u \lessdot v} \left(R^{\ell} f\right)(u)\right) \cdot \overline{\left(R^{\ell} f\right)(v)} \cdot \overline{\sum_{u \gtrdot v} \overline{\left(R^{\ell+1} f\right)(u)}}
\end{aligned}$$

(here, we have rewritten the summation signs $\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}}$ and $\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}}$ as $\sum\limits_{u \lessdot v}$ and $\sum\limits_{u \gtrdot v}$, since we understand the summation index u to range over $\widehat{P}$ by default).

Using (24), we can rewrite this as follows:

$$v_{\ell+1} = \left(\sum_{u \lessdot v} u_{\ell}\right) \cdot \overline{v_{\ell}} \cdot \overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}} \tag{29}$$

for each $v \in P$ and $\ell \in \mathbb{N}$ satisfying $R^{\ell+1} f \neq \bot$.

Next, we formally define the paths that will play a key role in the proof. A *path* means a sequence $(v_0, v_1, \ldots, v_k)$ of elements of $\widehat{P}$ satisfying $v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k$. We denote this path by $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$, and we will call it a *path from* v_0 *to* v_k (or, for short, a *path* $v_0 \to v_k$). The *vertices* of this path are defined to be the elements $v_0, v_1, \ldots, v_k$. We say that this path *starts* at v_0 and *ends* at v_k.

For each $v \in P$ and $\ell \in \mathbb{N}$, we set[18]

$$A_{\ell}^{v} := v_{\ell} \cdot \overline{\sum_{u \lessdot v} u_{\ell}} \qquad \text{and} \qquad V_{\ell}^{v} := \overline{\sum_{u \gtrdot v} \overline{u_{\ell}}} \cdot \overline{v_{\ell}}.$$

[18]We recall that the summation signs "$\sum\limits_{u \lessdot v}$" and "$\sum\limits_{u \gtrdot v}$" mean "$\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot v}}$" and "$\sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot v}}$", respectively.

[19] Furthermore, when $v \in \{0, 1\}$, we set

$$A_\ell^v := 1 \qquad \text{and} \qquad \overline{V}_\ell^v := 1 \tag{30}$$

for all $\ell \in \mathbb{N}$.

For any path $\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ and any $\ell \in \mathbb{N}$, we set

$$A_\ell^{\mathbf{p}} := A_\ell^{v_0} A_\ell^{v_1} \cdots A_\ell^{v_k} \qquad \text{and}$$
$$\overline{V}_\ell^{\mathbf{p}} := \overline{V}_\ell^{v_0} \overline{V}_\ell^{v_1} \cdots \overline{V}_\ell^{v_k}$$

(assuming that the factors on the right hand sides are well-defined).

If u and v are elements of $\widehat{P}$, and if $\ell \in \mathbb{N}$, then we set

$$A_\ell^{u \to v} := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A_\ell^{\mathbf{p}} \qquad \text{and} \tag{31}$$

$$\overline{V}_\ell^{u \to v} := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} \overline{V}_\ell^{\mathbf{p}} \tag{32}$$

(assuming that all addends on the right hand sides are well-defined).

Example 6.1. Let $P = [2] \times [2]$ and $f \in \mathbb{K}^{\widehat{P}}$ be as in Example 3.19. Then,

$$(1,1)^\sim = (2,2)\,, \qquad (1,2)^\sim = (2,1)\,, \qquad (2,1)^\sim = (1,2)\,, \qquad (2,2)^\sim = (1,1)\,,$$
$$(1,1)_0 = f\,(1,1) = w, \qquad (1,1)_1 = (Rf)\,(1,1) = a\overline{z}b,$$
$$(1,1)_2 = \left(R^2 f\right)(1,1) = a\overline{b}z \cdot \overline{x+y} \cdot b,$$
$$(1,2)_2 = \left(R^2 f\right)(1,2) = a \cdot \overline{x+y} \cdot y\,(\overline{x} + \overline{y})\,b.$$

There are only two paths from $(2,2)$ to $(1,1)$: namely, the path $((2,2) \gtrdot (1,2) \gtrdot (1,1))$ and the path $((2,2) \gtrdot (2,1) \gtrdot (1,1))$. Each of these two paths has three vertices. There are no paths from $(1,1)$ to $(2,2)$, since we don't have $(1,1) \geqslant (2,2)$. The only path from 0 to 0 is the trivial path (0).

[19] These elements A_ℓ^v and $\overline{V}_\ell^v$ are not always well-defined. For A_ℓ^v to be well-defined, we need to have $R^\ell f \neq \bot$, and we need the element $\sum_{u \lessdot v} u_\ell$ to be invertible. For $\overline{V}_\ell^v$ to be well-defined, we need to have $R^\ell f \neq \bot$, and we need the elements $\overline{u_\ell}$ (for $u \gtrdot v$) and $\sum_{u \gtrdot v} \overline{u_\ell}$ and v_ℓ to be invertible.

We have

$$A_0^{(1,1)} = (1,1)_0 \cdot \overline{\sum_{u \lessdot (1,1)} u_0} = (1,1)_0 \cdot \overline{0_0} = w \cdot \overline{a},$$

$$A_0^{(2,2)} = (2,2)_0 \cdot \overline{\sum_{u \lessdot (2,2)} u_0} = (2,2)_0 \cdot \overline{(1,2)_0 + (2,1)_0} = z \cdot \overline{y + x},$$

$$A_1^{(1,1)} = (1,1)_1 \cdot \overline{\sum_{u \lessdot (1,1)} u_1} = (1,1)_1 \cdot \overline{0_1} = a\overline{z}b \cdot \overline{a},$$

$$V_0^{(1,1)} = \overline{\sum_{u \gtrdot (1,1)} \overline{u_0}} \cdot \overline{(1,1)_0} = \overline{\overline{(1,2)_0} + \overline{(2,1)_0}} \cdot \overline{(1,1)_0} = \overline{\overline{y} + \overline{x}} \cdot \overline{w},$$

$$V_1^{(1,1)} = \overline{\sum_{u \gtrdot (1,1)} \overline{u_1}} \cdot \overline{(1,1)_1} = \overline{\overline{(1,2)_1} + \overline{(2,1)_1}} \cdot \overline{(1,1)_1}$$

$$= \overline{\overline{w\overline{y}\,(x+y)\,\overline{z}b} + \overline{w\overline{x}\,(x+y)\,\overline{z}b}} \cdot \overline{a\overline{z}b} = w \cdot \overline{a} \qquad \text{(after simplifications)}.$$

Furthermore, for any $\ell \in \mathbb{N}$, we have

$$A_\ell^{((2,2)\gtrdot(1,2)\gtrdot(1,1))} = A_\ell^{(2,2)} A_\ell^{(1,2)} A_\ell^{(1,1)};$$
$$A_\ell^{(2,2)\to(1,1)} = A_\ell^{((2,2)\gtrdot(1,2)\gtrdot(1,1))} + A_\ell^{((2,2)\gtrdot(2,1)\gtrdot(1,1))}$$
$$= A_\ell^{(2,2)} A_\ell^{(1,2)} A_\ell^{(1,1)} + A_\ell^{(2,2)} A_\ell^{(2,1)} A_\ell^{(1,1)}$$

(and similarly for V instead of A).

The letter ℓ will always stand for a nonnegative integer (but will not be fixed).

Remark 6.2. The elements A_ℓ^v and V_ℓ^v (for $v \in P$ and $\ell \in \mathbb{N}$) are not entirely new. They are closely connected with the down-transfer operator ∇ and the up-transfer operator Δ studied in [JosRob20, Definition 5.11]; to be specific, we have $A_\ell^v = \left(\nabla R^\ell f\right)(v)$ and $V_\ell^v = \left(\Delta \Theta R^\ell f\right)(v)$ using the notations of [JosRob20, Definition 5.11]. These operators ∇ and Δ have a long history, going back to Stanley's "transfer map" ϕ between the order polytope and the chain polytope of a poset (see [Stan86, Definition 3.1]). The down-transfer operator ∇ does indeed restrict to ϕ when $\mathbb{K}$ is an appropriate tropical semiring. For this reason, we have been informally referring to A_ℓ^v and V_ℓ^v as the *down-slack* and the *up-slack* of v at time ℓ (harkening back to the notion of slack from linear optimization). Arguably, the behavior of these operators when $\mathbb{K}$ is the tropical semiring is not very indicative of the general case.

When $\mathbb{K}$ is commutative, our A_0^v have also implicitly appeared in [MusRob17]: If $v = (i,j) \in P$, then $A_0^v = \overline{A_{ij}}$, where A_{ij} is defined as in [MusRob17, (1)].

7. Proof of reciprocity: simple lemmas

Throughout this section, we use the notations introduced in Section 6.

Let us prove some relations between the elements we have introduced. We begin with a well-definedness result:

Lemma 7.1. Let $\ell \in \mathbb{N}$ be such that $\ell \geqslant 1$ and $R^{\ell} f \neq \bot$. Assume furthermore that a is invertible. Let $v \in \widehat{P}$. Then:

(a) The element v_{ℓ} is well-defined and invertible.

(b) The element $v_{\ell-1}$ is well-defined and invertible.

(c) The element $A_{\ell-1}^{v}$ is well-defined and invertible.

(d) The element V_{ℓ}^{v} is well-defined and invertible.

Proof. We know that a is invertible. In other words, $f(0)$ is invertible (since $a = f(0)$). Also, $\ell - 1 \in \mathbb{N}$ (since $\ell \geqslant 1$) and $R^{\ell-1} f \neq \bot$ (since $R\left(R^{\ell-1} f\right) = R^{\ell} f \neq \bot = R(\bot)$). Hence, Corollary 3.22 (applied to $\ell - 1$ instead of ℓ) yields $\left(R^{\ell-1} f\right)(0) = f(0)$ and $\left(R^{\ell-1} f\right)(1) = f(1)$. Thus, $\left(R^{\ell-1} f\right)(0) = f(0) = a$, so that $\left(R^{\ell-1} f\right)(0)$ is invertible (since a is invertible).

(a) Clearly, $v_{\ell} = \left(R^{\ell} f\right)(v)$ is well-defined (since $R^{\ell} f \neq \bot$). We have $R\left(R^{\ell-1} f\right) = R^{\ell} f \neq \bot$. Hence, Lemma 3.29 (applied to $R^{\ell-1} f$ instead of f) yields that $\left(R\left(R^{\ell-1} f\right)\right)(v)$ is invertible (since $\left(R^{\ell-1} f\right)(0)$ is invertible). In other words, v_{ℓ} is invertible (since $v_{\ell} = \underbrace{\left(R^{\ell} f\right)}_{=R\left(R^{\ell-1} f\right)}(v) = \left(R\left(R^{\ell-1} f\right)\right)(v)$). This proves Lemma 7.1 **(a)**.

(b) Clearly, $v_{\ell-1} = \left(R^{\ell-1} f\right)(v)$ is well-defined (since $R^{\ell-1} f \neq \bot$). It remains to prove that $v_{\ell-1}$ is invertible. If v is 0 or 1, then this follows easily from part **(a)**[20]. Thus, for the rest of this proof, we WLOG assume that v is neither 0 nor 1. Hence, $v \in P$.

We have $R\left(R^{\ell-1} f\right) = R^{\ell} f \neq \bot$. Hence, Lemma 3.24 (applied to $R^{\ell-1} f$ instead of f) yields that $\left(R^{\ell-1} f\right)(v)$ is invertible. In other words, $v_{\ell-1}$ is invertible (since $v_{\ell-1} = \left(R^{\ell-1} f\right)(v)$). This proves Lemma 7.1 **(b)**.

(c) If $v \in \{0, 1\}$, then this follows from (30). Thus, we WLOG assume that $v \notin \{0, 1\}$. Hence, $v \in P$.

Thus, applying (29) to $\ell - 1$ instead of ℓ, we obtain

$$v_{\ell} = \left(\sum_{u \lessdot v} u_{\ell-1}\right) \cdot \overline{v_{\ell-1}} \cdot \overline{\sum_{u \gtrdot v} \overline{u_{\ell}}} \tag{33}$$

[20] *Proof.* Assume that v is 0 or 1. We WLOG assume that $v = 0$ (since the case $v = 1$ is analogous).

We must show that $v_{\ell-1}$ is invertible. However, (25) yields $0_{\ell} = a$. Similarly, $0_{\ell-1} = a$. Comparing these two equalities, we obtain $0_{\ell} = 0_{\ell-1}$. In other words, $v_{\ell} = v_{\ell-1}$ (since $v = 0$). But Lemma 7.1 **(a)** shows that v_{ℓ} is invertible. Thus, $v_{\ell-1}$ is invertible (since $v_{\ell} = v_{\ell-1}$). Qed.

(since $R^\ell f \neq \bot$). This equality shows that $\overline{v_{\ell-1}}$ and $\overline{\sum_{u \gtrdot v} \overline{u_\ell}}$ are well-defined. In other words, the elements $v_{\ell-1}$ and $\sum_{u \gtrdot v} \overline{u_\ell}$ are invertible. Also, v_ℓ is invertible (by Lemma 7.1 **(a)**).

Multiplying the equality (33) by $\left(\sum_{u \gtrdot v} \overline{u_\ell}\right) \cdot v_{\ell-1}$ on the right, we obtain

$$v_\ell \cdot \left(\sum_{u \gtrdot v} \overline{u_\ell}\right) \cdot v_{\ell-1} = \left(\sum_{u \lessdot v} u_{\ell-1}\right) \cdot \overline{v_{\ell-1}} \cdot \underbrace{\overline{\sum_{u \gtrdot v} \overline{u_\ell}} \cdot \left(\sum_{u \gtrdot v} \overline{u_\ell}\right)}_{=1} \cdot v_{\ell-1}$$

$$= \left(\sum_{u \lessdot v} u_{\ell-1}\right) \cdot \underbrace{\overline{v_{\ell-1}} \cdot v_{\ell-1}}_{=1} = \sum_{u \lessdot v} u_{\ell-1}.$$

The left hand side of this equality is a product of three invertible elements (since v_ℓ and $\sum_{u \gtrdot v} \overline{u_\ell}$ and $v_{\ell-1}$ are invertible), and thus must itself be invertible. Hence, the right hand side is invertible. In other words, $\sum_{u \lessdot v} u_{\ell-1}$ is invertible. Thus, the element $\overline{\sum_{u \lessdot v} u_{\ell-1}}$ is well-defined. This element is furthermore invertible (since an inverse is always invertible).

Now, the definition of $A_{\ell-1}^v$ yields $A_{\ell-1}^v = v_{\ell-1} \cdot \overline{\sum_{u \lessdot v} u_{\ell-1}}$. This shows that $A_{\ell-1}^v$ is well-defined (since $v_{\ell-1}$ and $\overline{\sum_{u \lessdot v} u_{\ell-1}}$ are well-defined) and invertible (since $v_{\ell-1}$ and $\overline{\sum_{u \lessdot v} u_{\ell-1}}$ are invertible). This proves Lemma 7.1 **(c)**.

(d) If $v \in \{0, 1\}$, then this follows from (30). Thus, we WLOG assume that $v \notin \{0, 1\}$. Hence, $v \in P$.

Lemma 7.1 **(a)** shows that v_ℓ is invertible. Hence, $\overline{v_\ell}$ is well-defined, and invertible as well (since an inverse is always invertible). Also, in the proof of Lemma 7.1 **(c)**, we have shown that $\overline{\sum_{u \gtrdot v} \overline{u_\ell}}$ is well-defined. Thus, $\overline{\sum_{u \gtrdot v} \overline{u_\ell}}$ is invertible (since an inverse is always invertible). Now, the definition of V_ℓ^v yields $V_\ell^v = \overline{\sum_{u \gtrdot v} \overline{u_\ell}} \cdot \overline{v_\ell}$. Hence, V_ℓ^v is well-defined (since $\overline{\sum_{u \gtrdot v} \overline{u_\ell}}$ and $\overline{v_\ell}$ are well-defined) and invertible (since $\overline{\sum_{u \gtrdot v} \overline{u_\ell}}$ and $\overline{v_\ell}$ are invertible). This proves Lemma 7.1 **(d)**. □

Next we show some simple recursions for $A_\ell^{s \to t}$ and $V_\ell^{s \to t}$:

Proposition 7.2. Let s and t be two distinct elements of $\widehat{P}$, and fix $\ell \in \mathbb{N}$. Then

$$A_\ell^{s \to t} = A_\ell^s \sum_{\substack{u \in \widehat{P}; \\ s \gtrdot u}} A_\ell^{u \to t} \tag{34}$$

$$= \sum_{\substack{u \in \widehat{P}; \\ u \gtrdot t}} A_\ell^{s \to u} A_\ell^t \tag{35}$$

and

$$V_\ell^{s\to t} = V_\ell^s \sum_{\substack{u\in\widehat{P};\\ s\gtrdot u}} V_\ell^{u\to t} \tag{36}$$

$$= \sum_{\substack{u\in\widehat{P};\\ u\gtrdot t}} V_\ell^{s\to u} V_\ell^t. \tag{37}$$

Here, we assume that all the terms in the respective equalities are well-defined.

Proof. Any path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ from s to t has at least two vertices (since $s \neq t$), and thus has a well-defined second vertex v_1. This second vertex v_1 satisfies $s \gtrdot v_1$ (since $s = v_0 \gtrdot v_1$). In other words, this second vertex v_1 is an element $u \in \widehat{P}$ satisfying $s \gtrdot u$.

Fix an element $u \in \widehat{P}$ satisfying $s \gtrdot u$. If $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ is a path from s to t satisfying $v_1 = u$, then $(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$ is a path from u to t (since $v_1 = u$ and $v_k = t$). Hence, we have found a map

from $\{$paths $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ from s to t satisfying $v_1 = u\}$
to $\{$paths from u to $t\}$

that sends each path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ to $(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$. This map is injective[21] and surjective[22]; hence, it is a bijection. We can use this bijection to substitute $(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$ for $\mathbf{p}$ in the sum $\sum\limits_{\mathbf{p} \text{ is a path from } u \text{ to } t} A_\ell^s A_\ell^{\mathbf{p}}$. We thus obtain

$$\begin{aligned}
&\sum_{\mathbf{p} \text{ is a path from } u \text{ to } t} A_\ell^s A_\ell^{\mathbf{p}} \\
&= \sum_{\substack{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_1=u}} \underbrace{A_\ell^s}_{\substack{=A_\ell^{v_0}\\ (\text{since } s=v_0)}} \underbrace{A_\ell^{(v_1\gtrdot v_2\gtrdot\cdots\gtrdot v_k)}}_{\substack{=A_\ell^{v_1}A_\ell^{v_2}\cdots A_\ell^{v_k}\\ (\text{by the definition of } A_\ell^{(v_1\gtrdot v_2\gtrdot\cdots\gtrdot v_k)})}} \\
&= \sum_{\substack{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_1=u}} \underbrace{A_\ell^{v_0} A_\ell^{v_1} A_\ell^{v_2} \cdots A_\ell^{v_k}}_{\substack{=A_\ell^{v_0}A_\ell^{v_1}\cdots A_\ell^{v_k}\\ =A_\ell^{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k)}\\ (\text{by the definition of } A_\ell^{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k)})}} \\
&= \sum_{\substack{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_1=u}} A_\ell^{(v_0\gtrdot v_1\gtrdot\cdots\gtrdot v_k)}.
\end{aligned} \tag{38}$$

Now, forget that we fixed u. We thus have proved (38) for each $u \in \widehat{P}$ satisfying $s \gtrdot u$.

[21]because a path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ from s to t can be reconstructed from its image $(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$ under this map (since its first vertex v_0 is forced to be s)

[22]Indeed, if $\mathbf{p} = (v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$ is a path from u to t, then $(s \gtrdot v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_k)$ is a path from s to t satisfying $v_1 = u$ (since $s \gtrdot u = v_1$), and it is clear that our map sends the latter path to $\mathbf{p}$.

The definition of $A_\ell^{s\to t}$ yields

$$\begin{aligned}
A_\ell^{s\to t} &= \sum_{\mathbf{p} \text{ is a path from } s \text{ to } t} A_\ell^{\mathbf{p}} \\
&= \underbrace{\sum_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t}}_{\substack{= \sum_{u \in \widehat{P};\ s \gtrdot u} \ \sum_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\ v_1 = u} \\ \text{(because any path } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ from } s \text{ to } t \text{ has a well-defined} \\ \text{second vertex } v_1 \text{, and this second vertex } v_1 \text{ satisfies } s \gtrdot v_1)}} A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)} \\
&\qquad \text{(here we have renamed the index } \mathbf{p} \text{ as } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)) \\
&= \sum_{\substack{u \in \widehat{P}; \\ s \gtrdot u}} \underbrace{\sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t; \\ v_1 = u}} A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)}}_{\substack{= \sum_{\mathbf{p} \text{ is a path from } u \text{ to } t} A_\ell^s A_\ell^{\mathbf{p}} \\ \text{(by (38))}}} \\
&= \sum_{\substack{u \in \widehat{P}; \\ s \gtrdot u}} \ \sum_{\mathbf{p} \text{ is a path from } u \text{ to } t} A_\ell^s A_\ell^{\mathbf{p}} = A_\ell^s \sum_{\substack{u \in \widehat{P}; \\ s \gtrdot u}} \underbrace{\sum_{\mathbf{p} \text{ is a path from } u \text{ to } t} A_\ell^{\mathbf{p}}}_{\substack{= A_\ell^{u\to t} \\ \text{(by the definition of } A_\ell^{u\to t})}} \\
&= A_\ell^s \sum_{\substack{u \in \widehat{P}; \\ s \gtrdot u}} A_\ell^{u\to t}.
\end{aligned}$$

This proves (34). The same argument (but with each A symbol replaced by an V symbol) proves (36).

Let us now prove (35). This proof will be very similar to the above proof of (34), but we will now classify paths from s to t according to their second-to-last vertex instead of their second vertex. Here are the details:

Any path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ from s to t has at least two vertices (since $s \neq t$), and thus has a well-defined second-to-last vertex v_{k-1}. This second-to-last vertex v_{k-1} satisfies $v_{k-1} \gtrdot t$ (since $v_{k-1} \gtrdot v_k = t$). In other words, this second-to-last vertex v_{k-1} is an element $u \in \widehat{P}$ satisfying $u \gtrdot t$.

Now, fix an element $u \in \widehat{P}$ satisfying $u \gtrdot t$. If $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ is a path from s to t satisfying $v_{k-1} = u$, then $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})$ is a path from s to u (since $v_0 = s$ and $v_{k-1} = u$). Hence, we have found a map

$$\begin{aligned}
&\text{from } \{\text{paths } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ from } s \text{ to } t \text{ satisfying } v_{k-1} = u\} \\
&\text{to } \{\text{paths from } s \text{ to } u\}
\end{aligned}$$

that sends each path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ to $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})$. This map is injective[23] and surjective[24]; hence, it is a bijection. We can use this bijection to substitute $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})$ for $\mathbf{p}$ in the sum $\sum\limits_{\mathbf{p} \text{ is a path from } s \text{ to } u} A_\ell^{\mathbf{p}} A_\ell^t$. We thus obtain

$$\begin{aligned}
&\sum_{\mathbf{p} \text{ is a path from } s \text{ to } u} A_\ell^{\mathbf{p}} A_\ell^t \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_{k-1}=u}} \underbrace{A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})}}_{\substack{= A_\ell^{v_0} A_\ell^{v_1} \cdots A_\ell^{v_{k-1}} \\ \text{(by the definition of } A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})})}} \underbrace{A_\ell^t}_{\substack{= A_\ell^{v_k} \\ \text{(since } t = v_k)}} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_{k-1}=u}} \underbrace{A_\ell^{v_0} A_\ell^{v_1} \cdots A_\ell^{v_{k-1}} A_\ell^{v_k}}_{\substack{= A_\ell^{v_0} A_\ell^{v_1} \cdots A_\ell^{v_k} \\ = A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)} \\ \text{(by the definition of } A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)})}} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_{k-1}=u}} A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)}. \qquad (39)
\end{aligned}$$

Now, forget that we fixed u. We thus have proved (39) for each $u \in \widehat{P}$ satisfying $u \gtrdot t$. The definition of $A_\ell^{s \to t}$ yields

$$\begin{aligned}
A_\ell^{s \to t} &= \sum_{\mathbf{p} \text{ is a path from } s \text{ to } t} A_\ell^{\mathbf{p}} \\
&= \underbrace{\sum_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t}}_{\substack{= \sum\limits_{\substack{u \in \widehat{P};\\ u \gtrdot t}} \sum\limits_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_{k-1}=u}} \\ \text{(because any path } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ from } s \text{ to } t \text{ has a well-defined} \\ \text{second-to-last vertex } v_{k-1}, \text{ and this vertex } v_{k-1} \text{ satisfies } v_{k-1} \gtrdot t)}} A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)} \\
&\qquad \text{(here we have renamed the index } \mathbf{p} \text{ as } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)) \\
&= \sum_{\substack{u \in \widehat{P};\\ u \gtrdot t}} \underbrace{\sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k) \text{ is a path from } s \text{ to } t;\\ v_{k-1}=u}} A_\ell^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)}}_{\substack{= \sum\limits_{\mathbf{p} \text{ is a path from } s \text{ to } u} A_\ell^{\mathbf{p}} A_\ell^t \\ \text{(by (39))}}} \\
&= \sum_{\substack{u \in \widehat{P};\\ u \gtrdot t}} \underbrace{\sum_{\mathbf{p} \text{ is a path from } s \text{ to } u} A_\ell^{\mathbf{p}}}_{\substack{= A_\ell^{s \to u} \\ \text{(by the definition of } A_\ell^{s \to u})}} A_\ell^t = \sum_{\substack{u \in \widehat{P};\\ u \gtrdot t}} A_\ell^{s \to u} A_\ell^t.
\end{aligned}$$

[23]because a path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$ from s to t can be reconstructed from its image $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{k-1})$ under this map (since its last vertex v_k is forced to be t)

[24]Indeed, if $\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_\ell)$ is a path from s to u, then $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_\ell \gtrdot t)$ is a path from s to t satisfying $v_\ell = u$ (since $v_\ell = u \gtrdot t$), and it is clear that our map sends the latter path to $\mathbf{p}$.

This establishes (35). The same argument (but with each A symbol replaced by an V symbol) yields (37). Thus, Proposition 7.2 is proven. $\square$

The next proposition uses the products V_ℓ^v and $A_{\ell-1}^v$ to rewrite the equality (29) (which is essentially the definition of birational rowmotion) in a slick way:

Proposition 7.3 (Transition equation in $\boldsymbol{A}$-$\boldsymbol{V}$-form)**.** Let $v \in \widehat{P}$ and $\ell \geqslant 1$ be such that $R^\ell f \neq \bot$. Assume that a is invertible. Then,

$$V_\ell^v = A_{\ell-1}^v.$$

Proof. If v is 0 or 1, then the equality $V_\ell^v = A_{\ell-1}^v$ holds because both of its sides are 1 (by (30)). Thus, we WLOG assume that v is neither 0 nor 1. Hence, $v \in P$. Thus, $P \neq \varnothing$.

Lemma 7.1 **(a)** yields that v_ℓ is well-defined and invertible. Lemma 7.1 **(c)** yields that $A_{\ell-1}^v$ is well-defined. Lemma 7.1 **(d)** yields that V_ℓ^v is well-defined.

We have $\ell - 1 \in \mathbb{N}$ (since $\ell \geqslant 1$) and $R^\ell f \neq \bot$. Hence, (29) (applied to $\ell - 1$ instead of ℓ) yields

$$v_\ell = \left(\sum_{u \lessdot v} u_{\ell-1} \right) \cdot \overline{v_{\ell-1}} \cdot \overline{\sum_{u \gtrdot v} \overline{u_\ell}}. \tag{40}$$

As in the proof of Lemma 7.1 **(c)**, we can see that $\sum\limits_{u \lessdot v} u_{\ell-1}$ is invertible. Taking reciprocals on both sides of (40), we obtain

$$\overline{v_\ell} = \overline{\left(\sum_{u \lessdot v} u_{\ell-1} \right) \cdot \overline{v_{\ell-1}} \cdot \overline{\sum_{u \gtrdot v} \overline{u_\ell}}} = \left(\sum_{u \gtrdot v} \overline{u_\ell} \right) \cdot v_{\ell-1} \cdot \overline{\sum_{u \lessdot v} u_{\ell-1}}$$

(by Proposition 2.3 **(c)**). Multiplying this equality by $\overline{\sum\limits_{u \gtrdot v} \overline{u_\ell}}$ on the left, we obtain

$$\overline{\sum_{u \gtrdot v} \overline{u_\ell}} \cdot \overline{v_\ell} = \underbrace{\overline{\sum_{u \gtrdot v} \overline{u_\ell}} \cdot \left(\sum_{u \gtrdot v} \overline{u_\ell} \right)}_{=1} \cdot v_{\ell-1} \cdot \overline{\sum_{u \lessdot v} u_{\ell-1}} = v_{\ell-1} \cdot \overline{\sum_{u \lessdot v} u_{\ell-1}}.$$

But the left hand side of this equality is V_ℓ^v (by the definition of V_ℓ^v), whereas the right hand side is $A_{\ell-1}^v$. Hence, this equality simplifies to $V_\ell^v = A_{\ell-1}^v$. This proves Proposition 7.3. $\square$

As a consequence of Proposition 7.3, we have:

Corollary 7.4. Let $\mathbf{p}$ be a path. Let $\ell \geqslant 1$ be such that $R^\ell f \neq \bot$. Assume that a is invertible. Then,

$$V_\ell^{\mathbf{p}} = A_{\ell-1}^{\mathbf{p}}.$$

Proof. Write the path $\mathbf{p}$ as $\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)$. The definition of $V_\ell^{\mathbf{p}}$ thus yields

$$V_\ell^{\mathbf{p}} = \underbrace{V_\ell^{v_0}}_{\substack{=A_{\ell-1}^{v_0} \\ \text{(by Proposition 7.3)}}} \underbrace{V_\ell^{v_1}}_{\substack{=A_{\ell-1}^{v_1} \\ \text{(by Proposition 7.3)}}} \cdots \underbrace{V_\ell^{v_k}}_{\substack{=A_{\ell-1}^{v_k} \\ \text{(by Proposition 7.3)}}}$$
$$= A_{\ell-1}^{v_0} A_{\ell-1}^{v_1} \cdots A_{\ell-1}^{v_k}.$$

However, the definition of $A_{\ell-1}^{\mathbf{p}}$ yields

$$A_{\ell-1}^{\mathbf{p}} = A_{\ell-1}^{v_0} A_{\ell-1}^{v_1} \cdots A_{\ell-1}^{v_k} \qquad \left(\text{since } \mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k)\right).$$

Comparing these two equalities, we obtain $V_\ell^{\mathbf{p}} = A_{\ell-1}^{\mathbf{p}}$. This proves Corollary 7.4. □

Corollary 7.5. Let $u, v \in \widehat{P}$. Let $\ell \in \mathbb{N}$ be such that $\ell \geqslant 1$ and $R^\ell f \neq \bot$. Assume that a is invertible. Then,

$$V_\ell^{u \to v} = A_{\ell-1}^{u \to v}. \tag{41}$$

Proof. The definition of $V_\ell^{u \to v}$ yields

$$V_\ell^{u \to v} = \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} \underbrace{V_\ell^{\mathbf{p}}}_{\substack{=A_{\ell-1}^{\mathbf{p}} \\ \text{(by Corollary 7.4)}}} = \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A_{\ell-1}^{\mathbf{p}}.$$

On the other hand, the definition of $A_{\ell-1}^{u \to v}$ yields

$$A_{\ell-1}^{u \to v} = \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A_{\ell-1}^{\mathbf{p}}.$$

Comparing these two equalities, we obtain $V_\ell^{u \to v} = A_{\ell-1}^{u \to v}$. This proves Corollary 7.5. □

The next theorem gives ways to recover the labels $u_\ell = \left(R^\ell f\right)(u)$ from some of the sums defined in (31) and (32).[25]

Theorem 7.6 (path formulas for rectangle)**.** Let $\ell \in \mathbb{N}$. Assume that a is invertible. Then:

(a) If $R^\ell f \neq \bot$ and $\ell \geqslant 1$, then each $u \in P$ satisfies

$$u_\ell = \overline{V_\ell^{1 \to u}} \cdot b$$

(and the inverse $\overline{V_\ell^{1 \to u}}$ is well-defined).

[25] The condition $\ell \geqslant 1$ in Theorem 7.6 **(a)** and **(c)** is meant to ensure that $V_\ell^{1 \to u}$ and $V_\ell^{(p,q) \to u}$ are invertible. It can be replaced by directly requiring the latter.

(b) If $R^{\ell+1} f \neq \bot$, then each $u \in P$ satisfies

$$u_\ell = A_\ell^{u \to 0} \cdot a.$$

(c) If $R^\ell f \neq \bot$ and $\ell \geqslant 1$, then each $u \in P$ satisfies

$$u_\ell = \overline{V_\ell^{(p,q) \to u}} \cdot b$$

(and the inverse $\overline{V_\ell^{(p,q) \to u}}$ is well-defined).

(d) If $R^{\ell+1} f \neq \bot$, then each $u \in P$ satisfies

$$u_\ell = A_\ell^{u \to (1,1)} \cdot a.$$

Proof of Theorem 7.6. **(a)** Assume that $R^\ell f \neq \bot$ and $\ell \geqslant 1$. Then, Lemma 7.1 **(d)** yields that the element V_ℓ^v is well-defined and invertible for each $v \in \widehat{P}$. Hence, the element $V_\ell^{\mathbf{p}}$ is well-defined for each path $\mathbf{p}$. Therefore, the element $V_\ell^{1 \to u}$ is well-defined for each $u \in P$.

Next, we will prove the equality

$$V_\ell^{1 \to u} = b\overline{u_\ell} \qquad \text{for each } u \in P. \tag{42}$$

(The $\overline{u_\ell}$ on the right hand side here is well-defined, since Lemma 7.1 **(a)** (applied to $v = u$) shows that u_ℓ is well-defined and invertible.)

Proof of (42). We utilize downwards induction on u. This is a version of strong induction in which we fix an element $v \in P$ and assume (as the induction hypothesis) that (42) holds for all $u \in P$ satisfying $u > v$. We will then prove that (42) also holds for $u = v$. Since the poset P is finite, this will entail that (42) holds for all $u \in P$.

So let us prove (42) by downwards induction on u:

Let $v \in P$. Assume (as the induction hypothesis) that (42) holds for all $u \in P$ satisfying $u > v$. In other words, we have $V_\ell^{1 \to u} = b\overline{u_\ell}$ for each $u \in P$ satisfying $u > v$. Thus, in particular, we have

$$V_\ell^{1 \to u} = b\overline{u_\ell} \qquad \text{for each } u \in P \text{ satisfying } u \gtrdot v. \tag{43}$$

Note also that the only path from 1 to 1 is the trivial path (1). Hence,

$$V_\ell^{1 \to 1} = V_\ell^{(1)} = V_\ell^1 = 1 = b\overline{1_\ell} \tag{44}$$

(since $1_\ell = b$).

However, $1 \neq v$ (since $1 \notin P$ and $v \in P$). Thus, (37) (applied to $s = 1$ and $t = v$) yields

$$V_\ell^{1\to v} = \underbrace{\sum_{\substack{u \in \widehat{P};\\ u > v}}}_{\substack{=\sum_{u>v} \\ \text{(since our sums} \\ \text{range over } \widehat{P} \text{ by} \\ \text{default)}}} \underbrace{V_\ell^{1\to u}}_{\substack{=b\overline{u_\ell} \\ \text{(indeed, this follows from (43) when } u \in P, \\ \text{and follows from (44) when } u=1; \\ \text{and there are no other possibilities, since } u > v \text{ rules out } u=0)}} V_\ell^v$$

$$= \sum_{u>v} b\overline{u_\ell} V_\ell^v = b \left(\sum_{u>v} \overline{u_\ell} \right) \underbrace{V_\ell^v}_{\substack{=\overline{\sum_{u>v} \overline{u_\ell}} \cdot \overline{v_\ell} \\ \text{(by the definition of } V_\ell^v)}} = b \underbrace{\left(\sum_{u>v} \overline{u_\ell} \right) \overline{\sum_{u>v} \overline{u_\ell}}}_{=1} \cdot \overline{v_\ell} = b\overline{v_\ell}.$$

In other words, (42) holds for $u = v$. This completes the induction step. Thus, we have proved (42) by induction. □

Note that 1_ℓ is invertible (by Lemma 7.1 **(a)**, applied to $v = 1$). In other words, b is invertible (since $1_\ell = b$).

Now, let $u \in P$. Then, b is invertible (as we just saw), and $\overline{u_\ell}$ is invertible (since any inverse is invertible). Thus, $b\overline{u_\ell}$ is invertible (since a product of two invertible elements is invertible). In other words, $V_\ell^{1\to u}$ is invertible (since (42) says that $V_\ell^{1\to u} = b\overline{u_\ell}$). Hence, $\overline{V_\ell^{1\to u}}$ is well-defined. Furthermore, we have $u_\ell = \overline{V_\ell^{1\to u}} \cdot b$, since

$$\overline{V_\ell^{1\to u}} \cdot b = \underbrace{\overline{b\overline{u_\ell}}}_{\substack{=u_\ell \overline{b} \\ \text{(by Proposition 2.3 (b))}}} \cdot b \qquad \text{(by (42))}$$

$$= u_\ell \underbrace{\overline{b} b}_{=1} = u_\ell.$$

This proves Theorem 7.6 **(a)**.

(b) This proof is rather similar to that of part **(a)**, but uses upwards induction instead of downwards induction (and applies (34) instead of (37)).

Here are the details:

Assume that $R^{\ell+1} f \neq \bot$. Then, Lemma 7.1 **(c)** (applied to $\ell + 1$ instead of ℓ) yields that the element A_ℓ^v is well-defined and invertible for each $v \in \widehat{P}$. Hence, the element $A_\ell^{\mathbf{p}}$ is well-defined for each path $\mathbf{p}$. Therefore, the element $A_\ell^{u\to 0}$ is well-defined for each $u \in P$.

Next, we will prove the equality

$$A_\ell^{u\to 0} = u_\ell \overline{a} \qquad \text{for each } u \in P. \tag{45}$$

(The $\overline{a}$ on the right hand side here is well-defined, since we assumed that a is invertible.)

Proof of (45). We prove the equality (45) by upwards induction on u. This is a version of strong induction in which we fix an element $v \in P$ and assume (as the induction hypothesis) that (45) holds for all $u \in P$ satisfying $u < v$. We will then prove that (45) also holds for $u = v$. Since the poset P is finite, this will entail that (45) holds for all $u \in P$.

So let us prove (45) by upwards induction on u:

Let $v \in P$. Assume (as the induction hypothesis) that (45) holds for all $u \in P$ satisfying $u < v$. In other words, we have $A_\ell^{u\to 0} = u_\ell \overline{a}$ for each $u \in P$ satisfying $u < v$. Thus, in particular, we have

$$A_\ell^{u\to 0} = u_\ell \overline{a} \qquad \text{for each } u \in P \text{ satisfying } u \lessdot v. \tag{46}$$

Note also that the only path from 0 to 0 is the trivial path (0). Hence,

$$A_\ell^{0\to 0} = A_\ell^{(0)} = A_\ell^{0} = 1 = 0_\ell \overline{a} \tag{47}$$

(since $0_\ell = a$).

However, $v \neq 0$ (since $v \in P$ and $0 \notin P$). Thus, (34) (applied to $s = v$ and $t = 0$) yields

$$\begin{aligned} A_\ell^{v\to 0} &= A_\ell^{v} \underbrace{\sum_{\substack{u \in \widehat{P};\\ v \gtrdot u}}}_{\substack{= \sum_{u \in \widehat{P};\ u \lessdot v} = \sum_{u \lessdot v} \\ \text{(since our sums} \\ \text{range over } \widehat{P} \text{ by} \\ \text{default)}}} \underbrace{A_\ell^{u\to 0}}_{\substack{= u_\ell \overline{a} \\ \text{(indeed, this follows from (46) when } u \in P, \\ \text{and follows from (47) when } u=0; \\ \text{and there are no other possibilities, since } v \gtrdot u \text{ rules out } u=1)}} \\ &= \underbrace{A_\ell^{v}}_{\substack{= v_\ell \cdot \overline{\sum_{u \lessdot v} u_\ell} \\ \text{(by the definition of } A_\ell^{v})}} \sum_{u \lessdot v} u_\ell \overline{a} = v_\ell \cdot \underbrace{\overline{\sum_{u \lessdot v} u_\ell} \cdot \sum_{u \lessdot v} u_\ell}_{=1} \overline{a} = v_\ell \overline{a}. \end{aligned}$$

In other words, (45) holds for $u = v$. This completes the induction step, and (45) is proven. ☐

Now, for each $u \in P$, we have $u_\ell = A_\ell^{u\to 0} \cdot a$, since

$$\begin{aligned} A_\ell^{u\to 0} \cdot a &= u_\ell \underbrace{\overline{a} \cdot a}_{=1} \qquad \text{(by (45))} \\ &= u_\ell. \end{aligned}$$

This proves Theorem 7.6 **(b)**.

(c) Assume that $R^\ell f \neq \bot$ and $\ell \geqslant 1$. Let $t \in P$. Every element of $\widehat{P}$ distinct from 1 is $\leqslant (p, q)$. Therefore, the only element $u \in \widehat{P}$ satisfying $1 \gtrdot u$ is the maximal element (p, q)

of P. Hence, $\sum\limits_{\substack{u\in\widehat{P};\\ 1\gtrdot u}} V_\ell^{u\to t} = V_\ell^{(p,q)\to t}$. Now, (36) (applied to $s=1$) yields

$$V_\ell^{1\to t} = \underbrace{V_\ell^1}_{\substack{=1\\ (\text{by (30)})}} \underbrace{\sum_{\substack{u\in\widehat{P};\\ 1\gtrdot u}} V_\ell^{u\to t}}_{=V_\ell^{(p,q)\to t}} = V_\ell^{(p,q)\to t}.$$

Forget that we fixed t. We thus have proved that $V_\ell^{1\to t} = V_\ell^{(p,q)\to t}$ for each $t\in P$. Renaming the index t as u in this statement, we obtain the following:

$$V_\ell^{1\to u} = V_\ell^{(p,q)\to u} \qquad \text{for each } u\in P. \tag{48}$$

Now, let $u\in P$. Then, Theorem 7.6 **(a)** yields

$$u_\ell = \overline{V_\ell^{1\to u}}\cdot b = \overline{V_\ell^{(p,q)\to u}}\cdot b \qquad (\text{by (48)}).$$

This proves Theorem 7.6 **(c)**.

(d) Assume that $R^{\ell+1}f\neq\bot$. Let $s\in P$. Every element of $\widehat{P}$ distinct from 0 is $\geqslant (1,1)$. Thus, the only element $u\in\widehat{P}$ satisfying $u\gtrdot 0$ is the minimal element $(1,1)$ of P. Hence,

$$\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} A_\ell^{s\to u}A_\ell^0 = A_\ell^{s\to(1,1)} \underbrace{A_\ell^0}_{\substack{=1\\ (\text{by (30)})}} = A_\ell^{s\to(1,1)}.$$

Now, (35) (applied to $t=0$) yields

$$A_\ell^{s\to 0} = \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} A_\ell^{s\to u}A_\ell^0 = A_\ell^{s\to(1,1)}.$$

Forget that we fixed s. We thus have proved that $A_\ell^{s\to 0} = A_\ell^{s\to(1,1)}$ for each $s\in P$. Renaming the index s as u in this statement, we obtain the following:

$$A_\ell^{u\to 0} = A_\ell^{u\to(1,1)} \qquad \text{for each } u\in P. \tag{49}$$

Now, let $u\in P$. Then, Theorem 7.6 **(b)** yields

$$u_\ell = A_\ell^{u\to 0}\cdot a = A_\ell^{u\to(1,1)}\cdot a \qquad (\text{by (49)}).$$

This proves Theorem 7.6 **(d)**. □

Remark 7.7. Corollary 7.5, Proposition 7.2 and parts **(a)** and **(b)** of Theorem 7.6 hold more generally if P is replaced by any finite poset (not necessarily a rectangle). The proofs we gave above work in that generality. Parts **(c)** and **(d)** of Theorem 7.6 can be similarly generalized as long as the poset P has a global maximum (for part **(c)**) and a global minimum (for part **(d)**); all we need to do is to replace (p,q) by the global maximum and $(1,1)$ by the global minimum. We will have no need for this generality, though.

8. Proof of reciprocity: the case $(i, j) = (1, 1)$

Now, we are mostly ready to prove that Theorem 4.8 holds in the case when $(i, j) = (1, 1)$. For reasons both technical and pedagogical, it is useful for us to dispose of this case now in order to have less work to do later. First, we prove Theorem 4.8 for $(i, j) = (1, 1)$ under the extra assumption that a is invertible:

Lemma 8.1. Assume that P is the $p \times q$-rectangle $[p] \times [q]$. Let $\ell \in \mathbb{N}$ be such that $\ell \geqslant 1$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^\ell f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Assume that a is invertible. Then,

$$\left(R^\ell f\right)(1,1) = a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \cdot b.$$

Proof. We use the notations from Section 6. Thus, $\left(R^\ell f\right)(1,1) = (1,1)_\ell$ and

$$\left(R^{\ell-1} f\right)(p,q) = (p,q)_{\ell-1} = A_{\ell-1}^{(p,q)\to(1,1)} \cdot a$$

(by Theorem 7.6 **(d)**, applied to $\ell - 1$ and (p, q) instead of ℓ and u). Solving this equation for $A_{\ell-1}^{(p,q)\to(1,1)}$, we obtain

$$A_{\ell-1}^{(p,q)\to(1,1)} = \left(R^{\ell-1} f\right)(p,q) \cdot \overline{a} \tag{50}$$

(since a is invertible). Note also that $R\left(R^{\ell-1} f\right) = R^\ell f \neq \bot$, and thus $\left(R^{\ell-1} f\right)(p,q)$ is invertible (by Lemma 3.24, applied to $R^{\ell-1} f$ and (p, q) instead of f and v).

Now,

$$\begin{aligned}
\left(R^\ell f\right)(1,1) &= (1,1)_\ell = \overline{V_\ell^{(p,q)\to(1,1)}} \cdot b && \text{(by Theorem 7.6 } \textbf{(c)}\text{, applied to } u = (1,1)) \\
&= \overline{A_{\ell-1}^{(p,q)\to(1,1)}} \cdot b && \left(\text{since (41) yields } V_\ell^{(p,q)\to(1,1)} = A_{\ell-1}^{(p,q)\to(1,1)}\right) \\
&= \underbrace{\overline{\left(R^{\ell-1} f\right)(p,q) \cdot \overline{a}}}_{\substack{= a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \\ \text{(since } \left(R^{\ell-1} f\right)(p,q) \text{ and } \overline{a} \\ \text{are invertible)}}} \cdot b && \text{(by (50))} \\
&= a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \cdot b.
\end{aligned}$$

This proves Lemma 8.1. □

Unfortunately, our proof of Lemma 8.1 made use of the requirement that a be invertible, since $V_\ell^{(p,q)\to(1,1)}$ and $A_{\ell-1}^{(p,q)\to(1,1)}$ would not be well-defined otherwise. In order to remove this requirement, we make use of a trick, in which we "temporarily" set the label $f(0)$ to 1 and then argue that this has a predictable effect on $(Rf)(1,1)$. This trick relies on the following:

Lemma 8.2. Let P be an arbitrary finite poset (not necessarily $[p] \times [q]$). Let $f, g \in \mathbb{K}^{\widehat{P}}$ be two $\mathbb{K}$-labelings such that $Rf \neq \bot$. Assume that

$$g(x) = f(x) \qquad \text{for each } x \in \widehat{P} \setminus \{0\}. \tag{51}$$

Assume furthermore that $g(0) = 1$. Set $a = f(0)$. Then:

(a) We have $Rg \neq \bot$.

(b) If $v \in P$ is not a minimal element of P, then $(Rf)(v) = (Rg)(v)$.

(c) If $v \in P$ is a minimal element of P, then $(Rf)(v) = a \cdot (Rg)(v)$.

Proof of Lemma 8.2. Pick a linear extension $(v_1, v_2, \ldots, v_m)$ of P. (We know from Theorem 1.5 that such a linear extension exists.)

For each $i \in \{0, 1, \ldots, m\}$, define a partial map

$$R_i := T_{v_{i+1}} \circ T_{v_{i+2}} \circ \cdots \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}.$$

Thus, in particular,

$$R_m = T_{v_{m+1}} \circ T_{v_{m+2}} \circ \cdots \circ T_{v_m} = (\text{empty composition}) = \text{id}$$

and

$$R_0 = T_{v_{0+1}} \circ T_{v_{0+2}} \circ \cdots \circ T_{v_m} = T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} = R$$

(by the definition of R).

For each $i \in \{0, 1, \ldots, m\}$, we set

$$f^{(i)} := R_{m-i} f \qquad \text{and} \qquad g^{(i)} := R_{m-i} g.$$

Each of $f^{(i)}$ and $g^{(i)}$ is either a $\mathbb{K}$-labeling in $\mathbb{K}^{\widehat{P}}$ or $\bot$; we will soon see that it is a $\mathbb{K}$-labeling.

The tuple $(v_1, v_2, \ldots, v_m)$ is a linear extension of P. Thus, for each $x \in P$, there exists a unique $i \in \{1, 2, \ldots, m\}$ that satisfies $x = v_i$. Let us denote this i by $\rho(x)$. Thus, the map

$$\rho : P \to \{1, 2, \ldots, m\}, \\ x \mapsto \rho(x)$$

is a bijection.

Let M denote the set of all minimal elements of P.

We now shall prove the following:

Claim 1: Let $i \in \{0, 1, \ldots, m\}$. Then, $f^{(i)} \neq \bot$ and $g^{(i)} \neq \bot$. Moreover, $g^{(i)}(1) = f^{(i)}(1)$ and $f^{(i)}(0) = a \cdot g^{(i)}(0)$. Furthermore, each $v \in P$ satisfies

$$f^{(i)}(v) = \begin{cases} g^{(i)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - i; \\ a \cdot g^{(i)}(v), & \text{otherwise.} \end{cases}$$

Proof of Claim 1. We proceed by induction on i:

Base case: The definition of $f^{(0)}$ yields $f^{(0)} = \underbrace{R_{m-0}}_{=R_m = \text{id}} f = f \neq \bot$. The definition of $g^{(0)}$ yields $g^{(0)} = \underbrace{R_{m-0}}_{=R_m = \text{id}} g = g \neq \bot$.

From $g^{(0)} = g$, we obtain $g^{(0)}(1) = g(1) = f(1)$ (by (51), applied to $x = 1$). In other words, $g^{(0)}(1) = f^{(0)}(1)$ (since $f^{(0)} = f$).

From $g^{(0)} = g$, we obtain $g^{(0)}(0) = g(0) = 1$, so that $a \cdot g^{(0)}(0) = a \cdot 1 = a = f(0) = f^{(0)}(0)$ (since $f = f^{(0)}$). In other words, $f^{(0)}(0) = a \cdot g^{(0)}(0)$.

Now, let $v \in P$. Then, (51) (applied to $x = v$) yields $g(v) = f(v)$ (since $v \in P \subseteq \widehat{P} \setminus \{0\}$). In view of $g^{(0)} = g$ and $f^{(0)} = f$, we can rewrite this as $g^{(0)}(v) = f^{(0)}(v)$. However, we have $\rho(v) \in \{1, 2, \ldots, m\}$ (by the definition of $\rho(v)$) and therefore $\rho(v) \leqslant m = m - 0$. Thus,

$$\begin{aligned} &\begin{cases} g^{(0)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - 0; \\ a \cdot g^{(0)}(v), & \text{otherwise} \end{cases} \\ &= g^{(0)}(v) \qquad \left(\text{since } v \notin M \text{ or } \rho(v) \leqslant m - 0 \text{ (because } \rho(v) \leqslant m - 0)\right) \\ &= f^{(0)}(v). \end{aligned}$$

Hence,

$$f^{(0)}(v) = \begin{cases} g^{(0)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - 0; \\ a \cdot g^{(0)}(v), & \text{otherwise.} \end{cases}$$

Forget that we fixed v. We thus have shown that each $v \in P$ satisfies

$$f^{(0)}(v) = \begin{cases} g^{(0)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - 0; \\ a \cdot g^{(0)}(v), & \text{otherwise.} \end{cases}$$

Since we also know that $f^{(0)} \neq \bot$ and $g^{(0)} \neq \bot$ and $g^{(0)}(1) = f^{(0)}(1)$ and $f^{(0)}(0) = a \cdot g^{(0)}(0)$, we have thus finished proving that Claim 1 holds for $i = 0$.

Induction step: Let $j \in \{0, 1, \ldots, m-1\}$. Assume (as the induction hypothesis) that Claim 1 holds for $i = j$. We must prove that Claim 1 holds for $i = j+1$. In other words, we must prove that $f^{(j+1)} \neq \bot$ and $g^{(j+1)} \neq \bot$ and $g^{(j+1)}(1) = f^{(j+1)}(1)$ and $f^{(j+1)}(0) = a \cdot g^{(j+1)}(0)$ and that each $v \in P$ satisfies

$$f^{(j+1)}(v) = \begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - (j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise.} \end{cases} \tag{52}$$

Our induction hypothesis tells us that Claim 1 holds for $i = j$. In other words, we have $f^{(j)} \neq \bot$ and $g^{(j)} \neq \bot$ and $g^{(j)}(1) = f^{(j)}(1)$ and $f^{(j)}(0) = a \cdot g^{(j)}(0)$, and each $v \in P$ satisfies

$$f^{(j)}(v) = \begin{cases} g^{(j)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-j; \\ a \cdot g^{(j)}(v), & \text{otherwise.} \end{cases} \tag{53}$$

Let $y := v_{m-j}$. Thus, y is an element of P. The definition of $\rho(y)$ yields that $\rho(y)$ is the unique $i \in \{1, 2, \ldots, m\}$ that satisfies $y = v_i$. Thus, $\rho(y) = m - j$ (since $y = v_{m-j}$).

It is easy to see that

$$R_{m-(j+1)} = T_y \circ R_{m-j} \tag{54}$$

[26].

It is easy to see that $f^{(j+1)} \neq \bot$ [27]. Furthermore, we have

$$f^{(j+1)} = T_y f^{(j)}$$

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[26] *Proof.* The definition of R_{m-j} yields $R_{m-j} = T_{v_{m-j+1}} \circ T_{v_{m-j+2}} \circ \cdots \circ T_{v_m}$. The definition of $R_{m-(j+1)}$ yields

$$\begin{aligned} R_{m-(j+1)} &= \underbrace{T_{v_{m-(j+1)+1}}}_{=T_{v_{m-j}}} \circ \underbrace{T_{v_{m-(j+1)+2}}}_{=T_{v_{m-j+1}}} \circ \cdots \circ T_{v_m} = T_{v_{m-j}} \circ T_{v_{m-j+1}} \circ \cdots \circ T_{v_m} \\ &= \underbrace{T_{v_{m-j}}}_{\substack{=T_y \\ (\text{since } v_{m-j}=y)}} \circ \underbrace{T_{v_{m-j+1}} \circ T_{v_{m-j+2}} \circ \cdots \circ T_{v_m}}_{=R_{m-j}} = T_y \circ R_{m-j}. \end{aligned}$$

This proves (54).

[27] *Proof.* The definition of $R_{m-(j+1)}$ yields

$$R_{m-(j+1)} = \underbrace{T_{v_{m-(j+1)+1}}}_{=T_{v_{m-j}}} \circ \underbrace{T_{v_{m-(j+1)+2}}}_{=T_{v_{m-j+1}}} \circ \cdots \circ T_{v_m} = T_{v_{m-j}} \circ T_{v_{m-j+1}} \circ \cdots \circ T_{v_m}.$$

However, the definition of R yields

$$\begin{aligned} R &= T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_m} = \left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right) \circ \underbrace{\left(T_{v_{m-j}} \circ T_{v_{m-j+1}} \circ \cdots \circ T_{v_m}\right)}_{=R_{m-(j+1)}} \\ &= \left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right) \circ R_{m-(j+1)}. \end{aligned}$$

Thus, if we had $R_{m-(j+1)} f = \bot$, then we would have

$$\begin{aligned} \underbrace{R}_{=\left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right) \circ R_{m-(j+1)}} f &= \left(\left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right) \circ R_{m-(j+1)}\right) f \\ &= \left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right) \left(\underbrace{R_{m-(j+1)} f}_{=\bot}\right) \\ &= \left(T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_{m-j-1}}\right)(\bot) = \bot, \end{aligned}$$

which would contradict $Rf \neq \bot$. Hence, we cannot have $R_{m-(j+1)} f = \bot$. Thus, $R_{m-(j+1)} f \neq \bot$. However, the definition of $f^{(j+1)}$ yields $f^{(j+1)} = R_{m-(j+1)} f \neq \bot$.

[28]. Similarly,

$$g^{(j+1)} = T_y g^{(j)}.$$

Next, we observe that for each $u \in \widehat{P}$ satisfying $u \gtrdot y$, we have

$$g^{(j)}(u) = f^{(j)}(u) \tag{55}$$

[29]. Furthermore, we have

$$g^{(j)}(y) = f^{(j)}(y) \tag{56}$$

[30].

However, recall that $f^{(j+1)} = T_y f^{(j)}$, so that $T_y f^{(j)} = f^{(j+1)} \neq \bot$. Thus, the expression

$$\left(\sum_{\substack{u\in\widehat{P};\\ u\lessdot y}} f^{(j)}(u)\right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot y}} \overline{f^{(j)}(u)}} \tag{57}$$

is well-defined (because if this expression was not well-defined, then the definition of the y-toggle T_y (Definition 3.12) would dictate that $T_y f^{(j)} = \bot$; but this would contradict

[28]*Proof.* The definition of $f^{(j)}$ yields $f^{(j)} = R_{m-j} f$. Hence, $R_{m-j} f = f^{(j)}$. The definition of $f^{(j+1)}$ yields

$$f^{(j+1)} = \underbrace{R_{m-(j+1)}}_{\substack{=T_y\circ R_{m-j}\\ \text{(by (54))}}} f = (T_y \circ R_{m-j}) f = T_y \underbrace{(R_{m-j} f)}_{=f^{(j)}} = T_y f^{(j)}.$$

[29]*Proof of (55):* Let $u \in \widehat{P}$ be such that $u \gtrdot y$. Then, $u \gtrdot y$, so that $u > y$. In other words, $y < u$. Thus, there exists an element of P that is smaller than u (namely, y). Hence, u cannot be a minimal element of P. In other words, $u \notin M$ (since M is the set of all minimal elements of P).

We must prove that $g^{(j)}(u) = f^{(j)}(u)$. If $u = 1$, then this follows directly from $g^{(j)}(1) = f^{(j)}(1)$. Thus, we WLOG assume that $u \neq 1$. Moreover, $u \neq 0$ (because if we had $u = 0$, then we would have $0 = u > y$, which would contradict the fact that 0 is not larger than any element of $\widehat{P}$). Combining $u \in \widehat{P}$ with $u \neq 0$ and $u \neq 1$, we obtain $u \in \widehat{P} \setminus \{0, 1\} = P$. Hence, $\rho(u)$ is well-defined.

We have $u \notin M$ or $\rho(u) \leqslant m - j$ (since $u \notin M$). Now, (53) (applied to $v = u$) yields

$$f^{(j)}(u) = \begin{cases} g^{(j)}(u), & \text{if } u \notin M \text{ or } \rho(u) \leqslant m-j; \\ a \cdot g^{(j)}(u), & \text{otherwise} \end{cases}$$
$$= g^{(j)}(u) \qquad (\text{since } u \notin M \text{ or } \rho(u) \leqslant m-j).$$

In other words, $g^{(j)}(u) = f^{(j)}(u)$. This proves (55).

[30]*Proof of (56):* We have $\rho(y) = m - j \leqslant m - j$. Therefore, $y \notin M$ or $\rho(y) \leqslant m - j$. Now, (53) (applied to $v = y$) yields

$$f^{(j)}(y) = \begin{cases} g^{(j)}(y), & \text{if } y \notin M \text{ or } \rho(y) \leqslant m-j; \\ a \cdot g^{(j)}(y), & \text{otherwise} \end{cases}$$
$$= g^{(j)}(y) \qquad (\text{since } y \notin M \text{ or } \rho(y) \leqslant m-j).$$

In other words, $g^{(j)}(y) = f^{(j)}(y)$. This proves (56).

$T_y f^{(j)} \neq \bot$). As a consequence, the expressions $\overline{f^{(j)}(y)}$ and $\overline{\sum\limits_{\substack{u \in \widehat{P};\\ u > y}} \overline{f^{(j)}(u)}}$ are also well-defined (since they are parts of the well-defined expression (57)). In view of (55) and (56), we can rewrite this as follows: The expressions $\overline{g^{(j)}(y)}$ and $\overline{\sum\limits_{\substack{u \in \widehat{P};\\ u > y}} \overline{g^{(j)}(u)}}$ are well-defined. Thus, the expression

$$\left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u)\right) \cdot \overline{g^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u > y}} \overline{g^{(j)}(u)}} \qquad (58)$$

is well-defined as well (since the expression $\sum\limits_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u)$ is clearly well-defined[31]). Consequently, the definition of the y-toggle T_y (Definition 3.12) yields $T_y g^{(j)} \neq \bot$. In other words, $g^{(j+1)} \neq \bot$ (since $g^{(j+1)} = T_y g^{(j)}$).

Next, it is easy to see that $g^{(j+1)}(1) = f^{(j+1)}(1)$ [32] and $f^{(j+1)}(0) = a \cdot g^{(j+1)}(0)$ [33].

We now prove that each $v \in P$ satisfies (52).

Proof of (52). Let $v \in P$. We must prove (52). We are in one of the following two cases:

Case 1: We have $v \neq y$.

Case 2: We have $v = y$.

Let us first consider Case 1. In this case, we have $v \neq y$. Hence, the statement "$\rho(v) \leqslant m - j$" is equivalent to "$\rho(v) \leqslant m - (j+1)$" [34].

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[31] because $g^{(j)} \neq \bot$

[32] *Proof.* Recall that $g^{(j)}(1) = f^{(j)}(1)$. However, $1 \neq y$ (since $1 \notin P$ but $y \in P$). Thus, Proposition 3.14 **(a)** (applied to y, $g^{(j)}$ and 1 instead of v, f and w) yields $\left(T_y g^{(j)}\right)(1) = g^{(j)}(1)$ (since $T_y g^{(j)} \neq \bot$). In view of $g^{(j+1)} = T_y g^{(j)}$, we can rewrite this as $g^{(j+1)}(1) = g^{(j)}(1)$. Also, Proposition 3.14 **(a)** (applied to y, $f^{(j)}$ and 1 instead of v, f and w) yields $\left(T_y f^{(j)}\right)(1) = f^{(j)}(1)$ (since $T_y f^{(j)} = f^{(j+1)} \neq \bot$). In view of $f^{(j+1)} = T_y f^{(j)}$, we can rewrite this as $f^{(j+1)}(1) = f^{(j)}(1)$. Hence, $f^{(j)}(1) = f^{(j+1)}(1)$. Combining what we have shown so far, we obtain

$$g^{(j+1)}(1) = g^{(j)}(1) = f^{(j)}(1) = f^{(j+1)}(1).$$

[33] *Proof.* Recall that $f^{(j)}(0) = a \cdot g^{(j)}(0)$. However, $0 \neq y$ (since $0 \notin P$ but $y \in P$). Thus, Proposition 3.14 **(a)** (applied to y, $g^{(j)}$ and 0 instead of v, f and w) yields $\left(T_y g^{(j)}\right)(0) = g^{(j)}(0)$ (since $T_y g^{(j)} \neq \bot$). In view of $g^{(j+1)} = T_y g^{(j)}$, we can rewrite this as $g^{(j+1)}(0) = g^{(j)}(0)$. Also, Proposition 3.14 **(a)** (applied to y, $f^{(j)}$ and 0 instead of v, f and w) yields $\left(T_y f^{(j)}\right)(0) = f^{(j)}(0)$ (since $T_y f^{(j)} = f^{(j+1)} \neq \bot$). In view of $f^{(j+1)} = T_y f^{(j)}$, we can rewrite this as $f^{(j+1)}(0) = f^{(j)}(0)$. Hence,

$$f^{(j+1)}(0) = f^{(j)}(0) = a \cdot \underbrace{g^{(j)}(0)}_{\substack{= g^{(j+1)}(0) \\ (\text{since } g^{(j+1)}(0) = g^{(j)}(0))}} = a \cdot g^{(j+1)}(0).$$

[34] *Proof.* Recall that $\rho(v)$ is the unique $i \in \{1, 2, \ldots, m\}$ that satisfies $v = v_i$ (by the definition of $\rho(v)$).

However, Proposition 3.14 **(a)** (applied to y, $g^{(j)}$ and v instead of v, f and w) yields $\left(T_y g^{(j)}\right)(v) = g^{(j)}(v)$ (since $v \neq y$ and $T_y g^{(j)} \neq \bot$). In view of $g^{(j+1)} = T_y g^{(j)}$, we can rewrite this as $g^{(j+1)}(v) = g^{(j)}(v)$. In other words, $g^{(j)}(v) = g^{(j+1)}(v)$. Also, Proposition 3.14 **(a)** (applied to y, $f^{(j)}$ and v instead of v, f and w) yields $\left(T_y f^{(j)}\right)(v) = f^{(j)}(v)$ (since $v \neq y$ and $T_y f^{(j)} \neq \bot$). In view of $f^{(j+1)} = T_y f^{(j)}$, we can rewrite this as $f^{(j+1)}(v) = f^{(j)}(v)$. Hence,

$$f^{(j+1)}(v) = f^{(j)}(v) = \begin{cases} g^{(j)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - j; \\ a \cdot g^{(j)}(v), & \text{otherwise} \end{cases} \qquad \text{(by (53))}$$
$$= \begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - j; \\ a \cdot g^{(j+1)}(v), & \text{otherwise} \end{cases} \qquad \left(\text{since } g^{(j)}(v) = g^{(j+1)}(v)\right)$$
$$= \begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m - (j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise} \end{cases}$$

(since the statement "$\rho(v) \leqslant m - j$" is equivalent to "$\rho(v) \leqslant m - (j+1)$"). In other words, (52) holds. Thus, (52) is proved in Case 1.

Let us now consider Case 2. In this case, we have $v = y$. Hence, $\rho(v) = \rho(y) = m - j > m - j - 1 = m - (j+1)$. Thus, we do not have $\rho(v) \leqslant m - (j+1)$. Hence, the statement "$v \notin M$ or $\rho(v) \leqslant m - (j+1)$" is equivalent to "$v \notin M$".

Recall that $T_y f^{(j)} \neq \bot$ and $T_y g^{(j)} \neq \bot$. Thus, Proposition 3.14 **(b)** (applied to $f^{(j)}$ and y instead of f and v) yields

$$\left(T_y f^{(j)}\right)(y) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} f^{(j)}(u)\right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}}.$$

In view of $f^{(j+1)} = T_y f^{(j)}$, we can rewrite this as

$$f^{(j+1)}(y) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} f^{(j)}(u)\right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}}. \qquad (59)$$

Hence, $v = v_{\rho(v)}$. Therefore, $v_{\rho(v)} = v \neq y = v_{m-j}$, so that $\rho(v) \neq m - j$ (because if we had $\rho(v) = m - j$, then we would have $v_{\rho(v)} = v_{m-j}$, which would contradict $v_{\rho(v)} \neq v_{m-j}$). In other words, we don't have $\rho(v) = m - j$.

Now, we have the following chain of equivalences:

$$\begin{aligned} (\rho(v) \leqslant m - j) &\iff (\rho(v) < m - j \text{ or } \rho(v) = m - j) \\ &\iff (\rho(v) < m - j) \qquad (\text{since we don't have } \rho(v) = m - j) \\ &\iff (\rho(v) \leqslant (m - j) - 1) \qquad (\text{since } \rho(v) \text{ and } m - j \text{ are integers}) \\ &\iff (\rho(v) \leqslant m - (j+1)) \qquad (\text{since } (m - j) - 1 = m - (j+1)). \end{aligned}$$

In other words, the statement "$\rho(v) \leqslant m - j$" is equivalent to "$\rho(v) \leqslant m - (j+1)$".

Also, recall that $T_y g^{(j)} \neq \bot$. Hence, Proposition 3.14 **(b)** (applied to $g^{(j)}$ and y instead of f and v) yields

$$\begin{aligned}\left(T_y g^{(j)}\right)(y) &= \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u)\right) \cdot \overline{\underbrace{g^{(j)}(y)}_{\substack{=\overline{f^{(j)}(y)}\\ \text{(by (56))}}}} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{\underbrace{g^{(j)}(u)}_{\substack{=\overline{f^{(j)}(u)}\\ \text{(by (55))}}}}} \\ &= \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u)\right) \cdot \overline{\overline{f^{(j)}(y)}} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{\overline{f^{(j)}(u)}}}.\end{aligned}$$

In view of $g^{(j+1)} = T_y g^{(j)}$, we can rewrite this as

$$g^{(j+1)}(y) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u)\right) \cdot \overline{\overline{f^{(j)}(y)}} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{\overline{f^{(j)}(u)}}}. \tag{60}$$

Now, we are in one of the following two subcases:

Subcase 2.1: We have $v \in M$.

Subcase 2.2: We have $v \notin M$.

Let us first consider Subcase 2.1. In this subcase, we have $v \in M$. In other words, $y \in M$ (since $v = y$). In other words, y is a minimal element of P (since M is the set of all minimal elements of P). Hence, the only $u \in \widehat{P}$ that satisfies $u \lessdot y$ is the element 0 of $\widehat{P}$. Thus,

$$\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} g^{(j)}(u) = g^{(j)}(0) \qquad \text{and} \qquad \sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} f^{(j)}(u) = f^{(j)}(0).$$

Now, from $v = y$, we obtain

$$\begin{aligned}f^{(j+1)}(v) &= f^{(j+1)}(y) \\ &= \underbrace{\left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot y}} f^{(j)}(u)\right)}_{\substack{=f^{(j)}(0)\\ =a \cdot g^{(j)}(0)}} \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{f^{(j)}(u)}} \qquad \text{(by (59))} \\ &= a \cdot g^{(j)}(0) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot y}} \overline{f^{(j)}(u)}}. \end{aligned} \tag{61}$$

On the other hand, we don't have $v \notin M$ (since $v \in M$). Now, recall that the statement "$v \notin M$ or $\rho(v) \leqslant m-(j+1)$" is equivalent to "$v \notin M$". Hence, we don't have "$v \notin M$ or $\rho(v) \leqslant m-(j+1)$" (since we don't have $v \notin M$). Therefore,

$$\begin{aligned}
&\begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-(j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise} \end{cases} \\
&= a \cdot g^{(j+1)}(v) = a \cdot g^{(j+1)}(y) \qquad\qquad (\text{since } v = y) \\
&= a \cdot \underbrace{\left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} g^{(j)}(u) \right)}_{= g^{(j)}(0)} \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}} \qquad\qquad (\text{by (60)}) \\
&= a \cdot g^{(j)}(0) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}}.
\end{aligned}$$

Comparing this with (61), we obtain

$$f^{(j+1)}(v) = \begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-(j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise.} \end{cases}$$

In other words, (52) holds. Thus, we have proved (52) in Subcase 2.1.

Now, let us consider Subcase 2.2. In this subcase, we have $v \notin M$. In other words, v is not a minimal element of P (since M is the set of all minimal elements of P). Thus, we don't have $0 \lessdot v$ in $\widehat{P}$. In other words, we don't have $0 \lessdot y$ in $\widehat{P}$ (since $v = y$).

For each $u \in \widehat{P}$ satisfying $u \lessdot y$, we have

$$g^{(j)}(u) = f^{(j)}(u) \tag{62}$$

[35]. Now, from $v = y$, we obtain

$$\begin{aligned}
f^{(j+1)}(v) &= f^{(j+1)}(y) \\
&= \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} f^{(j)}(u) \right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}} \qquad\qquad (\text{by (59)}).
\end{aligned}$$

[35] *Proof of (62):* Let $u \in \widehat{P}$ be such that $u \lessdot y$. Then, $u \lessdot y$, so that $u < y$. Thus, $u \neq 1$ (because if we had $u = 1$, then we would have $1 = u < y$, which would contradict the fact that 1 is not smaller than any element of $\widehat{P}$). Moreover, $u \neq 0$ (because if we had $u = 0$, then we would have $0 = u \lessdot y$, which would contradict the fact that we don't have $0 \lessdot y$ in $\widehat{P}$). Combining $u \in \widehat{P}$ with $u \neq 0$ and $u \neq 1$, we obtain $u \in \widehat{P} \setminus \{0,1\} = P$. Hence, $\rho(u)$ is well-defined.

The definition of $\rho(u)$ shows that $\rho(u)$ is the unique $i \in \{1,2,\ldots,m\}$ that satisfies $u = v_i$. Hence, $u = v_{\rho(u)}$.

Recall that $(v_1, v_2, \ldots, v_m)$ is a linear extension of P. Thus, any $k \in \{1,2,\ldots,m\}$ and $\ell \in \{1,2,\ldots,m\}$ satisfying $v_k < v_\ell$ must satisfy $k < \ell$ (by the definition of a linear extension). We can apply this to $k = \rho(u)$ and $\ell = m-j$ (since $v_{\rho(u)} = u < y = v_{m-j}$), and thus obtain $\rho(u) < m-j$.

Comparing this with

$$\begin{aligned}
&\begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-(j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise} \end{cases} \\
&= g^{(j+1)}(v) \qquad (\text{since } v \notin M \text{ or } \rho(v) \leqslant m-(j+1) \text{ (because } v \notin M)) \\
&= g^{(j+1)}(y) \qquad (\text{since } v = y) \\
&= \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} \underbrace{g^{(j)}(u)}_{\substack{= f^{(j)}(u) \\ (\text{by (62)})}} \right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}} \qquad (\text{by (60)}) \\
&= \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot y}} f^{(j)}(u) \right) \cdot \overline{f^{(j)}(y)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot y}} \overline{f^{(j)}(u)}},
\end{aligned}$$

we obtain

$$f^{(j+1)}(v) = \begin{cases} g^{(j+1)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-(j+1); \\ a \cdot g^{(j+1)}(v), & \text{otherwise.} \end{cases}$$

Thus, we have proved (52) in Subcase 2.2.

We have now proved (52) in both Subcases 2.1 and 2.2. Since these two Subcases cover all of Case 2, we thus have proved (52) in Case 2.

We have now proved (52) in both Cases 1 and 2. Therefore, (52) always holds. This completes the proof of (52). ☐

Altogether, we have now proved that $f^{(j+1)} \neq \bot$ and $g^{(j+1)} \neq \bot$ and $g^{(j+1)}(1) = f^{(j+1)}(1)$ and $f^{(j+1)}(0) = a \cdot g^{(j+1)}(0)$ and that each $v \in P$ satisfies (52). In other words, Claim 1 holds for $i = j+1$. This completes the induction step. Thus, Claim 1 is proven. ☐

In order to finish our proof of Lemma 8.2, we now apply Claim 1 to $i = m$:

Claim 1 (applied to $i = m$) shows that $f^{(m)} \neq \bot$ and $g^{(m)} \neq \bot$ and $g^{(m)}(1) = f^{(m)}(1)$ and $f^{(m)}(0) = a \cdot g^{(m)}(0)$, and that each $v \in P$ satisfies

$$f^{(m)}(v) = \begin{cases} g^{(m)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-m; \\ a \cdot g^{(m)}(v), & \text{otherwise.} \end{cases} \tag{63}$$

Hence, $\rho(u) \leqslant m-j$. Therefore, $u \notin M$ or $\rho(u) \leqslant m-j$. Now, (53) (applied to u instead of v) yields

$$\begin{aligned}
f^{(j)}(u) &= \begin{cases} g^{(j)}(u), & \text{if } u \notin M \text{ or } \rho(u) \leqslant m-j; \\ a \cdot g^{(j)}(u), & \text{otherwise} \end{cases} \\
&= g^{(j)}(u) \qquad (\text{since } u \notin M \text{ or } \rho(u) \leqslant m-j).
\end{aligned}$$

In other words, $g^{(j)}(u) = f^{(j)}(u)$. This proves (62).

The definition of $g^{(m)}$ yields $g^{(m)} = \underbrace{R_{m-m}}_{=R_0=R} g = Rg$. The definition of $f^{(m)}$ yields $f^{(m)} = \underbrace{R_{m-m}}_{=R_0=R} f = Rf$.

Now, the three parts of Lemma 8.2 easily follow:

(a) From $g^{(m)} = Rg$, we obtain $Rg = g^{(m)} \neq \bot$. This proves Lemma 8.2 **(a)**.

(b) Let $v \in P$ be not a minimal element of P. Thus, $v \notin M$ (since M is the set of all minimal elements of P). Moreover, from $f^{(m)} = Rf$, we obtain $Rf = f^{(m)}$, and thus

$$\begin{aligned}
(Rf)(v) = f^{(m)}(v) &= \begin{cases} g^{(m)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-m; \\ a \cdot g^{(m)}(v), & \text{otherwise} \end{cases} \qquad \text{(by (63))} \\
&= \underbrace{g^{(m)}}_{=Rg}(v) \qquad (\text{since } v \notin M \text{ or } \rho(v) \leqslant m-m \text{ (because } v \notin M)) \\
&= (Rg)(v).
\end{aligned}$$

This proves Lemma 8.2 **(b)**.

(c) Let $v \in P$ be a minimal element of P. Thus, $v \in M$ (since M is the set of all minimal elements of P). Hence, we don't have $v \notin M$.

Moreover, $\rho(v) \in \{1, 2, \ldots, m\}$ (by the definition of $\rho(v)$) and therefore $\rho(v) \geqslant 1 > 0 = m - m$. Thus, we don't have $\rho(v) \leqslant m - m$.

From $f^{(m)} = Rf$, we obtain $Rf = f^{(m)}$, and thus

$$\begin{aligned}
(Rf)(v) = f^{(m)}(v) &= \begin{cases} g^{(m)}(v), & \text{if } v \notin M \text{ or } \rho(v) \leqslant m-m; \\ a \cdot g^{(m)}(v), & \text{otherwise} \end{cases} \qquad \text{(by (63))} \\
&= a \cdot \underbrace{g^{(m)}}_{=Rg}(v) \qquad \left(\begin{array}{c} \text{since we don't have "} v \notin M \text{ or } \rho(v) \leqslant m-m \text{"} \\ \text{(because we don't have } v \notin M, \\ \text{and we don't have } \rho(v) \leqslant m-m) \end{array} \right) \\
&= a \cdot (Rg)(v).
\end{aligned}$$

This proves Lemma 8.2 **(c)**. $\square$

Let us now get rid of the "a is invertible" requirement in Lemma 8.1:

Lemma 8.3. Assume that P is the $p \times q$-rectangle $[p] \times [q]$. Let $\ell \in \mathbb{N}$ be such that $\ell \geqslant 1$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^\ell f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Then,

$$\left(R^\ell f\right)(1,1) = a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \cdot b.$$

Proof. If $\ell \geqslant 2$, then we can easily see that a is invertible[36]. Hence, if $\ell \geqslant 2$, then Lemma 8.3 follows immediately from Lemma 8.1. Thus, for the rest of this proof, we

[36] *Proof.* Assume that $\ell \geqslant 2$. Thus, $2 \leqslant \ell$. Hence, from $R^\ell f \neq \bot$, we obtain $R^2 f \neq \bot$ (by Lemma 3.23). Hence, Lemma 3.26 yields that $f(0)$ and $f(1)$ are invertible. In other words, a and b are invertible (since $a = f(0)$ and $b = f(1)$). This proves that a is invertible.

WLOG assume that we **don't** have $\ell \geqslant 2$. Hence, $\ell = 1$ (since $\ell \geqslant 1$). Therefore, $R^{\ell-1} = R^{1-1} = R^0 = \mathrm{id}$, so that $R^{\ell-1} f = \mathrm{id}\, f = f$ and therefore $\left(R^{\ell-1} f\right)(p,q) = f(p,q)$. Also, $R^\ell = R$ (since $\ell = 1$). Hence, $R = R^\ell$, so that $Rf = R^\ell f \neq \bot$.

Now, let $g \in \mathbb{K}^{\widehat{P}}$ be the $\mathbb{K}$-labeling that is obtained from f by replacing the label $f(0)$ by 1. Thus, we have

$$g(x) = f(x) \qquad \text{for each } x \in \widehat{P} \setminus \{0\}, \tag{64}$$

and we have $g(0) = 1$. Then, Lemma 8.2 **(a)** yields $Rg \neq \bot$. In other words, $R^1 g \neq \bot$.

Note that $(p,q) \in P \subseteq \widehat{P} \setminus \{0\}$. Hence, (64) (applied to $x = (p,q)$) yields $g(p,q) = f(p,q)$.

We have $1 \in \widehat{P} \setminus \{0\}$. Thus, applying (64) to $x = 1$, we obtain $g(1) = f(1) = b$, so that $b = g(1)$. Also, $1 = g(0)$, and clearly 1 is invertible. Hence, Lemma 8.1 (applied to 1, g and 1 instead of ℓ, f and a) yields

$$\left(R^1 g\right)(1,1) = 1 \cdot \overline{\left(R^{1-1} g\right)(p,q)} \cdot b = \overline{\left(R^{1-1} g\right)(p,q)} \cdot b.$$

In view of $R^1 = R$ and $\underbrace{R^{1-1}}_{=R^0 = \mathrm{id}} g = \mathrm{id}\, g = g$, we can rewrite this as

$$(Rg)(1,1) = \overline{g(p,q)} \cdot b.$$

However, $(1,1)$ is a minimal element of P. Thus, Lemma 8.2 **(c)** (applied to $v = (1,1)$) yields

$$(Rf)(1,1) = a \cdot \underbrace{(Rg)(1,1)}_{= \overline{g(p,q)} \cdot b} = a \cdot \overline{g(p,q)} \cdot b = a \cdot \overline{f(p,q)} \cdot b \qquad \left(\text{since } g(p,q) = f(p,q)\right).$$

In view of $R^\ell = R$ and $\left(R^{\ell-1} f\right)(p,q) = f(p,q)$, we can rewrite this as

$$\left(R^\ell f\right)(1,1) = a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \cdot b.$$

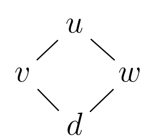
Thus, Lemma 8.3 is proven. □

This settles the easiest case of Theorem 4.8 – namely, the case $(i,j) = (1,1)$. To get a grip on the general case, we need more lemmas.

9. The conversion lemma

We continue using the notations from Section 6.

Lemma 9.1 (Four neighbors lemma)**.** Let u, v, w, d be four adjacent elements of P that are arranged as follows on the Hasse diagram of P:



(i.e., we have $d = (i,j)$, $v = (i+1,j)$, $w = (i,j+1)$ and $u = (i+1,j+1)$ for some $i \in [p-1]$ and some $j \in [q-1]$).

Assume that a is invertible. Let $\ell \geqslant 1$ be such that $R^{\ell+1} f \neq \bot$. Then:

(a) We have

$$\overline{v_\ell} \cdot V_\ell^d \cdot d_\ell = \overline{u_\ell} \cdot A_\ell^u \cdot w_\ell.$$

(b) We have

$$\overline{w_\ell} \cdot V_\ell^d \cdot d_\ell = \overline{u_\ell} \cdot A_\ell^u \cdot v_\ell.$$

Proof. **(a)** We have $R\left(R^\ell f\right) = R^{\ell+1} f \neq \bot = R(\bot)$ and thus $R^\ell f \neq \bot$. Hence, Lemma 7.1 **(a)** yields that v_ℓ is invertible. Similarly, w_ℓ and u_ℓ and d_ℓ are invertible. Also, Lemma 7.1 **(d)** (applied to d instead of v) yields that the element V_ℓ^d is well-defined and invertible. Moreover, Lemma 7.1 **(c)** (applied to u and $\ell+1$ instead of v and ℓ) yields that the element A_ℓ^u is well-defined and invertible.

The elements $s \in \widehat{P}$ that satisfy $s \gtrdot d$ are v and w. Hence, $\sum\limits_{s \gtrdot d} \overline{s_\ell} = \overline{v_\ell} + \overline{w_\ell}$ (where, of course, the sum ranges over $s \in \widehat{P}$). Now, the definition of V_ℓ^d yields

$$V_\ell^d = \overline{\sum_{s \gtrdot d} \overline{s_\ell}} \cdot \overline{d_\ell} = \overline{\overline{v_\ell} + \overline{w_\ell}} \cdot \overline{d_\ell} \tag{65}$$

(since $\sum\limits_{s \gtrdot d} \overline{s_\ell} = \overline{v_\ell} + \overline{w_\ell}$).

The elements $s \in \widehat{P}$ that satisfy $s \lessdot u$ are v and w. Hence, $\sum\limits_{s \lessdot u} s_\ell = v_\ell + w_\ell$. Now, the definition of A_ℓ^u yields

$$A_\ell^u = u_\ell \cdot \overline{\sum_{s \lessdot u} s_\ell} = u_\ell \cdot \overline{v_\ell + w_\ell} \tag{66}$$

(since $\sum\limits_{s \lessdot u} s_\ell = v_\ell + w_\ell$). Since this is well-defined, the element $v_\ell + w_\ell$ of $\mathbb{K}$ must be invertible. Also, we already know that v_ℓ and w_ℓ are invertible. Hence, Proposition 2.4 **(b)** (applied to v_ℓ and w_ℓ instead of a and b) yields that $\overline{v_\ell} + \overline{w_\ell}$ is invertible as well and its inverse is

$$\overline{\overline{v_\ell} + \overline{w_\ell}} = v_\ell \cdot \overline{v_\ell + w_\ell} \cdot w_\ell.$$

Now,

$$\overline{v_\ell} \cdot \underbrace{V_\ell^d}_{\substack{= \overline{\overline{v_\ell} + \overline{w_\ell}} \cdot \overline{d_\ell} \\ \text{(by (65))}}} \cdot d_\ell = \overline{v_\ell} \cdot \underbrace{\overline{\overline{v_\ell} + \overline{w_\ell}}}_{= v_\ell \cdot \overline{v_\ell + w_\ell} \cdot w_\ell} \cdot \underbrace{\overline{d_\ell} \cdot d_\ell}_{=1} = \underbrace{\overline{v_\ell} \cdot v_\ell}_{=1} \cdot \overline{v_\ell + w_\ell} \cdot w_\ell = \overline{v_\ell + w_\ell} \cdot w_\ell.$$

Comparing this with

$$\overline{u_\ell} \cdot \underbrace{A_\ell^u}_{\substack{= u_\ell \cdot \overline{v_\ell + w_\ell} \\ \text{(by (66))}}} \cdot w_\ell = \underbrace{\overline{u_\ell} \cdot u_\ell}_{=1} \cdot \overline{v_\ell + w_\ell} \cdot w_\ell = \overline{v_\ell + w_\ell} \cdot w_\ell,$$

we obtain $\overline{v_\ell} \cdot V_\ell^d \cdot d_\ell = \overline{u_\ell} \cdot A_\ell^u \cdot w_\ell$. Thus, Lemma 9.1 **(a)** is proved.

(b) This can be proved by the same argument that we used to prove part **(a)** (with the roles of v and w interchanged). □

We recall our conventions for drawing the $p \times q$-rectangle $P = [p] \times [q]$. In light of these conventions, we shall refer to the set $\{(k, q) \mid k \in [p]\}$ as the *northeastern edge* of P, and to the set $\{(i, 1) \mid i \in [p]\}$ as the *southwestern edge* of P.

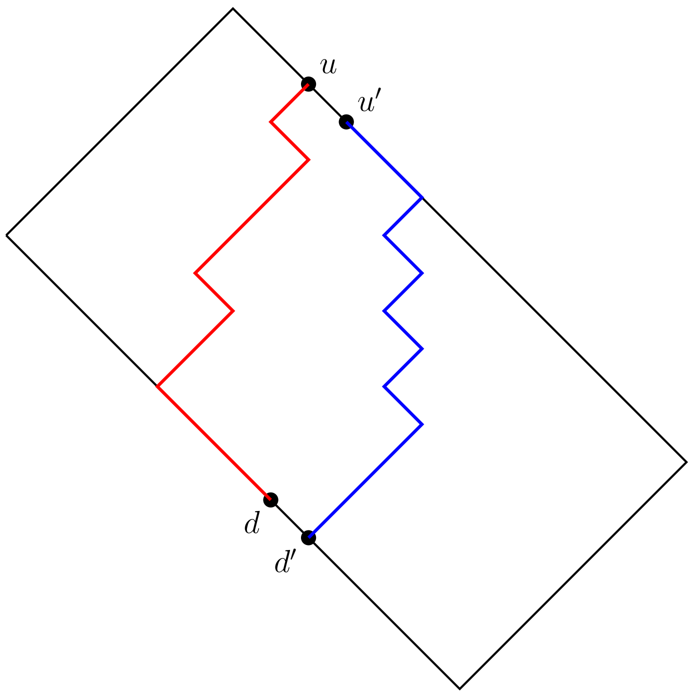
The next lemma is crucial, as it allows us to "convert" between A's and V's without changing the subscript.

Lemma 9.2 (Conversion lemma)**.** Let u and u' be two elements of the northeastern edge of P satisfying $u \gtrdot u'$ (that is, let $u = (k, q)$ and $u' = (k-1, q)$ for some $k \in \{2, 3, \ldots, p\}$). Let d and d' be two elements of the southwestern edge of P satisfying $d \gtrdot d'$ (that is, let $d = (i, 1)$ and $d' = (i-1, 1)$ for some $i \in \{2, 3, \ldots, p\}$).

Assume that a is invertible. Let $\ell \geqslant 1$ be such that $R^{\ell+1} f \neq \bot$. Then we have:

$$A_\ell^{u \to d} = V_\ell^{u' \to d'}.$$

Here is an illustration for this lemma:



(the red path indexes one addend in the sum $A_\ell^{u \to d} = \sum_{\mathbf{p} \text{ is a path from } u \text{ to } d} A_\ell^{\mathbf{p}}$, while the blue path contributes to the sum $V_\ell^{u' \to d'} = \sum_{\mathbf{p} \text{ is a path from } u' \text{ to } d'} V_\ell^{\mathbf{p}}$).

In the case when $\mathbb{K}$ is commutative, Lemma 9.2 was independently discovered by Johnson and Liu [JohLiu22]. More precisely, [JohLiu22, Lemma 4.1] extends it from sums over paths (such as $A_\ell^{u\to d}$ and $V_\ell^{u'\to d'}$) to sums over k-tuples of non-intersecting paths. It is unclear whether this extension can still be made when $\mathbb{K}$ is not commutative (what order should the A_ℓ^v's along different paths be multiplied in?), but the use of determinants likely precludes any noncommutative generalization of the proof in [JohLiu22].

Proof of Lemma 9.2. Let $\ell \in \mathbb{N}$. We "interpolate" between the paths from u to d and the paths from u' to d' using what we call "path-jump-paths". To define these formally, we introduce some more basic notations.

The first coordinate of any $x \in P$ will be denoted by $\mathfrak{first}\, x$. Thus, $\mathfrak{first}\,(i,j) = i$ for any $(i,j) \in P$.

Furthermore, for any $x = (i,j) \in P$, we define the *rank* of x to be the positive integer $i+j-1$. This rank will be denoted by $\mathfrak{rank}\, x$.

We define a new binary relation $\blacktriangleright$ on the set P as follows: If x and y are two elements of P, then the relation $x \blacktriangleright y$ means "$\mathfrak{rank}\, x = \mathfrak{rank}\, y + 1$ and $\mathfrak{first}\, x > \mathfrak{first}\, y$". In other words, the relation $x \blacktriangleright y$ means that

$$\text{if } x = (i,j)\,, \text{ then } y = (i-k, j+k-1) \text{ for some } k > 0.$$

Visually speaking, it means that y is one step southeast and a (nonnegative) amount of steps east of x (on the Hasse diagram).

We define a *path-jump-path* to be a tuple $\mathbf{p} = (v_0, v_1, \ldots, v_k)$ of elements of P along with a chosen number $i \in \{0, 1, \ldots, k-1\}$ such that the chain of relations

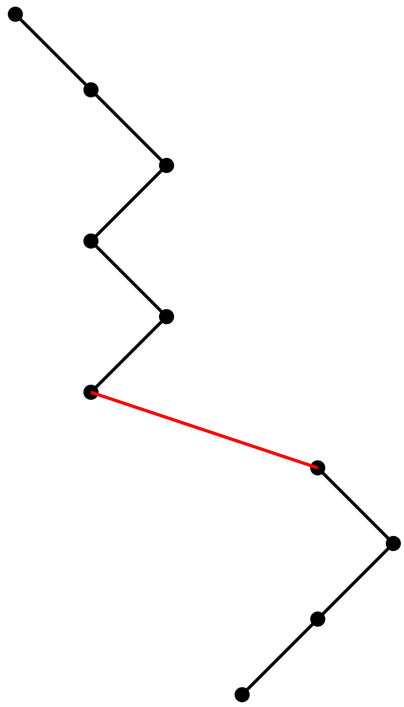
$$v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k$$

holds. We denote this path-jump-path simply by

$$\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)\,, \tag{67}$$

and we say that this path-jump-path $\mathbf{p}$ has *jump at* i. The elements $v_0, v_1, \ldots, v_k$ are called the *vertices* of this path-jump-path. The pairs (v_j, v_{j+1}) of consecutive vertices are called the *steps* of this path-jump-path. Such a step (v_j, v_{j+1}) is said to be a $\gtrdot$*-step* if $j \neq i$, and it is said to be a $\blacktriangleright$*-step* if $j = i$.

Here is an example of a path-jump-path, where the red edge is the $\blacktriangleright$-step:



(Note that two vertices x and y can satisfy $x \blacktriangleright y$ and $x \gtrdot y$ simultaneously. Thus, it can happen that several path-jump-paths with jumps at different i's contain the same vertices. We nevertheless do not consider these path-jump-paths to be identical, because we understand a path-jump-path like (67) to "remember" not only its vertices $v_0, v_1, \ldots, v_k$ but also the value of i.)

A *path-jump-path from u to d'* will mean a path-jump-path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$ such that $v_0 = u$ and $v_k = d'$.

We note that if two elements x and y of P satisfy $x \gtrdot y$ or $x \blacktriangleright y$, then

$$\mathfrak{rank}\, y = \mathfrak{rank}\, x - 1. \tag{68}$$

As a consequence of this fact, successive entries v_{j-1} and v_j in a path-jump-path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$ always satisfy $\mathfrak{rank}\,(v_j) = \mathfrak{rank}\,(v_{j-1}) - 1$ for each $j \in [k]$. In other words, the ranks of the vertices of a path-jump-path decrease by 1 at each step.

Hence, the difference in ranks between the first and final entries of a path-jump-path $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$ is one less than its number of entries:

$$\mathfrak{rank}\,(v_0) - \mathfrak{rank}\,(v_k) = k. \tag{69}$$

[*Proof of (69):* Let $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$ be a path-jump-path. Then, each $j \in [k]$ satisfies $v_{j-1} \gtrdot v_j$ or $v_{j-1} \blacktriangleright v_j$ (by the definition of a "path-jump-path"). Hence, each $j \in [k]$ satisfies $\mathfrak{rank}\,(v_j) = \mathfrak{rank}\,(v_{j-1}) - 1$ (by (68), applied to $x = v_{j-1}$ and

$y = v_j$). In other words, each $j \in [k]$ satisfies $1 = \operatorname{\mathfrak{rank}}(v_{j-1}) - \operatorname{\mathfrak{rank}}(v_j)$. Summing this equality over all $j \in [k]$, we obtain

$$\begin{aligned}\sum_{j\in[k]} 1 &= \sum_{j\in[k]} \left(\operatorname{\mathfrak{rank}}(v_{j-1}) - \operatorname{\mathfrak{rank}}(v_j)\right) \\ &= \left(\operatorname{\mathfrak{rank}}(v_0) - \operatorname{\mathfrak{rank}}(v_1)\right) + \left(\operatorname{\mathfrak{rank}}(v_1) - \operatorname{\mathfrak{rank}}(v_2)\right) + \cdots + \left(\operatorname{\mathfrak{rank}}(v_{k-1}) - \operatorname{\mathfrak{rank}}(v_k)\right) \\ &= \operatorname{\mathfrak{rank}}(v_0) - \operatorname{\mathfrak{rank}}(v_k) \qquad \text{(by the telescope principle)}.\end{aligned}$$

Hence, $\operatorname{\mathfrak{rank}}(v_0) - \operatorname{\mathfrak{rank}}(v_k) = \sum_{j\in[k]} 1 = |[k]| \cdot 1 = |[k]| = k$. This proves (69).]

Let $r := \operatorname{\mathfrak{rank}} u - \operatorname{\mathfrak{rank}}(d')$. Thus, any path-jump-path from u to d' must contain exactly $r + 1$ vertices[37]. In other words, any path-jump-path from u to d' must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$.

We have $R\left(R^\ell f\right) = R^{\ell+1} f \neq \bot = R(\bot)$ and thus $R^\ell f \neq \bot$. Hence, Lemma 7.1 **(a)** yields that v_ℓ is well-defined and invertible for each $v \in P$. Also, Lemma 7.1 **(d)** yields that $\mathcal{V}_\ell^v$ is well-defined and invertible for each $v \in P$. Moreover, Lemma 7.1 **(c)** (applied to $\ell + 1$ instead of ℓ) yields that A_ℓ^v is well-defined and invertible for each $v \in P$.

In this proof, we will not consider any $\mathbb{K}$-labelings other than $R^\ell f$. Thus, the only labels we will be using are the labels $v_\ell = \left(R^\ell f\right)(v)$ for $v \in \widehat{P}$. Thus, we agree to use the following shorthand notation: If $v \in \widehat{P}$, then the elements v_ℓ, $\mathcal{V}_\ell^v$ and A_ℓ^v of $\mathbb{K}$ will be denoted simply by v, $\mathcal{V}^v$ and A^v, respectively. In other words, **we shall omit subscripts when these subscripts are $\boldsymbol{\ell}$.** For instance, the product $A_\ell^u u_\ell \overline{u'_\ell}$ will thus be abbreviated as $A^u u \overline{u'}$.

For any path-jump-path

$$\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$$

that contains $r + 1$ vertices, we set

$$E_{\mathbf{p}} := A^{v_0} A^{v_1} \cdots A^{v_{i-1}} v_i \overline{v_{i+1}} \mathcal{V}^{v_{i+2}} \mathcal{V}^{v_{i+3}} \cdots \mathcal{V}^{v_r} \in \mathbb{K}.$$

(Here, as we have already announced, we are omitting a subscript under each symbol. All the omitted subscripts are ℓ – for example, "v_i" means $(v_i)_\ell$, and "v_{i+1}" means $(v_{i+1})_\ell$, and "A^{v_0}" means $A_\ell^{v_0}$, and so on. We will do the same with all expressions that follow.)

Now we claim the following (again omitting subscripts that are ℓ):

[37] *Proof.* Let $\mathbf{p}$ be a path-jump-path from u to d'. We must prove that $\mathbf{p}$ contains exactly $r + 1$ vertices. Write $\mathbf{p}$ in the form $\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$. Since $\mathbf{p}$ is a path-jump-path from u to d', we thus have $v_0 = u$ and $v_k = d'$. However, (69) yields $\operatorname{\mathfrak{rank}}(v_0) - \operatorname{\mathfrak{rank}}(v_k) = k$. Hence,

$$k = \operatorname{\mathfrak{rank}}\Bigg(\underbrace{v_0}_{=u}\Bigg) - \operatorname{\mathfrak{rank}}\Bigg(\underbrace{v_k}_{=d'}\Bigg) = \operatorname{\mathfrak{rank}} u - \operatorname{\mathfrak{rank}}(d') = r.$$

However, $\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$ shows that $\mathbf{p}$ contains exactly $k + 1$ vertices. In other words, $\mathbf{p}$ contains exactly $r + 1$ vertices (since $k = r$), qed.

Claim 1: We have

$$A^{u \to d} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } r-1}} E_{\mathbf{p}}.$$

Claim 2: We have

$$\forall^{u' \to d'} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 0}} E_{\mathbf{p}}.$$

Claim 3: For each $j \in \{0, 1, \ldots, r-2\}$, we have

$$\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j}} E_{\mathbf{p}} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j+1}} E_{\mathbf{p}}.$$

Before we prove these three claims, let us explain how Lemma 9.2 will follow from them:

$$\begin{aligned}
\forall_{\ell}^{u' \to d'} &= \forall^{u' \to d'} \\
&= \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 0}} E_{\mathbf{p}} && \text{(by Claim 2)} \\
&= \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 1}} E_{\mathbf{p}} && \text{(by Claim 3, applied to } j = 0) \\
&= \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 2}} E_{\mathbf{p}} && \text{(by Claim 3, applied to } j = 1) \\
&= \cdots \\
&= \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } r-1}} E_{\mathbf{p}} && \text{(by Claim 3, applied to } j = r-2) \\
&= A^{u \to d} \qquad \text{(by Claim 1)} \\
&= A_{\ell}^{u \to d}.
\end{aligned}$$

Hence, Lemma 9.2 will follow once Claims 1, 2 and 3 have been proved. Let us now prove these three claims:

Proof of Claim 1. We know that d lies on the southwestern edge of P. Hence, the only $s \in \widehat{P}$ satisfying $s \lessdot d$ is d' (since $d \gtrdot d'$). Therefore, $\sum_{\substack{s \in \widehat{P}; \\ s \lessdot d}} s_{\ell} = d'_{\ell}$. However, the definition of A_ℓ^d shows that $A_\ell^d = d_\ell \cdot \overline{\sum_{s\in\widehat{P};\ s\lessdot d} s_\ell} = d_\ell \overline{d'_\ell}$ (since $\sum_{s\in\widehat{P};\ s\lessdot d} s_\ell = d'_\ell$). Since we omit subscripts (when these subscripts are ℓ), we can rewrite this as

$$A^d = d\overline{d'}. \tag{70}$$

We know that any path-jump-path from u to d' must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$. If such a path-jump-path has jump at $r-1$, then it must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1} \blacktriangleright v_r)$; that is, its last step (v_{r-1}, v_r) is an $\blacktriangleright$-step. However, since it ends at d', we must have $v_r = d'$ and thus $v_{r-1} \blacktriangleright v_r = d'$. This entails $v_{r-1} = d$ (since the only $g \in P$ satisfying $g \blacktriangleright d'$ is d [38]), and therefore $(v_{r-1}, v_r) = (d, d')$ (since $v_r = d'$). In other words, the last step of this path-jump-path is (d, d').

We have thus shown that if a path-jump-path from u to d' has jump at $r-1$, then its last step is (d, d'). Hence, any path-jump-path from u to d' with jump at $r-1$ must have the form

$$(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1} \blacktriangleright d'),$$

where $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1})$ is a path from u to d. Conversely, any tuple of the latter form is a path-jump-path from u to d' with jump at $r-1$ (since $d \blacktriangleright d'$). Therefore, we can substitute $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1} \blacktriangleright d')$ for $\mathbf{p}$ in the sum $\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } r-1}} E_{\mathbf{p}}$. We thus

[38] *Proof.* Recall that d' lies on the southwestern edge of P. In other words, $d' = (i, 1)$ for some $i \in [p]$. Consider this i. Therefore, $d = (i+1, 1)$ (since d also lies on the southwestern edge of P and satisfies $d \gtrdot d'$).

From $d' = (i, 1)$, we obtain $\mathfrak{first}(d') = i$ and $\mathfrak{rank}(d') = i + 1 - 1 = i$.

Now, let $g \in P$ be such that $g \blacktriangleright d'$. By the definition of the relation $\blacktriangleright$, we thus have $\mathfrak{rank}\, g = \mathfrak{rank}(d') + 1$ and $\mathfrak{first}\, g > \mathfrak{first}(d')$. Hence, $\mathfrak{first}\, g > \mathfrak{first}(d') = i$ and $\mathfrak{rank}\, g = \underbrace{\mathfrak{rank}(d')}_{=i} + 1 = i + 1$.

Write g in the form $g = (i', j')$ for some $i' \in [p]$ and some $j' \in [q]$. Thus, $\mathfrak{first}\, g = i'$ and $\mathfrak{rank}\, g = i' + j' - 1$. Hence, $i' = \mathfrak{first}\, g > i$ and $i + 1 = \mathfrak{rank}\, g = \underbrace{i'}_{>i} + j' - 1 > i + j' - 1$. Subtracting i from both sides of the latter inequality, we obtain $1 > j' - 1$. Thus, $j' < 1 + 1 = 2$, so that $j' = 1$ (since $j' \in [q]$). Now, from $i + 1 = i' + \underbrace{j'}_{=1} - 1 = i' + 1 - 1 = i'$, we obtain $i' = i + 1$. Hence, $g = (i', j') = (i+1, 1)$ (since $i' = i+1$ and $j' = 1$). Comparing this with $d = (i+1, 1)$, we find $g = d$.

Forget that we fixed g. We thus have shown that if $g \in P$ satisfies $g \blacktriangleright d'$, then $g = d$. In other words, the only $g \in P$ satisfying $g \blacktriangleright d'$ is d (since it is easy to see that d does indeed satisfy $d \blacktriangleright d'$).

obtain

$$\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } r-1}} E_{\mathbf{p}} = \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} \underbrace{E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1} \blacktriangleright d')}}_{\substack{= A^{v_0} A^{v_1} \cdots A^{v_{r-2}} v_{r-1} \overline{d'} \\ \text{(by the definition of } E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1} \blacktriangleright d')})}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} A^{v_0} A^{v_1} \cdots A^{v_{r-2}} \underbrace{v_{r-1}}_{\substack{=d \\ \text{(since } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d)}} \overline{d'}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} A^{v_0} A^{v_1} \cdots A^{v_{r-2}} \underbrace{d \overline{d'}}_{\substack{=A^d \\ \text{(by (70))}}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} A^{v_0} A^{v_1} \cdots A^{v_{r-2}} \underbrace{A^d}_{\substack{=A^{v_{r-1}} \\ \text{(because } d = v_{r-1} \\ \text{(again since } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d))}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} \underbrace{A^{v_0} A^{v_1} \cdots A^{v_{r-2}} A^{v_{r-1}}}_{\substack{= A^{v_0} A^{v_1} \cdots A^{v_{r-1}} \\ = A^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1})} \\ \text{(by the definition of } A^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1})})}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \\ \text{is a path from } u \text{ to } d}} A^{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1})}$$

$$= \sum_{\mathbf{p} \text{ is a path from } u \text{ to } d} \underbrace{A^{\mathbf{p}}}_{=A_\ell^{\mathbf{p}}}$$

$$\left(\begin{array}{c} \text{here we have renamed the} \\ \text{summation index } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{r-1}) \text{ as } \mathbf{p} \end{array} \right)$$

$$= \sum_{\mathbf{p} \text{ is a path from } u \text{ to } d} A_\ell^{\mathbf{p}}$$

$$= A_\ell^{u \to d} \qquad \left(\text{by the definition of } A_\ell^{u \to d}\right)$$

$$= A^{u \to d}.$$

This proves Claim 1. ☐

Proof of Claim 2. This is mostly analogous to the above proof of Claim 1, but we nevertheless present the full argument for the sake of completeness.

We know that u' lies on the northeastern edge of P. Hence, the only $s \in \widehat{P}$ satisfying $s \gtrdot u'$ is u (since $u \gtrdot u'$). Therefore, $\sum_{\substack{s \in \widehat{P}; \\ s \gtrdot u'}} \overline{s_\ell} = \overline{u_\ell}$. Therefore,

$$\overline{\sum_{\substack{s \in \widehat{P}; \\ s \gtrdot u'}} \overline{s_\ell}} = \overline{\overline{u_\ell}} = u_\ell.$$

However, the definition of $V_\ell^{u'}$ shows that $V_\ell^{u'} = \overline{\sum_{\substack{s\in\widehat{P};\\ s\gtrdot u'}} \overline{s_\ell}} \cdot \overline{u'_\ell} = u_\ell \overline{u'_\ell}$ (since $\overline{\sum_{\substack{s\in\widehat{P};\\ s\gtrdot u'}} \overline{s_\ell}} = u_\ell$). Since we omit subscripts (when these subscripts are ℓ), we can rewrite this as

$$V^{u'} = u\overline{u'}. \tag{71}$$

We know that any path-jump-path from u to d' must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$. If such a path-jump-path has jump at 0, then it must have the form $(v_0 \blacktriangleright v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)$; that is, its first step (v_0, v_1) is an $\blacktriangleright$-step. However, since it starts at u, we must have $v_0 = u$ and thus $u = v_0 \blacktriangleright v_1$. This entails $v_1 = u'$ (since the only $g \in P$ satisfying $u \blacktriangleright g$ is u' [39]), and therefore $(v_0, v_1) = (u, u')$ (since $v_0 = u$). In other words, the first step of this path-jump-path is (u, u').

We have thus shown that if a path-jump-path from u to d' has jump at 0, then its first step is (u, u'). Hence, any path-jump-path from u to d' with jump at 0 must have the form

$$(u \blacktriangleright v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r),$$

where $(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)$ is a path from u' to d'. Conversely, any tuple of the latter form is a path-jump-path from u to d' with jump at 0 (since $u \blacktriangleright u'$). Therefore, we can

[39] *Proof.* Recall that u lies on the northeastern edge of P. In other words, $u = (i, q)$ for some $i \in [p]$. Consider this i. Therefore, $u' = (i-1, q)$ (since u' also lies on the northeastern edge of P and satisfies $u \gtrdot u'$).

From $u = (i, q)$, we obtain $\mathfrak{first}\, u = i$ and $\mathfrak{rank}\, u = i + q - 1$.

Now, let $g \in P$ be such that $u \blacktriangleright g$. By the definition of the relation $\blacktriangleright$, we thus have $\mathfrak{rank}\, u = \mathfrak{rank}\, g + 1$ and $\mathfrak{first}\, u > \mathfrak{first}\, g$. Hence, $\mathfrak{first}\, g < \mathfrak{first}\, u = i$.

Write g in the form $g = (i', j')$ for some $i' \in [p]$ and some $j' \in [q]$. Thus, $\mathfrak{first}\, g = i'$ and $\mathfrak{rank}\, g = i' + j' - 1$. Hence, $i' = \mathfrak{first}\, g < i$ and

$$i + q - 1 = \mathfrak{rank}\, u = \underbrace{\mathfrak{rank}\, g}_{=i'+j'-1} + 1 = i' + j' - 1 + 1 = \underbrace{i'}_{<i} + j' < i + j'.$$

Subtracting i from both sides of the latter inequality, we obtain $q - 1 < j'$. Thus, $j' > q - 1$, so that $j' = q$ (since $j' \in [q]$). Now, subtracting q from both sides of the equality $i + q - 1 = i' + \underbrace{j'}_{=q} = i' + q$, we obtain $i - 1 = i'$. In other words, $i' = i - 1$. Hence, $g = (i', j') = (i - 1, q)$ (since $i' = i - 1$ and $j' = q$). Comparing this with $u' = (i - 1, q)$, we find $g = u'$.

Forget that we fixed g. We thus have shown that if $g \in P$ satisfies $u \blacktriangleright g$, then $g = u'$. In other words, the only $g \in P$ satisfying $u \blacktriangleright g$ is u' (since it is easy to see that u' does indeed satisfy $u \blacktriangleright u'$).

substitute $(u \blacktriangleright v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)$ for $\mathbf{p}$ in the sum $\sum\limits_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 0}} E_{\mathbf{p}}$. We thus obtain

$$\begin{aligned}
\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } 0}} E_{\mathbf{p}} &= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \underbrace{E_{(u \blacktriangleright v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)}}_{\substack{= u \overline{v_1} \forall^{v_2} \forall^{v_3} \cdots \forall^{v_r} \\ \text{(by the definition of } E_{(u \blacktriangleright v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)})}} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} u \underbrace{\overline{v_1}}_{\substack{= \overline{u'} \\ \text{(because } v_1 = u' \\ \text{(since } (v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'))}} \forall^{v_2} \forall^{v_3} \cdots \forall^{v_r} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \underbrace{u \overline{u'}}_{\substack{= \forall^{u'} \\ \text{(by (71))}}} \forall^{v_2} \forall^{v_3} \cdots \forall^{v_r} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \underbrace{\forall^{u'}}_{\substack{= \forall^{v_1} \\ \text{(because } u' = v_1 \\ \text{(since } (v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'))}} \forall^{v_2} \forall^{v_3} \cdots \forall^{v_r} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \underbrace{\forall^{v_1} \forall^{v_2} \forall^{v_3} \cdots \forall^{v_r}}_{\substack{= \forall^{v_1} \forall^{v_2} \cdots \forall^{v_r} \\ = \forall^{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)} \\ \text{(by the definition of } \forall^{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)})}} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \forall^{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r)} \\
&= \sum_{\substack{(v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \\ \text{is a path from } u' \text{ to } d'}} \underbrace{\forall^{\mathbf{p}}}_{= \forall_{\ell}^{\mathbf{p}}} \\
&\qquad \left(\begin{array}{c} \text{here we have renamed the} \\ \text{summation index } (v_1 \gtrdot v_2 \gtrdot \cdots \gtrdot v_r) \text{ as } \mathbf{p} \end{array} \right) \\
&= \sum_{\mathbf{p} \text{ is a path from } u' \text{ to } d'} \forall_{\ell}^{\mathbf{p}} \\
&= \forall_{\ell}^{u' \to d'} \qquad \left(\text{by the definition of } \forall_{\ell}^{u' \to d'} \right) \\
&= \forall^{u' \to d'}.
\end{aligned}$$

This proves Claim 2. $\square$

Proving Claim 3 is a bit trickier. As an auxiliary result, we first show the following:

Claim 4: Let s and t be two elements of P. Then,

$$\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s \overline{x} \forall^{t} = \sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^{s} x \overline{t}. \tag{72}$$

Proof of Claim 4. We first observe that Claim 4 trivially holds if $\mathfrak{rank}\, s - \mathfrak{rank}\, t \neq 2$ [40]. Thus, for the rest of this proof, we WLOG assume that $\mathfrak{rank}\, s - \mathfrak{rank}\, t = 2$. In terms of the way that we draw our poset P, this means that the point s lies two rows above the point t.

The definition of V_ℓ^t yields $V_\ell^t = \overline{\sum_{x \gtrdot t} \overline{x_\ell}} \cdot \overline{t_\ell}$. Omitting the subscripts, we can rewrite this as

$$V^t = \overline{\sum_{x \gtrdot t} \overline{x}} \cdot \overline{t}. \tag{73}$$

The definition of A_ℓ^s yields $A_\ell^s = s_\ell \cdot \overline{\sum_{x \lessdot s} x_\ell}$. Omitting the subscripts, we can rewrite this as

$$A^s = s \cdot \overline{\sum_{x \lessdot s} x}. \tag{74}$$

Write $s \in P$ in the form $s = (i, j)$. Write $t \in P$ in the form $t = (i', j')$. From $s = (i, j)$, we obtain $\mathfrak{rank}\, s = i + j - 1$. Likewise, $\mathfrak{rank}\, t = i' + j' - 1$. Hence,

$$\mathfrak{rank}\, s - \mathfrak{rank}\, t = (i + j - 1) - (i' + j' - 1) = i + j - i' - j',$$

so that $i + j - i' - j' = \mathfrak{rank}\, s - \mathfrak{rank}\, t = 2$. Thus, $j' = i + j - i' - 2$.

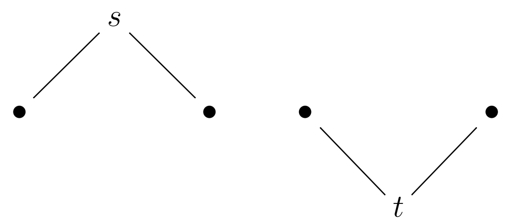
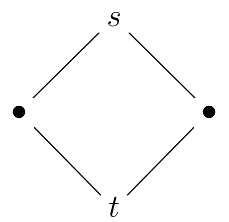
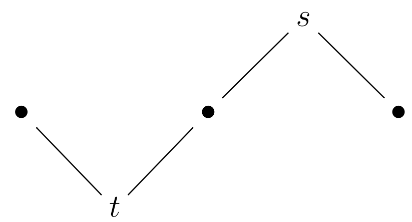
We are in one of the following three cases:

Case 1: We have $i' < i - 1$.

Case 2: We have $i' = i - 1$.

Case 3: We have $i' > i - 1$.

Representative examples for these three cases are illustrated in the following pictures:

Case 1	Case 2	Case 3
		

[40] *Proof.* Assume that $\mathfrak{rank}\, s - \mathfrak{rank}\, t \neq 2$.

We claim that there exists no $x \in P$ such that $s \blacktriangleright x \gtrdot t$. Indeed, assume the contrary. Thus, such an x does exist. Consider this x. Then, (68) (applied to s and x instead of x and y) yields $\mathfrak{rank}\, x = \mathfrak{rank}\, s - 1$ (since $s \blacktriangleright x$). Also, (68) (applied to t instead of y) yields $\mathfrak{rank}\, t = \mathfrak{rank}\, x - 1$ (since $x \gtrdot t$). Thus,

$$\mathfrak{rank}\, t = \underbrace{\mathfrak{rank}\, x}_{= \mathfrak{rank}\, s - 1} - 1 = \mathfrak{rank}\, s - 1 - 1 = \mathfrak{rank}\, s - 2,$$

so that $\mathfrak{rank}\, s - \mathfrak{rank}\, t = 2$; but this contradicts $\mathfrak{rank}\, s - \mathfrak{rank}\, t \neq 2$. This contradiction shows that our assumption was false. Hence, we have shown that there exists no $x \in P$ such that $s \blacktriangleright x \gtrdot t$. Thus, the sum $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t$ is empty. Similarly, we can see that the sum $\sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$ is empty. Thus, these two sums are both empty and therefore both equal 0. Hence, $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = \sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$. We have thus proved Claim 4 in the case when $\mathfrak{rank}\, s - \mathfrak{rank}\, t \neq 2$.

(the bullets signify the positions of potential neighbors of s and t; some of these positions may fall outside of P, but this does not disturb our argument). In terms of the way we draw our poset P, the three cases can be reformulated as "the point s lies further west than t" (Case 1), "the point s lies due north of t" (Case 2) and "the point s lies further east than t" (Case 3). Note that two elements $x, y \in P$ satisfy $x \blacktriangleright y$ if and only if y lies one step south and some arbitrary distance east of x in our pictures.

Let us first consider Case 1. In this case, we have $i' < i - 1$. Thus, $i' + 1 < i$. Hence, each element x of P that satisfies $x \gtrdot t$ must satisfy $s \blacktriangleright x$ automatically[41]. Therefore, the summation sign $\sum\limits_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}}$ can be simplified to $\sum\limits_{\substack{x \in P; \\ x \gtrdot t}}$. In turn, the latter summation sign $\sum\limits_{\substack{x \in P; \\ x \gtrdot t}}$ can be rewritten as $\sum\limits_{\substack{x \in \widehat{P}; \\ x \gtrdot t}}$ (because any $x \in \widehat{P}$ that satisfies $x \gtrdot t$ must automatically belong to P [42]). Hence, we obtain the following equality of summation signs:

$$\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} = \sum_{\substack{x \in P; \\ x \gtrdot t}} = \sum_{\substack{x \in \widehat{P}; \\ x \gtrdot t}} = \sum_{x \gtrdot t}$$

(since our sums are understood to range over $\widehat{P}$ by default). Thus,

$$\begin{aligned} \sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t &= \sum_{x \gtrdot t} s\overline{x}V^t = s \left(\sum_{x \gtrdot t} \overline{x} \right) \underbrace{V^t}_{\substack{= \overline{\sum_{x \gtrdot t} \overline{x} \cdot \overline{t}} \\ \text{(by (73))}}} = s \underbrace{\left(\sum_{x \gtrdot t} \overline{x} \right) \overline{\sum_{x \gtrdot t} \overline{x}}}_{=1} \cdot \overline{t} \\ &= s\overline{t}. \end{aligned} \tag{75}$$

Furthermore, $i' < i - 1$, so that $i - 1 > i'$. Hence, each element x of P that satisfies $s \gtrdot x$ must satisfy $x \blacktriangleright t$ automatically[43]. Therefore, the summation sign $\sum\limits_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}}$ can be simplified

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[41] *Proof.* Let $x \in P$ be such that $x \gtrdot t$. We must show that $s \blacktriangleright x$.

From $x \gtrdot t$, we obtain $\mathfrak{rank}\, t = \mathfrak{rank}\, x - 1$ (by (68), applied to $y = t$) and thus $\mathfrak{rank}\, x - \mathfrak{rank}\, t = 1$. Now,

$$\mathfrak{rank}\, s - \mathfrak{rank}\, x = \underbrace{(\mathfrak{rank}\, s - \mathfrak{rank}\, t)}_{=2} - \underbrace{(\mathfrak{rank}\, x - \mathfrak{rank}\, t)}_{=1} = 2 - 1 = 1.$$

Thus, $\mathfrak{rank}\, s = \mathfrak{rank}\, x + 1$.

Furthermore, x is either $(i'+1, j')$ or $(i', j'+1)$ (since $x \gtrdot t = (i', j')$). Hence, $\mathfrak{first}\, x \leqslant i' + 1 < i = \mathfrak{first}\, s$ (since $s = (i, j)$), so that $\mathfrak{first}\, s > \mathfrak{first}\, x$. Combining this with $\mathfrak{rank}\, s = \mathfrak{rank}\, x + 1$, we obtain $s \blacktriangleright x$ (by the definition of the relation $\blacktriangleright$).

[42] *Proof.* From $t = (i', j')$, we obtain $\mathfrak{first}\, t = i' < i - 1 < i \leqslant p = \mathfrak{first}\,(p, q)$. Hence, $\mathfrak{first}\, t \neq \mathfrak{first}\,(p, q)$, so that $t \neq (p, q)$. This shows that t is not a maximal element of P (since the only maximal element of P is (p, q)). In other words, we don't have $t \lessdot 1$ in $\widehat{P}$. In other words, we don't have $1 \gtrdot t$ in $\widehat{P}$. Hence, any $x \in \widehat{P}$ that satisfies $x \gtrdot t$ must automatically belong to P (since we have neither $1 \gtrdot t$ nor $0 \gtrdot t$).

[43] *Proof.* Let $x \in P$ be such that $s \gtrdot x$. We must show that $x \blacktriangleright t$.

From $s \gtrdot x$, we obtain $\mathfrak{rank}\, x = \mathfrak{rank}\, s - 1$ (by (68), applied to s and x instead of x and y). Hence, to $\sum\limits_{\substack{x \in P;\\ s \gtrdot x}}$. In turn, the latter summation sign $\sum\limits_{\substack{x \in P;\\ s \gtrdot x}}$ can be rewritten as $\sum\limits_{\substack{x \in \widehat{P};\\ s \gtrdot x}}$ (because any $x \in \widehat{P}$ that satisfies $s \gtrdot x$ must automatically belong to P [44]). Hence, we obtain the following equality of summation signs:

$$\sum_{\substack{x \in P;\\ s \gtrdot x \blacktriangleright t}} = \sum_{\substack{x \in P;\\ s \gtrdot x}} = \sum_{\substack{x \in \widehat{P};\\ s \gtrdot x}} = \sum_{\substack{x \in \widehat{P};\\ x \lessdot s}} = \sum_{x \lessdot s}$$

(since our sums are understood to range over $\widehat{P}$ by default). Thus,

$$\sum_{\substack{x \in P;\\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t} = \sum_{x \lessdot s} A^s x \overline{t} = \underbrace{A^s}_{\substack{= s \cdot \overline{\sum\limits_{x \lessdot s} x} \\ \text{(by (74))}}} \left(\sum_{x \lessdot s} x \right) \overline{t} = s \cdot \underbrace{\overline{\sum_{x \lessdot s} x} \left(\sum_{x \lessdot s} x \right)}_{=1} \overline{t} = s \overline{t}.$$

Comparing this with (75), we obtain $\sum\limits_{\substack{x \in P;\\ s \blacktriangleright x \gtrdot t}} s \overline{x} V^t = \sum\limits_{\substack{x \in P;\\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$. Thus, Claim 4 is proved in Case 1.

Let us now consider Case 2. In this case, we have $i' = i - 1$. Hence, $j' = i + j - \underbrace{i'}_{= i-1} - 2 = i + j - (i-1) - 2 = j - 1$. Thus, $t = (i', j') = (i-1, j-1)$ (since $i' = i - 1$ and $j' = j - 1$). Let $v := (i, j-1)$ and $w := (i-1, j)$. In our coordinate system, the four points

$$s = (i, j), \qquad t = (i-1, j-1), \qquad v = (i, j-1), \qquad w = (i-1, j)$$

are arranged in a 1×1-square, which looks as follows:

$$\begin{array}{ccccc} & & s & & \\ & \diagup & & \diagdown & \\ v & & & & w \\ & \diagdown & & \diagup & \\ & & t & & \end{array} . \tag{76}$$

Hence, v and w belong to P (since s and t belong to P), and furthermore, Lemma 9.1 **(b)** (applied to s and t instead of u and d) yields

$$\overline{w_\ell} \cdot V_\ell^t \cdot t_\ell = \overline{s_\ell} \cdot A_\ell^s \cdot v_\ell .$$

$\mathfrak{rank}\, s - \mathfrak{rank}\, x = 1$, so that

$$\mathfrak{rank}\, x - \mathfrak{rank}\, t = \underbrace{(\mathfrak{rank}\, s - \mathfrak{rank}\, t)}_{=2} - \underbrace{(\mathfrak{rank}\, s - \mathfrak{rank}\, x)}_{=1} = 2 - 1 = 1.$$

Thus, $\mathfrak{rank}\, x = \mathfrak{rank}\, t + 1$.

Furthermore, x is either $(i-1, j)$ or $(i, j-1)$ (since $(i, j) = s \gtrdot x$). Hence, $\mathfrak{first}\, x \geqslant i - 1 > i' = \mathfrak{first}\, t$ (since $t = (i', j')$). Combining this with $\mathfrak{rank}\, x = \mathfrak{rank}\, t + 1$, we obtain $x \blacktriangleright t$ (by the definition of the relation $\blacktriangleright$).

[44] *Proof.* From $s = (i, j)$, we obtain $\mathfrak{first}\, s = i > i - 1 > i' \geqslant 1 = \mathfrak{first}\,(1,1)$. Hence, $\mathfrak{first}\, s \neq \mathfrak{first}\,(1,1)$, so that $s \neq (1,1)$. This shows that s is not a minimal element of P (since the only minimal element of P is $(1,1)$). In other words, we don't have $s \gtrdot 0$ in $\widehat{P}$. Hence, any $x \in \widehat{P}$ that satisfies $s \gtrdot x$ must automatically belong to P (since we have neither $s \gtrdot 0$ nor $s \gtrdot 1$).

Since we are omitting subscripts, we can rewrite this as follows:

$$\overline{w} \cdot V^t \cdot t = \overline{s} \cdot A^s \cdot v.$$

The picture (76) shows that we have $s \blacktriangleright w$ but not $s \blacktriangleright v$. Hence, there is only one element $x \in P$ that satisfies $s \blacktriangleright x \gtrdot t$; namely, this element x is w. Hence,

$$\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = s\overline{w}V^t = s \cdot \overline{w} \cdot V^t \cdot \underbrace{1}_{= t \cdot \overline{t}} = s \cdot \underbrace{\overline{w} \cdot V^t \cdot t}_{= \overline{s} \cdot A^s \cdot v} \cdot \overline{t} = \underbrace{s \cdot \overline{s}}_{=1} \cdot A^s \cdot v \cdot \overline{t}$$
$$= A^s \cdot v \cdot \overline{t}. \qquad (77)$$

On the other hand, the picture (76) shows that we have $v \blacktriangleright t$ but not $w \blacktriangleright t$. Hence, there is only one element $x \in P$ that satisfies $s \gtrdot x \blacktriangleright t$; namely, this element x is v. Hence,

$$\sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t} = A^s v \overline{t} = A^s \cdot v \cdot \overline{t}.$$

Comparing this with (77), we obtain $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = \sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$. Thus, Claim 4 is proved in Case 2.

Let us finally consider Case 3. In this case, we have $i' > i - 1$. Thus, $i' \geqslant i$ (since i' and i are integers), so that $i \leqslant i'$. Note that $i = \mathfrak{first}\, s$ (since $s = (i, j)$) and $i' = \mathfrak{first}\, t$ (since $t = (i', j')$).

There exists no $x \in P$ satisfying $s \blacktriangleright x \gtrdot t$ [45]. Hence, the sum $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t$ is empty. Thus, $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = 0$.

Furthermore, there exists no $x \in P$ satisfying $s \gtrdot x \blacktriangleright t$ [46]. Hence, the sum $\sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$ is empty. Thus, $\sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t} = 0$.

Comparing this with $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = 0$, we obtain $\sum_{\substack{x \in P; \\ s \blacktriangleright x \gtrdot t}} s\overline{x}V^t = \sum_{\substack{x \in P; \\ s \gtrdot x \blacktriangleright t}} A^s x \overline{t}$. Thus, Claim 4 is proved in Case 3.

We have now proved Claim 4 in all three cases. □

We can now step to the proof of Claim 3:

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[45] *Proof.* Assume the contrary. Thus, there exists an $x \in P$ satisfying $s \blacktriangleright x \gtrdot t$. Consider this x.
We have $x \gtrdot t = (i', j')$; thus, x is either $(i'+1, j')$ or $(i', j'+1)$. Hence, $\mathfrak{first}\, x \geqslant i' \geqslant i = \mathfrak{first}\, s$. However, from $s \blacktriangleright x$, we obtain $\mathfrak{first}\, s > \mathfrak{first}\, x$ (by the definition of the relation $\blacktriangleright$). This contradicts $\mathfrak{first}\, x \geqslant \mathfrak{first}\, s$. This contradiction shows that our assumption was false, qed.

[46] *Proof.* Assume the contrary. Thus, there exists an $x \in P$ satisfying $s \gtrdot x \blacktriangleright t$. Consider this x.
We have $s \gtrdot x$, so that $x \lessdot s = (i, j)$; thus, x is either $(i-1, j)$ or $(i, j-1)$. Hence, $\mathfrak{first}\, x \leqslant i \leqslant i' = \mathfrak{first}\, t$. However, from $x \blacktriangleright t$, we obtain $\mathfrak{first}\, x > \mathfrak{first}\, t$ (by the definition of the relation $\blacktriangleright$). This contradicts $\mathfrak{first}\, x \leqslant \mathfrak{first}\, t$. This contradiction shows that our assumption was false, qed.

Proof of Claim 3. Let $j \in \{0, 1, \ldots, r-2\}$.

We know that any path-jump-path from u to d' must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$. If such a path-jump-path has jump at j, then it must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r)$. Thus,

$$\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j}} E_{\mathbf{p}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j}} \underbrace{E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r)}}_{\substack{= A^{v_0} A^{v_1} \cdots A^{v_{j-1}} v_j \overline{v_{j+1}} \forall^{v_{j+2}} \forall^{v_{j+3}} \cdots \forall^{v_r} \\ \text{(by the definition of } E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r)})}}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} v_j \overline{v_{j+1}} \forall^{v_{j+2}} \forall^{v_{j+3}} \cdots \forall^{v_r}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} \sum_{\substack{v_{j+1} \in P; \\ v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2}}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} v_j \overline{v_{j+1}} \forall^{v_{j+2}} \forall^{v_{j+3}} \cdots \forall^{v_r}$$

$$\left(\begin{array}{c} \text{here, we have broken up our} \\ \text{path-jump-path } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \gtrdot \cdots \gtrdot v_r) \\ \text{into two paths } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \text{ and } (v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{and an intermediate vertex } v_{j+1} \text{ satisfying } v_j \blacktriangleright v_{j+1} \gtrdot v_{j+2} \end{array}\right)$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} \underbrace{\sum_{\substack{x \in P; \\ v_j \blacktriangleright x \gtrdot v_{j+2}}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} v_j \overline{x} \forall^{v_{j+2}} \forall^{v_{j+3}} \cdots \forall^{v_r}}_{\substack{= \sum_{x \in P;\ v_j \blacktriangleright x \gtrdot v_{j+2}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} v_j \overline{x} \forall^{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r} \\ = A^{v_0} A^{v_1} \cdots A^{v_{j-1}} \sum_{x \in P;\ v_j \blacktriangleright x \gtrdot v_{j+2}} v_j \overline{x} \forall^{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}}}$$

(here we have renamed v_{j+1} as x in the inner sum)

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} \underbrace{\sum_{\substack{x \in P; \\ v_j \blacktriangleright x \gtrdot v_{j+2}}} v_j \overline{x} \forall^{v_{j+2}}}_{\substack{= \sum_{x \in P;\ v_j \gtrdot x \blacktriangleright v_{j+2}} A^{v_j} x \overline{v_{j+2}} \\ \text{(by Claim 4,} \\ \text{applied to } s = v_j \\ \text{and } t = v_{j+2})}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}$$

$$= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} \sum_{\substack{x \in P; \\ v_j \gtrdot x \blacktriangleright v_{j+2}}} A^{v_j} x \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}.$$

We know that any path-jump-path from u to d' must have the form

$(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_r)$. If such a path-jump-path has jump at $j+1$, then it must have the form $(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r)$. Thus,

$$\begin{aligned}
&\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j+1}} E_{\mathbf{p}} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j+1}} \underbrace{E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r)}}_{\substack{= A^{v_0} A^{v_1} \cdots A^{v_j} v_{j+1} \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r} \\ \text{(by the definition of } E_{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r)})}} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j+1}} A^{v_0} A^{v_1} \cdots A^{v_j} v_{j+1} \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} \sum_{\substack{v_{j+1} \in P; \\ v_j \gtrdot v_{j+1} \blacktriangleright v_{j+2}}} A^{v_0} A^{v_1} \cdots A^{v_j} v_{j+1} \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r} \\
&\qquad \left(\begin{array}{c} \text{here, we have broken up our} \\ \text{path-jump-path } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_{j+1} \blacktriangleright v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{into two paths } (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \text{ and } (v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{and an intermediate vertex } v_{j+1} \text{ satisfying } v_j \gtrdot v_{j+1} \blacktriangleright v_{j+2} \end{array}\right) \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} \underbrace{\sum_{\substack{x \in P; \\ v_j \gtrdot x \blacktriangleright v_{j+2}}} A^{v_0} A^{v_1} \cdots A^{v_j} x \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}}_{\substack{= \sum_{\substack{x \in P; \\ v_j \gtrdot x \blacktriangleright v_{j+2}}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} A^{v_j} x \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r} \\ = A^{v_0} A^{v_1} \cdots A^{v_{j-1}} \sum_{\substack{x \in P; \\ v_j \gtrdot x \blacktriangleright v_{j+2}}} A^{v_j} x \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}}} \\
&\qquad \text{(here we have renamed } v_{j+1} \text{ as } x \text{ in the inner sum)} \\
&= \sum_{\substack{(v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_j) \\ \text{is a path starting at } u}} \sum_{\substack{(v_{j+2} \gtrdot v_{j+3} \gtrdot \cdots \gtrdot v_r) \\ \text{is a path ending at } d'}} A^{v_0} A^{v_1} \cdots A^{v_{j-1}} \sum_{\substack{x \in P; \\ v_j \gtrdot x \blacktriangleright v_{j+2}}} A^{v_j} x \overline{v_{j+2}} \forall^{v_{j+3}} \forall^{v_{j+4}} \cdots \forall^{v_r}.
\end{aligned}$$

Comparing our last two equalities, we obtain

$$\sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j}} E_{\mathbf{p}} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path} \\ \text{from } u \text{ to } d' \\ \text{with jump at } j+1}} E_{\mathbf{p}}.$$

Thus, Claim 3 is proven. □

We have now proved all three Claims 1, 2 and 3. As we explained, this completes the proof of Lemma 9.2. □

Remark 9.3. Parts of the above proof of Lemma 9.2 can be rewritten in a more abstract (although probably not shorter) manner, avoiding the notion of a "path-jump-path" and the nested sums that appeared in our proof of Claim 3.

To rewrite the proof, we need the notion of $P \times P$-matrices. A $P \times P$*-matrix* is a matrix whose rows and columns are indexed not by integers but by elements of P. (That is, it is a family of elements of $\mathbb{K}$ indexed by pairs $(i,j) \in P \times P$.) If C is any $P \times P$-matrix, and if i and j are two elements of P, then the (i,j)-th entry of C is denoted by $C_{i,j}$. Addition and multiplication are defined for $P \times P$-matrices in the same way as they are for usual matrices. That is, for any $P \times P$-matrices C and D and any $(i,j) \in P \times P$, we have

$$(C+D)_{i,j} = C_{i,j} + D_{i,j} \qquad \text{and} \qquad (CD)_{i,j} = \sum_{k \in P} C_{i,k} D_{k,j}.$$

For any statement $\mathcal{A}$, we let $[\mathcal{A}]$ be the Iverson bracket (i.e., truth value) of $\mathcal{A}$. That is, $[\mathcal{A}] = 1$ if $\mathcal{A}$ is true, and $[\mathcal{A}] = 0$ if $\mathcal{A}$ is false.

Now, let $\ell \in \mathbb{N}$. Define three $P \times P$-matrices $\mathbf{A}$, $\mathbf{V}$ and $\mathbf{U}$ by

$$\begin{aligned} \mathbf{A}_{x,y} &:= A^x [x \gtrdot y], \\ \mathbf{V}_{x,y} &:= V^y [x \gtrdot y], \\ \mathbf{U}_{x,y} &:= x\overline{y} [x \blacktriangleright y] \qquad \text{for all } x, y \in P. \end{aligned}$$

Here, the relation $x \blacktriangleright y$ is defined as in the above proof of Lemma 9.2, and we are again omitting the "ℓ" subscripts, so (for instance) "$x\overline{y}$" actually means $x_\ell \overline{y_\ell}$.

Now, Claim 4 in our above proof of Lemma 9.2 can be rewritten in a nice and compact form as the equality

$$\mathbf{AU} = \mathbf{UV}.$$

From this, we easily obtain

$$\mathbf{A}^k \mathbf{U} = \mathbf{U} \mathbf{V}^k \qquad \text{for any } k \in \mathbb{N}. \tag{78}$$

This equality essentially replaces Claim 3 in the above proof.

Setting $k = \mathfrak{rank}\, u - \mathfrak{rank}\, d$ in (78), and comparing the (u, d')-entries of both sides, we quickly obtain $A^{u \to d} = V^{u' \to d'}$ (since $x \blacktriangleright d'$ holds only for $x = d$, and since $u \blacktriangleright x$ holds only for $x = u'$). This proves Lemma 9.2 again.

10. Proof of reciprocity: the case $j = 1$

Using the conversion lemma, we can now easily prove Theorem 4.8 in the case when $j = 1$:

Lemma 10.1. Assume that P is the $p \times q$-rectangle $[p] \times [q]$. Let $i \in [p]$. Let $\ell \in \mathbb{N}$ satisfy $\ell \geqslant i$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^\ell f \neq \bot$. Let $a = f(0)$ and

$b = f(1)$. Then, using the notations from Section 6, we have

$$(i,1)_\ell = a \cdot \overline{(p+1-i,\ q)_{\ell-i}} \cdot b.$$

Proof. We have $\ell \geqslant i \geqslant 1$ (since $i \in [p]$). Also, from $i \in [p]$ and $1 \in [q]$, we obtain $(i,1) \in [p] \times [q] = P$. Furthermore, from $i \in [p]$, we obtain $p+1-i \in [p]$. Combined with $q \in [q]$, this leads to $(p+1-i,\ q) \in [p] \times [q] = P$.

If $\ell = 1$, then the claim of Lemma 10.1 easily follows from Lemma 8.3[47]. Thus, for the rest of this proof, we WLOG assume that $\ell \neq 1$. Hence, $\ell \geqslant 2$ (since $\ell \geqslant 1$). Thus, $2 \leqslant \ell$. Hence, from $R^\ell f \neq \perp$, we obtain $R^2 f \neq \perp$ (by Lemma 3.23). Hence, Lemma 3.26 yields that $f(0)$ and $f(1)$ are invertible. In other words, a and b are invertible (since $a = f(0)$ and $b = f(1)$).

We have $\underbrace{\ell}_{\geqslant i} - i + 1 \geqslant i - i + 1 = 1$. Furthermore, $\ell - \underbrace{i}_{\geqslant 1} + 1 \leqslant \ell - 1 + 1 = \ell$ and thus $R^{\ell-i+1} f \neq \perp$ (by Lemma 3.23, since $R^\ell f \neq \perp$). Thus, Lemma 7.1 **(b)** (applied to $(p-i+1,\ q)$ and $\ell - i + 1$ instead of v and ℓ) yields that the element $(p-i+1,\ q)_{\ell-i}$ is well-defined and invertible.

Furthermore, Theorem 7.6 **(d)** (applied to $\ell - i$ and $(p-i+1,\ q)$ instead of ℓ and u) yields $(p-i+1,\ q)_{\ell-i} = A_{\ell-i}^{(p-i+1,\ q)\to(1,1)} \cdot a$ (since $R^{\ell-i+1} f \neq \perp$). Solving this for $A_{\ell-i}^{(p-i+1,\ q)\to(1,1)}$, we obtain

$$A_{\ell-i}^{(p-i+1,\ q)\to(1,1)} = (p-i+1,\ q)_{\ell-i} \cdot \overline{a}. \tag{80}$$

The right hand side of this equality is a product of two invertible elements (since both $(p-i+1,\ q)_{\ell-i}$ and $\overline{a}$ are invertible), and thus is invertible. Hence, the left hand side is

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[47] *Proof.* Assume that $\ell = 1$. Then, $1 = \ell \geqslant i$, so that $i \leqslant 1$ and therefore $i = 1$ (since $i \in [p]$). However, Lemma 8.3 yields

$$\left(R^\ell f\right)(1,1) = a \cdot \overline{\left(R^{\ell-1} f\right)(p,q)} \cdot b. \tag{79}$$

Since we are using the notations from Section 6, we have

$$(i,1)_\ell = \left(R^\ell f\right)(i,1) = \left(R^\ell f\right)(1,1) \qquad \text{(since } i = 1\text{)}$$

and

$$\begin{aligned}(p+1-i,\ q)_{\ell-i} &= \left(R^{\ell-i} f\right)(p+1-i,\ q) = \left(R^{\ell-1} f\right)\left(\underbrace{p+1-1}_{=p},\ q\right) \qquad \text{(since } i = 1\text{)} \\ &= \left(R^{\ell-1} f\right)(p,q).\end{aligned}$$

Thus, we can rewrite (79) as

$$(i,1)_\ell = a \cdot \overline{(p+1-i,\ q)_{\ell-i}} \cdot b.$$

Hence, Lemma 10.1 is proved under the assumption that $\ell = 1$.

invertible as well. Taking reciprocals on both sides of (80), we now obtain

$$\begin{aligned} \overline{A_{\ell-i}^{(p-i+1,\ q)\to(1,1)}} &= \overline{(p-i+1,\ q)_{\ell-i} \cdot \overline{a}} \\ &= a \cdot \overline{(p-i+1,\ q)_{\ell-i}}. \end{aligned} \tag{81}$$

Now, using Lemma 9.2 and Proposition 7.3, we can easily see the following: For each $k \in \{0, 1, \ldots, i-2\}$, we have

$$V_{\ell-k}^{(p-k,\ q)\to(i-k,\ 1)} = V_{\ell-(k+1)}^{(p-(k+1),\ q)\to(i-(k+1),\ 1)}. \tag{82}$$

[*Proof of (82):* Let $k \in \{0, 1, \ldots, i-2\}$. Then, $k \leqslant i - 2 < \underbrace{i}_{\leqslant \ell} -1 \leqslant \ell - 1$, so that $\ell - 1 > k$ and thus $\ell - k > 1$.

From $k < i - 1$, we also obtain $1 < i - k$, so that $1 \leqslant i - k - 1$ (since 1 and $i - k$ are integers). Also, $i \in [p]$, so that $i \leqslant p$ and thus $i - k \leqslant p - k$. Furthermore, $k \geqslant 0$, so that $p - k \leqslant p$.

Now, we have $i - k - 1 \in [p]$ (since $1 \leqslant i - k - 1$ and $i - k - 1 \leqslant i - k \leqslant p - k \leqslant p$), so that $(i - k - 1,\ 1) \in [p] \times [q] = P$.

Furthermore, we have $i - k \in [p]$ (since $1 \leqslant i - k - 1 \leqslant i - k$ and $i - k \leqslant p - k \leqslant p$), so that $(i - k,\ 1) \in [p] \times [q] = P$.

Furthermore, we have $p-k-1 \in [p]$ (since $1 \leqslant \underbrace{i}_{\leqslant p} -k-1 \leqslant p-k-1$ and $p-k-1 \leqslant p - k \leqslant p$), so that $(p - k - 1,\ q) \in [p] \times [q] = P$.

Furthermore, we have $p - k \in [p]$ (since $1 \leqslant i - k - 1 \leqslant i - k \leqslant p - k$ and $p - k \leqslant p$), so that $(p - k,\ q) \in [p] \times [q] = P$.

Also, $\ell - \underbrace{k}_{\geqslant 0} \leqslant \ell$ and therefore $R^{\ell-k} f \neq \bot$ (by Lemma 3.23, since $R^{\ell} f \neq \bot$). Hence, (41) (applied to $(p - k,\ q)$, $(i - k,\ 1)$ and $\ell - k$ instead of u, v and ℓ) yields

$$V_{\ell-k}^{(p-k,\ q)\to(i-k,\ 1)} = A_{\ell-k-1}^{(p-k,\ q)\to(i-k,\ 1)}.$$

However, $(p - k,\ q)$ and $(p - k - 1,\ q)$ are two elements of the northeastern edge of P satisfying $(p - k,\ q) \gtrdot (p - k - 1,\ q)$, whereas $(i - k,\ 1)$ and $(i - k - 1,\ 1)$ are two elements of the southwestern edge of P satisfying $(i - k,\ 1) \gtrdot (i - k - 1,\ 1)$. We furthermore have $\ell - k - 1 \geqslant 1$ (since $\ell - k > 1$) and $R^{\ell-k-1+1} f = R^{\ell-k} f \neq \bot$. Thus, Lemma 9.2 (applied to $(p - k,\ q)$, $(p - k - 1,\ q)$, $(i - k,\ 1)$, $(i - k - 1,\ 1)$ and $\ell - k - 1$ instead of u, u', d, d' and ℓ) yields

$$A_{\ell-k-1}^{(p-k,\ q)\to(i-k,\ 1)} = V_{\ell-k-1}^{(p-k-1,\ q)\to(i-k-1,\ 1)}.$$

Combining what we have shown, we now obtain

$$V_{\ell-k}^{(p-k,\ q)\to(i-k,\ 1)} = A_{\ell-k-1}^{(p-k,\ q)\to(i-k,\ 1)} = V_{\ell-k-1}^{(p-k-1,\ q)\to(i-k-1,\ 1)} = V_{\ell-(k+1)}^{(p-(k+1),\ q)\to(i-(k+1),\ 1)}$$

(since $\ell - k - 1 = \ell - (k + 1)$ and $p - k - 1 = p - (k + 1)$ and $i - k - 1 = i - (k + 1)$). This proves (82).]

Now,

$$\begin{aligned}
V_{\ell}^{(p,\ q)\to(i,\ 1)} &= V_{\ell-0}^{(p-0,\ q)\to(i-0,\ 1)} && (\text{since } p = p-0 \text{ and } i = i-0 \text{ and } \ell = \ell - 0) \\
&= V_{\ell-1}^{(p-1,\ q)\to(i-1,\ 1)} && (\text{by (82), applied to } k=0) \\
&= V_{\ell-2}^{(p-2,\ q)\to(i-2,\ 1)} && (\text{by (82), applied to } k=1) \\
&= \cdots \\
&= V_{\ell-(i-1)}^{(p-(i-1),\ q)\to(i-(i-1),\ 1)} && (\text{by (82), applied to } k=i-2) \\
&= V_{\ell-i+1}^{(p-i+1,\ q)\to(1,\ 1)} && \left(\begin{array}{c}\text{since } p-(i-1) = p-i+1 \\ \text{and } i-(i-1)=1 \\ \text{and } \ell-(i-1) = \ell-i+1\end{array}\right) \\
&= A_{\ell-i}^{(p-i+1,\ q)\to(1,\ 1)}
\end{aligned} \tag{83}$$

(by (41), applied to $\ell - i + 1$, $(p - i + 1,\ q)$ and $(1,\ 1)$ instead of ℓ, u and v).

However, Theorem 7.6 **(c)** (applied to $u = (i, 1)$) yields

$$\begin{aligned}
(i,1)_{\ell} &= \overline{V_{\ell}^{(p,\ q)\to(i,\ 1)}} \cdot b = \overline{A_{\ell-i}^{(p-i+1,\ q)\to(1,\ 1)}} \cdot b && (\text{by (83)}) \\
&= a \cdot \overline{(p-i+1,\ q)_{\ell-i}} \cdot b && (\text{by (81)}) \\
&= a \cdot \overline{(p+1-i,\ q)_{\ell-i}} \cdot b && (\text{since } p-i+1 = p+1-i).
\end{aligned}$$

This proves Lemma 10.1. ☐

In analogy to Lemma 10.1, we have the following:

Lemma 10.2. Assume that P is the $p \times q$-rectangle $[p] \times [q]$. Let $j \in [q]$. Let $\ell \in \mathbb{N}$ satisfy $\ell \geqslant j$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^{\ell} f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Then, using the notations from Section 6, we have

$$(1,j)_{\ell} = a \cdot \overline{(p,\ q+1-j)_{\ell-j}} \cdot b.$$

Proof. The two coordinates u and v of an element $(u, v) \in P$ play symmetric roles. Lemma 10.2 is just Lemma 10.1 with the roles of these two coordinates interchanged. Thus, the proof of Lemma 10.2 is analogous to the proof of Lemma 10.1. ☐

11. Proof of reciprocity: the general case

Somewhat surprisingly, the general case of Theorem 4.8 follows by a fairly straightforward induction argument from Lemma 10.1:

Proof of Theorem 4.8. We again use the notations from Section 6.

For any $(i, j) \in P$, we define $\mathfrak{tilt}(i, j)$ to be the positive integer $i + 2j$.

Our goal is to prove Theorem 4.8. In other words, our goal is to prove (27) for each $x = (i, j) \in P$ and $\ell \in \mathbb{N}$ satisfying $\ell - i - j + 1 \geqslant 0$ and $R^\ell f \neq \bot$. We will now prove this by strong induction on $\operatorname{tilt} x$.

Induction step: Fix $N \in \mathbb{N}$. Assume (as the induction hypothesis) that

> (27) holds for each $x = (i, j) \in P$ satisfying $\operatorname{tilt} x < N$ and each $\ell \in \mathbb{N}$ satisfying $\ell - i - j + 1 \geqslant 0$ and $R^\ell f \neq \bot$.

We now fix an element $v = (i, j) \in P$ satisfying $\operatorname{tilt} v = N$ and an $\ell \in \mathbb{N}$ satisfying $\ell - i - j + 1 \geqslant 0$ and $R^\ell f \neq \bot$. Our goal is to prove that (27) holds for $x = v$. In other words, our goal is to prove that $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$.

We have $N = \operatorname{tilt} \underbrace{v}_{=(i,j)} = \operatorname{tilt}(i, j) = i + 2j$ (by the definition of $\operatorname{tilt}(i, j)$). We are in one of the following six cases:

Case 1: We have $i = 1$.
Case 2: We have $j = 1$.
Case 3: We have $j = 2$ and $1 < i < p$.
Case 4: We have $j = 2$ and $i = p > 1$.
Case 5: We have $j > 2$ and $1 < i < p$.
Case 6: We have $j > 2$ and $i = p > 1$.

Let us first consider Case 1. In this case, we have $i = 1$. Thus, $v = (i, j) = (1, j)$ (since $i = 1$). The definition of $v^{\sim}$ thus yields $v^{\sim} = (p + 1 - 1,\ q + 1 - j) = (p,\ q + 1 - j)$. Also, $\ell - \underbrace{i}_{=1} - j + 1 = \ell - 1 - j + 1 = \ell - j$, so that $\ell - j = \ell - i - j + 1 \geqslant 0$. In other words, $\ell \geqslant j$. Hence, Lemma 10.2 yields

$$(1, j)_\ell = a \cdot \overline{(p,\ q + 1 - j)_{\ell - j}} \cdot b.$$

In view of $v = (1, j)$ and $v^{\sim} = (p,\ q + 1 - j)$ and $\ell - i - j + 1 = \ell - j$, we can rewrite this as $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$. Thus, $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$ is proved in Case 1.

Similarly (but using Lemma 10.1 instead of Lemma 10.2), we can obtain the same result (viz., $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$) in Case 2.

Next, let us analyze the four remaining cases: Cases 3, 4, 5 and 6. The most complex of these four cases is Case 5, so it is this case that we start with.

Thus, let us consider Case 5. In this case, we have $j > 2$ and $1 < i < p$. Recall that $v = (i, j)$. Define the four further pairs

$$m := (i, j - 1), \qquad u := (i + 1, j - 1),$$
$$s := (i, j - 2), \qquad t := (i - 1, j - 1).$$

The conditions $j > 2$ and $1 < i < p$ entail that all these four pairs m, u, s and t belong to $[p] \times [q] = P$. Here is how the five elements v, m, u, s, t of P are aligned on the Hasse diagram of P:

$$\begin{array}{ccccc} u & & & & v \\ & \diagdown & & \diagup & \\ & & m & & \\ & \diagup & & \diagdown & \\ s & & & & t \end{array} . \tag{84}$$

In particular, the two elements of P that cover m are u and v, whereas the two elements of P that are covered by m are s and t. Clearly, we can replace P by $\widehat{P}$ in this sentence (since 1 only covers those elements of P that are not covered by any element of P, and since 0 is covered only by those elements of P that do not cover any element of P). Thus, we obtain the following: The two elements of $\widehat{P}$ that cover m are u and v, whereas the two elements of $\widehat{P}$ that are covered by m are s and t.

Moreover, the map $P \to P,\ x \mapsto x^{\sim}$ (which can be visualized as "reflecting" each point in P around the center of the rectangle $[p] \times [q]$) "reverses" covering relations: That is, if two elements x and y of P satisfy $x \gtrdot y$, then $x^{\sim} \lessdot y^{\sim}$. Hence, the two elements of P that are covered by $m^{\sim}$ are $u^{\sim}$ and $v^{\sim}$ (since the two elements of P that cover m are u and v), whereas the two elements of P that cover $m^{\sim}$ are $s^{\sim}$ and $t^{\sim}$ (since the two elements of P that are covered by m are s and t). Clearly, we can replace P by $\widehat{P}$ in this sentence (since 1 only covers those elements of P that are not covered by any element of P, and since 0 is covered only by those elements of P that do not cover any element of P). Thus, we obtain the following: The two elements of $\widehat{P}$ that are covered by $m^{\sim}$ are $u^{\sim}$ and $v^{\sim}$, whereas the two elements of $\widehat{P}$ that cover $m^{\sim}$ are $s^{\sim}$ and $t^{\sim}$.

All in all, applying the map $P \to P,\ x \mapsto x^{\sim}$ to the diagram (84) yields

$$\begin{array}{ccccc} t^{\sim} & & & & s^{\sim} \\ & \diagdown & & \diagup & \\ & & m^{\sim} & & \\ & \diagup & & \diagdown & \\ v^{\sim} & & & & u^{\sim} \end{array} .$$

From $\ell - i - j + 1 \geqslant 0$, we obtain $\ell \geqslant \underbrace{i}_{>1} + \underbrace{j}_{>2} - 1 > 1 + 2 - 1 = 2$, so that $\ell \geqslant 2$. Therefore, $\ell - 1 \geqslant 1 \geqslant 0$ and thus $\ell - 1 \in \mathbb{N}$. Also, $\ell \geqslant 2$ entails $2 \leqslant \ell$.

Hence, from $R^{\ell} f \neq \bot$, we obtain $R^2 f \neq \bot$ (by Lemma 3.23). Therefore, Lemma 3.26 yields that $f(0)$ and $f(1)$ are invertible. In other words, a and b are invertible (since $a = f(0)$ and $b = f(1)$). Also, we have $R^{\ell-1} f \neq \bot$ (since $R\left(R^{\ell-1} f\right) = R^{\ell} f \neq \bot = R(\bot)$).

Set $k := i + j - 2$. Then, $k = \underbrace{i}_{\geqslant 1} + \underbrace{j}_{\geqslant 1} - 2 \geqslant 1 + 1 - 2 = 0$, so that $k \in \mathbb{N}$.

Now, it is easy to see that the four elements m, u, s and t of P satisfy

$$\operatorname{tilt} m < N, \qquad \operatorname{tilt} u < N, \qquad \operatorname{tilt} s < N, \qquad \operatorname{tilt} t < N$$

[48]. Hence, using the induction hypothesis, it is easy to see that the five equalities

$$m_\ell = a \cdot \overline{\widetilde{m_{\ell-k}}} \cdot b, \tag{85}$$

$$s_{\ell-1} = a \cdot \overline{\widetilde{s_{\ell-k}}} \cdot b, \tag{86}$$

$$t_{\ell-1} = a \cdot \overline{\widetilde{t_{\ell-k}}} \cdot b, \tag{87}$$

$$m_{\ell-1} = a \cdot \overline{\widetilde{m_{\ell-k-1}}} \cdot b, \tag{88}$$

$$u_\ell = a \cdot \overline{\widetilde{u_{\ell-k-1}}} \cdot b \tag{89}$$

hold[49].

[48] *Proof.* Recall that $\operatorname{tilt}(i', j') = i' + 2j'$ for each $(i', j') \in P$ (by the definition of $\operatorname{tilt}(i', j')$). Thus:

- From $m = (i, j-1)$, we obtain $\operatorname{tilt} m = i + 2(j-1) = i + 2j - 2 < i + 2j = N$.
- From $u = (i+1, j-1)$, we obtain $\operatorname{tilt} u = i + 1 + 2(j-1) = i + 2j - 1 < i + 2j = N$.
- From $s = (i, j-2)$, we obtain $\operatorname{tilt} s = i + 2(j-2) = i + 2j - 4 < i + 2j = N$.
- From $t = (i-1, j-1)$, we obtain $\operatorname{tilt} t = i - 1 + 2(j-1) = i + 2j - 3 < i + 2j = N$.

[49] *Proof.* The induction hypothesis tells us that we can apply (27) to m and $(i, j-1)$ instead of x and (i, j) (since $m = (i, j-1) \in P$ and $\operatorname{tilt} m < N$ and $\ell \in \mathbb{N}$ and $\ell - i - \underbrace{(j-1)}_{\leqslant j} + 1 \geqslant \ell - i - j + 1 \geqslant 0$ and $R^\ell f \neq \bot$). Thus, we obtain

$$m_\ell = a \cdot \overline{\widetilde{m_{\ell-i-(j-1)+1}}} \cdot b = a \cdot \overline{\widetilde{m_{\ell-k}}} \cdot b$$

(since $\ell - i - (j-1) + 1 = \ell - \underbrace{(i+j-2)}_{=k} = \ell - k$). This proves (85).

The induction hypothesis tells us that we can apply (27) to s and $(i, j-2)$ and $\ell - 1$ instead of x and (i, j) and ℓ (since $s = (i, j-2) \in P$ and $\operatorname{tilt} s < N$ and $\ell - 1 \in \mathbb{N}$ and $\ell - 1 - i - (j-2) + 1 = \ell - i - j + 2 \geqslant \ell - i - j + 1 \geqslant 0$ and $R^{\ell-1} f \neq \bot$). Thus, we obtain

$$s_{\ell-1} = a \cdot \overline{\widetilde{s_{(\ell-1)-i-(j-2)+1}}} \cdot b = a \cdot \overline{\widetilde{s_{\ell-k}}} \cdot b$$

(since $(\ell-1) - i - (j-2) + 1 = \ell - \underbrace{(i+j-2)}_{=k} = \ell - k$). This proves (86).

The induction hypothesis tells us that we can apply (27) to t and $(i-1, j-1)$ and $\ell - 1$ instead of x and (i, j) and ℓ (since $t = (i-1, j-1) \in P$ and $\operatorname{tilt} t < N$ and $\ell - 1 \in \mathbb{N}$ and $\ell - 1 - (i-1) - (j-1) + 1 = \ell - i - j + 2 \geqslant \ell - i - j + 1 \geqslant 0$ and $R^{\ell-1} f \neq \bot$). Thus, we obtain

$$t_{\ell-1} = a \cdot \overline{\widetilde{t_{(\ell-1)-(i-1)-(j-1)+1}}} \cdot b = a \cdot \overline{\widetilde{t_{\ell-k}}} \cdot b$$

(since $(\ell-1) - (i-1) - (j-1) + 1 = \ell - \underbrace{(i+j-2)}_{=k} = \ell - k$). This proves (87).

The induction hypothesis tells us that we can apply (27) to m and $(i, j-1)$ and $\ell - 1$ instead of x and (i, j) and ℓ (since $m = (i, j-1) \in P$ and $\operatorname{tilt} m < N$ and $\ell - 1 \in \mathbb{N}$ and $\ell - 1 - i - (j-1) + 1 = \ell - i - j + 1 \geqslant 0$ and $R^{\ell-1} f \neq \bot$). Thus, we obtain

$$m_{\ell-1} = a \cdot \overline{\widetilde{m_{(\ell-1)-i-(j-1)+1}}} \cdot b = a \cdot \overline{\widetilde{m_{\ell-k-1}}} \cdot b$$

(since $(\ell-1) - i - (j-1) + 1 = \ell - \underbrace{(i+j-2)}_{=k} - 1 = \ell - k - 1$). This proves (88).

We have $\ell - 1 \in \mathbb{N}$ and $R^{\ell-1+1} f = R^\ell f \neq \bot$. Hence, the equality (29) (applied to m and $\ell - 1$ instead of v and ℓ) yields

$$m_{\ell-1+1} = \left(\sum_{x \lessdot m} x_{\ell-1} \right) \cdot \overline{m_{\ell-1}} \cdot \overline{\sum_{x \gtrdot m} \overline{x_{\ell-1+1}}}$$

(here we have renamed the summation indices u from (29) as x, since the letter u is already being used for something else in our current setting). Since $\ell - 1 + 1 = \ell$, this can be simplified to

$$m_\ell = \left(\sum_{x \lessdot m} x_{\ell-1} \right) \cdot \overline{m_{\ell-1}} \cdot \overline{\sum_{x \gtrdot m} \overline{x_\ell}}. \tag{90}$$

However, recall that the two elements of $\widehat{P}$ that are covered by m are s and t. In other words, the two elements $x \in \widehat{P}$ satisfying $x \lessdot m$ are s and t. Hence, $\sum_{x \lessdot m} x_{\ell-1} = s_{\ell-1} + t_{\ell-1}$.

Also, recall that the two elements of $\widehat{P}$ that cover m are u and v. In other words, the two elements $x \in \widehat{P}$ satisfying $x \gtrdot m$ are u and v. Hence, $\sum_{x \gtrdot m} \overline{x_\ell} = \overline{u_\ell} + \overline{v_\ell}$.

Now we know that $\sum_{x \lessdot m} x_{\ell-1} = s_{\ell-1} + t_{\ell-1}$ and $\sum_{x \gtrdot m} \overline{x_\ell} = \overline{u_\ell} + \overline{v_\ell}$. Using these formulas, we can rewrite (90) as

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}}. \tag{91}$$

On the other hand, from $k = i + j - 2$, we obtain $\ell - k - 1 = \ell - (i + j - 2) - 1 = \ell - i - j + 1 \geqslant 0$. Thus, $\ell - k - 1 \in \mathbb{N}$. Also, $\ell - \underbrace{k}_{\geqslant 0} \leqslant \ell$, so that $R^{\ell-k} f \neq \bot$ (by Lemma 3.23, since $R^\ell f \neq \bot$). Hence, $R^{\ell-k-1+1} f = R^{\ell-k} f \neq \bot$. Hence, the equality (29) (applied to $m^\sim$ and $\ell - k - 1$ instead of v and ℓ) yields

$$m^\sim_{\ell-k-1+1} = \left(\sum_{x \lessdot m^\sim} x_{\ell-k-1} \right) \cdot \overline{m^\sim_{\ell-k-1}} \cdot \overline{\sum_{x \gtrdot m^\sim} \overline{x_{\ell-k-1+1}}}.$$

Since $\ell - k - 1 + 1 = \ell - k$, this can be simplified to

$$m^\sim_{\ell-k} = \left(\sum_{x \lessdot m^\sim} x_{\ell-k-1} \right) \cdot \overline{m^\sim_{\ell-k-1}} \cdot \overline{\sum_{x \gtrdot m^\sim} \overline{x_{\ell-k}}}. \tag{92}$$

The induction hypothesis tells us that we can apply (27) to u and $(i+1, j-1)$ instead of x and (i, j) (since $u = (i+1, j-1) \in P$ and $\operatorname{tilt} u < N$ and $\ell \in \mathbb{N}$ and $\ell - (i+1) - (j-1) + 1 = \ell - i - j + 1 \geqslant 0$ and $R^\ell f \neq \bot$). Thus, we obtain

$$u_\ell = a \cdot \overline{u^\sim_{\ell-(i+1)-(j-1)+1}} \cdot b = a \cdot \overline{u^\sim_{\ell-k-1}} \cdot b$$

(since $\ell - (i+1) - (j-1) + 1 = \ell - \underbrace{(i + j - 2)}_{=k} - 1 = \ell - k - 1$). This proves (89).

However, recall that the two elements of $\widehat{P}$ that are covered by $m^{\sim}$ are $u^{\sim}$ and $v^{\sim}$. In other words, the two elements $x \in \widehat{P}$ satisfying $x \lessdot m^{\sim}$ are $u^{\sim}$ and $v^{\sim}$. Hence, $\sum\limits_{x \lessdot m^{\sim}} x_{\ell-k-1} = u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}$.

Also, recall that the two elements of $\widehat{P}$ that cover $m^{\sim}$ are $s^{\sim}$ and $t^{\sim}$. In other words, the two elements $x \in \widehat{P}$ satisfying $x \gtrdot m^{\sim}$ are $s^{\sim}$ and $t^{\sim}$. Hence, $\sum\limits_{x \gtrdot m^{\sim}} \overline{x_{\ell-k}} = \overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}$.

We now know that $\sum\limits_{x \lessdot m^{\sim}} x_{\ell-k-1} = u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}$ and $\sum\limits_{x \gtrdot m^{\sim}} \overline{x_{\ell-k}} = \overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}$. In light of these two equalities, we can rewrite (92) as

$$m^{\sim}_{\ell-k} = \left(u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}\right) \cdot \overline{m^{\sim}_{\ell-k-1}} \cdot \overline{\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}}. \tag{93}$$

This entails that the inverses $\overline{\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}}$ and $\overline{m^{\sim}_{\ell-k-1}}$ are well-defined (since they appear on the right hand side of this equality). In other words, the elements $\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}$ and $m^{\sim}_{\ell-k-1}$ of $\mathbb{K}$ are invertible. Hence, their product $\left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1}$ is invertible as well.

Also, $\ell - k \geqslant 1$ (since $\ell - k - 1 \geqslant 0$) and $R^{\ell-k} f \neq \perp$. Hence, Lemma 7.1 **(a)** (applied to $\ell - k$ and $m^{\sim}$ instead of ℓ and v) shows that $m^{\sim}_{\ell-k}$ is well-defined and invertible. Now, multiplying both sides of (93) with $\left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1}$, on the right, we obtain

$$\begin{aligned}
& m^{\sim}_{\ell-k} \cdot \left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1} \\
&= \left(u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}\right) \cdot \overline{m^{\sim}_{\ell-k-1}} \cdot \underbrace{\overline{\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}} \cdot \left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right)}_{=1} \cdot m^{\sim}_{\ell-k-1} \\
&= \left(u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}\right) \cdot \underbrace{\overline{m^{\sim}_{\ell-k-1}} \cdot m^{\sim}_{\ell-k-1}}_{=1} = u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}.
\end{aligned}$$

Hence,

$$u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1} = m^{\sim}_{\ell-k} \cdot \left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1}.$$

This shows that $u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}$ is a product of three invertible elements (since $m^{\sim}_{\ell-k}$ and $\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}$ and $m^{\sim}_{\ell-k-1}$ are invertible). Thus, $u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}$ itself is invertible.

Taking reciprocals on both sides of (93), we obtain

$$\begin{aligned}
\overline{m^{\sim}_{\ell-k}} &= \overline{\left(u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}\right) \cdot \overline{m^{\sim}_{\ell-k-1}} \cdot \overline{\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}}} \\
&= \left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1} \cdot \overline{u^{\sim}_{\ell-k-1} + v^{\sim}_{\ell-k-1}}
\end{aligned} \tag{94}$$

(by Proposition 2.3 **(c)**).

Comparing (91) with (85), we obtain

$$a \cdot \overline{m_{\ell-k}^{\sim}} \cdot b = \left(\underbrace{s_{\ell-1}}_{\substack{=a\cdot \overline{s_{\ell-k}^{\sim}}\cdot b \\ \text{(by (86))}}} + \underbrace{t_{\ell-1}}_{\substack{=a\cdot \overline{t_{\ell-k}^{\sim}}\cdot b \\ \text{(by (87))}}} \right) \cdot \overline{\underbrace{m_{\ell-1}}_{\substack{=a\cdot \overline{m_{\ell-k-1}^{\sim}}\cdot b \\ \text{(by (88))}}}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}}$$

$$= \underbrace{\left(a \cdot \overline{s_{\ell-k}^{\sim}} \cdot b + a \cdot \overline{t_{\ell-k}^{\sim}} \cdot b\right)}_{=a\cdot\left(\overline{s_{\ell-k}^{\sim}}+\overline{t_{\ell-k}^{\sim}}\right)\cdot b} \cdot \underbrace{\overline{a \cdot \overline{m_{\ell-k-1}^{\sim}} \cdot b}}_{\substack{=\overline{b}\cdot m_{\ell-k-1}^{\sim}\cdot \overline{a} \\ \text{(by Proposition 2.3 } \mathbf{(c)}\text{,} \\ \text{since } a \text{ and } \overline{m_{\ell-k-1}^{\sim}} \text{ and } b \text{ are invertible)}}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}}$$

$$= a \cdot \left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot \underbrace{b \cdot \overline{b}}_{=1} \cdot m_{\ell-k-1}^{\sim} \cdot \overline{a} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}}$$

$$= a \cdot \left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim} \cdot \overline{a} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}}.$$

Multiplying both sides of this equality by $\overline{a}$ on the left and by $\overline{b}$ on the right (this is allowed, since a and b are invertible), we obtain

$$\overline{a} \cdot a \cdot \overline{m_{\ell-k}^{\sim}} \cdot b \cdot \overline{b} = \underbrace{\overline{a} \cdot a}_{=1} \cdot \left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim} \cdot \underbrace{\overline{a} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}} \cdot \overline{b}}_{\substack{=\overline{b\cdot(\overline{u_\ell}+\overline{v_\ell})\cdot a} \\ \text{(by Proposition 2.3 } \mathbf{(c)}\text{)}}}$$

$$= \left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim} \cdot \overline{b \cdot (\overline{u_\ell} + \overline{v_\ell}) \cdot a}.$$

Hence,

$$\begin{aligned}
&\left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim} \cdot \overline{b \cdot (\overline{u_\ell} + \overline{v_\ell}) \cdot a} \\
&= \underbrace{\overline{a} \cdot a}_{=1} \cdot \overline{m_{\ell-k}^{\sim}} \cdot \underbrace{b \cdot \overline{b}}_{=1} = \overline{m_{\ell-k}^{\sim}} \\
&= \left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim} \cdot \overline{u_{\ell-k-1}^{\sim} + v_{\ell-k-1}^{\sim}} \qquad \text{(by (94))}.
\end{aligned}$$

Cancelling the $\left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim}$ factors on the left of this equality (this is allowed, since $\left(\overline{s_{\ell-k}^{\sim}} + \overline{t_{\ell-k}^{\sim}}\right) \cdot m_{\ell-k-1}^{\sim}$ is invertible), we obtain

$$\overline{b \cdot (\overline{u_\ell} + \overline{v_\ell}) \cdot a} = \overline{u_{\ell-k-1}^{\sim} + v_{\ell-k-1}^{\sim}}.$$

Taking reciprocals on both sides, we find

$$b \cdot (\overline{u_\ell} + \overline{v_\ell}) \cdot a = u_{\ell-k-1}^{\sim} + v_{\ell-k-1}^{\sim}.$$

In other words,

$$b \cdot \overline{u_\ell} \cdot a + b \cdot \overline{v_\ell} \cdot a = u_{\ell-k-1}^{\sim} + v_{\ell-k-1}^{\sim} \tag{95}$$

(since $b \cdot (\overline{u_\ell} + \overline{v_\ell}) \cdot a = b \cdot \overline{u_\ell} \cdot a + b \cdot \overline{v_\ell} \cdot a$).

However, (89) yields

$$\overline{u_\ell} = \overline{a \cdot \overline{u_{\ell-k-1}^{\sim}} \cdot b} = \overline{b} \cdot u_{\ell-k-1}^{\sim} \cdot \overline{a} \qquad \text{(by Proposition 2.3 } \mathbf{(c)}\text{)}.$$

Thus,

$$b \cdot \overline{u_\ell} \cdot a = \underbrace{b \cdot \overline{b}}_{=1} \cdot u^{\sim}_{\ell-k-1} \cdot \underbrace{\overline{a} \cdot a}_{=1} = u^{\sim}_{\ell-k-1}.$$

Subtracting this equality from (95), we obtain

$$b \cdot \overline{v_\ell} \cdot a = v^{\sim}_{\ell-k-1}. \tag{96}$$

The left hand side of this equality is a product of three invertible elements (since b, $\overline{v_\ell}$ and a are invertible), and thus itself invertible. Hence, the right hand side is invertible as well. In other words, $v^{\sim}_{\ell-k-1}$ is invertible.

Taking reciprocals on both sides of (96), we now obtain $\overline{b \cdot \overline{v_\ell} \cdot a} = \overline{v^{\sim}_{\ell-k-1}}$. Hence,

$$\overline{v^{\sim}_{\ell-k-1}} = \overline{b \cdot \overline{v_\ell} \cdot a} = \overline{a} \cdot v_\ell \cdot \overline{b} \qquad \text{(by Proposition 2.3 } \textbf{(c)}\text{)}\,.$$

Thus,

$$a \cdot \overline{v^{\sim}_{\ell-k-1}} \cdot b = \underbrace{a \cdot \overline{a}}_{=1} \cdot v_\ell \cdot \underbrace{\overline{b} \cdot b}_{=1} = v_\ell.$$

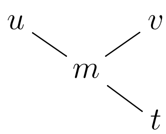
In other words,

$$v_\ell = a \cdot \overline{v^{\sim}_{\ell-k-1}} \cdot b = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b \qquad \text{(since } \ell - k - 1 = \ell - i - j + 1\text{)}\,.$$

Thus, $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$ is proved in Case 5.

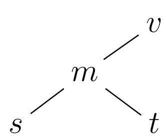
The arguments required to prove $v_\ell = a \cdot \overline{v^{\sim}_{\ell-i-j+1}} \cdot b$ in the Cases 3, 4 and 6 are similar to the one we have used in Case 5, but simpler in some ways. The specific differences are as follows:

- In Case 3, we have $s \notin P$. The "neighborhood" of m thus looks as follows:



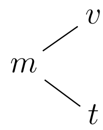
  (instead of looking as in (84)). This necessitates some changes to the proof; in particular, all addends that involve s or $s^{\sim}$ in any way need to be removed, along with the equality (86).

- In Case 6, we have $u \notin P$. The "neighborhood" of m thus looks as follows:



  (instead of looking as in (84)). This necessitates some changes to the proof; in particular, all addends that involve u or $u^{\sim}$ in any way need to be removed, along with the equality (89). (Subtraction is no longer required in this case.)

- In Case 4, we have $s \notin P$ and $u \notin P$. The "neighborhood" of m thus looks as follows:



(instead of looking as in (84)). This necessitates some changes to the proof; in particular, all addends that involve u or $u^{\sim}$ or s or $s^{\sim}$ in any way need to be removed, along with the equalities (86) and (89).

Thus, we have proved the equality $v_\ell = a \cdot \overline{v_{\ell-i-j+1}^{\sim}} \cdot b$ in all six Cases 1, 2, 3, 4, 5 and 6. Hence, this equality always holds. In other words, (27) holds for $x = v$. This completes the induction step. Thus, (27) is proved by induction. In other words, Theorem 4.8 is proven. □

As we have already seen (in Section 5), this entails that Theorem 4.7 is proven as well.

12. The case of a semiring

An attentive reader may have noticed that nowhere in the definitions of v-toggles and birational rowmotion do any subtraction sign appear. This means that all these definitions can be extended to the case when $\mathbb{K}$ is not a ring but a *semiring*.

A *semiring* is a set $\mathbb{K}$ equipped with a structure of an abelian semigroup $(\mathbb{K}, +)$ and the structure of a (not necessarily abelian) monoid $(\mathbb{K}, \cdot, 1)$ such that the distributive laws $(a+b)\,c = ac + bc$ and $a\,(b+c) = ab + ac$ are satisfied (where we use the shorthand notation xy for $x \cdot y$). Some standard concepts defined for rings can be straightforwardly generalized to semirings; in particular, any nonempty finite family $(a_i)_{i \in I}$ of elements of a semiring $\mathbb{K}$ has a well-defined sum $\sum_{i \in I} a_i$. Definition 2.2, too, applies verbatim to the case when $\mathbb{K}$ is a semiring instead of a ring. Thus, the definition of a v-toggle (Definition 3.12) and the definition of birational rowmotion (Definition 3.18) can be applied to a semiring $\mathbb{K}$ as well. We thus can wonder:

Question 12.1. Do twisted periodicity (Theorem 4.7) and reciprocity (Theorem 4.8) still hold if $\mathbb{K}$ is not a ring but merely a semiring?

If we assume that $\mathbb{K}$ is commutative, then the answer to this question is positive, for fairly simple general reasons (see [GriRob16, Remark 10]). However, no such general reasoning helps for noncommutative $\mathbb{K}$. Indeed, there are subtraction-free identities involving inverses that hold for all rings but fail for some semirings. One example is the identity $a \cdot \overline{a+b} \cdot b = b \cdot \overline{a+b} \cdot a$ from Proposition 2.4 **(a)**: David Speyer has constructed an example of a semiring $\mathbb{K}$ and two elements a and b of $\mathbb{K}$ such that $a + b$ is invertible (actually, $a + b = 1$ in his example), but this identity does not hold. See [Speyer21] for details.

Of course, this does not mean that the answer to Question 12.1 is negative; we are, in fact, inclined to suspect that the question has a positive answer. Our proofs of Lemma 10.1 and Lemma 10.2 apply in the semiring setting (i.e., when $\mathbb{K}$ is a semiring rather than a ring) without any need for changes; thus, Theorem 4.8 holds over any semiring $\mathbb{K}$ at least in the case when one of i and j is 1. Unfortunately, subtraction is used in the proof of Theorem 4.8, and we have so far been unable to excise it from the argument. (With a bit of thought, we can convince ourselves that subtraction is actually unnecessary if $p = 2$ or $q = 2$, so the first interesting case is obtained for $P = [3] \times [3]$.)

13. Other posets: conjectures and results

We now proceed to discuss the behavior of R on some other families of posets P. We no longer use the notations introduced in Section 6.

13.1. The Δ and ∇ triangles

When $p = q$, the $p \times q$-rectangle $[p] \times [q]$ becomes a square. By cutting this square in half along its horizontal axis, we obtain two triangles:

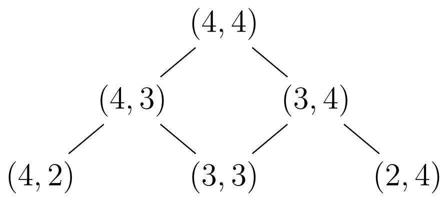
Definition 13.1. Let p be a positive integer. Define two subsets $\Delta(p)$ and $\nabla(p)$ of the $p \times p$-rectangle $[p] \times [p]$ by

$$\Delta(p) = \{(i,k) \in [p] \times [p] \ \mid \ i + k > p + 1\};$$
$$\nabla(p) = \{(i,k) \in [p] \times [p] \ \mid \ i + k < p + 1\}.$$

Each of these two subsets $\Delta(p)$ and $\nabla(p)$ inherits a poset structure from $[p] \times [p]$. In the following, we will consider $\Delta(p)$ and $\nabla(p)$ as posets using these structures.

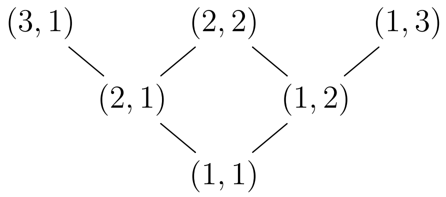
The Hasse diagrams of these posets $\Delta(p)$ and $\nabla(p)$ look like triangles; if we draw $[p] \times [p]$ as agreed in Convention 4.4, then $\Delta(p)$ is the "upper half" of the square $[p] \times [p]$, whereas $\nabla(p)$ is the "lower half" of this square.

Example 13.2. Here is the Hasse diagram of the poset $\Delta(4)$:



.

Here, on the other hand, is the Hasse diagram of the poset $\nabla(4)$:



.

Note that $\Delta(p) = \varnothing$ when $p = 1$.

Computations with SageMath [S+09] for $p = 3$ have made us suspect a periodicity-like phenomenon similar to Theorem 4.7:

Conjecture 13.3 (periodicity conjecture for **$\boldsymbol{\Delta}$**-triangle)**.** Let $p \geqslant 2$ be an integer. Assume that P is the poset $\Delta(p)$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^p f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Let $x \in \widehat{P}$. We define an element $x' \in \widehat{P}$ as follows:

- If $x = 0$ or $x = 1$, then we set $x' := x$.
- Otherwise, we write x in the form $x = (i, j)$, and we set $x' := (j, i)$.

Then, a and b are invertible, and we have

$$(R^p f)(x) = a\overline{b} \cdot f(x') \cdot \overline{a}b.$$

Conjecture 13.4 (periodicity conjecture for **$\boldsymbol{\nabla}$**-triangle)**.** The same holds if $P = \nabla(p)$ instead of $P = \Delta(p)$.

If true, these two conjectures would generalize [GriRob15, Theorem 65], where $\mathbb{K}$ is commutative.

13.2. The "right half" triangle

We can also cut the square $[p] \times [p]$ along its vertical axis:

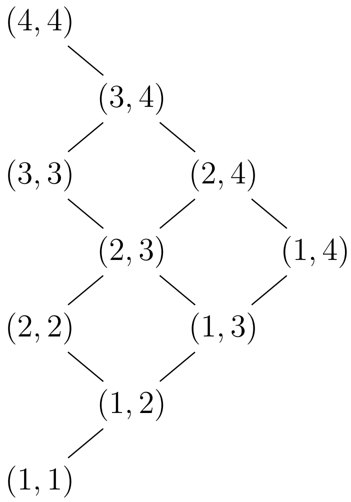
Definition 13.5. Let p be a positive integer. Define a subset $\operatorname{Tria}(p)$ of the $p \times p$-rectangle $[p] \times [p]$ by

$$\operatorname{Tria}(p) := \{(i,k) \in [p] \times [p] \mid i \leqslant k\}.$$

This subset $\operatorname{Tria}(p)$ inherits a poset structure from $[p] \times [p]$.

The Hasse diagram of this poset $\operatorname{Tria}(p)$ has the shape of a triangle; if we draw $[p] \times [p]$ as agreed in Convention 4.4, then $\operatorname{Tria}(p)$ is the "right half" of the square $[p] \times [p]$.

Example 13.6. Here is the Hasse diagram of the poset $\operatorname{Tria}(4)$:



.

The inequality $i \leqslant k$ in Definition 13.5 could just as well be replaced by the reverse inequality $i \geqslant k$; the resulting poset would be isomorphic to $\operatorname{Tria}(p)$. But we have to agree on something.

Now, we again suspect a periodicity-like phenomenon:

Conjecture 13.7 (periodicity conjecture for "right half" triangle)**.** Let p be a positive integer. Assume that P is the poset $\operatorname{Tria}(p)$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^{2p} f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Let $x \in \widehat{P}$. Then, a and b are invertible, and we have

$$\left(R^{2p} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a} b.$$

If true, this conjecture would generalize [GriRob15, Theorem 58], where $\mathbb{K}$ is commutative.

In a sense, we can "almost" prove Conjecture 13.7: Namely, the proof of its commutative case ([GriRob15, Theorem 58]) given in [GriRob15] can be adapted to the case of a general ring $\mathbb{K}$, as long as the number 2 is invertible in $\mathbb{K}$. The latter condition has all the earmarks of a technical assumption that should not matter for the validity of the result;

unfortunately, however, we are not aware of a rigorous argument that would allow us to dispose of such an assumption in the noncommutative case.

13.3. Trapezoids

Nathan Williams's conjecture [GriRob15, Conjecture 75], too, seems to extend to the noncommutative setting:

Conjecture 13.8 (periodicity conjecture for the trapezoid)**.** Let p be an integer > 1. Let $s \in \mathbb{N}$. Assume that P is the subposet

$$\{(i, k) \in [p] \times [p] \ \mid \ i + k > p + 1 \text{ and } i \leqslant k \text{ and } k \geqslant s\}$$

of $[p] \times [p]$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labeling such that $R^p f \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Let $x \in \widehat{P}$. Then, a and b are invertible, and we have

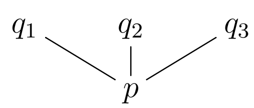
$$(R^p f)(x) = a\overline{b} \cdot f(x) \cdot \overline{a} b.$$

Again, this has been verified using SageMath for certain values of p and s and some randomly chosen $\mathbb{K}$-labelings with $\mathbb{K} = \mathbb{Q}^{3\times 3}$. Even for commutative $\mathbb{K}$, a proof is yet to be found, although significant advances have been recently made (see [Johnso23, Chapter 4][50]).

13.4. Ill-behaved posets

The above results and conjectures may suggest that every finite poset P for which birational rowmotion R has finite order when $\mathbb{K}$ is commutative must also satisfy a similar (if slightly more complicated) property when $\mathbb{K}$ is noncommutative. In particular, one might expect that if some positive integer m satisfies $R^m = \mathrm{id}$ (as rational maps) for all fields $\mathbb{K}$, then $R^m f = f$ should also hold for all noncommutative rings $\mathbb{K}$ and all $\mathbb{K}$-labelings $f \in \mathbb{K}^{\widehat{P}}$ that satisfy $f(0) = f(1) = 1$ (the latter condition ensures, e.g., that the $a\overline{b}$ and $\overline{a}b$ factors in Theorem 4.7 can be removed). However, this expectation is foiled by the following example:

Example 13.9. Let P be the four-element poset $\{p, q_1, q_2, q_3\}$ with order relation defined by setting $p < q_i$ for each $i \in \{1, 2, 3\}$. This poset has Hasse diagram



.

It is known (see [GriRob16, Example 18] or [GriRob16, Corollary 76]) that the birational rowmotion R of this poset P satisfies $R^6 = \mathrm{id}$ (as rational maps) if $\mathbb{K}$ is a field. In

[50] See also [DWYWZ20] for a proof on the level of order ideals.

other words, if $\mathbb{K}$ is a field, and if $f \in \mathbb{K}^{\widehat{P}}$ is a $\mathbb{K}$-labeling such that $R^6 f \neq \bot$, then $R^6 f = f$. But nothing like this holds when $\mathbb{K}$ is a noncommutative ring. For instance, if we let $\mathbb{K}$ be the matrix ring $\mathbb{Q}^{2\times 2}$, and if we define a $\mathbb{K}$-labeling $f \in \mathbb{K}^{\widehat{P}}$ by

$$f(0) = I_2 \qquad \text{(the identity matrix in } \mathbb{K}\text{)},$$
$$f(1) = I_2, \qquad f(p) = I_2, \qquad f(q_1) = I_2,$$
$$f(q_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f(q_3) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then $R^m f$ is distinct from f (and also distinct from $\bot$) for all positive integers m.

Proof sketch. Let $\mathbb{K}$ be the matrix ring $\mathbb{Q}^{2\times 2}$. For any row vector $(y, z) \in \mathbb{Q}^2$, we define a $\mathbb{K}$-labeling $f_{(y,z)} \in \mathbb{K}^{\widehat{P}}$ by setting

$$f_{(y,z)}(0) = f_{(y,z)}(1) = f_{(y,z)}(p) = I_2;$$
$$f_{(y,z)}(q_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad f_{(y,z)}(q_1) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}; \qquad f_{(y,z)}(q_3) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

Using direct computation (by hand or using SageMath [S+09]), we can see the following:

Claim 1: For any $y, z \in \mathbb{Q}$, we have $R^6 f_{(y,z)} = f_{(y',z')}$, where

$$y' := \frac{5y + 4z}{9} \qquad \text{and} \qquad z' := \frac{4y + 5z}{9}.$$

We define a $\mathbb{Q}$-linear map $\Phi : \mathbb{Q}^2 \to \mathbb{Q}^2$ that sends each row vector $(y, z) \in \mathbb{Q}^2$ to (y', z'), where y' and z' are as in Claim 1. Then, Claim 1 says that $R^6 f_v = f_{\Phi(v)}$ for each $v \in \mathbb{Q}^2$. Thus, for each $v \in \mathbb{Q}^2$ and each $i \in \mathbb{N}$, we have $R^{6i} f_v = f_{\Phi^i(v)}$.

However, the endomorphism Φ of $\mathbb{Q}^2$ is diagonalizable with eigenvalues 1 and $\dfrac{1}{9}$. Hence, if a vector $(y, z) \in \mathbb{Q}^2$ satisfies $y \neq z$, then its iterative images $\Phi^0(y, z)$, $\Phi^1(y, z)$, $\Phi^2(y, z)$, ... are (pairwise) distinct. Therefore, if $y, z \in \mathbb{Q}^2$ satisfy $y \neq z$, then the $\mathbb{K}$-labelings $R^{6i} f_{(y,z)} = f_{\Phi^i(y,z)}$ are (pairwise) distinct and thus, in particular, distinct from $f_{(y,z)}$. Therefore, in this case, $R^m f_{(y,z)}$ is distinct from $f_{(y,z)}$ for any positive integer m (because if we had $R^m f_{(y,z)} = f_{(y,z)}$ for some m, then we would also have $R^{6m} f_{(y,z)} = f_{(y,z)}$). In particular, by taking $y = 0$ and $z = 1$, we obtain the specific labeling constructed in Example 13.9. □

Example 13.10. Let P be the four-element poset $\{p_1, p_2, q_1, q_2\}$ with order relation defined by setting $p_i < q_j$ for each i, j. It follows from [GriRob16, Proposition 74 **(b)** and Proposition 61] that the birational rowmotion R of this poset P satisfies $R^6 = \mathrm{id}$ (as rational maps) if $\mathbb{K}$ is a field. On the other hand, if $\mathbb{K}$ is the matrix ring $\mathbb{Q}^{2\times 2}$, then we can easily find a $\mathbb{K}$-labeling f of P such that $R^m f \neq f$ for all $1 \leqslant m \leqslant 10\,000$ (and probably for all positive m, but we have not verified this formally), despite $f(0)$ and $f(1)$ both being the identity matrix I_2.

14. A note on general posets

We finish with some curiosities. While Theorem 4.8 is specific to rectangles, its $(i,j) = (1,1)$ case can be generalized to arbitrary finite posets P in the following form:

Proposition 14.1. Let P be any finite poset. Let $f \in \mathbb{K}^{\widehat{P}}$ be a labeling of P such that $Rf \neq \bot$. Let $a = f(0)$ and $b = f(1)$. Then,

$$b \cdot \sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} \cdot a = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u), \tag{97}$$

assuming that the inverses $\overline{(Rf)(u)}$ on the left-hand side are well-defined.

Proof. Even though we are not requiring P to be a rectangle, we shall use some of the notations introduced in Section 6. Specifically, we shall use the notation x_ℓ defined in (24), the notion of a "path", and the notations A_ℓ^v, $\forall_\ell^v$, $A_\ell^{\mathbf{p}}$, $\forall_\ell^{\mathbf{p}}$, $A_\ell^{u\to v}$ and $\forall_\ell^{u\to v}$ defined afterwards. Every $u \in \widehat{P}$ satisfies

$$\begin{aligned} u_0 &= \Big(\underbrace{R^0 f}_{\substack{=f\\ (\text{since } R^0 = \mathrm{id})}}\Big)(u) \qquad \text{(by (24))} \\ &= f(u) \end{aligned} \tag{98}$$

and

$$\begin{aligned} u_1 &= \Big(\underbrace{R^1}_{=R} f\Big)(u) \qquad \text{(by (24))} \\ &= (Rf)(u). \end{aligned} \tag{99}$$

We assume that the inverses $\overline{(Rf)(u)}$ on the left-hand side of (97) are well-defined (since the claim of Proposition 14.1 requires this). In other words, we assume that

$$(Rf)(u) \text{ is invertible for every } u \in \widehat{P} \text{ satisfying } u \gtrdot 0. \tag{100}$$

It is easy to see that Proposition 14.1 holds when $P = \varnothing$ [51]. Thus, we WLOG

[51] *Proof.* Assume that $P = \varnothing$. Thus, $\widehat{P} = \{0,1\}$ with $0 < 1$. Hence, the only $u \in \widehat{P}$ satisfying $u \gtrdot 0$ is 1. Therefore, $\sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} = \overline{(Rf)(1)} = \overline{b}$ (since Proposition 3.21 yields $(Rf)(1) = f(1) = b$). Moreover, the only $u \in \widehat{P}$ satisfying $u \lessdot 1$ is 0 (since $\widehat{P} = \{0,1\}$ with $0 < 1$). Thus, $\sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u) = f(0) = a$. Now,

$$b \cdot \underbrace{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)}}_{=\overline{b}} \cdot a = \underbrace{b \cdot \overline{b}}_{=1} \cdot a = a = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u).$$

Hence, Proposition 14.1 is proved (under the assumption that $P = \varnothing$).

assume that $P \neq \varnothing$. Hence, Lemma 3.25 yields that $f(1)$ is invertible. In other words, b is invertible (since $b = f(1)$).

Furthermore, we can easily see that a is invertible[52].

In Remark 7.7, we have observed that Corollary 7.5, Proposition 7.2 and parts **(a)** and **(b)** of Theorem 7.6 hold for our poset P (even though P is not necessarily a rectangle). In particular, Corollary 7.5 (applied to $\ell = 1$, $u = 1$ and $v = 0$) yields that

$$V_1^{1\to 0} = A_0^{1\to 0} \tag{102}$$

(since $1 \geqslant 1$ and $R^1 f = Rf \neq \bot$).

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[52] *Proof.* Proposition 1.9 **(a)** yields that the poset P has a minimal element. Consider such an element, and denote it by p. Then, p is a minimal element of P, and therefore satisfies $0 \lessdot p$ in $\widehat{P}$ (by Remark 3.3 **(a)**). Hence, $p > 0$ in $\widehat{P}$. Therefore, (100) (applied to $u = p$) yields that $(Rf)(p)$ is invertible. However, Proposition 3.20 (applied to $v = p$) yields

$$(Rf)(p) = \left(\sum_{\substack{u\in\widehat{P};\\ u\lessdot p}} f(u)\right) \cdot \overline{f(p)} \cdot \overline{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}}. \tag{101}$$

Thus, the two elements $f(p)$ and $\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}$ of $\mathbb{K}$ are invertible (since their inverses appear in (101)).

However, p is a minimal element of P. Thus, no $u \in P$ satisfies $u \lessdot p$. Therefore, the only $u \in \widehat{P}$ that satisfies $u \lessdot p$ is 0. Therefore, $\sum_{\substack{u\in\widehat{P};\\ u\lessdot p}} f(u) = f(0) = a$. Thus, we can rewrite the equality (101) as

$$(Rf)(p) = a \cdot \overline{f(p)} \cdot \overline{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}}.$$

Multiplying both sides of this equality by $\left(\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}\right) \cdot f(p)$ on the right, we obtain

$$\begin{aligned}(Rf)(p) \cdot \left(\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}\right) \cdot f(p) &= a \cdot \overline{f(p)} \cdot \underbrace{\overline{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}} \cdot \left(\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}\right)}_{=1} \cdot f(p) \\ &= a \cdot \underbrace{\overline{f(p)} \cdot f(p)}_{=1} = a.\end{aligned}$$

The left hand side of this equality is a product of three invertible elements of $\mathbb{K}$ (since the elements $(Rf)(p)$, $\sum_{\substack{u\in\widehat{P};\\ u\gtrdot p}} \overline{(Rf)(u)}$ and $f(p)$ are invertible), and thus itself must be invertible. Hence, the right hand side is invertible as well. In other words, a is invertible.

Now, $1 \geqslant 1$ and $R^1 f = Rf \neq \bot$. Hence, Theorem 7.6 **(a)** (applied to $\ell = 1$) shows that each $u \in P$ satisfies

$$u_1 = \overline{V_1^{1\to u}} \cdot b. \tag{103}$$

Hence, for each $u \in \widehat{P}$ satisfying $u \gtrdot 0$, we have

$$b \cdot \overline{(Rf)(u)} = V_1^{1\to u} \tag{104}$$

[53]. Thus,

$$\begin{aligned} b \cdot \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} \overline{(Rf)(u)} \cdot a &= \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} \underbrace{b \cdot \overline{(Rf)(u)}}_{\substack{=V_1^{1\to u}\\ (\text{by (104)})}} \cdot a \\ &= \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} V_1^{1\to u} \cdot a. \end{aligned} \tag{105}$$

However, $V_1^0 = 1$ (by definition of V_1^0). Now, (37) (applied to $\ell = 1$ and $s = 1$ and $t = 0$) yields

$$V_1^{1\to 0} = \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} V_1^{1\to u} \underbrace{V_1^0}_{=1} = \sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} V_1^{1\to u}. \tag{106}$$

---

[53] *Proof of (104):* Let $u \in \widehat{P}$ satisfy $u \gtrdot 0$. We must prove (104). We note that $(Rf)(u)$ is invertible (by (100)), so that $\overline{(Rf)(u)}$ is well-defined.

We are in one of the following two cases:

Case 1: We have $u = 1$.

Case 2: We have $u \neq 1$.

Let us first consider Case 1. In this case, we have $u = 1$. Thus, $(Rf)(u) = (Rf)(1) = f(1)$ (by Proposition 3.21). Hence, $(Rf)(u) = f(1) = b$. Thus, $b \cdot \overline{(Rf)(u)} = b \cdot \overline{b} = 1$.

However, the only path from 1 to 1 is the trivial path (1). Thus, $V_1^{1\to 1} = V_1^{(1)} = V_1^1 = 1$ (by the definition of V_1^1). Comparing this with $b \cdot \overline{(Rf)(u)} = 1$, we obtain $b \cdot \overline{(Rf)(u)} = V_1^{1\to 1}$. In other words, $b \cdot \overline{(Rf)(u)} = V_1^{1\to u}$ (since $1 = u$). Thus, (104) is proved in Case 1.

Let us now consider Case 2. In this case, we have $u \neq 1$. Also, we have $u > 0$ (since $u \gtrdot 0$), so that $u \neq 0$. Combining $u \in \widehat{P}$ with $u \neq 0$ and $u \neq 1$, we obtain $u \in \widehat{P} \setminus \{0, 1\} = P$. Hence, (103) yields $u_1 = \overline{V_1^{1\to u}} \cdot b$. In view of (99), we can rewrite this as $(Rf)(u) = \overline{V_1^{1\to u}} \cdot b$. Since $(Rf)(u)$ is invertible, we can take inverses on both sides of this equality. We thus obtain

$$\begin{aligned} \overline{(Rf)(u)} &= \overline{\overline{V_1^{1\to u}} \cdot b} = \overline{b} \cdot \underbrace{\overline{\overline{V_1^{1\to u}}}}_{=V_1^{1\to u}} \qquad \left(\text{since } \overline{V_1^{1\to u}} \text{ and } b \text{ are invertible}\right) \\ &= \overline{b} \cdot V_1^{1\to u}. \end{aligned}$$

Thus,

$$b \cdot \overline{(Rf)(u)} = \underbrace{b \cdot \overline{b}}_{=1} \cdot V_1^{1\to u} = V_1^{1\to u}.$$

Thus, (104) is proved in Case 2.

We have now proved (104) in both Cases 1 and 2. Hence, (104) always holds.

Thus, (105) becomes

$$b\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}}\overline{(Rf)(u)}\cdot a=\underbrace{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}}V_1^{1\to u}}_{\substack{=V_1^{1\to 0}\\ \text{(by (106))}}}\cdot a=\underbrace{V_1^{1\to 0}}_{\substack{=A_0^{1\to 0}\\ \text{(by (102))}}}\cdot a$$

$$=A_0^{1\to 0}\cdot a. \tag{107}$$

However, $R^{0+1}f=R^1f=Rf\neq\bot$. Hence, Theorem 7.6 **(b)** (applied to $\ell=0$) shows that each $u\in P$ satisfies

$$u_0=A_0^{u\to 0}\cdot a. \tag{108}$$

Hence, for each $u\in\widehat{P}$ satisfying $u\lessdot 1$, we have

$$f(u)=A_0^{u\to 0}\cdot a \tag{109}$$

[54]. Thus,

$$\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}}\underbrace{f(u)}_{\substack{=A_0^{u\to 0}\cdot a\\ \text{(by (109))}}}=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}}A_0^{u\to 0}\cdot a. \tag{110}$$

However, (34) (applied to $\ell=0$ and $s=1$ and $t=0$) yields

$$A_0^{1\to 0}=\underbrace{A_0^1}_{\substack{=1\\ \text{(by the definition of } A_0^1)}}\sum_{\substack{u\in\widehat{P};\\ 1\gtrdot u}}A_0^{u\to 0}=\sum_{\substack{u\in\widehat{P};\\ 1\gtrdot u}}A_0^{u\to 0}=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}}A_0^{u\to 0}$$

(since the condition "$1\gtrdot u$" under the summation sign is equivalent to "$u\lessdot 1$"). Thus,

$$A_0^{1\to 0}\cdot a=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}}A_0^{u\to 0}\cdot a=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}}f(u)$$

[54] *Proof of (109):* Let $u\in\widehat{P}$ satisfy $u\lessdot 1$. We must prove (109).

We are in one of the following two cases:

Case 1: We have $u=0$.

Case 2: We have $u\neq 0$.

Let us first consider Case 1. In this case, we have $u=0$. Thus, $f(u)=f(0)=a$.

However, the only path from 0 to 0 is the trivial path (0). Thus, $A_0^{0\to 0}=A_0^{(0)}=A_0^0=1$ (by the definition of A_0^0). From $u=0$, we obtain $A_0^{u\to 0}\cdot a=\underbrace{A_0^{0\to 0}}_{=1}\cdot a=a$. Comparing this with $f(u)=a$, we obtain $f(u)=A_0^{u\to 0}\cdot a$. Thus, (109) is proved in Case 1.

Let us now consider Case 2. In this case, we have $u\neq 0$. Also, we have $u<1$ (since $u\lessdot 1$), so that $u\neq 1$. Combining $u\in\widehat{P}$ with $u\neq 0$ and $u\neq 1$, we obtain $u\in\widehat{P}\setminus\{0,1\}=P$. Hence, (108) yields $u_0=A_0^{u\to 0}\cdot a$. In view of (98), we can rewrite this as $f(u)=A_0^{u\to 0}\cdot a$. Thus, (109) is proved in Case 2.

We have now proved (109) in both Cases 1 and 2. Hence, (109) always holds.

(by (110)). Therefore, (107) can be rewritten as

$$b \cdot \sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} \cdot a = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u).$$

Proposition 14.1 is thus proven. □

Proposition 14.2. Let P be any finite poset. Let $f \in \mathbb{K}^{\widehat{P}}$ be a labeling of P such that $Rf \neq \bot$ and $f(0) = f(1) = 1$. Then,

$$\sum_{\substack{u,v \in \widehat{P};\\ u \lessdot v}} (Rf)(u) \cdot \overline{(Rf)(v)} = \sum_{\substack{u,v \in \widehat{P};\\ u \lessdot v}} f(u) \cdot \overline{f(v)},$$

assuming that the inverses $\overline{(Rf)(v)}$ on the left-hand side are well-defined.

Proposition 14.2 is essentially saying that the sum $\sum_{\substack{u,v \in \widehat{P};\\ u \lessdot v}} f(u) \cdot \overline{f(v)}$ is an invariant under birational rowmotion R when $f(0) = f(1) = 1$. This is a noncommutative analogue of the conservation of the "superpotential" $\mathcal{F}_G(X)$ of an R-system ([GalPyl19, Proposition 5.2]). We do not know whether such invariants exist in the general case.

Proof of Proposition 14.2. We have $f(0) = f(1) = 1$. Thus, $1 = f(0)$ and $1 = f(1)$. Hence, Proposition 14.1 (applied to $a = 1$ and $b = 1$) yields

$$1 \cdot \sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} \cdot 1 = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u).$$

This obviously simplifies to

$$\sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u). \tag{111}$$

Proposition 3.21 yields $(Rf)(0) = f(0) = 1$. Also, from $f(1) = 1$, we obtain $\overline{f(1)} = \overline{1} = 1$.

Now, let $v \in P$. Then, Proposition 3.20 yields

$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P};\\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P};\\ u \gtrdot v}} \overline{(Rf)(u)}}.$$

Multiplying both sides of this equality by $\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}$ on the right, we obtain

$$\begin{aligned}
(Rf)(v)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)} &= \left(\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f(u)\right)\cdot\overline{f(v)}\cdot\underbrace{\overline{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}}\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}}_{=1} \\
&= \left(\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f(u)\right)\cdot\overline{f(v)} \\
&= \sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f(u)\cdot\overline{f(v)}. \qquad (112)
\end{aligned}$$

Forget that we fixed v. We thus have proved (112) for each $v\in P$.

Now,

$$\begin{aligned}
&\sum_{\substack{u,v\in\widehat{P};\\ u\lessdot v}} (Rf)(u)\cdot\overline{(Rf)(v)} \\
&= \sum_{\substack{u,v\in\widehat{P};\\ v\gtrdot u}} (Rf)(u)\cdot\overline{(Rf)(v)} \qquad \left(\begin{array}{c}\text{since the condition ``}u\lessdot v\text{''}\\ \text{is equivalent to ``}v\gtrdot u\text{''}\end{array}\right)\\
&= \underbrace{\sum_{\substack{v,u\in\widehat{P};\\ u\gtrdot v}}}_{=\sum_{v\in\widehat{P}}\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}} (Rf)(v)\cdot\overline{(Rf)(u)} \qquad \left(\begin{array}{c}\text{here, we have renamed the}\\ \text{summation indices } u \text{ and } v \text{ as } v \text{ and } u\end{array}\right)\\
&= \sum_{v\in\widehat{P}}\ \sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}} (Rf)(v)\cdot\overline{(Rf)(u)}\\
&= \sum_{v\in P\cup\{0,1\}}\ \sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}} (Rf)(v)\cdot\overline{(Rf)(u)} \qquad \left(\text{since } \widehat{P}=P\cup\{0,1\}\right)\\
&= \underbrace{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}} (Rf)(0)\cdot\overline{(Rf)(u)}}_{=(Rf)(0)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}}\overline{(Rf)(u)}} + \underbrace{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 1}} (Rf)(1)\cdot\overline{(Rf)(u)}}_{\substack{=(\text{empty sum})\\ (\text{since there exists no } u\in\widehat{P}\\ \text{satisfying } u\gtrdot 1)}} + \sum_{v\in P}\ \underbrace{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}} (Rf)(v)\cdot\overline{(Rf)(u)}}_{=(Rf)(v)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}}\\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addends}\\ \text{for } v=0 \text{ and for } v=1 \text{ from the sum}\end{array}\right)\\
&= (Rf)(0)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}}\overline{(Rf)(u)} + \underbrace{(\text{empty sum})}_{=0} + \sum_{v\in P}(Rf)(v)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}\\
&= \underbrace{(Rf)(0)}_{=1}\cdot\underbrace{\sum_{\substack{u\in\widehat{P};\\ u\gtrdot 0}}\overline{(Rf)(u)}}_{\substack{=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}} f(u)\\ (\text{by (111)})}} + \underbrace{\sum_{v\in P}(Rf)(v)\cdot\sum_{\substack{u\in\widehat{P};\\ u\gtrdot v}}\overline{(Rf)(u)}}_{\substack{=\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f(u)\cdot\overline{f(v)}\\ (\text{by (112)})}}\\
&= \sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}} f(u) + \sum_{v\in P}\ \sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f(u)\cdot\overline{f(v)}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\underbrace{\sum_{\substack{u,v\in\widehat{P};\\ u\lessdot v}}}_{=\sum_{v\in\widehat{P}}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}}} f\left(u\right)\cdot\overline{f\left(v\right)} \\
&=\sum_{v\in\widehat{P}}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)} \\
&=\sum_{v\in P\cup\{0,1\}}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)} \qquad \left(\text{since } \widehat{P}=P\cup\{0,1\}\right) \\
&=\underbrace{\sum_{\substack{u\in\widehat{P};\\ u\lessdot 0}} f\left(u\right)\cdot\overline{f\left(0\right)}}_{\substack{=(\text{empty sum})\\ (\text{since there exists no } u\in\widehat{P}\\ \text{satisfying } u\lessdot 0)}} + \sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}} f\left(u\right)\cdot\underbrace{\overline{f\left(1\right)}}_{=1} + \sum_{v\in P}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)} \\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have split off the addends}\\ \text{for } v=0 \text{ and for } v=1 \text{ from the sum}\end{array}\right) \\
&=\underbrace{(\text{empty sum})}_{=0} + \sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}} f\left(u\right) + \sum_{v\in P}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)} \\
&=\sum_{\substack{u\in\widehat{P};\\ u\lessdot 1}} f\left(u\right) + \sum_{v\in P}\sum_{\substack{u\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)},
\end{aligned}$$

we obtain

$$\sum_{\substack{u,v\in\widehat{P};\\ u\lessdot v}} \left(Rf\right)\left(u\right)\cdot\overline{\left(Rf\right)\left(v\right)} = \sum_{\substack{u,v\in\widehat{P};\\ u\lessdot v}} f\left(u\right)\cdot\overline{f\left(v\right)}.$$

This proves Proposition 14.2. □

References

[AyKlSc12] Arvind Ayyer, Steven Klee, Anne Schilling, *Combinatorial Markov chains on linear extensions*, Journal of Algebraic Combinatorics **39**, September 2013, DOI 10.1007/s10801-013-0470-9. Also available as arXiv:1205.7074v3.

[BerRet15] Arkady Berenstein, Vladimir Retakh, *Noncommutative marked surfaces*, Advances in Mathematics **328** (2018), pp. 1010–1087. Also available as arXiv:1510.02628v4.

[BrSchr74] Andries E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centr. report ZW24, Amsterdam (Jun. 1974).

[DefKra21] Colin Defant, Noah Kravitz, *Friends and strangers walking on graphs*, Combinatorial Theory **1** (2021), #6.

[DWYWZ20] Quang Vu Dao, Julian Wellman, Calvin Yost-Wolff, Sylvester W. Zhang, *Rowmotion Orbits of Trapezoid Posets*, The Electronic Journal of Combinatorics **29**, Issue 2 (2022), #P2.29.

[EinPro13] David Einstein, James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, Algebraic Combinatorics **4** (2021) no. 2, pp. 201–224.

[Etienn84] Gwihen Etienne, *Linear extensions of finite posets and a conjecture of G. Kreweras on permutations*, Discrete Mathematics **52** (1984), Issue 1, pp. 107–111.

[GalPyl19] Pavel Galashin, Pavlo Pylyavskyy, *R-systems*, Selecta Mathematica **25** (2019), Article 22.

[GonKon21] Alexander Goncharov, Maxim Kontsevich, *Spectral description of non-commutative local systems on surfaces and non-commutative cluster varieties*, 2021, arXiv:2108.04168v3.

[GriRob14] Darij Grinberg, Tom Roby, *Iterative properties of birational rowmotion*, 15 August 2022, arXiv:1402.6178v7.

[GriRob15] Darij Grinberg, Tom Roby, *Iterative Properties of Birational Rowmotion II: Rectangles and Triangles*, Electronic Journal of Combinatorics **22** (2015), Paper #P3.40.

[GriRob16] Darij Grinberg, Tom Roby, *Iterative Properties of Birational Rowmotion I: Generalities and Skeletal Posets*, Electronic Journal of Combinatorics **23** (2016), Paper #P1.33.

[GriRob23] Darij Grinberg, Tom Roby, *Birational rowmotion over noncommutative rings*, Séminaire Lotharingien de Combinatoire, **89B**.50 (2023), 12 pp. (FPSAC 2023)..

[Gyoja86] Akihiko Gyoja, *A q-analogue of Young symmetrizer*, Osaka J. Math. **23** (1986), pp. 841–852.

[IyuShk14] Natalia Iyudu, Stanislav Shkarin, *The proof of the Kontsevich periodicity conjecture on noncommutative birational transformations*, Duke Math. J. **164**(13), pp. 2539–2575 (1 October 2015).

[Johnso23] Joseph William Norman Johnson, *Problems in Dynamical Algebraic Combinatorics and Algebraic Statistics*, dissertation at North Carolina State University, 2023.

[JohLiu22] Joseph Johnson, Ricky Ini Liu, *Birational rowmotion and the octahedron recurrence*, arXiv:2204.04255v1.

[JosRob20] Michael Joseph, Tom Roby, *Birational and noncommutative lifts of antichain toggling and rowmotion*, Algebraic Combinatorics **3** (2020), issue 4, pp. 955–984.

[JosRob21] Michael Joseph, Tom Roby, *A birational lifting of the Stanley-Thomas word on products of two chains*, Discrete Mathematics & Theoretical Computer Science **23** (2021) no. 1, Combinatorics (August 18, 2021) dmtcs:8367.

[Kirill00] Anatol N. Kirillov, *Introduction to tropical combinatorics*, Physics and combinatorics: Proceedings of the Nagoya 2000 International Workshop, held 21 - 26 August 2000 in Nagoya University. Edited by Anatol N Kirillov (Nagoya University) & Nadejda Liskova. Published by World Scientific Publishing Co. Pte. Ltd., 2001. ISBN #9789812810007, pp. 82–150.

[KirBer95] Anatol N. Kirillov, Arkadiy D. Berenstein, *Groups generated by involutions, Gelfand-Tsetlin patterns, and combinatorics of Young tableaux*, Algebra i Analiz **7** (1995), issue 1, pp. 92–152. A preprint is available at: `https://pages.uoregon.edu/arkadiy/bk1.pdf`

[MusRob17] Gregg Musiker, Tom Roby, *Paths to Understanding Birational Rowmotion on Products of Two Chains*, Algebraic Combinatorics **2** (2019) no. 2, pp. 275–304.

[Naatz00] Michael Naatz, *The graph of linear extensions revisited*, Siam J. Discrete Math. **13** (2000), No. 3, pp. 354–369.

[ProRob13] James Propp and Tom Roby, *Homomesy in products of two chains*, DMTCS proc. FPSAC 2013, `https://www2.math.uconn.edu/~troby/ceFPSAC.pdf`

[ProRob14] James Propp and Tom Roby, *Homomesy in products of two chains*, The Electronic Journal of Combinatorics **22**, Issue 3 (2015), Paper #P3.4. A preprint appeared as `arXiv:1310.5201v6`.

[Roby15] Tom Roby, *Dynamical Algebraic Combinatorics and the Homomesy Phenomenon*, in: A. Beveridge, J. Griggs, L. Hogben, G. Musiker, P. Tetali, eds., Recent Trends in Combinatorics (IMA Volume in Mathematics and its Applications), Springer, 2016. `https://www2.math.uconn.edu/~troby/homomesyIMA2015Revised.pdf`

[Rupel17] Dylan Rupel, *Rank Two Non-Commutative Laurent Phenomenon and Pseudo-Positivity*, Algebraic Combinatorics **2** (2019), issue 6, pp. 1239–1273.

[S+09] W. A. Stein et al., *Sage Mathematics Software (Version 9.4)*, The Sage Development Team, 2021, `http://www.sagemath.org`.

[Sage08] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, 2008. `http://combinat.sagemath.org`

[Speyer21] David E Speyer, *MathOverflow post #401273*, 2021, https://mathoverflow.net/q/401273 .

[Stan86] Richard P. Stanley, *Two Poset Polytopes*, Discrete & Computational Geometry, 1986, Volume 1, Issue 1, pp. 9–23.

[Stan11] Richard P. Stanley, *Enumerative Combinatorics, volume 1*, Second edition, version of 15 July 2011. Available at http://math.mit.edu/~rstan/ec/. See http://math.mit.edu/~rstan/ec/ for errata.

[StWi11] Jessica Striker, Nathan Williams, *Promotion and Rowmotion*, European Journal of Combinatorics 33 (2012), pp. 1919–1942, DOI 10.1016/j.ejc.2012.05.003.
https://doi.org/10.1016/j.ejc.2012.05.003

[ThoWil19] Hugh Thomas and Nathan Williams, *Rowmotion in slow motion*, Proceedings of the London Mathematical Society, **119**(5) (2019) 1149–1178.

[Volk06] Alexandre Yu. Volkov, *On the Periodicity Conjecture for Y-systems*, Communications Math. Physics 276 (2007), pp. 509–517, DOI 10.1007/s00220-007-0343-y.
A preprint of this paper is also available under the name *On Zamolodchikov's Periodicity Conjecture* as arXiv:hep-th/0606094v1:
http://arxiv.org/abs/hep-th/0606094v1