On the representation theory of finite $\mathcal{J}$-trivial monoids

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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "On the representation theory of finite $\mathcal{J}$-trivial monoids" by the numbers under which they appear in this article (specifically, in its version of 4 March 2011, posted on arXiv under the identifier arXiv:1010.3455v3).

6. Errata

- **Page 3:** Replace "and illustrates them" by "and illustrate them".
- **Page 4:** "satistfies" should be "satisfies".
- **Page 4:** Here it is claimed that "Groups are an example of a variety of monoids, as are all of the classes of monoids described in this paper". The latter part of this sentence is not completely true: The class of all ordered monoids with 1 on top is not a variety, since a quotient of such a monoid can fail to be in this class (unless, I assume, you regard it as a variety of ordered monoids). An example is provided by Examples 2.4 and 2.5 in this very paper (since the monoid M there is $\mathcal{J}$-trivial and thus is a quotient of an ordered monoid with 1 on top, but itself is not ordered).
- **Page 5:** Replace "1 is the largest element of these (pre)-orders" by "in every of these preorders, we have $x \leq 1$ for every $x \in M$". (Speaking of "the largest element" is mildly ambiguous, because a preorder can have several elements each of which is "the largest".)
- **Page 5, Proposition 2.2:** Replace "the partial order $\leq$ is finer than $\leq_{\mathcal{K}}$" by "the partial order $\leq_{\mathcal{K}}$ is finer than $\leq$".
- **Page 6, §2.1:** Here it is claimed that "any $\mathcal{R}$-trivial monoid can be represented as a monoid of regressive functions on some finite poset P". This is correct if one uses the convention that functions act on values **from the right** (i.e., the value of an element x under a function f is written $x.f$ rather than $f(x)$) and compose accordingly (i.e., the composition fg of two functions f and g is the function which sends every x to $(x.f).g$). This is a nonstandard convention, and ought to be explained early on in the paper. (It is explained in §2.5, but it is used in §2.1 already.)

- **Page 8, §2.5:** In the sentence starting with "When P is a chain on N elements", as well as in the next sentence, replace each "N" by "n".
- **Page 11, proof of Lemma 3.6:** This proof can be simplified:

  Assume that $e \leq_{\mathcal{J}} y$. Then, $e = ayb$ for some $a, b \in M$. Applying Lemma 3.5 to yb instead of b, we find $e = eyb$. Hence, $e \leq_{\mathcal{J}} ey \leq_{\mathcal{J}} e$, which entails $e = ey$ since M is $\mathcal{J}$-trivial. Moreover, applying Lemma 3.5 to ay instead of a, we find $e = aye$. Hence, $e \leq_{\mathcal{J}} ye \leq_{\mathcal{J}} e$, thus $e = ye$. The converse implications hold by the definition of $\leq_{\mathcal{J}}$.
- **Page 12, proof of Corollary 3.8:** It also needs to be proved that $\mathcal{C} \subseteq \operatorname{rad} \mathbb{K}M$. This is a consequence of Corollary 3.7: Indeed, Corollary 3.7 shows that the quotient algebra $\mathbb{K}M / \operatorname{rad} \mathbb{K}M$ is commutative (being isomorphic to the semigroup algebra $\mathbb{K}E(M)$ of the commutative monoid $(E(M), \star)$), and thus the commutator ideal $\mathcal{C}$ of $\mathbb{K}M$ must be annihilated by the canonical projection $\mathbb{K}M \to \mathbb{K}M / \operatorname{rad} \mathbb{K}M$. But this means that this ideal $\mathcal{C}$ is contained in $\operatorname{rad} \mathbb{K}M$.
- **Page 12, Example 3.9:** Replace "$\mathbb{C}$" by "$\mathbb{K}$" throughout this example.
- **Page 12, Example 3.9:** Replace "algebra morphisms from $H_0(W) \to H_0\left(W_{I \setminus \{i\}}\right)$" by "algebra morphisms from $\mathbb{K}H_0(W) \to \mathbb{K}H_0\left(W_{I \setminus \{i\}}\right)$".
- **Page 13, (3.7):** It is worth saying that g_e is understood to lie in $\mathbb{K}M$ (not in $\mathbb{K}M / \operatorname{rad} \mathbb{K}M$).
- **Page 14:** "For any number a denote by $\lceil a \rceil$ the smallest integer larger than a." should be "For any number a denote by $\lceil a \rceil$ the smallest integer larger than or equal to a.". (Otherwise, the statement "there is an N such that $u_N = 1$" in the proof of Proposition 3.12 does not hold.)
- **Page 14, proof of Proposition 3.12:** After "Define $u_{n+1} = \left\lceil \frac{u_n}{2} \right\rceil$", add ", so that Lemma 3.14 yields $(y_n (y_n - 1))^{u_n} = 0$ by induction on n".
- **Page 15, proof of Proposition 3.15:** In the first sentence ("First it is clear that the f_i are pairwise orthogonal idempotents"), remove the word "idempotents". Indeed, the idempotency of the f_i will only be shown later.

  Namely, the idempotency of the f_i follows from the equality (3.17) proved in the next paragraph. Indeed, this equality shows that the element $\phi\left(\left(1 - \sum_{j<i} f_i\right) g_j \left(1 - \sum_{j<i} f_i\right)\right)$ is idempotent (since g_j is idempotent). Therefore, $P\left(\left(1 - \sum_{j<i} f_i\right) g_j \left(1 - \sum_{j<i} f_i\right)\right)$ must be idempotent as well (since $P(x)$ is idempotent whenever $\phi(x)$ is idempotent).

- **Page 15, proof of Proposition 3.15:** In the last sentence, the claim that "the coefficient of e_i in f_i must be 1" doesn't look that obvious to me. I understand why this coefficient equals the coefficient of e_i in $P(g_i)$ (because we have $\left(1 - \sum_{j<i} f_i\right) g_j \left(1 - \sum_{j<i} f_i\right) \equiv g_i$ modulo the ideal span $\{x \mid x <_{\mathcal{J}} e_i\}$) and thus also equals the coefficient of e_i in $P(e_i)$ (since $g_i \equiv e_i$ modulo the same ideal). But in order to see that the latter coefficient is 1, I need to use the fact that $P(x) = x$ whenever x is idempotent. This is itself quite easy, but should be stated as a lemma.

- **Page 16, Proposition 3.17:** I think "decreasing" should be "(weakly) increasing" both times here.

- **Page 17, proof of Theorem 3.23:** "are two ideals" $\to$ "are two right ideals". (Or is there a reason why they are two-sided ideals?)

- **Page 21:** "An algebra is called *split basic*" $\to$ "An algebra A is called *split basic*".

7. Addenda and remarks

7.1. Page 6, §2.1.

Let us prove some of the claims that are left unproven in this section.

First of all, the following very easy fact is used without proof:

Lemma 7.1. Let M be a monoid[1]. Let $x \in M$. Let $N \in \mathbb{N}$ be such that $x^N = x^{N+1}$. Then, $x^N = x^{N+1} = x^{N+2} = \cdots$.

Proof of Lemma 7.1. We have

$$x^{N+m} = x^{N+m+1} \qquad \text{for every } m \in \mathbb{N}. \tag{1}$$

[2] Combining these equalities for all $m \in \mathbb{N}$, we obtain $x^N = x^{N+1} = x^{N+2} = \cdots$. This proves Lemma 7.1. □

[1]We follow the conventions of the paper. In particular, "monoid" means "finite monoid" for us.

[2]*Proof of (1):* We shall prove (1) by induction over m:

Induction base: We have $x^{N+0} = x^N = x^{N+1} = x^{N+0+1}$ (since $N+1 = N+0+1$). In other words, (1) holds for $m = 0$. This completes the induction base.

Induction step: Let μ be a positive integer. Assume that (1) holds for $m = \mu - 1$. We now need to prove that (1) holds for $m = \mu$ as well.

We know that (1) holds for $m = \mu - 1$. In other words, $x^{N+(\mu-1)} = x^{N+(\mu-1)+1}$. Now, Next, we shall prove a fact which is almost obvious and is often used without explicit mention:

Lemma 7.2. Let M be a monoid.

(a) If M is $\mathcal{J}$-trivial, then M is $\mathcal{R}$-trivial.
(b) If M is $\mathcal{J}$-trivial, then M is $\mathcal{L}$-trivial.
(c) If M is $\mathcal{R}$-trivial, then M is $\mathcal{H}$-trivial.
(d) If M is $\mathcal{L}$-trivial, then M is $\mathcal{H}$-trivial.
(e) If M is $\mathcal{J}$-trivial, then M is $\mathcal{H}$-trivial.

Proof of Lemma 7.2. **(a)** Assume that M is $\mathcal{J}$-trivial. Then, all $\mathcal{J}$-classes are of cardinality one. In other words, any two $\mathcal{J}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \ \mathcal{J} \ y \text{ must satisfy } x = y. \tag{2}$$

Now, let x and y be two elements of M satisfying $x \ \mathcal{R} \ y$. Then, $xM = yM$ (since $x \ \mathcal{R} \ y$ holds if and only if $xM = yM$). Hence, $M \underbrace{xM}_{=yM} = MyM$. In other words, $x \ \mathcal{J} \ y$ (since $x \ \mathcal{J} \ y$ holds if and only if $MxM = MyM$). Therefore, $x = y$ (by (2)).

Let us now forget that we fixed x and y. We thus have shown that every two elements x and y of M satisfying $x \ \mathcal{R} \ y$ must satisfy $x = y$. In other words, any two $\mathcal{R}$-equivalent elements of M are identical. In other words, all $\mathcal{R}$-classes are of cardinality one. In other words, M is $\mathcal{R}$-trivial. This proves Lemma 7.2 **(a)**.

(b) Assume that M is $\mathcal{J}$-trivial. Then, all $\mathcal{J}$-classes are of cardinality one. In other words, any two $\mathcal{J}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \ \mathcal{J} \ y \text{ must satisfy } x = y. \tag{3}$$

Now, let x and y be two elements of M satisfying $x \ \mathcal{L} \ y$. Then, $Mx = My$ (since $x \ \mathcal{L} \ y$ holds if and only if $Mx = My$). Hence, $\underbrace{Mx}_{=My} M = MyM$. In other words, $x \ \mathcal{J} \ y$ (since $x \ \mathcal{J} \ y$ holds if and only if $MxM = MyM$). Therefore, $x = y$ (by (3)).

$x^{N+(\mu-1)}x = x^{N+(\mu-1)+1} = x^{N+\mu}$ (since $N + (\mu - 1) + 1 = N + \mu$). Hence,

$$x^{N+\mu} = \underbrace{x^{N+(\mu-1)}}_{\substack{=x^{N+(\mu-1)+1}=x^{N+\mu} \\ (\text{since } N+(\mu-1)+1=N+\mu)}} x = x^{N+\mu}x = x^{N+\mu+1}.$$

In other words, (1) holds for $m = \mu$. This completes the induction step. Thus, the induction proof of (1) is complete.

Let us now forget that we fixed x and y. We thus have shown that every two elements x and y of M satisfying $x \mathcal{L} y$ must satisfy $x = y$. In other words, any two $\mathcal{L}$-equivalent elements of M are identical. In other words, all $\mathcal{L}$-classes are of cardinality one. In other words, M is $\mathcal{L}$-trivial. This proves Lemma 7.2 **(b)**.

(c) Assume that M is $\mathcal{R}$-trivial. Then, all $\mathcal{R}$-classes are of cardinality one. In other words, any two $\mathcal{R}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \mathcal{R} y \text{ must satisfy } x = y. \tag{4}$$

Now, let x and y be two elements of M satisfying $x \mathcal{H} y$. Then, $x \mathcal{R} y$ and $x \mathcal{L} y$ (because $x \mathcal{H} y$ holds if and only if ($x \mathcal{R} y$ and $x \mathcal{L} y$)). Hence, $x = y$ (by (4)).

Let us now forget that we fixed x and y. We thus have shown that every two elements x and y of M satisfying $x \mathcal{H} y$ must satisfy $x = y$. In other words, any two $\mathcal{H}$-equivalent elements of M are identical. In other words, all $\mathcal{H}$-classes are of cardinality one. In other words, M is $\mathcal{H}$-trivial. This proves Lemma 7.2 **(c)**.

(d) Assume that M is $\mathcal{L}$-trivial. Then, all $\mathcal{L}$-classes are of cardinality one. In other words, any two $\mathcal{L}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \mathcal{L} y \text{ must satisfy } x = y. \tag{5}$$

Now, let x and y be two elements of M satisfying $x \mathcal{H} y$. Then, $x \mathcal{R} y$ and $x \mathcal{L} y$ (because $x \mathcal{H} y$ holds if and only if ($x \mathcal{R} y$ and $x \mathcal{L} y$)). Hence, $x = y$ (by (5)).

Let us now forget that we fixed x and y. We thus have shown that every two elements x and y of M satisfying $x \mathcal{H} y$ must satisfy $x = y$. In other words, any two $\mathcal{H}$-equivalent elements of M are identical. In other words, all $\mathcal{H}$-classes are of cardinality one. In other words, M is $\mathcal{H}$-trivial. This proves Lemma 7.2 **(d)**.

(e) Assume that M is $\mathcal{J}$-trivial. Then, M is $\mathcal{L}$-trivial (by Lemma 7.2 **(b)**). Hence, M is $\mathcal{H}$-trivial (by Lemma 7.2 **(d)**). This proves Lemma 7.2 **(e)**.

Now, Lemma 7.2 is proven. □

Next, we show an auxiliary lemma that is nearly trivial:

Lemma 7.3. Let M be a monoid. Let x, u and v be three elements of M such that $x = uxv$. Then,

$$x = u^m x v^m \qquad \text{for every } m \in \mathbb{N}. \tag{6}$$

Proof of Lemma 7.3. We will prove (6) by induction over m:

Induction base: We have $\underbrace{u^0}_{=1} x \underbrace{v^0}_{=1} = x$, thus $x = u^0 x v^0$. In other words, (6) holds for $m = 0$. This completes the induction base.

Induction step: Let $k \in \mathbb{N}$. Assume that (6) holds for $m = k$. We now need to prove that (6) holds for $m = k + 1$.

We know that (6) holds for $m = k$. In other words, we have $x = u^k x v^k$. Thus,

$$x = u^k \underbrace{x}_{=uxv} v^k = \underbrace{u^k u}_{=u^{k+1}} x \underbrace{v v^k}_{=v^{k+1}} = u^{k+1} x v^{k+1}.$$

In other words, (6) holds for $m = k + 1$. This completes the induction step. Thus, the induction proof of (6) is complete. Hence, Lemma 7.3 is proved. □

Next, let us show a slightly less trivial (but still easy) fact. In fact, we are going to prove the claim made in §2.1 that "The class of $\mathcal{H}$-trivial monoids coincides with that of *aperiodic* monoids". This fact is the equivalence of the assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ in the following lemma:

Lemma 7.4. Let M be a monoid. Then, the following four assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ are equivalent:

- *Assertion $\mathcal{A}_1$:* The monoid M is $\mathcal{H}$-trivial.
- *Assertion $\mathcal{A}_2$:* The monoid M is aperiodic.
- *Assertion $\mathcal{A}_3$:* For every four elements x, y, b and v of M satisfying $x = yv$ and $y = bx$, we have $x = y$.
- *Assertion $\mathcal{A}_4$:* For every four elements x, y, a and u of M satisfying $x = uy$ and $y = xa$, we have $x = y$.

Proof of Lemma 7.4. Let us recall the definition of the relation $\mathcal{H}$: If x and y are two elements of M, then

$$x \mathcal{H} y \text{ holds if and only if } (x \mathcal{R} y \text{ and } x \mathcal{L} y). \tag{7}$$

Let us also recall the definition of the relation $\mathcal{L}$: If x and y are two elements of M, then

$$x \mathcal{L} y \text{ holds if and only if } Mx = My. \tag{8}$$

Finally, let us recall the definition of the relation $\mathcal{R}$: If x and y are two elements of M, then

$$x \mathcal{R} y \text{ holds if and only if } xM = yM. \tag{9}$$

Now, we shall prove the implications $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$, $\mathcal{A}_2 \Longrightarrow \mathcal{A}_3$, $\mathcal{A}_3 \Longrightarrow \mathcal{A}_4$ and $\mathcal{A}_4 \Longrightarrow \mathcal{A}_1$.

Proof of the implication $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$: Assume that Assertion $\mathcal{A}_1$ holds. We shall show that Assertion $\mathcal{A}_2$ holds.

We know that Assertion $\mathcal{A}_1$ holds. In other words, the monoid M is $\mathcal{H}$-trivial. In other words, all $\mathcal{H}$-classes are of cardinality one. In other words, any two $\mathcal{H}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \ \mathcal{H} \ y \text{ must satisfy } x = y. \tag{10}$$

Now, let $x \in M$. We know that M is finite. Thus, $|M| \in \mathbb{N}$. Let ϕ be the map

$$\{1, 2, \ldots, |M| + 1\} \to M, \quad i \mapsto x^i.$$

There exist two distinct elements p and q of $\{1, 2, \ldots, |M| + 1\}$ satisfying $\phi(p) = \phi(q)$ [3]. Consider these p and q. We WLOG assume that $p \leq q$ (otherwise, we can simply switch p with q). Thus, $p < q$ (since p and q are distinct), and therefore $p \leq q - 1$ (since p and q are integers). Hence, $(q-1) - p \in \mathbb{N}$. Thus, $q - (p+1) = (q-1) - p \in \mathbb{N}$. Hence, $x^{q-(p+1)}$ is a well-defined element of M.

We have $\phi(p) = x^p$ (by the definition of ϕ) and $\phi(q) = x^q$ (by the definition of ϕ). From $\phi(p) = x^p$, we obtain $x^p = \phi(p) = \phi(q) = x^q$.

But $q = (q - (p+1)) + (p+1)$ and thus

$$x^q = x^{(q-(p+1))+(p+1)} = \underbrace{x^{q-(p+1)}}_{\in M} x^{p+1} \qquad (\text{since } q - (p+1) \in \mathbb{N})$$
$$\in Mx^{p+1}.$$

Hence, $x^p = x^q \in Mx^{p+1}$, so that $M \underbrace{x^p}_{\in Mx^{p+1}} \subseteq \underbrace{MM}_{\subseteq M} x^{p+1} \subseteq Mx^{p+1}$. Combined with $M \underbrace{x^{p+1}}_{=xx^p} = \underbrace{Mx}_{\subseteq M} x^p \subseteq Mx^p$, this yields $Mx^p = Mx^{p+1}$. But $x^p \ \mathcal{L} \ x^{p+1}$ holds if and only if $Mx^p = Mx^{p+1}$ (because of (8), applied to x^p and x^{p+1} instead of x and y). Thus, we have $x^p \ \mathcal{L} \ x^{p+1}$ (since $Mx^p = Mx^{p+1}$).

Also, $q = (p+1) + (q - (p+1))$ and thus

$$x^q = x^{(p+1)+(q-(p+1))} = x^{p+1} \underbrace{x^{q-(p+1)}}_{\in M} \qquad (\text{since } q - (p+1) \in \mathbb{N})$$
$$\in x^{p+1}M.$$

Hence, $x^p = x^q \in x^{p+1}M$, so that $\underbrace{x^p}_{\in x^{p+1}M} M \subseteq x^{p+1} \underbrace{MM}_{\subseteq M} \subseteq x^{p+1}M$. Combined with $\underbrace{x^{p+1}}_{=x^p x} M = x^p \underbrace{xM}_{\subseteq M} \subseteq x^p M$, this yields $x^p M = x^{p+1} M$. But $x^p \ \mathcal{R} \ x^{p+1}$ holds

[3] *Proof.* Assume the contrary. Thus, there exist no two distinct elements p and q of $\{1, 2, \ldots, |M| + 1\}$ satisfying $\phi(p) = \phi(q)$. In other words, any two distinct elements p and q of $\{1, 2, \ldots, |M| + 1\}$ satisfy $\phi(p) \neq \phi(q)$. In other words, the map ϕ is injective. Hence, there exists an injective map $\{1, 2, \ldots, |M| + 1\} \to M$ (namely, ϕ). Consequently, $|M| \geq |\{1, 2, \ldots, |M| + 1\}| = |M| + 1 > |M|$. This is absurd. This contradiction proves that our assumption was wrong, qed.

if and only if $x^p M = x^{p+1} M$ (because of (9), applied to x^p and x^{p+1} instead of x and y). Thus, we have $x^p \mathcal{R} x^{p+1}$ (since $x^p M = x^{p+1} M$).

Finally, $x^p \mathcal{H} x^{p+1}$ holds if and only if $\left(x^p \mathcal{R} x^{p+1} \text{ and } x^p \mathcal{L} x^{p+1}\right)$ (because of (7), applied to x^p and x^{p+1} instead of x and y). Hence, we have $x^p \mathcal{H} x^{p+1}$ (since $x^p \mathcal{R} x^{p+1}$ and $x^p \mathcal{L} x^{p+1}$). Thus, $x^p = x^{p+1}$ (by (10), applied to x^p and x^{p+1} instead of x and y). Thus, there exists some positive integer N such that $x^N = x^{N+1}$ (namely, $N = p$).

Now, let us forget that we fixed x. We thus have shown that for every $x \in M$, there exists some positive integer N such that $x^N = x^{N+1}$. In other words, the monoid M is aperiodic. In other words, Assertion $\mathcal{A}_2$ holds. This proves the implication $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$.

Proof of the implication $\mathcal{A}_2 \Longrightarrow \mathcal{A}_3$: Assume that Assertion $\mathcal{A}_2$ holds. We shall show that Assertion $\mathcal{A}_3$ holds.

Assertion $\mathcal{A}_2$ holds. In other words, the monoid M is aperiodic. In other words, for every $x \in M$,

$$\text{there exists some positive integer } N \text{ such that } x^N = x^{N+1}. \tag{11}$$

Now, let x, y, b and v be four elements of M satisfying $x = yv$ and $y = bx$. We are going to prove that $x = y$.

There exists some positive integer N such that $b^N = b^{N+1}$ (according to (11), applied to b instead of x). Consider this N.

We have $x = \underbrace{y}_{=bx} v = bxv$. Hence, Lemma 7.3 (applied to $u = b$) shows that

$$x = b^m x v^m \qquad \text{for every } m \in \mathbb{N}. \tag{12}$$

Applying (12) to $m = N$, we obtain $x = \underbrace{b^N}_{=b^{N+1} = bb^N} xv^N = bb^N xv^N$. Compared with $y = b \underbrace{x}_{=b^N xv^N} = bb^N xv^N$, this yields $x = y$.

Let us now forget that we fixed x, y, b and v. We thus have proven that for every four elements x, y, b and v of M satisfying $x = yv$ and $y = bx$, we have $x = y$. In other words, Assertion $\mathcal{A}_3$ holds. This proves the implication $\mathcal{A}_2 \Longrightarrow \mathcal{A}_3$.

Proof of the implication $\mathcal{A}_3 \Longrightarrow \mathcal{A}_4$: Assume that Assertion $\mathcal{A}_3$ holds. We shall show that Assertion $\mathcal{A}_4$ holds.

Let x, y, a and u be four elements of M satisfying $x = uy$ and $y = xa$. Recall that Assertion $\mathcal{A}_3$ holds. Hence, Assertion $\mathcal{A}_3$ (applied to y, x, u and a instead of x, y, b and v) yields $y = x$. In other words, $x = y$.

Let us now forget that we fixed x, y, a and u. We thus have shown that for every four elements x, y, a and u of M satisfying $x = uy$ and $y = xa$, we have $x = y$. In other words, Assertion $\mathcal{A}_4$ holds. This proves the implication $\mathcal{A}_3 \Longrightarrow \mathcal{A}_4$.

Proof of the implication $\mathcal{A}_4 \Longrightarrow \mathcal{A}_1$: Assume that Assertion $\mathcal{A}_4$ holds. We shall show that Assertion $\mathcal{A}_1$ holds.

Let x and y be two elements of M such that $x \ \mathcal{H} \ y$. Recall that $x \ \mathcal{H} \ y$ holds if and only if ($x \ \mathcal{R} \ y$ and $x \ \mathcal{L} \ y$) (because of (7)). Hence, we must have ($x \ \mathcal{R} \ y$ and $x \ \mathcal{L} \ y$) (since we have $x \ \mathcal{H} \ y$). Thus, $x \ \mathcal{R} \ y$ and $x \ \mathcal{L} \ y$.

We know that $x \ \mathcal{L} \ y$ holds if and only if $Mx = My$ (according to (8)). Thus, we must have $Mx = My$ (since $x \ \mathcal{L} \ y$ holds). Hence, $x = \underbrace{1}_{\in M} x \in Mx = My$. In other words, there exists an $u \in M$ such that $x = uy$. Consider this u.

We know that $x \ \mathcal{R} \ y$ holds if and only if $xM = yM$ (according to (9)). Thus, we must have $xM = yM$ (since $x \ \mathcal{R} \ y$ holds). Hence, $y = y \underbrace{1}_{\in M} \in yM = xM$. In other words, there exists an $a \in M$ such that $y = xa$. Consider this a.

Now, Assertion $\mathcal{A}_4$ yields $x = y$ (since $x = uy$ and $y = xa$).

Let us now forget that we fixed x and y. We thus have proven that every two elements x and y of M satisfying $x \ \mathcal{H} \ y$ must satisfy $x = y$. In other words, any two $\mathcal{H}$-equivalent elements of M are identical. In other words, all $\mathcal{H}$-classes are of cardinality one. In other words, M is $\mathcal{H}$-trivial. In other words, Assertion $\mathcal{A}_1$ holds. This proves the implication $\mathcal{A}_4 \Longrightarrow \mathcal{A}_1$.

We have thus proven the four implications $\mathcal{A}_1 \Longrightarrow \mathcal{A}_2$, $\mathcal{A}_2 \Longrightarrow \mathcal{A}_3$, $\mathcal{A}_3 \Longrightarrow \mathcal{A}_4$ and $\mathcal{A}_4 \Longrightarrow \mathcal{A}_1$. Combined, these implications yield the equivalence $\mathcal{A}_1 \Longleftrightarrow \mathcal{A}_2 \Longleftrightarrow \mathcal{A}_3 \Longleftrightarrow \mathcal{A}_4$. This proves Lemma 7.4. □

Next, let us prove another elementary result, which is used in §2.3 (in the sentence "Since M is finite, this implies that M is $\mathcal{J}$-trivial (see [Pin10a, Chapter V, Theorem 1.9])"):

Lemma 7.5. Let M be a monoid which is $\mathcal{R}$-trivial and $\mathcal{L}$-trivial. Then, M is $\mathcal{J}$-trivial.

Proof of Lemma 7.5. Let us recall the definition of the relation $\mathcal{J}$: If x and y are two elements of M, then

$$x \ \mathcal{J} \ y \text{ holds if and only if } MxM = MyM. \tag{13}$$

Let us also recall the definition of the relation $\mathcal{L}$: If x and y are two elements of M, then

$$x \ \mathcal{L} \ y \text{ holds if and only if } Mx = My. \tag{14}$$

Finally, let us recall the definition of the relation $\mathcal{R}$: If x and y are two elements of M, then

$$x \ \mathcal{R} \ y \text{ holds if and only if } xM = yM. \tag{15}$$

We know that M is $\mathcal{R}$-trivial. Thus, all $\mathcal{R}$-classes are of cardinality one. In other words, any two $\mathcal{R}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \ \mathcal{R} \ y \text{ must satisfy } x = y. \tag{16}$$

We know that M is $\mathcal{L}$-trivial. Thus, all $\mathcal{L}$-classes are of cardinality one. In other words, any two $\mathcal{L}$-equivalent elements of M are identical. In other words,

$$\text{every two elements } x \text{ and } y \text{ of } M \text{ satisfying } x \mathrel{\mathcal{L}} y \text{ must satisfy } x = y. \tag{17}$$

But the monoid M is $\mathcal{R}$-trivial, and thus $\mathcal{H}$-trivial (according to Lemma 7.2 **(c)**). In other words, the Assertion $\mathcal{A}_1$ of Lemma 7.4 holds. Hence, the Assertion $\mathcal{A}_2$ of Lemma 7.4 holds as well (since the Assertions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$ of Lemma 7.4 are equivalent (according to Lemma 7.4)). In other words, the monoid M is aperiodic. In other words, for every $x \in M$,

$$\text{there exists some positive integer } N \text{ such that } x^N = x^{N+1}. \tag{18}$$

Let x and y be two elements of M such that $x \mathrel{\mathcal{J}} y$. Recall that $x \mathrel{\mathcal{J}} y$ holds if and only if $MxM = MyM$ (because of (13)). Hence, we must have $MxM = MyM$ (since we have $x \mathrel{\mathcal{J}} y$).

Now, $x = \underbrace{1}_{\in M} x \underbrace{1}_{\in M} \in MxM = MyM$. In other words, there exist $u \in M$ and $v \in M$ such that $x = uyv$. Consider these u and v.

We also have $y = \underbrace{1}_{\in M} y \underbrace{1}_{\in M} \in MyM = MxM$ (since $MxM = MyM$). In other words, there exists $a \in M$ and $b \in M$ such that $y = axb$. Consider these a and b.

Now, $x = u \underbrace{y}_{=axb} v = uaxbv = (ua)\,x\,(bv)$. Thus, Lemma 7.3 (applied to ua and bv instead of u and v) shows that

$$x = (ua)^m\,x\,(bv)^m \qquad \text{for every } m \in \mathbb{N}. \tag{19}$$

Now, there exists some positive integer N such that $(ua)^N = (ua)^{N+1}$ (according to (18), applied to ua instead of x). Let us denote this N by α. Thus, α is a positive integer such that $(ua)^\alpha = (ua)^{\alpha+1}$. Now, (19) (applied to $m = \alpha$) yields $x = \underbrace{(ua)^\alpha}_{=(ua)^{\alpha+1}=(ua)(ua)^\alpha} x\,(bv)^\alpha = (ua)\,(ua)^\alpha\,x\,(bv)^\alpha$. Compared with $(ua) \underbrace{x}_{=(ua)^\alpha x (bv)^\alpha} = (ua)\,(ua)^\alpha\,x\,(bv)^\alpha$, this yields $x = (ua)\,x$. Thus, $x = (ua)\,x = \underbrace{u}_{\in M} ax \in Max$, so that $M \underbrace{x}_{\in Max} \subseteq \underbrace{MM}_{\subseteq M} ax \subseteq Max$. Combined with $\underbrace{Ma}_{\subseteq M} x \subseteq Mx$, this yields $Max = Mx$. But $ax \mathrel{\mathcal{L}} x$ holds if and only if $Max = Mx$ (according to (14), applied to ax and x instead of x and y). Hence, $ax \mathrel{\mathcal{L}} x$ (because $Max = Mx$). Hence, (17) (applied to ax and x instead of x and y) yields $ax = x$.

Furthermore, there exists some positive integer N such that $(bv)^N = (bv)^{N+1}$ (according to (18), applied to bv instead of x). Let us denote this N by β. Thus, β is a positive integer such that $(bv)^\beta = (bv)^{\beta+1}$. Now, (19) (applied to

$m = \beta$) yields $x = (ua)^{\beta} x \underbrace{(bv)^{\beta}}_{=(bv)^{\beta+1} = (bv)^{\beta}(bv)} = (ua)^{\beta} x (bv)^{\beta} (bv)$. Compared with $\underbrace{x}_{=(ua)^{\beta} x (bv)^{\beta}} (bv) = (ua)^{\beta} x (bv)^{\beta} (bv)$, this yields $x = x(bv)$. Thus, $x = x(bv) = xb \underbrace{v}_{\in M} \in xbM$, so that $\underbrace{x}_{\in xbM} M \subseteq xb \underbrace{MM}_{\subseteq M} \subseteq xbM$. Combined with $x \underbrace{bM}_{\subseteq M} \subseteq xM$, this yields $xbM = xM$. But $xb \mathrel{\mathcal{R}} x$ holds if and only if $xbM = xM$ (according to (15), applied to xb and x instead of x and y). Hence, $xb \mathrel{\mathcal{R}} x$ (because $xbM = xM$). Hence, (16) (applied to xb and x instead of x and y) yields $xb = x$.

Now, $ax = x$ and $xb = x$. Recall now that $y = \underbrace{ax}_{=x} b = xb = x$. Hence, $x = y$.

Let us now forget that we fixed x and y. We thus have shown that every two elements x and y of M satisfying $x \mathrel{\mathcal{J}} y$ must satisfy $x = y$. In other words, any two $\mathcal{J}$-equivalent elements of M are identical. In other words, all $\mathcal{J}$-classes are of cardinality one. In other words, M is $\mathcal{J}$-trivial. This proves Lemma 7.5. □

Finally, let us prove yet another elementary result about the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ (which is, to my knowledge, not used in the paper, but still interesting):[4]

Proposition 7.6. Let M be a monoid. Let a and b be two elements of M. Assume that there exists a $c \in M$ such that $a \mathrel{\mathcal{R}} c$ and $c \mathrel{\mathcal{L}} b$. Then, there exists a $d \in M$ such that $a \mathrel{\mathcal{L}} d$ and $d \mathrel{\mathcal{R}} b$.

Proof of Proposition 7.6. Let us recall the definition of the relation $\mathcal{L}$: If x and y are two elements of M, then

$$x \mathrel{\mathcal{L}} y \text{ holds if and only if } Mx = My. \tag{20}$$

Also, let us recall the definition of the relation $\mathcal{R}$: If x and y are two elements of M, then

$$x \mathrel{\mathcal{R}} y \text{ holds if and only if } xM = yM. \tag{21}$$

Now, we have assumed that there exists a $c \in M$ such that $a \mathrel{\mathcal{R}} c$ and $c \mathrel{\mathcal{L}} b$. Consider this c.

We know that $a \mathrel{\mathcal{R}} c$ holds if and only if $aM = cM$ (by (21), applied to $x = a$ and $y = c$). Thus, we have $aM = cM$ (since $a \mathrel{\mathcal{R}} c$ holds).

We know that $c \mathrel{\mathcal{L}} b$ holds if and only if $Mc = Mb$ (by (20), applied to $x = c$ and $y = b$). Thus, we have $Mc = Mb$ (since $c \mathrel{\mathcal{L}} b$ holds).

We have $a = a \underbrace{1}_{\in M} \in aM = cM$. Thus, there exists an $x \in M$ such that $a = cx$. Consider this x.

[4] Proposition 7.6 is one part of Proposition 1.6 in Chapter V of [Pin10a] (in the case of monoids, rather than arbitrary semigroups). (It is arguably the harder part.)

We have $c = \underbrace{1}_{\in M} c \in Mc = Mb$. Thus, there exists a $y \in M$ such that $c = yb$. Consider this y.

We have $c = c \underbrace{1}_{\in M} \in cM = aM$ (since $aM = cM$). Thus, there exists an $x' \in M$ such that $c = ax'$. Consider this x'.

We have $b = \underbrace{1}_{\in M} b \in Mb = Mc$ (since $Mc = Mb$). Thus, there exists a $y' \in M$ such that $b = y'c$. Consider this y'.

Now, define an element f of M by $f = bx$. Then, $f = \underbrace{b}_{=y'c} x = y' \underbrace{cx}_{\substack{=a \\ (\text{since } a=cx)}} = y'a$. Thus, $M \underbrace{f}_{=y'a} = \underbrace{My'}_{\subseteq M} a \subseteq Ma$.

On the other hand, $a = \underbrace{c}_{=yb} x = y \underbrace{bx}_{=f} = yf$. Hence, $M \underbrace{a}_{=yf} = \underbrace{My}_{\subseteq M} f \subseteq Mf$. Combined with $Mf \subseteq Ma$, this yields $Ma = Mf$.

We know that $a \mathcal{L} f$ holds if and only if $Ma = Mf$ (by (20), applied to $x = a$ and $y = f$). Thus, $a \mathcal{L} f$ holds (since $Ma = Mf$).

Furhermore, $b = y' \underbrace{c}_{=ax'} = \underbrace{y'a}_{\substack{=f \\ (\text{since } f=y'a)}} x' = fx'$. Thus, $\underbrace{b}_{=fx'} M = f \underbrace{x'M}_{\subseteq M} \subseteq fM$. Combined with $\underbrace{f}_{=bx} M = b \underbrace{xM}_{\subseteq M} \subseteq bM$, this yields $fM = bM$.

We know that $f \mathcal{R} b$ holds if and only if $fM = bM$ (by (21), applied to $x = f$ and $y = b$). Thus, $f \mathcal{R} b$ holds (since $fM = bM$).

Now, we know that $a \mathcal{L} f$ and $f \mathcal{R} b$. Thus, there exists a $d \in M$ such that $a \mathcal{L} d$ and $d \mathcal{R} b$ (namely, $d = f$). This proves Proposition 7.6. $\square$

Note that Proposition 7.6 does not require the finiteness of M.