The V/L recursion for Macdonald's 7th Variation Schur polynomials

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Abstract. We generalize and prove the recursive relation

$$S_\lambda(V) = \sum_{L \subseteq V \text{ line}} S_\lambda(V \mathbin{/\!/} L)$$

conjectured by I. G. Macdonald for his "7th variation" of the Schur functions. This variation is a family of polynomials over a finite field that mimic the (straight and skew) Schur polynomials using powers of the Frobenius.

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In [Macdon92], Ian Macdonald discusses nine variations on the Schur functions. One of them – the 7th Variation in his numbering – can be described as a "function field version", in that it is defined over a finite field $\mathbb{F}_q$ and exhibits $\operatorname{GL}(V)$-symmetry rather than merely S_n-symmetry. Macdonald proves several properties of this family, including Jacobi–Trudi formulas, a sum-over-tableaux formula, and a dual-Cauchy-like identity.

In this note, we shall prove a further property, which Macdonald left unproved [Macdon92, (7.25 ?)]: a recursion for the 7th Variation Schur polynomials $S_\lambda(V)$. We will in fact generalize it to the skew version $S_{\lambda/\mu}(V)$ of these polynomials, showing that any finite-dimensional vector subspace V of a commutative F-algebra and any two partitions λ and μ of length $< \dim V$ satisfy

$$S_{\lambda/\mu}(V) = \sum_{\substack{L \subseteq V \text{ line (i.e.,} \\ \text{1-dimensional subspace)}}} S_{\lambda/\mu}(V \mathbin{/\!/} L),$$

where we assume that **A** is an integral domain with an invertible Frobenius morphism (so that $S_{\lambda/\mu}$ is well-defined) (Theorem 1.3). Here, $V \mathbin{//} L$ denotes the internal quotient of V by L, consisting of the products of elements of all cosets of L in V. Along the way, we will show a Pieri-like recursion (Lemma 2.11). After proving the above recursion, we will use it to show the explicit formula

$$S_\lambda(V) = \sum_{\substack{V=V_0>V_1>\cdots>V_n=0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i}(V_{i-1} \mathbin{//} V_i)$$

(Theorem 3.1), spelling out the argument left to the reader by Macdonald in [Macdon92]. (In the appendix, we will also spell out some other implicit arguments from [Macdon92].)

> **Update:** After posting the first version of this note, I have learned that a slight generalization of Theorem 1.3 was proved by Hoang in 2025 [Hoang25, Theorem 1.24]. He also gave a proof of (an equivalent form of) Theorem 3.1 in [Hoang25, Theorem 1.26]. I was unaware of his work when I wrote this note, and would like to thank him for informing me of it and making it available. The proofs I give below are largely the same, although more detailed and more explicit about the algebraic infrastructure built upon.

1. Definitions and results

We will study the 7th Variation of Schur functions [Macdon92, 7th Variation]. First, we shall recall the relevant definitions (following [Macdon92, errata] for the right level of generality).

1.1. Macdonald's 7th Variation

We let $\mathbb{N} := \{0, 1, 2, \ldots\}$, and we set $[n] := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$.

Fix a finite field $F := \mathbb{F}_q$ and any commutative F-algebra **A**. If $x_1, x_2, \ldots, x_n \in$ **A** are elements, and if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, then we define the *α-alternant*

$$A_\alpha := \det\left(x_i^{q^{\alpha_j}}\right)_{i,j\in[n]} \tag{1}$$

in **A**. We also define the special n-tuple

$$\delta := (n-1, n-2, \ldots, 1, 0) \in \mathbb{N}^n.$$

We define the addition of n-tuples in $\mathbb{N}^n$ entrywise (i.e., if $\alpha, \beta \in \mathbb{N}^n$ are two n-tuples, then $\alpha + \beta$ denotes the n-tuple whose i-th entry is the sum of the i-th entries of α and β). Recall that a partition is a weakly decreasing finite tuple of positive integers. Any partition of length $\leq n$ is identified with an n-tuple by inserting trailing zeroes at its end. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$ and any n elements $x_1, x_2, \ldots, x_n \in \mathbf{A}$, we set

$$S_\lambda (x_1, x_2, \ldots, x_n) := A_{\lambda+\delta} / A_\delta,$$

where the quotient is taken "universally" (i.e., we treat the $x_1, x_2, \ldots, x_n$ as independent indeterminates, then compute the quotient $A_{\lambda+\delta}/A_\delta$ in the polynomial ring $F[x_1, x_2, \ldots, x_n]$, and only then substitute the original values of $x_1, x_2, \ldots, x_n$ back). This is a polynomial in $x_1, x_2, \ldots, x_n$, and can be easily seen to be GL_n-invariant, i.e., we have

$$S_\lambda (x_1, x_2, \ldots, x_n) = S_\lambda (y_1, y_2, \ldots, y_n) \tag{2}$$

whenever $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are two bases of the same F-vector subspace of $\mathbf{A}$ (see [Macdon92, first paragraph on page 24] for the proof). Thus, for any finite-dimensional F-vector subspace V of $\mathbf{A}$ and any partition λ of length $\leq \dim V$, we can set

$$S_\lambda (V) := S_\lambda (x_1, x_2, \ldots, x_n), \tag{3}$$

where $(x_1, x_2, \ldots, x_n)$ is an arbitrary basis of V (the result will not depend on the choice of this basis). We also set

$$S_\lambda (V) := 0 \qquad \text{if } \lambda \text{ is a partition of length } > \dim V. \tag{4}$$

These $S_\lambda (V)$ are Macdonald's *7th variation* of the Schur functions. He introduced them in [Macdon92, 7th Variation] and proved several of their properties, in particular defining a skew version $S_{\lambda/\mu} (V)$, to which we shall come later. The problem we are interested in involves subspaces and "internal quotients", which we shall introduce now.

1.2. Internal quotients

If U is a finite-dimensional F-vector subspace of $\mathbf{A}$, then the polynomial

$$f_U (t) := \prod_{u \in U} (t + u) \in \mathbf{A}[t]$$

is a q-polynomial, i.e., it is an F-linear combination of the monomials t^{q^i} for $i \in \mathbb{N}$ (see [Macdon92, (7.7) and (7.8)] or [Grinbe16, Theorem 1.6]). In particular, it satisfies $f_U (x + y) = f_U (x) + f_U (y)$ in the polynomial ring $\mathbf{A}[x, y]$ as well as $f_U (\lambda t) = \lambda f_U (t)$ for each $\lambda \in F$ (since each $\lambda \in F$ satisfies $\lambda^q = \lambda$). Thus, this polynomial f_U defines an F-linear map

$$\begin{aligned} \widetilde{f_U} : \mathbf{A} &\to \mathbf{A}, \\ v &\mapsto f_U (v), \end{aligned}$$

which contains U in its kernel (because for each $v \in U$, we have $\widetilde{f_U}(v) = f_U(v) = \prod_{u \in U} \underbrace{(v+u)}_{=0 \text{ for } u=-v} = 0$). When $\mathbf{A}$ is an integral domain, the kernel of this map $\widetilde{f_U}$ is precisely U.

Now, if $U \subseteq V$ are two finite-dimensional F-vector subspaces of $\mathbf{A}$, then we define the *internal quotient* $V \mathbin{//} U$ to be the image $\widetilde{f_U}(V)$ of the subspace V under the F-linear map $\widetilde{f_U} : \mathbf{A} \to \mathbf{A}$. This is an F-vector subspace of $\mathbf{A}$ again. If $\mathbf{A}$ is an integral domain, then this internal quotient $V \mathbin{//} U$ is isomorphic to the actual quotient V/U (by the first isomorphism theorem, since the F-linear map $\widetilde{f_U}\mid_V : V \to \mathbf{A}$ has kernel U and image $V \mathbin{//} U$); thus, Macdonald simply denotes it by V/U. We shall stick with the safer notation $V \mathbin{//} U$, however. Either way, the design behind $V \mathbin{//} U$ is for $V \mathbin{//} U$ to act as a "canonical internal copy" of the quotient space V/U inside $\mathbf{A}$, in which each coset of U in V is replaced by the product of all vectors in this coset. The magic of the Frobenius endomorphism $v \mapsto v^q$ ensures that this replacement does not disturb the linear structure.

1.3. Macdonald's conjectural recursion

A *line* in an F-vector space W means a 1-dimensional vector subspace of W. The set of all lines in W is known as the *projective space* $\mathbb{P}(W)$.

Now Macdonald's conjecture [Macdon92, (7.25 ?)] says the following:

Theorem 1.1. Assume that $\mathbf{A}$ is an integral domain. Let V be a finite-dimensional F-vector subspace of $\mathbf{A}$. Let λ be a partition of length $< \dim V$. Then,

$$S_\lambda(V) = \sum_{L \subseteq V \text{ line}} S_\lambda(V \mathbin{//} L).$$

(The sum ranges over all lines L in V.)

Macdonald proved this for $\lambda = (1^r)$ in [Macdon95, §I.2, Example 26 (d)].

We shall prove Theorem 1.1 in the general case, along with a generalization to skew partitions. Before we do any of this, however, we need to introduce more notations.

1.4. On the Frobenius morphism

The skew version of Macdonald's 7th Variation requires a certain technical condition on $\mathbf{A}$, defined in terms of the so-called Frobenius morphism. Let us briefly recall it.

The *Frobenius morphism* φ of the commutative F-algebra $\mathbf{A}$ is defined to be the map

$$\varphi : \mathbf{A} \to \mathbf{A},$$
$$a \mapsto a^q.$$

It is well-known that φ is an F-algebra endomorphism (since $F = \mathbb{F}_q$). Note that

$$\varphi^i(a) = a^{q^i} \qquad \text{for each } a \in \mathbf{A} \text{ and } i \in \mathbb{N}. \tag{5}$$

If $\mathbf{A}$ is an integral domain, then the Frobenius morphism φ is injective (since $a^q = 0$ entails $a = 0$ in this case). However, φ is often not surjective (since not every element of $\mathbf{A}$ is a q-th power). The next remark shows some ways to find commutative F-algebras whose φ is bijective:

Remark 1.2.

(a) Let $\mathbb{N}[1/q]$ denote the set of all nonnegative rational numbers of the form i/q^j with $i, j \in \mathbb{N}$. We call these numbers *nonnegative q-local integers*. Their set $\mathbb{N}[1/q]$ is a monoid under addition (and even a commutative semiring).

If $\mathbf{A}$ is a polynomial ring $F[x_1, x_2, \ldots, x_n]$ over F, then the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is not surjective (unless $n = 0$), but $\mathbf{A}$ can be embedded into a larger commutative F-algebra $\widehat{\mathbf{A}}$ whose Frobenius morphism $\varphi : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ is invertible. This larger algebra $\widehat{\mathbf{A}}$ can be defined informally as the ring of all "polynomials" in $x_1, x_2, \ldots, x_n$ over F, where the exponents are not restricted to nonnegative integers but can be any nonnegative q-local integers. Formally speaking, this is the monoid algebra (over F) of the additive monoid $(\mathbb{N}[1/q])^n$ (where we rename each element $(a_1, a_2, \ldots, a_n) \in (\mathbb{N}[1/q])^n$ of this monoid as the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$). The embedding of $\mathbf{A}$ into $\widehat{\mathbf{A}}$ is the obvious one (sending each x_i to x_i). Note that $\widehat{\mathbf{A}}$ is an integral domain (this can be proved in the same way as for usual polynomial rings over a field).

(b) More generally, if $\mathbf{A}$ is any commutative F-algebra whose Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is injective, then we can construct a commutative F-algebra $\widehat{\mathbf{A}}$ that contains $\mathbf{A}$ as a subalgebra and has an invertible Frobenius morphism $\varphi : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$. Namely, we define $\widehat{\mathbf{A}}$ as a direct limit of an infinite sequence

$$\mathbf{A} \xrightarrow{\varphi} \mathbf{A} \xrightarrow{\varphi} \mathbf{A} \xrightarrow{\varphi} \cdots$$

of copies of $\mathbf{A}$, where each copy embeds into the next via the F-algebra morphism φ. We call this $\widehat{\mathbf{A}}$ the *perfect closure* of $\mathbf{A}$ (see, e.g., [Leptie22, Proposition 1.4]). If $\mathbf{A}$ is a polynomial ring $F[x_1, x_2, \ldots, x_n]$, then this $\widehat{\mathbf{A}}$ is isomorphic to the monoid algebra $\widehat{\mathbf{A}}$ from Remark 1.2 **(a)**. In general, if $\mathbf{A}$ is an integral domain, then $\widehat{\mathbf{A}}$ is an integral domain as well.

1.5. The skew 7th Variation

We need some more notations before we can state our result.

For every $r \in \mathbb{N}$ and any finite-dimensional F-vector subspace V of $\mathbf{A}$, we set

$$H_r(V) := S_{(r)}(V)$$

and

$$E_r(V) := S_{(1^r)}(V), \qquad \text{where } (1^r) := \Big(\underbrace{1, 1, \ldots, 1}_{r \text{ times}}\Big).$$

Note that if $r > \dim V$, then $E_r(V) = S_{(1^r)}(V) = 0$ by (4), since (1^r) is a partition of length r. These polynomials $H_r(V)$ and $E_r(V)$ are analogues of the complete homogeneous and elementary symmetric polynomials, respectively. We also set $H_r(V) := 0$ and $E_r(V) := 0$ for all negative r.

Macdonald makes the following definition: If $\mathbf{A}$ is a commutative F-algebra whose Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible, and if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ are two partitions, and if V is a finite-dimensional F-vector subspace of $\mathbf{A}$, then the *skew 7th Variation Schur polynomial* of V corresponding to λ / μ is defined by

$$S_{\lambda/\mu}(V) := \det\left(\left(\varphi^{\mu_j - j + 1} H_{\lambda_i - \mu_j - i + j}(V)\right)_{i,j \in [k]}\right). \tag{6}$$

Here, the power $\varphi^{\mu_j - j + 1}$ of φ is well-defined even if the exponent is negative, since φ is invertible. It is easy to see (see [Macdon92, errata, "page 26, (7.11)"]) that the right hand side of (6) does not depend on k (as long as λ and μ have length $\leq k$ each). It is furthermore easy to see that

$$S_{\lambda/\mu}(V) = 0 \qquad \text{unless } \mu \subseteq \lambda \tag{7}$$

(a particular case of [Macdon92, (7.12)]); this means that the "interesting" skew 7th Variation Schur polynomials are indexed by skew partitions. If the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is not invertible, then we can still define $S_{\lambda/\mu}(V)$ in the perfect closure $\widehat{\mathbf{A}}$ of $\mathbf{A}$, provided that φ is injective.

As usual, we let $\varnothing$ denote the empty partition $()$. If λ is any partition and V is any finite-dimensional F-vector subspace of $\mathbf{A}$, then

$$S_{\lambda/\varnothing}(V) = S_\lambda(V). \tag{8}$$

(This follows by comparing (6) with [Macdon92, (7.10)] if the length of λ is $\leq \dim V$. Otherwise, both sides of this equality are 0, since [Macdon92, (7.12)] shows that $S_{\lambda/\varnothing}(V) = 0$ (because $\lambda_1' - \varnothing_1' = \lambda_1' = \ell(\lambda) > \dim V$) whereas (4) yields $S_\lambda(V) = 0$.)

We are now able to state the skew generalization of Theorem 1.1 that constitutes the main result of this paper:

Theorem 1.3. Assume that $\mathbf{A}$ is an integral domain such that the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible. Let V be a finite-dimensional F-vector subspace of $\mathbf{A}$. Let λ and μ be two partitions of length $< \dim V$ each. Then,

$$S_{\lambda/\mu}(V) = \sum_{L \subseteq V \text{ line}} S_{\lambda/\mu}(V \mathbin{/\!/} L).$$

Applying Theorem 1.3 to $\mu = \varnothing$ (the empty partition), and simplifying using (8), we obtain Theorem 1.1 in the case when $\mathbf{A}$ is an integral domain whose Frobenius morphism φ is invertible. From this case, we can easily deduce the general case (see Section A.2 in the Appendix). Our main quest is thus to prove Theorem 1.3.

2. Lemmas and the proof

2.1. A few elementary lemmas

In preparation for the proof, we first establish some lemmas about finite fields and permutations. The first one helps us simplify (or complicate, depending on one's viewpoint) sums over lines in V:

Lemma 2.1. Let V be a finite-dimensional F-vector space. Let A be an F-vector space. Let b_L be an element of A for each line $L \subseteq V$. Then,

$$\sum_{L \subseteq V \text{ line}} b_L = - \sum_{w \in V \setminus \{0\}} b_{\operatorname{span}(w)}.$$

Proof. We have $q = 0$ in F (since $F = \mathbb{F}_q$), and thus $q - 1 = -1$ in F. Moreover, for each line $L \subseteq V$, we have $|L| = q$ (since a line is a 1-dimensional F-vector space) and $0 \in L$, and therefore

$$|L \setminus \{0\}| = |L| - 1 = q - 1 \tag{9}$$

(since $|L| = q$).

For each vector $w \in V \setminus \{0\}$, the span $\operatorname{span}(w)$ is a line in V. Thus,

$$\sum_{w \in V \setminus \{0\}} b_{\operatorname{span}(w)} = \sum_{L \subseteq V \text{ line}} \sum_{\substack{w \in V \setminus \{0\};\\ \operatorname{span}(w) = L}} \underbrace{b_{\operatorname{span}(w)}}_{\substack{= b_L \\ (\text{since } \operatorname{span}(w) = L)}} = \sum_{L \subseteq V \text{ line}} \underbrace{\sum_{\substack{w \in V \setminus \{0\};\\ \operatorname{span}(w) = L}} b_L}_{\substack{= \sum_{w \in L \setminus \{0\}} b_L \\ = |L \setminus \{0\}| b_L}}$$
$$= \sum_{L \subseteq V \text{ line}} \underbrace{|L \setminus \{0\}|}_{\substack{= q-1 \\ (\text{by } (9))}} b_L = \underbrace{(q-1)}_{= -1 \text{ in } F} \sum_{L \subseteq V \text{ line}} b_L = - \sum_{L \subseteq V \text{ line}} b_L.$$

This yields the lemma. □

Our next lemma is so trivial it is barely worth the name. Following Macdonald, we use the notation $\pi(U)$ for the product $\prod_{u \in U \setminus \{0\}} u$ of all nonzero vectors in a finite-dimensional F-vector subspace U of $\mathbf{A}$. The lemma gives an explicit formula for this product when U is a line:

Lemma 2.2. Let $v \in \mathbf{A}$ be nonzero. Then,

$$\pi(\operatorname{span}(v)) = -v^{q-1}. \tag{10}$$

Proof. By its definition, $\pi(\operatorname{span}(v))$ is the product of all nonzero vectors in $\operatorname{span}(v)$.

However, the elements of $\operatorname{span}(v)$ are precisely the vectors αv for $\alpha \in F$, and furthermore these vectors are all distinct (since v is nonzero). Hence, the nonzero vectors in $\operatorname{span}(v)$ are precisely the vectors αv for $\alpha \in F \setminus \{0\}$, and furthermore these vectors are all distinct. Therefore, their product is

$$\prod_{\alpha \in F \setminus \{0\}} (\alpha v) = \left(\prod_{\alpha \in F \setminus \{0\}} \alpha \right) v^{|F \setminus \{0\}|}.$$

However, Wilson's theorem for finite fields says that $\prod_{\alpha \in F \setminus \{0\}} \alpha = -1$ [1]. Moreover, $|F \setminus \{0\}| = q - 1$ (since $|F| = q$), so that $v^{|F \setminus \{0\}|} = v^{q-1}$. Hence,

$$\prod_{\alpha \in F \setminus \{0\}} (\alpha v) = \underbrace{\left(\prod_{\alpha \in F \setminus \{0\}} \alpha \right)}_{=-1} \underbrace{v^{|F \setminus \{0\}|}}_{=v^{q-1}} = -v^{q-1}.$$

Since we previously saw that the product of the nonzero vectors in $\operatorname{span}(v)$ is $\prod_{\alpha \in F \setminus \{0\}} (\alpha v)$, we thus conclude that this product is $-v^{q-1}$. In other words, $\pi(\operatorname{span}(v)) = -v^{q-1}$. This proves Lemma 2.2. □

The next three lemmas are staples in combinatorial number theory:

[1] We recall the proof: In the product $\prod_{\alpha \in F \setminus \{0\}} \alpha$, each factor α can be paired with its inverse α^{-1} unless it equals this inverse (i.e., unless $\alpha = \alpha^{-1}$). Thus, all the factors of this product cancel out except for those that satisfy $\alpha = \alpha^{-1}$. But the latter factors are precisely 1 and -1 (since $\alpha = \alpha^{-1}$ is equivalent to $\alpha^2 = 1$, that is, $\alpha^2 - 1 = 0$, that is, $(\alpha - 1)(\alpha + 1) = 0$, that is, $\alpha \in \{1, -1\}$), and multiply to -1 (this holds even if these two factors are equal, which happens when $\operatorname{char} F = 2$). Hence, the entire product $\prod_{\alpha \in F \setminus \{0\}} \alpha$ simplifies to -1.

Lemma 2.3. Let $i \in \mathbb{N}$ be such that $i < q-1$. Then, $\sum_{\alpha \in F} \alpha^i = 0$.

Proof. Well-known, but let us give a proof nevertheless. If $i = 0$, then $\sum_{\alpha \in F} \underbrace{\alpha^i}_{=\alpha^0 = 1} = \sum_{\alpha \in F} 1 = |F| = q = 0$ in F. Thus, we WLOG assume that $i > 0$. Hence, the polynomial $x^{i+1} - x \in F[x]$ is a nonzero polynomial of degree $i+1$. Therefore, this polynomial has at most $i+1$ roots in F. Since $i + 1 < q$ (because $i < q-1$), this shows that this polynomial has fewer roots in F than F has elements. Thus, there exists some $u \in F$ that is not a root of this polynomial. Consider this u. Then, $u^{i+1} - u \neq 0$, so that $u\left(u^i - 1\right) = u^{i+1} - u \neq 0$. Thus, $u \neq 0$ and $u^i - 1 \neq 0$.

Now, $u \neq 0$ shows that u is invertible in the field F, and thus the map $F \to F$, $\alpha \mapsto u\alpha$ is a bijection. Hence, substituting $u\alpha$ for α in the sum $\sum_{\alpha \in F} \alpha^i$, we obtain $\sum_{\alpha \in F} \alpha^i = \sum_{\alpha \in F} \underbrace{(u\alpha)^i}_{=u^i\alpha^i} = u^i \sum_{\alpha \in F} \alpha^i$. Therefore,

$$\left(u^i - 1\right) \sum_{\alpha \in F} \alpha^i = u^i \sum_{\alpha \in F} \alpha^i - \sum_{\alpha \in F} \alpha^i = 0$$

(since $\sum_{\alpha \in F} \alpha^i = u^i \sum_{\alpha \in F} \alpha^i$). We can divide this by $u^i - 1$ (since $u^i - 1 \neq 0$), and obtain $\sum_{\alpha \in F} \alpha^i = 0$. This proves Lemma 2.3. □

Lemma 2.4. Let $P \in \mathbf{A}[t_1, t_2, \ldots, t_n]$ be a polynomial of total degree $< n(q-1)$ in n variables $t_1, t_2, \ldots, t_n$ over $\mathbf{A}$. Then,

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in F} P(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0.$$

Proof. The claim depends $\mathbf{A}$-linearly on P. Thus, we can WLOG assume that P is just a monomial $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ with $i_1 + i_2 + \cdots + i_n < n(q-1)$ (since any polynomial of total degree $< n(q-1)$ in $t_1, t_2, \ldots, t_n$ is an $\mathbf{A}$-linear combination of such monomials). Assume this, and consider these exponents $i_1, i_2, \ldots, i_n$. Then, at least one $k \in [n]$ satisfies $i_k < q-1$ (since $i_1 + i_2 + \cdots + i_n < n(q-1)$) and therefore $\sum_{\alpha_k \in F} \alpha_k^{i_k} = \sum_{\alpha \in F} \alpha^{i_k} = 0$ (by Lemma 2.3, applied to $i = i_k$). But $P = t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$, and therefore

$$\sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in F} P(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n \in F} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_n^{i_n} = \prod_{j=1}^{n} \underbrace{\sum_{\alpha_j \in F} \alpha_j^{i_j}}_{\substack{=0 \text{ for } j=k \\ \left(\text{since } \sum_{\alpha_k \in F} \alpha_k^{i_k} = 0\right)}} = 0$$

(since a product is 0 if at least one of its factors is 0). This proves the lemma. □

Lemma 2.5. Let $a_1, a_2, \ldots, a_k$ be k nonnegative integers, where $k > 0$. Let V be an n-dimensional F-vector subspace of $\mathbf{A}$, where $n > k$. Then,

$$\sum_{w \in V \setminus \{0\}} w^{(q-1)\left(q^{a_1}+q^{a_2}+\cdots+q^{a_k}\right)} = 0.$$

Proof. Since $k > 0$, we have $(q-1)\left(q^{a_1} + q^{a_2} + \cdots + q^{a_k}\right) > 0$ and thus

$$0^{(q-1)\left(q^{a_1}+q^{a_2}+\cdots+q^{a_k}\right)} = 0.$$

Hence, we can replace the $\sum\limits_{w \in V \setminus \{0\}}$ sign in Lemma 2.5 by a $\sum\limits_{w \in V}$ sign without changing the sum. It thus remains to prove

$$\sum_{w \in V} w^{(q-1)\left(q^{a_1}+q^{a_2}+\cdots+q^{a_k}\right)} = 0. \tag{11}$$

For each $a \in \mathbb{N}$, the map

$$\begin{aligned} \mathbf{A} &\to \mathbf{A}, \\ v &\mapsto v^{q^a} \end{aligned}$$

is an F-algebra endomorphism of $\mathbf{A}$ (indeed, it is the a-th power of the Frobenius morphism φ of $\mathbf{A}$). Thus, for each $a \in \mathbb{N}$ and any $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ and $x_1, x_2, \ldots, x_n \in \mathbf{A}$, we have

$$\begin{aligned} &\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\right)^{q^a} \\ &= \alpha_1 x_1^{q^a} + \alpha_2 x_2^{q^a} + \cdots + \alpha_n x_n^{q^a}. \end{aligned} \tag{12}$$

Now, fix a basis $(x_1, x_2, \ldots, x_n)$ of the F-vector space V. Then, each $w \in V$ can be uniquely written as $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ with $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$. Hence,

$$\begin{aligned}
&\sum_{w\in V} w^{(q-1)\left(q^{a_1}+q^{a_2}+\cdots+q^{a_k}\right)} \\
&= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in F} \underbrace{\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\right)^{(q-1)\left(q^{a_1}+q^{a_2}+\cdots+q^{a_k}\right)}}_{=\prod_{j=1}^{k}\left(\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n\right)^{(q-1)q^{a_j}}} \\
&= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in F} \prod_{j=1}^{k} \underbrace{\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\right)^{(q-1)q^{a_j}}}_{=\left(\left(\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n\right)^{q^{a_j}}\right)^{q-1}} \\
&= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in F} \prod_{j=1}^{k} \left(\underbrace{\left(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\right)^{q^{a_j}}}_{\substack{=\alpha_1 x_1^{q^{a_j}}+\alpha_2 x_2^{q^{a_j}}+\cdots+\alpha_n x_n^{q^{a_j}} \\ \text{(by (12), applied to } a=a_j\text{)}}} \right)^{q-1} \\
&= \sum_{\alpha_1,\alpha_2,\ldots,\alpha_n\in F} \prod_{j=1}^{k} \left(\alpha_1 x_1^{q^{a_j}} + \alpha_2 x_2^{q^{a_j}} + \cdots + \alpha_n x_n^{q^{a_j}}\right)^{q-1} = 0
\end{aligned}$$

(by Lemma 2.4, applied to the polynomial $P = \prod_{j=1}^{k} \left(t_1 x_1^{q^{a_j}} + t_2 x_2^{q^{a_j}} + \cdots + t_n x_n^{q^{a_j}}\right)^{q-1} \in \mathbf{A}\left[t_1, t_2, \ldots, t_n\right]$, which has total degree $\underbrace{k}_{<n}(q-1) < n(q-1)$). This proves (11) and thus the lemma. □

The following purely combinatorial lemma will help us simplify a determinant:

Lemma 2.6. Let $n \in \mathbb{N}$. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ be n integers, and let $\beta_1 > \beta_2 > \cdots > \beta_n$ be n further integers. Assume that

$$\alpha_i - \beta_i \in \{0, 1\} \qquad \text{for all } i \in [n]. \tag{13}$$

Let $\sigma \in S_n$ be a permutation such that $\sigma \neq \text{id}$. Then, there exists an $i \in [n]$ such that $\alpha_i - \beta_{\sigma(i)} \notin \{0, 1\}$.

Proof. We have assumed that $\sigma \neq \text{id}$. Thus, there exists some $k \in [n]$ such that $\sigma(k) \neq k$. Consider the **smallest** such k. Then,

$$\sigma(j) = j \qquad \text{for each } j < k \tag{14}$$

(since our k is the **smallest** k). Hence, if we had $\sigma(k) < k$, then we would have $\sigma(\sigma(k)) = \sigma(k)$ (by (14), applied to $j = \sigma(k)$), therefore $\sigma(k) = k$ (because

σ is injective); but this would contradict $\sigma(k) \neq k$. Hence, we cannot have $\sigma(k) < k$. Thus, we have $\sigma(k) \geq k$ and therefore $\sigma(k) > k$ (since $\sigma(k) \neq k$). That is, $\sigma(k) \geq k+1$. Hence, $k+1 \leq \sigma(k) \leq n$, so that $k+1 \in [n]$. Thus, from $\beta_1 > \beta_2 > \cdots > \beta_n$, we obtain $\beta_{\sigma(k)} \leq \beta_{k+1}$ (since $k+1 \leq \sigma(k)$) and $\beta_{k+1} < \beta_k$. Also, $\alpha_k > \alpha_{k+1}$ (which follows from $\alpha_1 > \alpha_2 > \cdots > \alpha_n$).

We want to find an $i \in [n]$ such that $\alpha_i - \beta_{\sigma(i)} \notin \{0,1\}$. If $\alpha_k - \beta_{\sigma(k)} \notin \{0,1\}$, then we can just take $i = k$ and be done. Thus, we WLOG assume that $\alpha_k - \beta_{\sigma(k)} \in \{0,1\}$. Thus, $0 \leq \alpha_k - \beta_{\sigma(k)} \leq 1$. In other words,

$$\beta_{\sigma(k)} \leq \alpha_k \leq \beta_{\sigma(k)} + 1. \tag{15}$$

But (13) yields $\alpha_k - \beta_k \in \{0,1\}$. Hence, $0 \leq \alpha_k - \beta_k \leq 1$. In other words,

$$\beta_k \leq \alpha_k \leq \beta_k + 1.$$

The same argument (applied to $k+1$ instead of k) yields

$$\beta_{k+1} \leq \alpha_{k+1} \leq \beta_{k+1} + 1.$$

If we had $\sigma(k) > k+1$, then we would have $\beta_{\sigma(k)} < \beta_{k+1}$ (since $\beta_1 > \beta_2 > \cdots > \beta_n$) and therefore $\beta_{\sigma(k)} + 1 \leq \beta_{k+1}$, which would entail $\alpha_k \leq \beta_{\sigma(k)} + 1 \leq \beta_{k+1} \leq \alpha_{k+1}$; but this would contradict $\alpha_k > \alpha_{k+1}$. Hence, we cannot have $\sigma(k) > k+1$. Thus, we have $\sigma(k) \leq k+1$. Combined with $\sigma(k) \geq k+1$, this leads to $\sigma(k) = k+1$. Therefore, we can rewrite the chain of inequalities (15) as

$$\beta_{k+1} \leq \alpha_k \leq \beta_{k+1} + 1.$$

Thus, $\alpha_k \leq \beta_{k+1} + 1 \leq \beta_k$ (since $\beta_{k+1} < \beta_k$). Combining this with $\beta_k \leq \alpha_k$, we obtain

$$\alpha_k = \beta_k.$$

Let $j := \sigma^{-1}(k)$. Then, $j \in [n]$ and $\sigma(j) = k$. If we had $j < k$, then we would have $\sigma(j) = j$ (by (14)) and therefore $k = \sigma(j) = j < k$, which is a contradiction. Thus, we must have $j \geq k$. Since $\sigma(j) = k \neq \sigma(k)$, we furthermore have $j \neq k$, and thus $j > k$ (since $j \geq k$). Hence, $\alpha_j < \alpha_k$ (since $\alpha_1 > \alpha_2 > \cdots > \alpha_n$). Thus, $\alpha_j < \alpha_k = \beta_k = \beta_{\sigma(j)}$ (since $k = \sigma(j)$). Therefore, $\alpha_j - \beta_{\sigma(j)} < 0$, so that $\alpha_j - \beta_{\sigma(j)} \notin \{0,1\}$. Thus, there exists an $i \in [n]$ such that $\alpha_i - \beta_{\sigma(i)} \notin \{0,1\}$ (namely, $i = j$). This completes the proof of Lemma 2.6. □

2.2. The $\widetilde{S}_{\lambda/\mu}$

We now come back to the properties of the 7th Variation. More precisely, we define a variant of it (a 7.5th Variation perhaps) in the skew case.

Convention 2.7. For this whole section, we assume that the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible.

For any two partitions λ and μ, we define

$$\widetilde{S}_{\lambda/\mu}(U) := \det\left(\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)\right)_{i,j\in[r]} \in \mathbf{A}, \tag{16}$$

where $r \in \mathbb{N}$ is chosen such that both $\ell(\lambda)$ and $\ell(\mu)$ are $\leq r$ (the exact choice of r is immaterial, because if $\lambda_r = \mu_r = 0$, then the r-th row of the matrix $\left(\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)\right)_{i,j\in[r]}$ is $(0, 0, \ldots, 0, 1)$, and thus Laplace expansion along this row shows that

$$\det\left(\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)\right)_{i,j\in[r]} = \det\left(\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)\right)_{i,j\in[r-1]}$$

in this case). This is similar to the formula [Macdon92, (7.11′)] for $S_{\lambda'/\mu'}(U)$, but probably not directly related. However, it has some properties in common. First, we have an analogue of [Macdon92, (7.12)]:

Lemma 2.8. Let λ and μ be two partitions. Let U be an n-dimensional F-vector subspace of $\mathbf{A}$. Then,

$$\widetilde{S}_{\lambda/\mu}(U) = 0 \text{ unless } 0 \leq \lambda_i - \mu_i \leq n \text{ for all } i \geq 1. \tag{17}$$

Proof. This is entirely analogous to [Macdon92, (7.12)]. (See [Macdon92, errata, proof of (6.10)] for a proof template that can be used almost verbatim here – just replace λ' and μ' by λ and μ, and replace $e_{\lambda'_i - \mu'_j - i + j}\left(x \mid \tau^{-\mu'_j + j - 1} a\right)$ by $\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)$.) □

As a particular case of Lemma 2.8, we see that

$$\widetilde{S}_{\lambda/\mu}(U) = 0 \text{ unless } \mu \subseteq \lambda. \tag{18}$$

Next, we have an analogue of [Macdon92, (7.22)]:[2]

Lemma 2.9. Let λ and μ be two partitions. Let U be a line (i.e., a 1-dimensional vector subspace) in $\mathbf{A}$. Recall the notation $\pi(U)$ for the product of all nonzero vectors in U. Then:

[2]Unlike Macdonald, we use λ/μ (not $\lambda - \mu$) to denote the skew partition consisting of two partitions μ and λ.

We also use the standard notation λ_i for the i-th entry of a partition λ. (If i surpasses the length of λ, then this entry is 0 by definition.)

Recall that a skew partition λ/μ is said to be a *vertical strip* if its Young diagram has no two distinct cells in the same row. This is equivalent to requiring that each $i \geq 1$ satisfies $\mu_i \leq \lambda_i \leq \mu_i + 1$.

(a) If λ/μ is a vertical strip, then we have

$$\widetilde{S}_{\lambda/\mu}(U) = \prod_{\substack{i\geq 1;\\ \lambda_i>\mu_i}} \varphi^{\lambda_i - i}(-\pi(U)). \tag{19}$$

(b) Otherwise, we have

$$\widetilde{S}_{\lambda/\mu}(U) = 0. \tag{20}$$

Proof. **(b)** Assume that λ/μ is not a vertical strip. Then, some $i \geq 1$ satisfies $\lambda_i < \mu_i$ or $\lambda_i > \mu_i + 1$. In other words, some $i \geq 1$ does **not** satisfy $0 \leq \lambda_i - \mu_i \leq 1$. Hence, Lemma 2.8 (applied to $n = 1$) shows that $\widetilde{S}_{\lambda/\mu}(U) = 0$. This proves Lemma 2.9 **(b)**.

(a) Let us first compute all the $E_r(U)$ for $r \in \mathbb{Z}$:

- We have $E_0(U) = 1$ (since each finite-dimensional vector subspace V of $\mathbf{A}$ satisfies $E_0(V) = S_{(1^0)}(V) = S_\varnothing(V) = 1$).
- From [Macdon92, (7.20)], we know that $\pi(U) = (-1)^1 E_1(U)$ (since $\dim U = 1$), and thus

$$E_1(U) = (-1)^1 \pi(U) = -\pi(U).$$

- Recall that if V is a finite-dimensional vector subspace of $\mathbf{A}$, then $E_r(V) = 0$ for all $r > \dim V$ and also for all $r < 0$. In other words, if V is a finite-dimensional vector subspace of $\mathbf{A}$, then

$$E_r(V) = 0 \qquad \text{for all } r \notin \{0, 1, \ldots, \dim V\}.$$

Applying this to $V = U$ (which has dimension $\dim U = 1$), we conclude that

$$E_r(U) = 0 \qquad \text{for all } r \notin \{0, 1\}. \tag{21}$$

Now assume that λ/μ is a vertical strip. Thus,

$$\lambda_i - \mu_i \in \{0, 1\} \qquad \text{for each } i \geq 1. \tag{22}$$

Let $n \in \mathbb{N}$ be such that both $\ell(\lambda)$ and $\ell(\mu)$ are $\leq n$. Then, the definition of $\widetilde{S}_{\lambda/\mu}(U)$ yields

$$\begin{aligned}\widetilde{S}_{\lambda/\mu}(U) &= \det\left(\varphi^{\lambda_i - i} E_{\lambda_i - \mu_j - i + j}(U)\right)_{i,j\in[n]} \\ &= \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U) \end{aligned} \tag{23}$$

(by the definition of a determinant).

We shall now show that the only nonzero addend in the sum on the right hand side is the addend for $\sigma = \operatorname{id}$. Indeed, let $\sigma \in S_n$ be a permutation such that $\sigma \neq \operatorname{id}$. We have $\lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_n - n$ (since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) and $\mu_1 - 1 > \mu_2 - 2 > \cdots > \mu_n - n$ (similarly) and $(\lambda_i - i) - (\mu_i - i) = \lambda_i - \mu_i \in \{0, 1\}$ for each $i \in [n]$ (by (22)). Hence, Lemma 2.6 (applied to $\alpha_i = \lambda_i - i$ and $\beta_i = \mu_i - i$) shows that there exists an $i \in [n]$ such that $(\lambda_i - i) - \left(\mu_{\sigma(i)} - \sigma(i)\right) \notin \{0, 1\}$. Consider this i. Then,

$$\lambda_i - \mu_{\sigma(i)} - i + \sigma(i) = (\lambda_i - i) - \left(\mu_{\sigma(i)} - \sigma(i)\right) \notin \{0, 1\}.$$

Thus, (21) yields $E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U) = 0$, so that $\varphi^{\lambda_i - i} E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U) = \varphi^{\lambda_i - i} 0 = 0$ as well.

Thus, we have found an $i \in [n]$ such that $\varphi^{\lambda_i - i} E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U) = 0$. In other words, the product $\prod_{i=1}^{n} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U)$ has at least one factor equal to 0. Thus, this whole product is 0.

Forget that we fixed σ. We thus have shown that if $\sigma \in S_n$ is a permutation such that $\sigma \neq \operatorname{id}$, then the product $\prod_{i=1}^{n} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_{\sigma(i)} - i + \sigma(i)}(U)$ is 0. Consequently, in the sum on the right hand side of (23), all addends with $\sigma \neq \operatorname{id}$ are 0. Thus, (23) simplifies to

$$\begin{aligned} \widetilde{S}_{\lambda/\mu}(U) &= \underbrace{(-1)^{\operatorname{id}}}_{=1} \prod_{i=1}^{n} \varphi^{\lambda_i - i} \underbrace{E_{\lambda_i - \mu_{\operatorname{id}(i)} - i + \operatorname{id}(i)}(U)}_{\substack{= E_{\lambda_i - \mu_i - i + i}(U) \\ = E_{\lambda_i - \mu_i}(U)}} = \prod_{i=1}^{n} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_i}(U) \\ &= \prod_{\substack{i \in [n]; \\ \lambda_i \neq \mu_i}} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_i}(U) \end{aligned} \tag{24}$$

(here, we have removed all the factors with $\lambda_i = \mu_i$ from the product, since they all equal $\varphi^{\mu_i - i} \underbrace{E_{\mu_i - \mu_i}(U)}_{= E_0(U) = 1} = \varphi^{\mu_i - i} 1 = 1$).

But (22) shows that each $i \in [n]$ satisfies $\lambda_i - \mu_i \in \{0, 1\}$, so that $\lambda_i \geq \mu_i$. Hence, the condition $\lambda_i \neq \mu_i$ under the product sign in (24) is equivalent to $\lambda_i > \mu_i$. That is, we can rewrite (24) as

$$\widetilde{S}_{\lambda/\mu}(U) = \prod_{\substack{i \in [n]; \\ \lambda_i > \mu_i}} \varphi^{\lambda_i - i} E_{\lambda_i - \mu_i}(U). \tag{25}$$

Moreover, if $i \in [n]$ satisfies $\lambda_i > \mu_i$, then it satisfies $\lambda_i - \mu_i = 1$ (since (22) yields $\lambda_i - \mu_i \in \{0, 1\}$, but $\lambda_i - \mu_i$ cannot be 0 because of $\lambda_i > \mu_i$) and thus

$E_{\lambda_i-\mu_i}(U) = E_1(U) = -\pi(U)$. Hence, we can rewrite (25) as

$$\widetilde{S}_{\lambda/\mu}(U) = \prod_{\substack{i\in[n];\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U)).$$

Comparing this with

$$\begin{aligned}
&\prod_{\substack{i\geq 1;\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U)) \\
&= \left(\prod_{\substack{i\in[n];\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U))\right) \underbrace{\left(\prod_{\substack{i>n;\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U))\right)}_{\substack{=1 \\ \text{(indeed, this is an empty product,} \\ \text{since each } i>n \text{ satisfies } \lambda_i=0 \\ \text{(because } \ell(\lambda)\leq n\text{) and } \mu_i=0 \text{ (since } \ell(\mu)\leq n\text{)} \\ \text{and thus cannot satisfy } \lambda_i>\mu_i\text{)}}} \\
&= \prod_{\substack{i\in[n];\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U)),
\end{aligned}$$

we obtain

$$\widetilde{S}_{\lambda/\mu}(U) = \prod_{\substack{i\geq 1;\\ \lambda_i>\mu_i}} \varphi^{\lambda_i-i}(-\pi(U)).$$

This proves Lemma 2.9 **(a)**. □

Most importantly, we have a formula akin to [Macdon92, (7.18)]:

Lemma 2.10. Assume that $\mathbf{A}$ is an integral domain such that the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible. Let λ and μ be two partitions. Let $U \subseteq V$ be two finite-dimensional F-vector subspaces of $\mathbf{A}$. Then,

$$S_{\lambda/\mu}(V \,//\, U) = \sum_{\nu} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{\dim(V/U)} \widetilde{S}_{\lambda/\nu}(U), \tag{26}$$

where the sum ranges over all partitions ν.

Proof. For any finite-dimensional F-vector subspace W of $\mathbf{A}$, we define the two $\mathbb{Z}\times\mathbb{Z}$-matrices

$$\mathbf{H}(W) := \left(\varphi^{i+1} H_{j-i}(W)\right)_{i,j\in\mathbb{Z}} \qquad \text{and}$$
$$\mathbf{E}(W) := \left((-1)^{j-i} \varphi^{j} E_{j-i}(W)\right)_{i,j\in\mathbb{Z}}$$

over $\mathbf{A}$. From [Macdon92, (7.9)], we know that these two matrices are upper-triangular and mutually inverse. In particular, $(\mathbf{H}(U))^{-1} = \mathbf{E}(U)$.

Moreover, from [Macdon92, (7.17) (ii)], we have

$$\mathbf{H}(V) = \mathbf{H}(V \mathbin{//} U) \cdot \varphi^{\dim(V/U)}(\mathbf{H}(U)),$$

where an algebra morphism such as $\varphi^{\dim(V/U)}$ is applied to a matrix by applying it to each entry of the matrix. Thus,

$$\begin{aligned}
\mathbf{H}(V \mathbin{//} U) &= \mathbf{H}(V) \cdot \left(\varphi^{\dim(V/U)}(\mathbf{H}(U))\right)^{-1} \\
&= \mathbf{H}(V) \cdot \varphi^{\dim(V/U)}\left(\underbrace{(\mathbf{H}(U))^{-1}}_{=\mathbf{E}(U)}\right) \\
&\qquad \text{(since } \varphi \text{ is an algebra morphism)} \\
&= \mathbf{H}(V) \cdot \varphi^{\dim(V/U)}(\mathbf{E}(U)). \qquad (27)
\end{aligned}$$

From this equality, we proceed using the Cauchy–Binet theorem, analogously to [Macdon92, proof of (7.18)] (see [Macdon92, errata, "page 19, proof of (6.13)"] for the details of that proof), to prove (26). See the Appendix (Section A.1) for the details of this argument. □

2.3. Proof of the recursion

One last formula – a sort of Pieri rule – will be crucial to our proof of Theorem 1.3:

Lemma 2.11. Assume that $\mathbf{A}$ is an integral domain such that the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible.

Let $n \in \mathbb{N}$. Let V be an n-dimensional F-vector subspace of $\mathbf{A}$. Let λ and μ be two partitions of length $< n$. Let $\ell \in V$ be nonzero, and let $L = \operatorname{span}(\ell) \subseteq V$. For any vertical strip λ/ν, we set

$$\mathbf{q}(\lambda, \nu) := (q-1) \sum_{\substack{i \geq 1; \\ \lambda_i > \nu_i}} q^{\lambda_i + n - 1 - i}.$$

Then,

$$S_{\lambda/\mu}(V \mathbin{//} L) = \sum_{\lambda/\nu \text{ is a vertical strip}} (-1)^{|\lambda| - |\nu|} \ell^{\mathbf{q}(\lambda,\nu)} S_{\nu/\mu}(V).$$

(The sum is understood to range over all partitions ν such that λ/ν is a vertical strip.)

Proof. We have $\dim L = 1$ (since L is the span of the nonzero vector ℓ) and $\dim V = n$, so that

$$\dim(V/L) = \underbrace{\dim V}_{=n} - \underbrace{\dim L}_{=1} = n-1.$$

Moreover, from $L = \operatorname{span}(\ell)$, we obtain (using the notation $\pi(L)$ defined previously)

$$\pi(L) = \pi(\operatorname{span}(\ell)) = -\ell^{q-1} \tag{28}$$

(by Lemma 2.2, applied to $v = \ell$, since ℓ is nonzero).

Applying (26) to $U = L$, we obtain

$$\begin{aligned} S_{\lambda/\mu}(V /\!/ L) &= \sum_{\nu} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \underbrace{\varphi^{\dim(V/L)}}_{\substack{=\varphi^{n-1} \\ (\text{since } \dim(V/L)=n-1)}} \widetilde{S}_{\lambda/\nu}(L) \\ &= \sum_{\nu} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{n-1} \widetilde{S}_{\lambda/\nu}(L). \end{aligned} \tag{29}$$

If ν is a partition for which λ/ν is **not** a vertical strip, then (20) (applied to ν and L instead of μ and U) yields $\widetilde{S}_{\lambda/\nu}(L) = 0$. Thus, in the sum on the right hand side of (29), all addends corresponding to such partitions ν vanish. Consequently, we can simplify the sum by removing all these addends, and obtain

$$S_{\lambda/\mu}(V /\!/ L) = \sum_{\substack{\lambda/\nu \text{ is a} \\ \text{vertical strip}}} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{n-1} \widetilde{S}_{\lambda/\nu}(L). \tag{30}$$

Now, let ν be a partition such that λ/ν is a vertical strip. Then, (19) (applied to L and ν instead of U and μ) yields

$$\begin{aligned} \widetilde{S}_{\lambda/\nu}(L) &= \prod_{\substack{i\geq 1; \\ \lambda_i > \nu_i}} \varphi^{\lambda_i - i} \left(- \underbrace{\pi(L)}_{\substack{=-\ell^{q-1} \\ (\text{by } (28))}} \right) \\ &= \prod_{\substack{i\geq 1; \\ \lambda_i > \nu_i}} \varphi^{\lambda_i - i} \left(- \left(-\ell^{q-1} \right) \right) = \prod_{\substack{i\geq 1; \\ \lambda_i > \nu_i}} \varphi^{\lambda_i - i} \left(\ell^{q-1} \right). \end{aligned}$$

Applying the algebra morphism φ^{n-1} to both sides of this equality, we find

$$\varphi^{n-1}\widetilde{S}_{\lambda/\nu}(L) = \prod_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} \underbrace{\varphi^{n-1}\varphi^{\lambda_i-i}\left(\ell^{q-1}\right)}_{\substack{=\varphi^{n-1+\lambda_i-i}\left(\ell^{q-1}\right) \\ =\left(\ell^{q-1}\right)^{q^{n-1+\lambda_i-i}} \\ \text{(by (5))}}} = \prod_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} \left(\ell^{q-1}\right)^{q^{n-1+\lambda_i-i}} = \left(\ell^{q-1}\right)^{\sum\limits_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} q^{n-1+\lambda_i-i}}$$
$$= \ell^{(q-1)\sum\limits_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} q^{n-1+\lambda_i-i}} = \ell^{\mathbf{q}(\lambda,\nu)} \tag{31}$$

(since $(q-1)\sum\limits_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} \underbrace{q^{n-1+\lambda_i-i}}_{=q^{\lambda_i+n-1-i}} = (q-1)\sum\limits_{\substack{i\geq 1;\\ \lambda_i>\nu_i}} q^{\lambda_i+n-1-i} = \mathbf{q}(\lambda,\nu)$).

Forget that we fixed ν. We thus have shown that (31) holds for each partition ν such that λ/ν is a vertical strip. Thus, (30) becomes

$$S_{\lambda/\mu}(V\ //\ L) = \sum_{\substack{\lambda/\nu \text{ is a}\\ \text{vertical strip}}} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V)\cdot \underbrace{\varphi^{n-1}\widetilde{S}_{\lambda/\nu}(L)}_{\substack{=\ell^{\mathbf{q}(\lambda,\nu)}\\ \text{(by (31))}}}$$
$$= \sum_{\substack{\lambda/\nu \text{ is a}\\ \text{vertical strip}}} (-1)^{|\lambda|-|\nu|} \ell^{\mathbf{q}(\lambda,\nu)} S_{\nu/\mu}(V).$$

This proves Lemma 2.11. □

Proof of Theorem 1.3. Let $n = \dim V$. Thus, $|V| = |F|^n = q^n$ (since $|F| = q$), so that $|V\setminus\{0\}| = q^n - 1$. The partitions λ and μ have length $< \dim V = n$; thus, $n > 0$, so that $q^n \equiv 0 \bmod q$ and therefore $q^n = 0$ in F. Hence, $q^n - 1 = -1$ in F.

For any vertical strip λ/ν, define $\mathbf{q}(\lambda,\nu)$ as in Lemma 2.11. It is clear that $\mathbf{q}(\lambda,\lambda) = 0$, since the sum is empty for $\nu = \lambda$.

By Lemma 2.1 (applied to $A = \mathbf{A}$ and $b_L = S_{\lambda/\mu}(V\ //\ L)$), we have

$$\sum_{L\subseteq V \text{ line}} S_{\lambda/\mu}(V\ //\ L)$$
$$= -\sum_{w\in V\setminus\{0\}} \underbrace{S_{\lambda/\mu}(V\ //\ \operatorname{span}(w))}_{\substack{=\sum\limits_{\lambda/\nu \text{ is a vertical strip}} (-1)^{|\lambda|-|\nu|} w^{\mathbf{q}(\lambda,\nu)} S_{\nu/\mu}(V)\\ \text{(by Lemma 2.11, applied to } \ell=w \text{ and } L=\operatorname{span}(w))}}$$
$$= -\sum_{w\in V\setminus\{0\}} \sum_{\lambda/\nu \text{ is a vertical strip}} (-1)^{|\lambda|-|\nu|} w^{\mathbf{q}(\lambda,\nu)} S_{\nu/\mu}(V)$$
$$= -\sum_{\lambda/\nu \text{ is a vertical strip}} (-1)^{|\lambda|-|\nu|} \left(\sum_{w\in V\setminus\{0\}} w^{\mathbf{q}(\lambda,\nu)}\right) S_{\nu/\mu}(V) \tag{32}$$

(here, we have interchanged the summation signs).

Now, let $\nu \neq \lambda$ be a partition such that λ / ν is a vertical strip. Then, $\nu \subseteq \lambda$, so that the partition ν has length $< n$ (since λ has length $< n$). Moreover, from $\nu \neq \lambda$, we conclude that at least one positive integer $i \geq 1$ satisfies $\nu_i \neq \lambda_i$ and therefore $\nu_i < \lambda_i$ (since $\nu \subseteq \lambda$ entails $\nu_i \leq \lambda_i$), that is, $\lambda_i > \nu_i$.

Let $i_1, i_2, \ldots, i_k$ be all the positive integers $i \geq 1$ satisfying $\lambda_i > \nu_i$ (listed without repetitions, so there are k of them). Then, each of these positive integers $i_1, i_2, \ldots, i_k$ must be $\leq n-1$ (since both λ and ν have length $< n$, so that each $i \geq n$ must satisfy $\lambda_i = 0 = \nu_i$, and therefore each positive integer $i \geq 1$ satisfying $\lambda_i > \nu_i$ must be $\leq n-1$). Hence, there are at most $n-1$ of them. In other words, $k \leq n-1 < n$. That is, $n > k$. But we must also have $k > 0$ (since we have shown that at least one positive integer $i \geq 1$ satisfies $\lambda_i > \nu_i$).

For each $j \in [k]$, set $a_j := \lambda_{i_j} + n - 1 - i_j$. This is a nonnegative integer, because we have $i_j \leq n-1$ (since each of the positive integers $i_1, i_2, \ldots, i_k$ is $\leq n-1$) and thus $a_j = \lambda_{i_j} + n - 1 - \underbrace{i_j}_{\leq n-1} \geq \lambda_{i_j} + n - 1 - (n-1) = \lambda_{i_j} \geq 0$.

Hence, Lemma 2.5 shows that

$$\sum_{w \in V \setminus \{0\}} w^{(q-1)(q^{a_1} + q^{a_2} + \cdots + q^{a_k})} = 0. \tag{33}$$

Since

$$\begin{aligned} & q^{a_1} + q^{a_2} + \cdots + q^{a_k} \\ & = \sum_{j=1}^{k} q^{a_j} = \sum_{j=1}^{k} q^{\lambda_{i_j} + n - 1 - i_j} \qquad \left(\text{since } a_j = \lambda_{i_j} + n - 1 - i_j\right) \\ & = \sum_{\substack{i \geq 1; \\ \lambda_i > \nu_i}} q^{\lambda_i + n - 1 - i} \qquad \left(\begin{array}{c} \text{since } i_1, i_2, \ldots, i_k \text{ are precisely the} \\ \text{positive integers } i \geq 1 \text{ satisfying } \lambda_i > \nu_i \end{array}\right), \end{aligned}$$

we have

$$(q-1)\left(q^{a_1} + q^{a_2} + \cdots + q^{a_k}\right) = (q-1) \sum_{\substack{i \geq 1; \\ \lambda_i > \nu_i}} q^{\lambda_i + n - 1 - i} = \mathbf{q}(\lambda, \nu)$$

(by the definition of $\mathbf{q}(\lambda, \nu)$). Thus, we can rewrite (33) as

$$\sum_{w \in V \setminus \{0\}} w^{\mathbf{q}(\lambda, \nu)} = 0. \tag{34}$$

Forget that we fixed ν. We thus have proved (34) for each partition $\nu \neq \lambda$ such that λ / ν is a vertical strip. Thus, on the right hand side of (32), all addends of the outer sum are 0 except for the addend for $\nu = \lambda$. Hence, (32) simplifies to

$$\sum_{L \subseteq V \text{ line}} S_{\lambda/\mu}(V \mathbin{//} L) = -\underbrace{(-1)^{|\lambda|-|\lambda|}}_{=(-1)^0=1} \left(\sum_{w \in V \setminus \{0\}} \underbrace{w^{\mathbf{q}(\lambda,\lambda)}}_{\substack{=1 \\ (\text{since } \mathbf{q}(\lambda,\lambda)=0)}} \right) S_{\lambda/\mu}(V)$$

$$= -\underbrace{\left(\sum_{w \in V \setminus \{0\}} 1 \right)}_{\substack{=|V \setminus \{0\}| \\ =q^n-1 \\ =-1 \text{ in } F}} S_{\lambda/\mu}(V) = -(-1)\, S_{\lambda/\mu}(V) = S_{\lambda/\mu}(V).$$

This proves Theorem 1.3. □

3. Macdonald's sum-over-flags formula

3.1. The sum-over-flags formula

Macdonald viewed Theorem 1.1 as a stepping stone towards an explicit sum-over-flags formula for the 7th Variation Schur functions ([Macdon92, (7.24 ?)]):

Theorem 3.1. Assume that $\mathbf{A}$ is an integral domain. Let $n \in \mathbb{N}$. Let V be an n-dimensional F-vector subspace of $\mathbf{A}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $\leq n$. Then,

$$S_\lambda(V) = \sum_{\substack{V=V_0>V_1>\cdots>V_n=0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i}(V_{i-1} \mathbin{//} V_i).$$

Since Macdonald does not explain how this can be derived from Theorem 1.1, we shall do this here. We do still need some preparations.

3.2. Lemmas

If V is a finite-dimensional F-vector subspace of $\mathbf{A}$, then $\pi(V)$ shall denote the product $\prod_{u \in V \setminus \{0\}} u$ of all nonzero vectors in V. This product will play an important role in the next few lemmas.

Lemma 3.2. Let n be a positive integer. Assume that $\mathbf{A}$ is an integral domain. Let V be an n-dimensional F-vector subspace of $\mathbf{A}$. Let $V = V_0 > V_1 > \cdots > V_n = 0$ be a complete flag in V. Then,

$$\pi(V) = \prod_{i=1}^{n} \pi(V_{i-1} \mathbin{//} V_i).$$

Proof. For each $i \in [n]$, we have $\pi\left(V_{i-1} \mathbin{//} V_i\right) = \prod_{u \in V_{i-1} - V_i} u$ (by [Macdon92, (7.21)], applied to $U = V_{i-1}$ and $U' = V_i$). Multiplying these n equalities, we find

$$\prod_{i=1}^{n} \pi\left(V_{i-1} \mathbin{//} V_i\right) = \underbrace{\prod_{i=1}^{n} \prod_{u \in V_{i-1} - V_i}}_{\substack{= \prod_{u \in V_0 - V_n} \\ \text{(since } V_0 - V_n \text{ is the union of the} \\ \text{disjoint sets } V_{i-1} - V_i \text{ for all } i \in [n])}} u = \prod_{u \in V_0 - V_n} u = \prod_{u \in V - \{0\}} u$$

(since $V_0 = V$ and $V_n = 0 = \{0\}$). But we also have $\pi(V) = \prod_{u \in V - \{0\}} u$ by the definition of $\pi(V)$. Comparing these two equalities, we find $\pi(V) = \prod_{i=1}^{n} \pi\left(V_{i-1} \mathbin{//} V_i\right)$. This proves Lemma 3.2. □

Lemma 3.3. Let U be a 1-dimensional F-vector subspace of $\mathbf{A}$. Let r be a positive integer. Then,

$$\pi(U) \cdot \varphi\left(H_{r-1}(U)\right) = -H_r(U).$$

Proof. The first equality in [Macdon92, proof of (7.22)] says that

$$H_r(U) = (-1)^r \prod_{j=1}^{r} \varphi^{j-1} \pi(U). \tag{35}$$

The same argument (applied to $r-1$ instead of r) yields

$$H_{r-1}(U) = (-1)^{r-1} \prod_{j=1}^{r-1} \varphi^{j-1} \pi(U).$$

Applying the map φ to this equality, we find

$$\begin{aligned}
\varphi\left(H_{r-1}(U)\right) &= \varphi\left((-1)^{r-1} \prod_{j=1}^{r-1} \varphi^{j-1} \pi(U)\right) \\
&= (-1)^{r-1} \prod_{j=1}^{r-1} \underbrace{\varphi\left(\varphi^{j-1} \pi(U)\right)}_{= \varphi^{j} \pi(U)} \qquad \text{(since } \varphi \text{ is a ring morphism)} \\
&= (-1)^{r-1} \prod_{j=1}^{r-1} \varphi^{j} \pi(U) = (-1)^{r-1} \prod_{j=2}^{r} \varphi^{j-1} \pi(U)
\end{aligned}$$

(here, we have substituted $j-1$ for j in the product). Thus,

$$\begin{aligned}
\pi(U)\cdot\varphi(H_{r-1}(U)) &= \underbrace{\pi(U)}_{\substack{=\varphi^0\pi(U)\\ =\varphi^{1-1}\pi(U)}}\cdot\underbrace{(-1)^{r-1}}_{=-(-1)^r}\prod_{j=2}^{r}\varphi^{j-1}\pi(U)\\
&= -(-1)^r\cdot\underbrace{\varphi^{1-1}\pi(U)\cdot\prod_{j=2}^{r}\varphi^{j-1}\pi(U)}_{=\prod_{j=1}^{r}\varphi^{j-1}\pi(U)}\\
&= -\underbrace{(-1)^r\prod_{j=1}^{r}\varphi^{j-1}\pi(U)}_{\substack{=H_r(U)\\ \text{(by (35))}}} = -H_r(U).
\end{aligned}$$

This proves Lemma 3.3. □

Lemma 3.4. Let n be a positive integer. Let V be an n-dimensional F-vector subspace of $\mathbf{A}$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition with $\lambda_n \geq 1$. Let $\lambda \ominus 1$ be the partition $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1)$. Then,

$$S_\lambda(V) = (-1)^n\,\pi(V)\cdot(S_{\lambda\ominus 1}(V))^q.$$

Proof. More generally, for any n-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, we set $\beta \ominus 1 := (\beta_1 - 1, \beta_2 - 1, \ldots, \beta_n - 1)$. This is an n-tuple of integers, but belongs to $\mathbb{N}^n$ if all entries of β are ≥ 1. In particular, $\lambda \ominus 1 \in \mathbb{N}^n$, since each $j \in [n]$ satisfies $\lambda_j \geq \lambda_n \geq 1$.

Let $\mathbf{B}$ be the polynomial ring $F[x_1, x_2, \ldots, x_n]$. Let W be the F-vector subspace $\operatorname{span}(x_1, x_2, \ldots, x_n)$ of $\mathbf{B}$. Then, the definition of $S_\lambda(x_1, x_2, \ldots, x_n)$ yields

$$S_\lambda(x_1, x_2, \ldots, x_n) = A_{\lambda+\delta}/A_\delta, \tag{36}$$

where the α-alternants A_α for all $\alpha \in \mathbb{N}^n$ are defined in (1). Similarly,

$$S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n) = A_{(\lambda\ominus 1)+\delta}/A_\delta. \tag{37}$$

Now, define an n-tuple $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$ by $\beta := \lambda + \delta$, so that $\beta_j = \lambda_j + n - j$ for each $j \in [n]$. From $\lambda + \delta = \beta$, we obtain

$$A_{\lambda+\delta} = A_\beta = \det\left(x_i^{q^{\beta_j}}\right)_{i,j\in[n]} \tag{38}$$

by (1). Moreover, each $j \in [n]$ satisfies $\beta_j = \underbrace{\lambda_j}_{\geq 1} + \underbrace{n - j}_{\geq 0} \geq 1$. Thus, each $i, j \in [n]$ satisfy

$$x_i^{q^{\beta_j}} = x_i^{q^{\beta_j - 1} q} = \left(x_i^{q^{\beta_j - 1}} \right)^q = \varphi \left(x_i^{q^{\beta_j - 1}} \right)$$

(by the definition of φ). Hence, we can rewrite (38) as

$$A_{\lambda+\delta} = \det \left(\varphi \left(x_i^{q^{\beta_j - 1}} \right) \right)_{i,j \in [n]} = \varphi \left(\det \left(\left(x_i^{q^{\beta_j - 1}} \right)_{i,j \in [n]} \right) \right) \qquad (39)$$

(since φ is a ring morphism and thus commutes with determinants).

However, $(\lambda \ominus 1) + \delta = \underbrace{(\lambda + \delta)}_{=\beta} \ominus 1 = \beta \ominus 1$. Hence,

$$A_{(\lambda \ominus 1)+\delta} = A_{\beta \ominus 1} = \det \left(\left(x_i^{q^{\beta_j - 1}} \right)_{i,j \in [n]} \right) \qquad \text{(by (1))} .$$

In view of this, we can rewrite (39) as

$$A_{\lambda+\delta} = \varphi \left(A_{(\lambda \ominus 1)+\delta} \right) .$$

Thus, (36) becomes

$$S_\lambda (x_1, x_2, \ldots, x_n) = \underbrace{A_{\lambda+\delta}}_{=\varphi\left(A_{(\lambda \ominus 1)+\delta}\right)} / A_\delta = \varphi \left(A_{(\lambda \ominus 1)+\delta} \right) / A_\delta = \frac{\varphi \left(A_{(\lambda \ominus 1)+\delta} \right)}{\varphi (A_\delta)} \cdot \frac{\varphi (A_\delta)}{A_\delta}. \qquad (40)$$

Applying the same argument to the partition $(1^n) = \left(\underbrace{1, 1, \ldots, 1}_{n \text{ times}} \right)$ instead of λ, we obtain

$$S_{(1^n)} (x_1, x_2, \ldots, x_n) = \frac{\varphi \left(A_{((1^n) \ominus 1)+\delta} \right)}{\varphi (A_\delta)} \cdot \frac{\varphi (A_\delta)}{A_\delta} = \underbrace{\frac{\varphi (A_\delta)}{\varphi (A_\delta)}}_{=1} \cdot \frac{\varphi (A_\delta)}{A_\delta} \qquad \left(\text{since } \underbrace{((1^n) \ominus 1)}_{=0} + \delta = \delta \right) = \frac{\varphi (A_\delta)}{A_\delta}.$$

Thus,

$$\begin{aligned}\frac{\varphi(A_\delta)}{A_\delta} &= S_{(1^n)}(x_1, x_2, \ldots, x_n) = S_{(1^n)}(W) \\ &\qquad \left(\begin{array}{c}\text{by the definition of } S_{(1^n)}(W), \\ \text{since } (x_1, x_2, \ldots, x_n) \text{ is a basis of } W\end{array}\right) \\ &= E_n(W) \qquad (\text{by the definition of } E_n(W)) \\ &= (-1)^n \pi(W)\end{aligned}$$

(since [Macdon92, (7.20)] says that $\pi(W) = (-1)^n E_n(W)$). Substituting this into (40), we obtain

$$S_\lambda(x_1, x_2, \ldots, x_n) = \frac{\varphi\left(A_{(\lambda\ominus 1)+\delta}\right)}{\varphi(A_\delta)} \cdot (-1)^n \pi(W).$$

In view of

$$\begin{aligned}\frac{\varphi\left(A_{(\lambda\ominus 1)+\delta}\right)}{\varphi(A_\delta)} &= \frac{A^q_{(\lambda\ominus 1)+\delta}}{A^q_\delta} \qquad (\text{by the definition of } \varphi) \\ &= \left(\underbrace{A_{(\lambda\ominus 1)+\delta} / A_\delta}_{\substack{= S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n) \\ (\text{by (37)})}}\right)^q \\ &= \left(S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n)\right)^q,\end{aligned}$$

we can rewrite this further as

$$\begin{aligned}S_\lambda(x_1, x_2, \ldots, x_n) &= \left(S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n)\right)^q \cdot (-1)^n \pi(W) \\ &= (-1)^n \pi(W) \cdot \left(S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n)\right)^q. \qquad (41)\end{aligned}$$

Now, pick a basis $(v_1, v_2, \ldots, v_n)$ of the n-dimensional F-vector space V. Let us substitute $v_1, v_2, \ldots, v_n$ for $x_1, x_2, \ldots, x_n$ on both sides of (41). This substitution transforms $S_\lambda(x_1, x_2, \ldots, x_n)$ into $S_\lambda(V)$ (since $S_\lambda(V) = S_\lambda(v_1, v_2, \ldots, v_n)$ was defined by substituting $v_1, v_2, \ldots, v_n$ for $x_1, x_2, \ldots, x_n$ in the polynomial $A_{\lambda+\delta}/A_\delta = S_\lambda(x_1, x_2, \ldots, x_n)$), and transforms $S_{\lambda\ominus 1}(x_1, x_2, \ldots, x_n)$ into $S_{\lambda\ominus 1}(V)$ (for analogous reasons); furthermore, it transforms $\pi(W)$ into $\pi(V)$ (since this substitution turns the F-linear combinations of the $x_1, x_2, \ldots, x_n$ into the corresponding F-linear combinations of $v_1, v_2, \ldots, v_n$, and is injective[3]). Hence, the result of applying this substitution to (41) is

$$S_\lambda(V) = (-1)^n \pi(V) \cdot \left(S_{\lambda\ominus 1}(V)\right)^q.$$

This proves Lemma 3.4. □

[3]In more detail: Let $\psi : \mathbf{B} \to \mathbf{A}$ be the map that substitutes $v_1, v_2, \ldots, v_n$ for $x_1, x_2, \ldots, x_n$ in any given polynomial $f \in \mathbf{B} = F[x_1, x_2, \ldots, x_n]$. Then, ψ is an F-algebra morphism, and sends $x_1, x_2, \ldots, x_n$ to $v_1, v_2, \ldots, v_n$. Hence, ψ restricts to an isomorphism from the F-vector

3.3. Proof of the formula

Proof of Theorem 3.1. We use strong induction on $|\lambda| + n$. The *base case* ($|\lambda| + n = 0$) is entirely trivial (because in this case, $n = 0$, so that $V = 0$ and $\lambda = \varnothing$). For the *induction step*, we fix a positive integer N. We assume (as the induction hypothesis) that Theorem 3.1 holds whenever $|\lambda| + n < N$. We must now prove Theorem 3.1 whenever $|\lambda| + n = N$.

So we fix some n and λ satisfying $|\lambda| + n = N$. Then, $|\lambda| + n = N > 0$, so that $n > 0$ (since $n = 0$ would force $\lambda = \varnothing$ and thus $|\lambda| + n = |\varnothing| + 0 = |\varnothing| = 0$). Hence, λ_n is well-defined. We are in one of the following two cases:

Case 1: We have $\lambda_n = 0$.

Case 2: We have $\lambda_n \geq 1$.

Let us consider Case 1. In this case, we have $\lambda_n = 0$. Thus, the partition λ has length $< n = \dim V$. Hence, Theorem 1.1 yields

$$S_\lambda (V) = \sum_{L \subseteq V \text{ line}} S_\lambda (V \mathbin{/\!/} L). \tag{43}$$

Now, let L be any line in V. Then, $\dim L = 1$. But what we said about internal quotients long ago[4] entails that $V \mathbin{/\!/} L \cong V / L$ as F-vector spaces, via the canonical isomorphism

$$\begin{aligned} V / L &\to V \mathbin{/\!/} L, \\ \overline{v} &\mapsto \widetilde{f_L} (v) = f_L (v). \end{aligned} \tag{44}$$

space W to V (since ψ sends the basis $(x_1, x_2, \ldots, x_n)$ of W to the basis $(v_1, v_2, \ldots, v_n)$ of V). Therefore, ψ also restricts to a bijection from $W \setminus \{0\}$ to $V \setminus \{0\}$. In other words, the map

$$\begin{aligned} W \setminus \{0\} &\to V \setminus \{0\}, \\ u &\mapsto \psi (u) \end{aligned} \tag{42}$$

is a bijection.

But the definition of $\pi (V)$ yields $\pi (V) = \prod_{u \in V \setminus \{0\}} u$. Similarly, $\pi (W) = \prod_{u \in W \setminus \{0\}} u$. Applying the map ψ to the latter equality, we find

$$\begin{aligned} \psi (\pi (W)) &= \psi \left(\prod_{u \in W \setminus \{0\}} u \right) = \prod_{u \in W \setminus \{0\}} \psi (u) \qquad \text{(since } \psi \text{ is an } F\text{-algebra morphism)} \\ &= \prod_{u \in V \setminus \{0\}} u \end{aligned}$$

(here, we have substituted u for $\psi (u)$ in the product, since the map (42) is a bijection). Comparing this with $\pi (V) = \prod_{u \in V \setminus \{0\}} u$, we obtain $\psi (\pi (W)) = \pi (V)$. In other words, the substitution of $v_1, v_2, \ldots, v_n$ for $x_1, x_2, \ldots, x_n$ transforms $\pi (W)$ into $\pi (V)$ (since this substitution is precisely ψ).

[4]specifically: the fact that if $\mathbf{A}$ is an integral domain, and if $U \subseteq V$ are two finite-dimensional F-vector subspaces of $\mathbf{A}$, then the internal quotient $V \mathbin{/\!/} U$ is isomorphic to the actual quotient V / U

Thus, $\dim(V \mathbin{/\!/} L) = \dim(V/L) = \underbrace{\dim V}_{=n} - \underbrace{\dim L}_{=1} = n-1$. In other words, the vector space $V \mathbin{/\!/} L$ is $(n-1)$-dimensional. Hence, by the induction hypothesis, we can apply Theorem 3.1 to $V \mathbin{/\!/} L$ and $n-1$ instead of V and n (since $\lambda_n = 0$ yields $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$). We thus obtain

$$S_\lambda(V \mathbin{/\!/} L) = \sum_{\substack{V/\!/L = W_0 > W_1 > \cdots > W_{n-1} = 0 \\ \text{is a complete flag in } V/\!/L}} \prod_{i=1}^{n-1} H_{\lambda_i}(W_{i-1} \mathbin{/\!/} W_i). \tag{45}$$

However, it is well-known (the "lattice isomorphism theorem") that there is an inclusion-respecting bijection[5]

$$\{\text{subspaces } U \text{ of } V \text{ such that } V \geq U \geq L\} \to \{\text{subspaces of } V/L\}, \qquad U \mapsto U/L,$$

which furthermore reduces the dimension of the subspace by 1 (meaning that $\dim(U/L) = \dim U - 1$ for any U).

We can use the isomorphism (44) to replace V/L by $V \mathbin{/\!/} L$ here; thus we obtain an inclusion-respecting bijection

$$\{\text{subspaces } U \text{ of } V \text{ such that } V \geq U \geq L\} \to \{\text{subspaces of } V \mathbin{/\!/} L\}, \qquad U \mapsto \widetilde{f}_L(U) = U \mathbin{/\!/} L,$$

which again reduces the dimension of the subspace by 1 (since $\dim(U \mathbin{/\!/} L) = \dim(U/L) = \dim U - 1$ for any U).

Since this bijection is inclusion-respecting and reduces dimension by 1, we thus obtain a bijection

$$\{\text{complete flags } V = V_0 > V_1 > \cdots > V_{n-1} = L > 0 \text{ in } V\}$$
$$\to \{\text{complete flags } V \mathbin{/\!/} L = W_0 > W_1 > \cdots > W_{n-1} = 0 \text{ in } V \mathbin{/\!/} L\}$$

that sends each flag $V = V_0 > V_1 > \cdots > V_{n-1} = L > 0$ to the flag $V \mathbin{/\!/} L = W_0 > W_1 > \cdots > W_{n-1} = 0$ given by $W_i = V_i \mathbin{/\!/} L$. Using this bijection to reindex the sum in (45), we can rewrite (45) as

$$S_\lambda(V \mathbin{/\!/} L) = \sum_{\substack{V = V_0 > V_1 > \cdots > V_{n-1} = L > 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n-1} H_{\lambda_i}((V_{i-1} \mathbin{/\!/} L) \mathbin{/\!/} (V_i \mathbin{/\!/} L)). \tag{46}$$

However, [Macdon92, (7.16)] says that if U, V, T are three subspaces of $\mathbf{A}$ satisfying $T \leq U \leq V$, then

$$V \mathbin{/\!/} U = (V \mathbin{/\!/} T) \mathbin{/\!/} (U \mathbin{/\!/} T).$$

[5] Here and in the following, the word "subspace" means "F-vector subspace". A map ϕ whose domain and target consist of sets is said to be *inclusion-respecting* if it has the property that if two sets A and B in its domain satisfy $A \subseteq B$, then $\phi(A) \subseteq \phi(B)$.

Hence, for each $i \in [n-1]$ and each complete flag $V = V_0 > V_1 > \cdots > V_{n-1} = L > 0$ in V, we obtain

$$V_{i-1} \mathbin{/\!/} V_i = (V_{i-1} \mathbin{/\!/} L) \mathbin{/\!/} (V_i \mathbin{/\!/} L) \tag{47}$$

(since $L \leq V_i \leq V_{i-1}$). Thus, (46) becomes

$$\begin{aligned} S_\lambda (V \mathbin{/\!/} L) &= \sum_{\substack{V=V_0>V_1>\cdots>V_{n-1}=L>0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n-1} H_{\lambda_i} \left(\underbrace{(V_{i-1} \mathbin{/\!/} L) \mathbin{/\!/} (V_i \mathbin{/\!/} L)}_{\substack{=V_{i-1} /\!/ V_i \\ \text{(by (47))}}} \right) \\ &= \sum_{\substack{V=V_0>V_1>\cdots>V_{n-1}=L>0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n-1} H_{\lambda_i} (V_{i-1} \mathbin{/\!/} V_i) . \end{aligned} \tag{48}$$

Forget that we fixed L. We thus have proved (48) for each line L in V. Substituting (48) into (43), we find

$$\begin{aligned} S_\lambda (V) &= \underbrace{\sum_{L \subseteq V \text{ line}} \ \sum_{\substack{V=V_0>V_1>\cdots>V_{n-1}=L>0 \\ \text{is a complete flag in } V}}}_{\substack{= \sum_{\substack{V=V_0>V_1>\cdots>V_{n-1}>0 \\ \text{is a complete flag in } V}} \\ = \sum_{\substack{V=V_0>V_1>\cdots>V_n=0 \\ \text{is a complete flag in } V}} \\ \text{(here, we have set } V_n=0 \text{ for the sake of uniformity)}}} \prod_{i=1}^{n-1} H_{\lambda_i} (V_{i-1} \mathbin{/\!/} V_i) \\ &= \sum_{\substack{V=V_0>V_1>\cdots>V_n=0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n-1} H_{\lambda_i} (V_{i-1} \mathbin{/\!/} V_i) . \end{aligned} \tag{49}$$

However, $\lambda_n = 0$, and thus $H_{\lambda_n} (V_{n-1} \mathbin{/\!/} V_n) = H_0 (V_{n-1} \mathbin{/\!/} V_n) = 1$ (since $H_0 (U) = 1$ for any subspace U of $\mathbf{A}$). Hence, the product $\prod_{i=1}^{n-1} H_{\lambda_i} (V_{i-1} \mathbin{/\!/} V_i)$ does not change if we extend it to encompass $i = n$ (since the extra factor that it thus gains is $H_{\lambda_n} (V_{n-1} \mathbin{/\!/} V_n) = 1$). Thus, by extending this product in this way, we can rewrite (49) as

$$S_\lambda (V) = \sum_{\substack{V=V_0>V_1>\cdots>V_n=0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i} (V_{i-1} \mathbin{/\!/} V_i) .$$

In other words, Theorem 3.1 holds for our n and λ. This completes the induction step in Case 1.

Let us now consider Case 2. In this case, we have $\lambda_n \geq 1$. Hence, Lemma 3.4 yields

$$S_\lambda(V) = (-1)^n \pi(V) \cdot \left(S_{\lambda \ominus 1}(V)\right)^q, \tag{50}$$

where $\lambda \ominus 1$ has been defined in Lemma 3.4. Since $|\lambda \ominus 1| = |\lambda| - n < |\lambda|$ (because $n > 0$), we have $|\lambda \ominus 1| + n < |\lambda| + n$. Thus, the induction hypothesis shows that Theorem 3.1 holds for $\lambda \ominus 1$ instead of λ. That is, we have

$$S_{\lambda \ominus 1}(V) = \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i) \tag{51}$$

(since $\lambda \ominus 1 = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_n - 1)$). Taking this equality to the q-th power, we find

$$\begin{aligned}
\left(S_{\lambda \ominus 1}(V)\right)^q &= \left(\sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i) \right)^q \\
&= \varphi \left(\sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i) \right) \\
&\qquad \left(\text{by the definition of } \varphi\right) \\
&= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} \varphi\left(H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i)\right) \\
&\qquad \left(\text{since } \varphi \text{ is an } F\text{-algebra morphism}\right).
\end{aligned}$$

Substituting this into (50), we obtain

$$\begin{aligned}
S_\lambda(V) &= (-1)^n \pi(V) \cdot \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} \varphi\left(H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i)\right) \\
&= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \underbrace{(-1)^n}_{= \prod_{i=1}^{n} (-1)} \underbrace{\pi(V)}_{\substack{= \prod_{i=1}^{n} \pi(V_{i-1} // V_i) \\ \text{(by Lemma 3.2)}}} \prod_{i=1}^{n} \varphi\left(H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i)\right) \\
&= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \underbrace{\prod_{i=1}^{n} (-1) \cdot \prod_{i=1}^{n} \pi(V_{i-1} \mathbin{//} V_i) \cdot \prod_{i=1}^{n} \varphi\left(H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i)\right)}_{= \prod_{i=1}^{n} \left(-\pi(V_{i-1} // V_i) \cdot \varphi\left(H_{\lambda_i - 1}(V_{i-1} // V_i)\right)\right)} \\
&= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} \left(-\pi(V_{i-1} \mathbin{//} V_i) \cdot \varphi\left(H_{\lambda_i - 1}(V_{i-1} \mathbin{//} V_i)\right)\right).
\end{aligned}$$

However, if $V = V_0 > V_1 > \cdots > V_n = 0$ is a complete flag in V, and if $i \in [n]$ is arbitrary, then the F-vector subspace $V_{i-1} \mathbin{/\!/} V_i$ of $\mathbf{A}$ is 1-dimensional[6], and thus satisfies

$$\pi\left(V_{i-1} \mathbin{/\!/} V_i\right) \cdot \varphi\left(H_{\lambda_i - 1}\left(V_{i-1} \mathbin{/\!/} V_i\right)\right) = -H_{\lambda_i}\left(V_{i-1} \mathbin{/\!/} V_i\right)$$

(by Lemma 3.3, applied to $U = V_{i-1} \mathbin{/\!/} V_i$ and $r = \lambda_i$), so that

$$-\pi\left(V_{i-1} \mathbin{/\!/} V_i\right) \cdot \varphi\left(H_{\lambda_i - 1}\left(V_{i-1} \mathbin{/\!/} V_i\right)\right) = H_{\lambda_i}\left(V_{i-1} \mathbin{/\!/} V_i\right). \tag{52}$$

Hence, we can continue our computation of $S_\lambda(V)$ as follows:

$$\begin{aligned} S_\lambda(V) &= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} \underbrace{\left(-\pi\left(V_{i-1} \mathbin{/\!/} V_i\right) \cdot \varphi\left(H_{\lambda_i - 1}\left(V_{i-1} \mathbin{/\!/} V_i\right)\right)\right)}_{\substack{= H_{\lambda_i}(V_{i-1} /\!/ V_i) \\ \text{(by (52))}}} \\ &= \sum_{\substack{V = V_0 > V_1 > \cdots > V_n = 0 \\ \text{is a complete flag in } V}} \prod_{i=1}^{n} H_{\lambda_i}\left(V_{i-1} \mathbin{/\!/} V_i\right). \end{aligned}$$

In other words, Theorem 3.1 holds for our n and λ. This completes the induction step in Case 2.

Hence, the induction step is completed in both cases. Thus, the inductive proof of Theorem 3.1 is complete. □

A. Appendix: Folklore proofs

A.1. Appendix: Details for the proof of Lemma 2.10

In our above proof of Lemma 2.10, we have said that (26) can be derived from (27) using the Cauchy–Binet theorem. Let us explain in detail how this derivation proceeds. We will need a tailored variant of the Cauchy–Binet theorem, which we shall first derive from a more classical version.

For the rest of this section, we fix a commutative ring R.

We shall use matrices whose rows and columns can be indexed by arbitrary objects, not just numbers. An *$I \times J$-matrix* (where I and J are two sets) is a matrix whose rows are indexed by the elements of I and whose columns are indexed by the elements of J. The space of all $I \times J$-matrices over R will be called $R^{I \times J}$. If $A = \left(a_{i,j}\right)_{i \in I,\ j \in J}$ is any $I \times J$-matrix, and if $(i_1, i_2, \ldots, i_k) \in I^k$ and $(j_1, j_2, \ldots, j_\ell) \in J^\ell$ are any finite lists of elements of I and J, respectively (for some $k, \ell \in \mathbb{N}$), then $\operatorname{sub}_{i_1, i_2, \ldots, i_k}^{j_1, j_2, \ldots, j_\ell} A$ shall denote the $k \times \ell$-matrix $\left(a_{i_x, j_y}\right)_{x \in [k],\ y \in [\ell]}$.

[6] Indeed, the same reasoning that gave us $\dim(V \mathbin{/\!/} L) = \dim(V/L)$ above can be used to show that $\dim(V_{i-1} \mathbin{/\!/} V_i) = \dim(V_{i-1}/V_i)$. But $\dim(V_{i-1}/V_i) = 1$ since $V = V_0 > V_1 > \cdots > V_n = 0$ is a complete flag. Hence, $\dim(V_{i-1} \mathbin{/\!/} V_i) = \dim(V_{i-1}/V_i) = 1$, so that $V_{i-1} \mathbin{/\!/} V_i$ is 1-dimensional.

One of the forms of the *Cauchy–Binet theorem* (see, e.g., [Grinbe20, Corollary 7.182]) says the following:

Proposition A.1. Let $A \in R^{n \times p}$ and $B \in R^{p \times m}$ be two matrices over R. Let $u \in \mathbb{N}$. Let $(i_1, i_2, \ldots, i_u) \in [n]^u$ and $(j_1, j_2, \ldots, j_u) \in [m]^u$ be two u-tuples of elements of $[n]$ and $[m]$, respectively. Then,

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) = \sum_{g_1<g_2<\cdots<g_u} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right), \tag{53}$$

where the sum ranges over all strictly increasing u-tuples $(g_1, g_2, \ldots, g_u) \in [p]^u$.

By relabelling the rows and columns of both matrices A and B here, we can rewrite this result in the following equivalent form:

Proposition A.2. Let N and M be two finite sets, and let P be a finite totally ordered set. Let $A \in R^{N \times P}$ and $B \in R^{P \times M}$ be two matrices over R. Let $u \in \mathbb{N}$. Let $(i_1, i_2, \ldots, i_u) \in N^u$ and $(j_1, j_2, \ldots, j_u) \in M^u$ be two u-tuples of elements of N and M, respectively. Then,

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) = \sum_{g_1<g_2<\cdots<g_u} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right), \tag{54}$$

where the sum ranges over all strictly increasing u-tuples $(g_1, g_2, \ldots, g_u) \in P^u$.

Proof. Just rename the elements of N as $1, 2, \ldots, n$, rename the elements of M as $1, 2, \ldots, m$, and rename the elements of P as $1, 2, \ldots, p$ in increasing order. Then, the claimed equality (54) becomes precisely (53). □

Reversing the order of the totally ordered set P, we can replace the condition $g_1 < g_2 < \cdots < g_u$ under the summation sign in (54) by the opposite condition $g_1 > g_2 > \cdots > g_u$ (that is, we can sum over the strictly decreasing u-tuples instead of the strictly increasing ones). Thus, we transform Proposition A.2 into the following proposition:

Proposition A.3. Let N and M be two finite sets, and let P be a finite totally ordered set. Let $A \in R^{N \times P}$ and $B \in R^{P \times M}$ be two matrices over R. Let $u \in \mathbb{N}$. Let $(i_1, i_2, \ldots, i_u) \in N^u$ and $(j_1, j_2, \ldots, j_u) \in M^u$ be two u-tuples of elements of N and M, respectively. Then,

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) = \sum_{g_1>g_2>\cdots>g_u} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right)\cdot\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right), \tag{55}$$

where the sum ranges over all strictly decreasing u-tuples $(g_1, g_2, \ldots, g_u) \in P^u$.

Now, let us restrict this result to upper-triangular matrices.

First we recall how they are defined: If P is a totally ordered set, then a $P \times P$-matrix $C = (c_{i,j})_{i\in P,\ j\in P} \in R^{P\times P}$ is said to be *upper-triangular* if its entries satisfy $c_{i,j} = 0$ whenever $i > j$.

Now, if the matrices A and B in Proposition A.3 are upper-triangular, then the sum on the right hand side of (55) can be made significantly shorter by removing many vanishing addends:

Proposition A.4. Let P be a finite totally ordered set. Let $A \in R^{P\times P}$ and $B \in R^{P\times P}$ be two upper-triangular matrices over R. Let $u \in \mathbb{N}$. Let $(i_1, i_2, \ldots, i_u) \in P^u$ and $(j_1, j_2, \ldots, j_u) \in P^u$ be two u-tuples of elements of P. Assume that $i_1 > i_2 > \cdots > i_u$ and $j_1 > j_2 > \cdots > j_u$. Then,

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) = \sum_{\substack{g_1>g_2>\cdots>g_u;\\ i_k\leq g_k\leq j_k \text{ for all } k\in[u]}} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right)\cdot\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right). \tag{56}$$

Proof. Write the $P \times P$-matrices A and B as $A = (a_{i,j})_{i\in P,\ j\in P}$ and $B = (b_{i,j})_{i\in P,\ j\in P}$.

Let $(g_1, g_2, \ldots, g_u) \in P^u$ be a strictly decreasing u-tuple of elements of P. Thus, $g_1 > g_2 > \cdots > g_u$.

We will use the following classical and easy fact (see, e.g., [Grinbe20, Exercise 6.47 **(a)**]): If $C = (c_{x,y})_{x,y\in[u]} \in R^{u\times u}$ is a $u \times u$-matrix, and if X and Y are two subsets of $[u]$ satisfying $|X| + |Y| > u$ and $\left(c_{x,y} = 0 \text{ for all } x \in X \text{ and } y \in Y\right)$, then $\det C = 0$. We will refer to this fact as the *too-many-zeroes lemma*. This lemma gives us two consequences in our specific situation:

- If some $k \in [u]$ satisfies $i_k > g_k$, then

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) = 0. \tag{57}$$

  [*Proof:* Assume that some $k \in [u]$ satisfies $i_k > g_k$. Consider this k.

We have $\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A = \left(a_{i_x,g_y}\right)_{x,y\in[u]}$ (since $A = \left(a_{i,j}\right)_{i\in P,\ j\in P}$).

Now, let $x \in \{1,2,\ldots,k\}$ and $y \in \{k,k+1,\ldots,u\}$ be arbitrary. We shall show that $a_{i_x,g_y} = 0$.

Indeed, from $x \in \{1,2,\ldots,k\}$, we obtain $x \leq k$ and thus $i_x \geq i_k$ (since $i_1 > i_2 > \cdots > i_u$). Moreover, from $y \in \{k,k+1,\ldots,u\}$, we obtain $y \geq k$, hence $k \leq y$ and thus $g_k \geq g_y$ (since $g_1 > g_2 > \cdots > g_u$). Thus, $i_x \geq i_k > g_k \geq g_y$, so that $a_{i_x,g_y} = 0$ (since the matrix $A = \left(a_{i,j}\right)_{i\in P,\ j\in P}$ is upper-triangular).

Forget that we fixed x and y. We thus have shown that $a_{i_x,g_y} = 0$ for all $x \in \{1,2,\ldots,k\}$ and $y \in \{k,k+1,\ldots,u\}$. Hence, the too-many-zeroes lemma (applied to the matrix $\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A = \left(a_{i_x,g_y}\right)_{x,y\in[u]}$ instead of $C = \left(c_{x,y}\right)_{x,y\in[u]}$, and to the sets $\{1,2,\ldots,k\}$ and $\{k,k+1,\ldots,u\}$ instead of X and Y) yields $\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) = 0$ (since $\underbrace{\left|\{1,2,\ldots,k\}\right|}_{=k} + \underbrace{\left|\{k,k+1,\ldots,u\}\right|}_{=u-k+1} = k + (u-k+1) = u+1 > u$). This proves (57).]

- If some $k \in [u]$ satisfies $g_k > j_k$, then

$$\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right) = 0. \tag{58}$$

[*Proof:* This is analogous to the proof of (57) (but now using B, $b_{i,j}$, g_x and j_y instead of A, $a_{i,j}$, i_x and g_y).]

- Thus, if we don't have ($i_k \leq g_k \leq j_k$ for all $k \in [u]$), then

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right) = 0. \tag{59}$$

[*Proof:* Assume that we don't have ($i_k \leq g_k \leq j_k$ for all $k \in [u]$). Hence, there exists some $k \in [u]$ such that we don't have $i_k \leq g_k \leq j_k$. Consider this k. Thus, we have $i_k > g_k$ or $g_k > j_k$ (since we don't have $i_k \leq g_k \leq j_k$). In the former case, we have

$$\underbrace{\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right)}_{\substack{=0 \\ \text{(by (57), since } i_k > g_k)}} \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right) = 0.$$

In the latter case, we have

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \underbrace{\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right)}_{\substack{=0 \\ \text{(by (58), since } g_k > j_k)}} = 0.$$

Thus, $\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right) = 0$ is proved in both cases. This completes the proof of (59).]

Forget that we fixed $(g_1, g_2, \ldots, g_u)$. We thus have proved (59) for each strictly decreasing u-tuple $(g_1, g_2, \ldots, g_u) \in P^u$ that does not satisfy $(i_k \leq g_k \leq j_k$ for all $k \in [u])$.

Now, (55) (applied to $N = P$ and $M = P$) becomes

$$\begin{aligned}
&\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) \\
&= \sum_{g_1 > g_2 > \cdots > g_u} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right) \\
&= \sum_{\substack{g_1 > g_2 > \cdots > g_u; \\ i_k \leq g_k \leq j_k \text{ for all } k \in [u]}} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right) \cdot \det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right)
\end{aligned}$$

(here, we have removed all addends that don't satisfy $(i_k \leq g_k \leq j_k$ for all $k \in [u])$ from our sum, since (59) shows that all these addends are 0). This proves (56). □

We shall now extend (56) to infinite matrices. It is not always possible to multiply two $P \times P$-matrices $A, B \in R^{P \times P}$ when the set P is infinite; for example, the product $(1)_{i,j \in \mathbb{Z}} \cdot (1)_{i,j \in \mathbb{Z}}$ makes no sense because its entries would be the divergent infinite sums $\sum\limits_{k \in \mathbb{Z}} 1$. However, in some cases, such products are defined. One such case is when the matrices are upper-triangular and the set P is equipped with an interval-finite total order. We recall the definition:

If P is a totally ordered set, and if $i, j \in P$ are two elements, then the *interval* $[i, j]_P$ is defined to be the set $\{k \in P \mid i \leq k \leq j\}$. A totally ordered set P is said to be *interval-finite* if for any two elements $i, j \in P$, the interval $[i, j]_P = \{k \in P \mid i \leq k \leq j\}$ is finite. If P is an interval-finite totally ordered set, and if $A = (a_{i,j})_{i \in P,\ j \in P} \in R^{P \times P}$ and $B = (b_{i,j})_{i \in P,\ j \in P} \in R^{P \times P}$ are two upper-triangular $P \times P$-matrices, then the product AB is well-defined, since its (i, j)-th entry (for all $i, j \in P$) is

$$\sum_{k \in P} \underbrace{a_{i,k} b_{k,j}}_{\substack{=0 \text{ unless } k \in [i,j]_P \\ (\text{because } a_{i,k} = 0 \text{ when } k < i, \\ \text{whereas } b_{k,j} = 0 \text{ when } k > j)}} = \underbrace{\sum_{k \in [i,j]_P} a_{i,k} b_{k,j}}_{\substack{\text{a finite sum,} \\ \text{since } [i,j]_P \text{ is finite}}} .$$

Moreover, this product AB is itself upper-triangular (since $[i, j]_P = \varnothing$ unless $i \leq j$).

We can now generalize (56) to infinite matrices:

Proposition A.5. Let P be an interval-finite totally ordered set. Let $A \in R^{P \times P}$ and $B \in R^{P \times P}$ be two upper-triangular matrices over R. Let $u \in \mathbb{N}$. Let

$(i_1, i_2, \ldots, i_u) \in P^u$ and $(j_1, j_2, \ldots, j_u) \in P^u$ be two u-tuples of elements of P. Assume that $i_1 > i_2 > \cdots > i_u$ and $j_1 > j_2 > \cdots > j_u$. Then,

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)\right) = \sum_{\substack{g_1>g_2>\cdots>g_u;\\ i_k\leq g_k\leq j_k \text{ for all } k\in[u]}} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right)\cdot\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right). \tag{60}$$

Proof. If $u = 0$, then the claim is trivial (since the determinant of a 0×0-matrix is 1). Thus, we WLOG assume that $u \geq 1$.

Pick two elements $m_- \in P$ and $m_+ \in P$ such that all the $2u$ elements $i_1, i_2, \ldots, i_u, j_1, j_2, \ldots, j_u$ belong to the interval $[m_-, m_+]_P$ (this is always possible: just let m_- be the smallest of these $2u$ elements, and m_+ be the largest of them). This interval $[m_-, m_+]_P$ is finite (since P is interval-finite). Moreover, the definition of m_- and m_+ yields $(i_1, i_2, \ldots, i_u) \in [m_-, m_+]_P^u$ and $(j_1, j_2, \ldots, j_u) \in [m_-, m_+]_P^u$.

Write the $P \times P$-matrices A and B as $A = (a_{i,j})_{i\in P,\ j\in P}$ and $B = (b_{i,j})_{i\in P,\ j\in P}$.

For each upper-triangular $P \times P$-matrix $C = (c_{i,j})_{i\in P,\ j\in P} \in R^{P\times P}$, we let $\widetilde{C}$ be the $[m_-, m_+]_P \times [m_-, m_+]_P$-matrix $(c_{i,j})_{i,j\in[m_-,m_+]_P} \in R^{[m_-,m_+]_P\times[m_-,m_+]_P}$. (This is simply the submatrix of C in which only the rows and the columns indexed by the elements of $[m_-, m_+]_P$ have been kept.) Thus, $\widetilde{A} = (a_{i,j})_{i,j\in[m_-,m_+]_P}$ and $\widetilde{B} = (b_{i,j})_{i,j\in[m_-,m_+]_P}$. Clearly, these matrices $\widetilde{A}$ and $\widetilde{B}$ are upper-triangular (since A and B are).

It is easy to see that

$$\widetilde{AB} = \widetilde{A} \cdot \widetilde{B}. \tag{61}$$

[7].

[7]*Proof:* Let $i, j \in [m_-, m_+]_P$. Then, the (i, j)-th entry of $\widetilde{AB}$ is the sum $\sum_{k\in P} a_{i,k}b_{k,j}$, whereas the (i, j)-th entry of $\widetilde{A} \cdot \widetilde{B}$ is the sum $\sum_{k\in[m_-,m_+]_P} a_{i,k}b_{k,j}$. But these two sums are equal, since

$$\begin{aligned}\sum_{k\in P} a_{i,k}b_{k,j} &= \sum_{\substack{k\in P;\\ k<m_-}} \underbrace{a_{i,k}}_{\substack{=0\\ (\text{since } k<m_-\leq i\\ (\text{because } i\in[m_-,m_+]_P),\\ \text{hence } i>k,\\ \text{but } A \text{ is upper-triangular})}} b_{k,j} + \underbrace{\sum_{\substack{k\in P;\\ m_-\leq k\leq m_+}}}_{=\sum_{k\in[m_-,m_+]_P}} a_{i,k}b_{k,j} + \sum_{\substack{k\in P;\\ k>m_+}} a_{i,k} \underbrace{b_{k,j}}_{\substack{=0\\ (\text{since } k>m_+\geq j\\ (\text{because } j\in[m_-,m_+]_P),\\ \text{but } B \text{ is upper-triangular})}} \\ &\qquad \left(\begin{array}{c}\text{because each } k\in P \text{ satisfies}\\ \text{either } k<m_- \text{ or } m_-\leq k\leq m_+ \text{ or } k>m_+\end{array}\right) \\ &= \underbrace{\sum_{\substack{k\in P;\\ k<m_-}} 0b_{k,j}}_{=0} + \sum_{k\in[m_-,m_+]_P} a_{i,k}b_{k,j} + \underbrace{\sum_{\substack{k\in P;\\ k>m_+}} a_{i,k}0}_{=0} = \sum_{k\in[m_-,m_+]_P} a_{i,k}b_{k,j}.\end{aligned}$$

But we can apply (56) to $[m_-, m_+]_P$, $\widetilde{A}$ and $\widetilde{B}$ instead of P, A and B (since $[m_-, m_+]_P$ is finite). Thus we obtain

$$\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}\left(\widetilde{A}\cdot\widetilde{B}\right)\right) = \sum_{\substack{g_1>g_2>\cdots>g_u;\\ i_k\leq g_k\leq j_k \text{ for all } k\in[u]}} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u}\widetilde{A}\right)\cdot\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u}\widetilde{B}\right). \qquad (62)$$

Here, the indices $g_1, g_2, \ldots, g_u$ in the sum are supposed to belong to $[m_-, m_+]_P$, but we could just as well relax this requirement and instead demand them to belong to P, because the second condition "$i_k \leq g_k \leq j_k$ for all $k \in [u]$" would force them to belong to $[m_-, m_+]_P$ anyway (indeed, if $i_k \leq g_k \leq j_k$ for all $k \in [u]$, then each $k \in [u]$ satisfies $m_- \leq i_k \leq g_k \leq j_k \leq m_+$ and thus $g_k \in [m_-, m_+]_P$). Thus, the sum on the right hand side of (62) ranges over the exact same u-tuples $(g_1, g_2, \ldots, g_u)$ as the sum on the right hand side of (60). Moreover, each such u-tuple $(g_1, g_2, \ldots, g_u)$ satisfies

$$\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u}\widetilde{A} = \operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A$$

(since $\widetilde{A}$ is a submatrix of A that preserves the same indexing as A: i.e., the (i,j)-th entry of $\widetilde{A}$ equals the (i,j)-th entry of A whenever the former is defined) and

$$\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u}\widetilde{B} = \operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B$$

(similarly). Hence, we can rewrite (62) as

$$\begin{aligned}&\det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}\left(\widetilde{A}\cdot\widetilde{B}\right)\right)\\ &= \sum_{\substack{g_1>g_2>\cdots>g_u;\\ i_k\leq g_k\leq j_k \text{ for all } k\in[u]}} \det\underbrace{\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u}\widetilde{A}\right)}_{=\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A}\cdot\det\underbrace{\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u}\widetilde{B}\right)}_{=\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B}\\ &= \sum_{\substack{g_1>g_2>\cdots>g_u;\\ i_k\leq g_k\leq j_k \text{ for all } k\in[u]}} \det\left(\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{g_1,g_2,\ldots,g_u} A\right)\cdot\det\left(\operatorname{sub}_{g_1,g_2,\ldots,g_u}^{j_1,j_2,\ldots,j_u} B\right).\end{aligned}$$

In order to prove (60), it thus remains to show that

$$\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB) = \operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}\left(\widetilde{A}\cdot\widetilde{B}\right). \qquad (63)$$

But this is easy: We have

$$\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}\left(\widetilde{AB}\right) = \operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB)$$

So we have showed that the (i,j)-th entries of the matrices $\widetilde{AB}$ and $\widetilde{A}\cdot\widetilde{B}$ are equal. Since we have showed this for all $i, j \in [m_-, m_+]_P$, we thus conclude that $\widetilde{AB} = \widetilde{A}\cdot\widetilde{B}$.

(since $\widetilde{AB}$ is a submatrix of AB that preserves the same indexing as AB: i.e., the (i,j)-th entry of $\widetilde{AB}$ equals the (i,j)-th entry of AB whenever the former is defined) and thus

$$\operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}(AB) = \operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u} \underbrace{\left(\widetilde{AB}\right)}_{\substack{=\widetilde{A}\cdot\widetilde{B} \\ \text{(by (61))}}} = \operatorname{sub}_{i_1,i_2,\ldots,i_u}^{j_1,j_2,\ldots,j_u}\left(\widetilde{A}\cdot\widetilde{B}\right).$$

This proves (63). Thus, our proof of (60) is complete. □

Let us also record a simple property of determinants:

Lemma A.6. Let $u \in \mathbb{N}$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_u)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_u)$ be two partitions of length $\leq u$. Let $\left(c_{i,j}\right)_{i,j\in[u]} \in R^{u\times u}$ be any $u \times u$-matrix. Then,

$$\det\left(\left((-1)^{\lambda_i-\nu_j-i+j}\, c_{i,j}\right)_{i,j\in[u]}\right) = (-1)^{|\lambda|-|\nu|} \det\left(\left(c_{i,j}\right)_{i,j\in[u]}\right).$$

Proof. For any $i, j \in [u]$, we have

$$(-1)^{\lambda_i-\nu_j-i+j} = (-1)^{(\lambda_i-i)+\left(j-\nu_j\right)} = (-1)^{\lambda_i-i}\,(-1)^{j-\nu_j}. \tag{64}$$

But the matrix $\left((-1)^{\lambda_i-i}\,(-1)^{j-\nu_j}\, c_{i,j}\right)_{i,j\in[u]}$ is obviously obtained from the matrix $\left(c_{i,j}\right)_{i,j\in[u]}$ by the following operations:

1. Scale the i-th row by the factor $(-1)^{\lambda_i-i}$ for each $i \in [u]$.
2. Scale the j-th column by the factor $(-1)^{j-\nu_j}$ for each $j \in [u]$.

The effect of these operations on the determinant of the matrix is that the determinant gets multiplied by $\left(\prod\limits_{i=1}^{u} (-1)^{\lambda_i-i}\right)\left(\prod\limits_{j=1}^{u} (-1)^{j-\nu_j}\right)$ (because when we scale a row or a column of a matrix by a factor λ, the determinant of this matrix gets multiplied by λ). Thus,

$$\det\left(\left((-1)^{\lambda_i-i}\,(-1)^{j-\nu_j}\, c_{i,j}\right)_{i,j\in[u]}\right) = \left(\prod_{i=1}^{u} (-1)^{\lambda_i-i}\right)\left(\prod_{j=1}^{u} (-1)^{j-\nu_j}\right) \det\left(\left(c_{i,j}\right)_{i,j\in[u]}\right).$$

In view of

$$\underbrace{\left(\prod_{i=1}^{u}(-1)^{\lambda_i-i}\right)}_{\substack{=(-1)^{(\lambda_1-1)+(\lambda_2-2)+\cdots+(\lambda_u-u)}\\=(-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_u)-(1+2+\cdots+u)}}}\underbrace{\left(\prod_{j=1}^{u}(-1)^{j-\nu_j}\right)}_{\substack{=(-1)^{(1-\nu_1)+(2-\nu_2)+\cdots+(u-\nu_u)}\\=(-1)^{(1+2+\cdots+u)-(\nu_1+\nu_2+\cdots+\nu_u)}}}$$
$$= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_u)-(1+2+\cdots+u)}(-1)^{(1+2+\cdots+u)-(\nu_1+\nu_2+\cdots+\nu_u)}$$
$$= (-1)^{((\lambda_1+\lambda_2+\cdots+\lambda_u)-(1+2+\cdots+u))+((1+2+\cdots+u)-(\nu_1+\nu_2+\cdots+\nu_u))}$$
$$= (-1)^{(\lambda_1+\lambda_2+\cdots+\lambda_u)-(\nu_1+\nu_2+\cdots+\nu_u)} = (-1)^{|\lambda|-|\nu|}$$

(because $\lambda_1+\lambda_2+\cdots+\lambda_u = |\lambda|$ and $\nu_1+\nu_2+\cdots+\nu_u = |\nu|$), we can rewrite this as

$$\det\left(\left((-1)^{\lambda_i-i}(-1)^{j-\nu_j}c_{i,j}\right)_{i,j\in[u]}\right) = (-1)^{|\lambda|-|\nu|}\det\left(\left(c_{i,j}\right)_{i,j\in[u]}\right).$$

Using (64), we can furthermore rewrite this as

$$\det\left(\left((-1)^{\lambda_i-\nu_j-i+j}c_{i,j}\right)_{i,j\in[u]}\right) = (-1)^{|\lambda|-|\nu|}\det\left(\left(c_{i,j}\right)_{i,j\in[u]}\right).$$

Thus, Lemma A.6 is proved. □

We can now finish the proof of Lemma 2.10:

Details for the proof of Lemma 2.10. We must derive (26) from (27).

Pick $u \in \mathbb{N}$ such that both $\ell(\lambda)$ and $\ell(\mu)$ are $\leq u$. Thus, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_u)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_u)$.

Let R be the commutative ring $\mathbf{A}$. Clearly, the totally ordered set $\mathbb{Z}$ is interval-finite. Recall that both matrices $\mathbf{H}(V)$ and $\mathbf{E}(U)$ are upper-triangular matrices in $\mathbf{A}^{\mathbb{Z}\times\mathbb{Z}} = R^{\mathbb{Z}\times\mathbb{Z}}$. Hence, the matrix $\varphi^{\dim(V/U)}(\mathbf{E}(U))$ is upper-triangular as well (since $\varphi^{\dim(V/U)}$ is a ring endomorphism). Moreover, from $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_u$ (since μ is a partition), we obtain $\mu_1 - 1 > \mu_2 - 2 > \cdots > \mu_u - u$. Similarly, $\lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_u - u$.

Thus, we can apply (60) to $P = \mathbb{Z}$ and $A = \mathbf{H}(V)$ and $B = \varphi^{\dim(V/U)}(\mathbf{E}(U))$

and $i_x = \mu_x - x$ and $j_y = \lambda_y - y$. As a result, we obtain

$$\begin{aligned}
&\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}\left(\mathbf{H}(V)\cdot\varphi^{\dim(V/U)}(\mathbf{E}(U))\right)\right)\\
&=\sum_{\substack{g_1>g_2>\cdots>g_u;\\ \mu_k-k\leq g_k\leq\lambda_k-k \text{ for all } k\in[u]}}\det\left(\operatorname{sub}^{g_1,g_2,\ldots,g_u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}(\mathbf{H}(V))\right)\\
&\qquad\cdot\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{g_1,g_2,\ldots,g_u}\left(\varphi^{\dim(V/U)}(\mathbf{E}(U))\right)\right)\\
&=\sum_{\substack{\nu_1\geq\nu_2\geq\cdots\geq\nu_u;\\ \mu_k-k\leq \nu_k-k\leq\lambda_k-k \text{ for all } k\in[u]}}\det\left(\operatorname{sub}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}(\mathbf{H}(V))\right)\\
&\qquad\cdot\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\varphi^{\dim(V/U)}(\mathbf{E}(U))\right)\right)
\end{aligned}\tag{65}$$

(here, we have substituted $\nu_k - k$ for g_k in the sum, noting that this substitution transforms the chain of inequalities $g_1 > g_2 > \cdots > g_u$ into $\nu_1 - 1 > \nu_2 - 2 > \cdots > \nu_u - u$, which is equivalent to $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_u$ because $\nu_1, \nu_2, \ldots, \nu_u$ are integers). Of course, the condition "$\mu_k - k \leq \nu_k - k \leq \lambda_k - k$ for all $k \in [u]$" under the summation sign in (65) is equivalent to "$\mu_k \leq \nu_k \leq \lambda_k$ for all $k \in [u]$"; thus the summation sign can be rewritten as follows:

$$\sum_{\substack{\nu_1\geq\nu_2\geq\cdots\geq\nu_u;\\ \mu_k-k\leq \nu_k-k\leq\lambda_k-k \text{ for all } k\in[u]}} = \sum_{\substack{\nu_1\geq\nu_2\geq\cdots\geq\nu_u;\\ \mu_k\leq \nu_k\leq\lambda_k \text{ for all } k\in[u]}} = \sum_{\substack{(\nu_1,\nu_2,\ldots,\nu_u)\\ \text{is a partition of length } \leq u;\\ \mu_k\leq \nu_k\leq\lambda_k \text{ for all } k\in[u]}}$$

(because the condition "$\mu_k \leq \nu_k \leq \lambda_k$ for all $k \in [u]$" forces all ν_k to be nonnegative[8], and then the condition "$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_u$" is simply saying that $(\nu_1, \nu_2, \ldots, \nu_u)$ is a partition of length $\leq u$). Thus, we can rewrite (65) as

$$\begin{aligned}
&\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}\left(\mathbf{H}(V)\cdot\varphi^{\dim(V/U)}(\mathbf{E}(U))\right)\right)\\
&=\sum_{\substack{(\nu_1,\nu_2,\ldots,\nu_u)\\ \text{is a partition of length } \leq u;\\ \mu_k\leq \nu_k\leq\lambda_k \text{ for all } k\in[u]}}\det\left(\operatorname{sub}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}(\mathbf{H}(V))\right)\\
&\qquad\cdot\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\varphi^{\dim(V/U)}(\mathbf{E}(U))\right)\right).
\end{aligned}$$

[8] since it entails $\nu_k \geq \mu_k \geq 0$ for each $k \in [u]$

In view of (27), we can furthermore rewrite this as

$$\det\left(\operatorname{sub}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}\left(\mathbf{H}\left(V\mathbin{/\!/}U\right)\right)\right)$$

$$= \underbrace{\sum_{\substack{(\nu_1,\nu_2,\ldots,\nu_u) \\ \text{is a partition of length } \leq u; \\ \mu_k\leq\nu_k\leq\lambda_k \text{ for all } k\in[u]}}}_{\substack{=\sum_{\substack{\nu=(\nu_1,\nu_2,\ldots,\nu_u) \\ \text{is a partition of length } \leq u; \\ \mu\subseteq\nu\subseteq\lambda}} \\ \text{(since the condition ``}\mu_k\leq\nu_k\leq\lambda_k \text{ for all } k\in[u]\text{''} \\ \text{is equivalent to ``}\mu\subseteq\nu\subseteq\lambda\text{'' when } \nu=(\nu_1,\nu_2,\ldots,\nu_u))}} \det\left(\operatorname{sub}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{H}\left(V\right)\right)\right)$$

$$\cdot \underbrace{\det\left(\operatorname{sub}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}\left(\varphi^{\dim(V/U)}\left(\mathbf{E}\left(U\right)\right)\right)\right)}_{\substack{=\varphi^{\dim(V/U)}\left(\det\left(\operatorname{sub}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}(\mathbf{E}(U))\right)\right) \\ \text{(since } \varphi^{\dim(V/U)} \text{ is a ring morphism, and thus commutes} \\ \text{with taking determinants and submatrices)}}}$$

$$= \sum_{\substack{\nu=(\nu_1,\nu_2,\ldots,\nu_u) \\ \text{is a partition of length } \leq u; \\ \mu\subseteq\nu\subseteq\lambda}} \det\left(\operatorname{sub}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{H}\left(V\right)\right)\right)$$

$$\cdot\, \varphi^{\dim(V/U)}\left(\det\left(\operatorname{sub}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}\left(\mathbf{E}\left(U\right)\right)\right)\right). \qquad (66)$$

Furthermore, if $\nu = (\nu_1, \nu_2, \ldots, \nu_u)$ is any partition of length $\leq u$, then

$$\operatorname{sub}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{H}\left(V\right)\right) = \left(\varphi^{\mu_x-x+1}H_{(\nu_y-y)-(\mu_x-x)}\left(V\right)\right)_{x,y\in[u]}$$

$$\left(\text{since } \mathbf{H}\left(V\right) = \left(\varphi^{i+1}H_{j-i}\left(V\right)\right)_{i,j\in\mathbb{Z}}\right)$$

$$= \left(\varphi^{\mu_x-x+1}H_{\nu_y-\mu_x-y+x}\left(V\right)\right)_{x,y\in[u]}$$

and thus

$$\det\left(\operatorname{sub}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}^{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{H}\left(V\right)\right)\right)$$

$$= \det\left(\left(\varphi^{\mu_x-x+1}H_{\nu_y-\mu_x-y+x}\left(V\right)\right)_{x,y\in[u]}\right)$$

$$= \det\left(\left(\varphi^{\mu_j-j+1}H_{\nu_i-\mu_j-i+j}\left(V\right)\right)_{j,i\in[u]}\right)$$

$$= \det\left(\left(\varphi^{\mu_j-j+1}H_{\nu_i-\mu_j-i+j}\left(V\right)\right)_{i,j\in[u]}\right)$$

$$\left(\begin{array}{c}\text{since the determinant of a matrix} \\ \text{equals the determinant of its transpose}\end{array}\right)$$

$$= S_{\nu/\mu}\left(V\right) \qquad \text{(by (6))}. \qquad (67)$$

The same argument (but with V and ν replaced by $V \mathbin{//} U$ and λ) shows that

$$\begin{aligned}&\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\mu_1-1,\mu_2-2,\ldots,\mu_u-u}\left(\mathbf{H}\left(V \mathbin{//} U\right)\right)\right)\\&=S_{\lambda/\mu}\left(V \mathbin{//} U\right).\end{aligned} \tag{68}$$

Furthermore, if $\nu = (\nu_1, \nu_2, \ldots, \nu_u)$ is any partition of length $\leq u$, then

$$\begin{aligned}&\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{E}\left(U\right)\right)\\&=\left((-1)^{(\lambda_y-y)-(\nu_x-x)}\,\varphi^{\lambda_y-y}E_{(\lambda_y-y)-(\nu_x-x)}\left(U\right)\right)_{x,y\in[u]}\\&\qquad\left(\text{since }\mathbf{E}\left(U\right)=\left((-1)^{j-i}\,\varphi^{j}E_{j-i}\left(U\right)\right)_{i,j\in\mathbb{Z}}\right)\\&=\left((-1)^{\lambda_y-\nu_x-y+x}\,\varphi^{\lambda_y-y}E_{\lambda_y-\nu_x-y+x}\left(U\right)\right)_{x,y\in[u]}\end{aligned}$$

and thus

$$\begin{aligned}&\det\left(\operatorname{sub}^{\lambda_1-1,\lambda_2-2,\ldots,\lambda_u-u}_{\nu_1-1,\nu_2-2,\ldots,\nu_u-u}\left(\mathbf{E}\left(U\right)\right)\right)\\&=\det\left(\left((-1)^{\lambda_y-\nu_x-y+x}\,\varphi^{\lambda_y-y}E_{\lambda_y-\nu_x-y+x}\left(U\right)\right)_{x,y\in[u]}\right)\\&=\det\left(\left((-1)^{\lambda_i-\nu_j-i+j}\,\varphi^{\lambda_i-i}E_{\lambda_i-\nu_j-i+j}\left(U\right)\right)_{j,i\in[u]}\right)\\&=\det\left(\left((-1)^{\lambda_i-\nu_j-i+j}\,\varphi^{\lambda_i-i}E_{\lambda_i-\nu_j-i+j}\left(U\right)\right)_{i,j\in[u]}\right)\\&\qquad\left(\begin{array}{c}\text{since the determinant of a matrix}\\\text{equals the determinant of its transpose}\end{array}\right)\\&=(-1)^{|\lambda|-|\nu|}\underbrace{\det\left(\left(\varphi^{\lambda_i-i}E_{\lambda_i-\nu_j-i+j}\left(U\right)\right)_{i,j\in[u]}\right)}_{\substack{=\widetilde{S}_{\lambda/\nu}(U)\\\text{(by (16))}}}\\&\qquad\left(\text{by Lemma A.6, applied to }c_{i,j}=\varphi^{\lambda_i-i}E_{\lambda_i-\nu_j-i+j}\left(U\right)\right)\\&=(-1)^{|\lambda|-|\nu|}\,\widetilde{S}_{\lambda/\nu}\left(U\right).\end{aligned} \tag{69}$$

Using (67), (68) and (69), we can rewrite the equality (66) as

$$S_{\lambda/\mu}(V \,/\!/\, U) = \sum_{\substack{\nu=(\nu_1,\nu_2,\ldots,\nu_u) \\ \text{is a partition of length } \leq u; \\ \mu\subseteq\nu\subseteq\lambda}} S_{\nu/\mu}(V) \cdot \underbrace{\varphi^{\dim(V/U)}\left((-1)^{|\lambda|-|\nu|}\widetilde{S}_{\lambda/\nu}(U)\right)}_{\substack{=(-1)^{|\lambda|-|\nu|}\varphi^{\dim(V/U)}\widetilde{S}_{\lambda/\nu}(U) \\ (\text{since } \varphi^{\dim(V/U)} \text{ is a ring morphism})}}$$

$$= \sum_{\substack{\nu=(\nu_1,\nu_2,\ldots,\nu_u) \\ \text{is a partition of length } \leq u; \\ \mu\subseteq\nu\subseteq\lambda}} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{\dim(V/U)}\widetilde{S}_{\lambda/\nu}(U)$$

$$= \sum_{\substack{\nu \text{ is a partition;} \\ \mu\subseteq\nu\subseteq\lambda}} (-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{\dim(V/U)}\widetilde{S}_{\lambda/\nu}(U) \qquad (70)$$

(here, we have removed the condition "ν has length $\leq u$" from the summation sign, since this condition is automatically implied by the condition "$\nu \subseteq \lambda$"). This is almost the desired formula (26). The only difference is that the right hand side of (26) contains more addends than the right hand side of (70), since the partition ν is not required to satisfy $\mu \subseteq \nu \subseteq \lambda$ in (26). However, this difference is immaterial: If a partition ν does not satisfy $\mu \subseteq \nu \subseteq \lambda$, then

- it either fails to satisfy $\mu \subseteq \nu$, in which case we have $S_{\nu/\mu}(V) = 0$ by (7) and therefore
$$(-1)^{|\lambda|-|\nu|} \underbrace{S_{\nu/\mu}(V)}_{=0} \cdot \varphi^{\dim(V/U)}\widetilde{S}_{\lambda/\nu}(U) = 0;$$

- or it fails to satisfy $\nu \subseteq \lambda$, in which case we have $\widetilde{S}_{\lambda/\nu}(U) = 0$ by (18) and therefore
$$(-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{\dim(V/U)} \underbrace{\widetilde{S}_{\lambda/\nu}(U)}_{=0} = 0.$$

In both cases, we obtain $(-1)^{|\lambda|-|\nu|} S_{\nu/\mu}(V) \cdot \varphi^{\dim(V/U)}\widetilde{S}_{\lambda/\nu}(U) = 0$. Thus, all addends on the right hand side of (26) that don't appear on the right hand side of (70) are 0, and therefore do not affect the sum. Consequently, the two right hand sides are equal. Thus, (26) follows from (70), so that the proof of Lemma 2.10 is complete. □

A.2. Appendix: Proof of Theorem 1.1

To prove Theorem 1.1, we need two simple functoriality lemmas. The first is a functoriality for the S_λ:

Lemma A.7. Let **A** and **B** be two commutative F-algebras. Let $\psi : \mathbf{B} \to \mathbf{A}$ be an F-algebra morphism, and let V be a finite-dimensional F-vector subspace of **B**. Assume that the restriction $\psi \mid_V$ is injective. Let λ be a partition. Then,

$$\psi\left(S_\lambda\left(V\right)\right) = S_\lambda\left(\psi\left(V\right)\right).$$

Proof. Pick a basis $(v_1, v_2, \ldots, v_n)$ of the F-vector space V. Then, the n-tuple $(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$ is a basis of $\psi(V)$ (since the restriction $\psi \mid_V$ is injective). Hence, $\dim(\psi(V)) = n = \dim V$.

We are in one of the following two cases:

Case 1: We have $\ell(\lambda) > n$.

Case 2: We have $\ell(\lambda) \leq n$.

Consider Case 1 first. In this case, $\ell(\lambda) > n$. Hence, λ has length $\ell(\lambda) > n = \dim V$. Thus, (4) shows that $S_\lambda(V) = 0$. But λ also has length $\ell(\lambda) > n = \dim(\psi(V))$, and therefore (4) shows that $S_\lambda(\psi(V)) = 0$. Hence, Lemma A.7 holds in Case 1 (since both $S_\lambda(V)$ and $S_\lambda(\psi(V))$ are 0).

Let us now consider Case 2. In this case, $\ell(\lambda) \leq n$. Now, (3) yields $S_\lambda(V) = S_\lambda(v_1, v_2, \ldots, v_n)$, and similarly

$$S_\lambda(\psi(V)) = S_\lambda(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$$

(since $(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$ is a basis of $\psi(V)$).

However, $S_\lambda(v_1, v_2, \ldots, v_n)$ is defined by substituting $v_1, v_2, \ldots, v_n$ for $x_1, x_2, \ldots, x_n$ into a certain polynomial $A_{\lambda+\delta}/A_\delta \in F[x_1, x_2, \ldots, x_n]$; likewise, $S_\lambda(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$ is defined by substituting $\psi(v_1), \psi(v_2), \ldots, \psi(v_n)$ into the same polynomial. Hence,

$$S_\lambda(\psi(v_1), \psi(v_2), \ldots, \psi(v_n)) = \psi(S_\lambda(v_1, v_2, \ldots, v_n))$$

(since ψ is an F-algebra morphism and thus commutes with polynomials). In other words, $S_\lambda(\psi(V)) = \psi(S_\lambda(V))$ (since $S_\lambda(\psi(V)) = S_\lambda(\psi(v_1), \psi(v_2), \ldots, \psi(v_n))$ and $S_\lambda(V) = S_\lambda(v_1, v_2, \ldots, v_n)$). Thus, Lemma A.7 is proved in Case 2.

We have now proved Lemma A.7 in both Cases 1 and 2. Hence, Lemma A.7 always holds. □

Our next functoriality lemma says that internal quotients are functorial with respect to F-algebra morphisms that are injective on the relevant subspaces:

Lemma A.8. Let **A** and **B** be two commutative F-algebras that are integral domains. Let $\psi : \mathbf{B} \to \mathbf{A}$ be an F-algebra morphism, and let $U \subseteq W$ be two finite-dimensional F-vector subspaces of **B**. Assume that the restriction $\psi \mid_W$ is injective. Then, $\psi \mid_{W // U}$ is an F-vector space isomorphism from $W \mathbin{//} U$ to $\psi(W) \mathbin{//} \psi(U)$.

Proof. We assumed that the restriction $\psi\mid_W$ is injective. Hence, the restriction $\psi\mid_U$ is injective as well (since $U \subseteq W$). Thus, the map $U \to \psi(U)$, $u \mapsto \psi(u)$ is a bijection.

By definition of internal quotients, we have

$$\begin{aligned} W \mathbin{/\!/} U = \widetilde{f}_U(W) &= \left\{ \widetilde{f}_U(w) \ \mid \ w \in W \right\} \\ &= \left\{ \prod_{u \in U} (w+u) \ \mid \ w \in W \right\} \end{aligned} \tag{71}$$

(since each $w \in W$ satisfies $\widetilde{f}_U(w) = f_U(w) = \prod_{u \in U} (w+u)$ by the definition of f_U) and similarly

$$\psi(W) \mathbin{/\!/} \psi(U) = \left\{ \prod_{u \in \psi(U)} (w+u) \ \mid \ w \in \psi(W) \right\}. \tag{72}$$

Applying the map ψ to both sides of (71), we find

$$\begin{aligned} \psi(W \mathbin{/\!/} U) &= \psi\left(\left\{ \prod_{u \in U} (w+u) \ \mid \ w \in W \right\} \right) \\ &= \left\{ \psi\left(\prod_{u \in U} (w+u) \right) \ \mid \ w \in W \right\}. \end{aligned} \tag{73}$$

However, each $w \in W$ satisfies

$$\begin{aligned} &\psi\left(\prod_{u \in U} (w+u) \right) \\ &= \prod_{u \in U} (\psi(w) + \psi(u)) \qquad \text{(since } \psi \text{ is an } F\text{-algebra morphism)} \\ &= \prod_{u \in \psi(U)} (\psi(w) + u) \end{aligned}$$

(here, we have substituted u for $\psi(u)$ in the product, since the map $U \to \psi(U)$, $u \mapsto \psi(u)$ is a bijection). Thus, we can rewrite (73) as

$$\begin{aligned} \psi(W \mathbin{/\!/} U) &= \left\{ \prod_{u \in \psi(U)} (\psi(w) + u) \ \mid \ w \in W \right\} \\ &= \left\{ \prod_{u \in \psi(U)} (w+u) \ \mid \ w \in \psi(W) \right\} \end{aligned}$$

(here, we have substituted w for $\psi(w)$ in the set, since $\psi(W)$ is the set of all $\psi(w)$ with $w \in W$). Comparing this with (72), we obtain

$$\psi(W \mathbin{//} U) = \psi(W) \mathbin{//} \psi(U).$$

Thus, the restriction $\psi \mid_{W//U}$ is a well-defined and surjective map from $W \mathbin{//} U$ to $\psi(W) \mathbin{//} \psi(U)$. Of course, this restriction is furthermore F-linear (since ψ is F-linear). It remains to prove that it is injective; then (combined with F-linearity and surjectivity) it will automatically follow that $\psi \mid_{W//U}$ is an isomorphism.

So let us prove that $\psi \mid_{W//U}$ is injective. Since $\psi \mid_{W//U}$ is F-linear, it suffices to show that $\operatorname{Ker}(\psi \mid_{W//U}) = 0$.

So let $r \in \operatorname{Ker}(\psi \mid_{W//U})$. Thus, $r \in W \mathbin{//} U$ and $\psi(r) = 0$. Since $r \in W \mathbin{//} U = \widetilde{f_U}(W)$, we can write r as $r = \widetilde{f_U}(w)$ for some $w \in W$. Consider this w. Then,

$$r = \widetilde{f_U}(w) = f_U(w) = \prod_{u \in U} (w + u)$$

(by the definition of f_U), and thus

$$\psi(r) = \psi\left(\prod_{u \in U} (w + u)\right) = \prod_{u \in U} (\psi(w) + \psi(u))$$

(since ψ is an F-algebra morphism), so that $\prod\limits_{u \in U} (\psi(w) + \psi(u)) = \psi(r) = 0$. Since $\mathbf{A}$ is an integral domain, this entails that $\psi(w) + \psi(u) = 0$ for some $u \in U$ (because a product in an integral domain can only be 0 if one of its factors is 0). Consider this u. Thus, $\psi(w) + \psi(u) = 0$, so that $\psi(w) = -\psi(u) = \psi(-u)$ (since ψ is an F-algebra morphism). Since $\psi \mid_W$ is injective (and since $w \in W$ and $-\underbrace{u}_{\in U} \in -U \subseteq U \subseteq W$), we thus conclude that $w = -\underbrace{u}_{\in U} \in -U \subseteq U \subseteq \operatorname{Ker} \widetilde{f_U}$ (since we know that $\widetilde{f_U}$ always contains U in its kernel). Thus, $\widetilde{f_U}(w) = 0$, so that $r = \widetilde{f_U}(w) = 0$.

Forget that we fixed r. We thus have shown that $r = 0$ for each $r \in \operatorname{Ker}(\psi \mid_{W//U})$. Hence, $\operatorname{Ker}(\psi \mid_{W//U}) = 0$, so that $\psi \mid_{W//U}$ is injective. As we explained above, this completes the proof of Lemma A.8. □

We can now derive Theorem 1.1 from Theorem 1.3:

Proof of Theorem 1.1. We cannot directly apply Theorem 1.3, since we have not assumed that the Frobenius morphism $\varphi : \mathbf{A} \to \mathbf{A}$ is invertible. Instead, we shall apply Theorem 1.3 to a new F-algebra $\widehat{\mathbf{B}}$ whose Frobenius morphism is invertible, and then transport the result to $\mathbf{A}$ via an F-algebra morphism.

Here are the details: Let $n = \dim V$, and pick a basis $(v_1, v_2, \ldots, v_n)$ of the F-vector space V. Let $\mathbf{B}$ be the polynomial ring $F[x_1, x_2, \ldots, x_n]$ in n indeterminates. By its universal property, there is an F-algebra morphism $\psi : \mathbf{B} \to \mathbf{A}$ that sends each x_i to v_i. Consider this ψ. Let W be the F-vector subspace of

B spanned by $x_1, x_2, \ldots, x_n$ (that is, the space of all homogeneous polynomials of degree 1 in **B**). Then, the F-linear map ψ sends the basis $(x_1, x_2, \ldots, x_n)$ of W to the basis $(v_1, v_2, \ldots, v_n)$ of V. Thus, ψ restricts to a vector space isomorphism from W to V. In particular, the restriction $\psi\mid_W$ is injective, and we have $\psi(W) = V$. Moreover, the lines $L \subseteq V$ are exactly the images of the lines $M \subseteq W$ under this isomorphism from W to V, and this yields a 1-to-1 correspondence $M \mapsto \psi(M)$ between the lines $M \subseteq W$ and the lines $L \subseteq V$.

However, Remark 1.2 **(a)** (applied to **B** instead of **A**) shows that **B** can be embedded into a larger commutative F-algebra $\widehat{\mathbf{B}}$ whose Frobenius morphism $\varphi : \widehat{\mathbf{B}} \to \widehat{\mathbf{B}}$ is invertible. Consider this larger algebra $\widehat{\mathbf{B}}$. Moreover, $\widehat{\mathbf{B}}$ is an integral domain (again by Remark 1.2 **(a)**). Hence, we can apply Theorem 1.3 to $\widehat{\mathbf{B}}$, W and $\varnothing$ instead of **A**, V and μ. This yields

$$S_{\lambda/\varnothing}(W) = \sum_{L \subseteq W \text{ line}} S_{\lambda/\varnothing}(W /\!/ L) = \sum_{M \subseteq W \text{ line}} S_{\lambda/\varnothing}(W /\!/ M).$$

In view of (8), we can rewrite this as

$$S_\lambda(W) = \sum_{M \subseteq W \text{ line}} S_\lambda(W /\!/ M). \tag{74}$$

This is an equality in $\widehat{\mathbf{B}}$, thus an equality in **B** (since both of its sides lie in the subring **B** of $\widehat{\mathbf{B}}$).

Let us now apply the F-algebra morphism $\psi : \mathbf{B} \to \mathbf{A}$ to both sides of this equality. Thus, we find

$$\begin{aligned}\psi(S_\lambda(W)) &= \psi\left(\sum_{M \subseteq W \text{ line}} S_\lambda(W /\!/ M)\right) \\ &= \sum_{M \subseteq W \text{ line}} \psi(S_\lambda(W /\!/ M)) \end{aligned} \tag{75}$$

(since ψ is F-linear). However, Lemma A.7 (applied to W instead of V) yields $\psi(S_\lambda(W)) = S_\lambda(\psi(W))$ (since the restriction $\psi\mid_W$ is injective). In view of $\psi(W) = V$, we rewrite this as $\psi(S_\lambda(W)) = S_\lambda(V)$. Comparing this with (75), we obtain

$$S_\lambda(V) = \sum_{M \subseteq W \text{ line}} \psi(S_\lambda(W /\!/ M)). \tag{76}$$

Now, let $M \subseteq W$ be a line. Then, Lemma A.8 (applied to $U = M$) yields that $\psi\mid_{W /\!/ M}$ is an F-vector space isomorphism from $W /\!/ M$ to $\psi(W) /\!/ \psi(M)$. Hence, in particular, the restriction $\psi\mid_{W/\!/M}$ is injective, and we have $\psi(W /\!/ M) = \underbrace{\psi(W)}_{=V} /\!/ \psi(M) = V /\!/ \psi(M)$. Therefore, Lemma A.7 (applied to $W /\!/ M$ instead of V) yields

$$\psi(S_\lambda(W /\!/ M)) = S_\lambda\left(\underbrace{\psi(W /\!/ M)}_{=V /\!/ \psi(M)}\right) = S_\lambda(V /\!/ \psi(M)).$$

Forget that we fixed M. We thus have proved that

$$\psi\left(S_{\lambda}\left(W \mathbin{//} M\right)\right) = S_{\lambda}\left(V \mathbin{//} \psi\left(M\right)\right) \qquad \text{for each line } M \subseteq W.$$

Thus, we can rewrite (76) as

$$S_{\lambda}\left(V\right) = \sum_{M \subseteq W \text{ line}} S_{\lambda}\left(V \mathbin{//} \psi\left(M\right)\right) = \sum_{L \subseteq V \text{ line}} S_{\lambda}\left(V \mathbin{//} L\right)$$

(here, we have substituted L for $\psi\left(M\right)$ in the sum, since we have a 1-to-1 correspondence $M \mapsto \psi\left(M\right)$ between the lines $M \subseteq W$ and the lines $L \subseteq V$). Thus, Theorem 1.1 is proved. □

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