Hopf algebras, from basics to applications to renormalization

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Revised and updated version, may 2006

arXiv:math/0408405v2

Errata and questions (version 2)

- There are two different references called [EGK1].

- **Introduction:** In the second line of page 2, you have a redundant comma ("element of $\mathcal{A}_-$, , and φ_+ is a map").

- **I.1, before Proposition I.1.1:** You write: "for any bilinear map f from $A \times B$ into C". I would consider replacing the "into" by a "to" here, since some people read "into" as "injective".

- **Proof of Proposition I.1.2:** You claim that the map j "is easily seen to be injective". While the proof of this injectivity indeed looks easy when one reads it (see e. g. http://mathoverflow.net/questions/72013/homa-c-homb-d-injects-into-homab-cd-when-why), I am not sure whether this proof is that easy to come up with on one's own. You might want to give a few hints here...

- **I.2.1:** The definitions of a left ideal and of a right ideal should be interchanged: You define a subspace $J \subseteq A$ to be a left ideal if $m(J \otimes A)$ is included in J, and a right ideal if $m(A \otimes J)$ is included in J; but this should be exactly the other way round. (Besides, there is a closing bracket missing between "$m(J \otimes A + A \otimes J)$" and "is included in J".)

- **I.2.3:** You write: "A *left A-module* is a k-vector space M together with a map". Maybe replace "map" by "k-linear map" here, unless you think this is clear to the reader anyway.

- **I.2.3:** In the second commutative diagram on page 7, replace the A in the lower right corner by an M.

- **I.2.3:** You write: "A left module M is *simple* if it does not contain any submodule different from $\{0\}$ or M itself." You should add "and is nonzero" here, because otherwise the trivial module $\{0\}$ would be a simple module, making many of the following results wrong.

- **Proof of Proposition I.2.1:** You write: "and by simplicity of M the map :

$$\phi_m : A \to M$$
$$a \mapsto a.m$$

gives rise to an morphism of left A-modules from A/J_m onto M". I don't think you use the simplicity of M here - unless the "onto M" part means that this map is surjective (but it is useless to state this here, since you state it again one line below). Thus I would propose removing "by simplicity of M" from this sentence, and replacing the "onto" by a "to". Also, "an morphism" should be "a morphism".

- **Proof of Proposition I.2.2:** Here you write: "write (thanks to semi-simplicity) $M = N \oplus T$ where T is another A-submodule of M". This is indeed a consequence of semi-simplicity, but not of the way you have defined semi-simplicity! You defined a module to be *semi-simple* if it can be written as a direct sum of simple modules. But what you use here is that a module is semi-simple iff every submodule of it is a direct addend. The equivalence is not completely trivial (although not too hard).

- **Corollary I.2.3:** Some people might misinterpret the word "into" as a statement that the map is injective (which is wrong). I would replace this word by "to".

- **I.2.4:** You write: "The *radical* $\operatorname{rad} M$ of a left module is by definition [...]". I would replace "a left module" by "a left module M" here (otherwise, the letter M is never defined).

- **Remark at the end of I.2.4:** Here you write "Jacobson ideal" twice. This should be "Jacobson radical". Also, replace $\operatorname{Rad} A$ by $\operatorname{rad} A$ (to make the notation compatible with the rest of your text).

- **Proof of Lemma I.2.9:** Replace "for any $i \in \{0, ..., n\}$" by "for any $i \in \{1, ..., n\}$".

- **Proof of Proposition I.2.8:** Replace "a finite-dimensional primitive ideal" by "a finite-<u>co</u>dimensional primitive ideal".

- **Proof of Proposition I.2.8:** Here you write: "But A''_M is a matrix algebra over D". I don't find this that obvious - what you are using here is that M is a finite-dimensional D-module, and every finite-dimensional D-module is free (which is because D is a skew field, and because Gaussian elimination and most of the linear algebra based on it work over skew fields just as well as over fields), so that M is free.

- **Proof of Proposition I.2.8:** Replace "according to lemma I.2.7" by "according to lemma I.2.9".

- **I.3.1:** On the first line of I.3.1, you write: "Coalgebras are objects wich [...]". There is an obvious typo here.

- **I.3.1:** Between the two commutative diagrams on page 11, you write: "Coalgebra C is *co-unital* if moreover there is a co-unit ε such that the following diagram commutes :". For the sake of completeness, I would replace "co-unit ε" by "co-unit $\varepsilon : C \to k$" here.

- **I.3.1:** In the definition of subcoalgebras (as well as left coideals, right coideals and two-sided coideals), you write: "is contained in $J \otimes J$ (resp. $C \otimes J$, $J \otimes C$, $J \otimes C + C \otimes J$) is included in J". Clearly, the "is included in J" part of this sentence should be removed.

- **Proposition I.3.1, 1):** I don't think the product of C^* is really the "transpose" of the coproduct of C. The coproduct of C is $\Delta : C \to C \otimes C$, and thus its transpose is $\Delta^* : (C \otimes C)^* \to C^*$. To get the product of C^*, we have to compose this with the injection $C^* \otimes C^* \to (C \otimes C)^*$.

- **Proposition I.3.1, 1):** Replace "co-unity" by "co-unit".
- **Proof of Proposition I.3.1:** Replace "$\Delta x \subset J \otimes C + C \otimes J$" by "$\Delta x \in J \otimes C + C \otimes J$".
- **Proposition I.3.2:** I am not sure about this, but I think that this proposition is false. More precisely, all the "if" parts are correct (cf. Sweedler, Proposition 1.4.3 b) and further), but the "only if" parts are not (or at least they don't seem correct to me). Also I think the words "Dually we have the following" before this proposition are misleading - this proposition does not follow from Proposition I.3.1 by duality.

  I think I have a counterexample to the "only if" part: Let C be a connected filtered coalgebra with $\operatorname{Prim} C$ (the space of primitive elements of C) infinite-dimensional (for instance, take C to be the tensor Hopf algebra of a vector space of dimension ≥ 2, or the shuffle Hopf algebra of an infinite-dimensional vector space). Let $(e_i)_{i \in J}$ be a basis of $\operatorname{Prim} C$, let x be an object not in J, and let e_x be the unity 1 of C (of course, C, being connected filtered, has a unity). Then, $(e_i)_{i \in J \cup \{x\}}$ is a basis of the subspace $\operatorname{Prim} C + k \cdot 1$ of C. Extend this basis to a basis $(e_i)_{i \in I}$ of C (with $I \supseteq J \cup \{x\}$). Now define a $g_i \in C$ for every $i \in I$ as follows:

  $$g_i = \begin{cases} 1, \text{ if } i = x \text{ (in this case, } e_i = 1 \text{ as well);} \\ 1 + e_i, \text{ if } i \neq x \end{cases}.$$

  It is easy to see that $(g_i)_{i \in I}$ is still a basis of C. Now, let $(f_i)_{i \in I}$ be the dual "basis" of C^* to the basis $(g_i)_{i \in I}$ of C (this means that f_i is the projection on the g_i-coordinate for every $i \in I$); of course, $(f_i)_{i \in I}$ is not really a basis, but at least a linearly independent subset.

  Now define a subspace K of C^* by $K = \langle f_i \mid i \in I \rangle$. Then, clearly, $K^\perp = 0$ is a two-sided coideal of C (which also satisfies $\varepsilon\left(K^\perp\right) = 0$, but this doesn't even matter, since you don't require coideals to satisfy $\varepsilon\left(K^\perp\right) = 0$). However, K is not a subalgebra. This is seen as follows:

  Every $i \in J$ satisfies $\Delta(g_i) = g_x \otimes g_i + g_i \otimes g_x - g_x \otimes g_x$ (in fact, this is just another way to state $\Delta(1 + e_i) = 1 \otimes (1 + e_i) + (1 + e_i) \otimes 1 - 1 \otimes 1$, which in turn is just another way to say that e_i is primitive). Thus, $(f_x * f_x)(g_i) = -1$. Since this holds for every $i \in J$, and J is infinite (because $\operatorname{Prim} C$ is infinite-dimensional), this shows that $f_x * f_x$ cannot lie in K (since K is the space of all linear maps $C \to k$ which are finite linear combinations of coordinate maps). This means that K is not a subalgebra.

  Or is it? I don't feel particularly sure of any counterexamples I produce, as I know that 50% of them are wrong.
- **Proposition I.3.2:** Replace "rightt" by "right".
- **I.3.1:** I believe that the paragraph directly after Proposition I.3.2 (this is the paragraph beginning with "The linear dual $(C \otimes C)^*$ naturally contains [...]"

and ending with "[...] implies that u is a unit") should rather be placed before Proposition I.3.1. It defines the algebra C^* used in Propositions I.3.1 and I.3.2.

- **I.3.1:** In the definition of the tensor product of two coalgebras (in the very last paragraph of page 13), you write: "Let C and D be unital k-coalgebras". The "unital" should be "co-unital" here. Also, in the same paragraph, "co-unity" should be "co-unit".

- **I.3.2:** When defining the notion of a subcomodule, you write: "$\Phi(C) \subset C \otimes N$". This should be $\Phi(N) \subset C \otimes N$.

- **I.3.2:** In the middle of page 14, the formula

$$(\Phi \otimes I) \circ \Phi(m) = \sum_{(x)} m_{1:1} \otimes m_{1:2} \otimes m_0 = \sum_{(m)} m_1 \otimes m_{0:1} \otimes m_{0:0} = (I \otimes \Delta) \circ \Phi(m)$$

has three typos. It should be

$$(\Delta \otimes I) \circ \Phi(m) = \sum_{(m)} m_{1:1} \otimes m_{1:2} \otimes m_0 = \sum_{(m)} m_1 \otimes m_{0:1} \otimes m_{0:0} = (I \otimes \Phi) \circ \Phi(m).$$

- **Proposition I.3.3:** "if and only is" should be "if and only if".

- **Proof of Theorem I.3.4:** In this proof, you seem to assume in that M is a right comodule (rather than a left one). (There is only one exception: that is when you write "Let us show that N is a left subcomodule of M".)

- **Proof of Theorem I.3.6:** Here you write: "$E = C^*/N^\perp$ is a finite-dimensional left module over $C^\perp$". Clearly you mean C^* instead of $C^\perp$.

- **Proof of Lemma I.3.9:** In the formula which defines the form f_γ, replace the word "si" by "if" (two times).

- **Proof of Lemma I.3.9:** At the very end of this proof, replace "y_γ in in D" by "y_γ is in D".

- **Proof of Proposition I.3.10:** In this proof you seem to use that R is the *direct* sum of the simple subcoalgebras of C. Why is that obvious? In my opinion, this requires a further lemma: that any sum of pairwise distinct simple subcoalgebras of C must be a direct sum. This, in turn, is a particular case of another theorem[1]: that any sum of subcoalgebras of C all of whose pairwise intersections are 0 must be a direct sum. This theorem is proven by reducing it to the case of finitely many subcoalgebras, and then proving it by induction over the number of subcoalgebras (using Lemma I.3.9 in the induction step).

- **Proof of Proposition I.3.11:** Replace "(lemma I.2.8)" by "(corollary I.2.10)".

[1] which can be seen as a dual to the Chinese Remainder Theorem

- **Proof of Proposition I.3.12:** I don't think the equality

$$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z) = \left(X^{\perp} Y^{\perp} Z^{\perp}\right)^{\perp}$$

is that much obvious. The definition only yields $(X \wedge Y) \wedge Z = \left(\left(\left(X^{\perp} Y^{\perp}\right)^{\perp}\right)^{\perp} Z^{\perp}\right)^{\perp}$ and $X \wedge (Y \wedge Z) = \left(X^{\perp}\left(\left(Y^{\perp} Z^{\perp}\right)^{\perp}\right)^{\perp}\right)^{\perp}$, but it is not clear why these things are the same as $\left(X^{\perp} Y^{\perp} Z^{\perp}\right)^{\perp}$. Am I missing something?
Here is how I would prove Proposition I.3.12 1):
Let π_X, π_Y and π_Z be the canonical projections from C onto $C \diagup X$, $C \diagup Y$ and $C \diagup Z$, respectively. Let $\pi_{X \wedge Y}$ be the canonical projection from C onto $C \diagup (X \wedge Y)$.
There is a well-known fact in linear algebra that if A, B, A' and B' are four vector spaces and $f : A \to A'$ and $g : B \to B'$ are two linear maps, then $\operatorname{Ker}(f \otimes g) = (\operatorname{Ker} f) \otimes B + A \otimes (\operatorname{Ker} g)$. Applied to C, C, $C \diagup X$, $C \diagup Y$, π_X and π_Y in lieu of A, B, A', B', f and g, this yields $\operatorname{Ker}(\pi_X \otimes \pi_Y) = \underbrace{(\operatorname{Ker} \pi_X)}_{=X} \otimes C + C \otimes \underbrace{(\operatorname{Ker} \pi_Y)}_{=Y} = X \otimes C + C \otimes Y$.
The definition of $X \wedge Y$ rewrites as $X \wedge Y = \Delta^{-1}(X \otimes C + C \otimes Y)$. Thus,

$$X \wedge Y = \Delta^{-1}\Bigg(\underbrace{X \otimes C + C \otimes Y}_{=\operatorname{Ker}(\pi_X \otimes \pi_Y)}\Bigg) = \Delta^{-1}(\operatorname{Ker}(\pi_X \otimes \pi_Y)) = \operatorname{Ker}((\pi_X \otimes \pi_Y) \circ \Delta).$$

Thus the map $(\pi_X \otimes \pi_Y) \circ \Delta : C \to (C \diagup X) \otimes (C \diagup Y)$ factors through $C \diagup (X \wedge Y)$. In other words, there exists a map $\overline{\Delta} : C \diagup (X \wedge Y) \to (C \diagup X) \otimes (C \diagup Y)$ such that $(\pi_X \otimes \pi_Y) \circ \Delta = \overline{\Delta} \circ \pi_{X \wedge Y}$. Moreover, this map $\overline{\Delta}$ is injective[2]. The map $\overline{\Delta} \otimes \operatorname{id} : (C \diagup (X \wedge Y)) \otimes (C \diagup Z) \to (C \diagup X) \otimes (C \diagup Y) \otimes (C \diagup Z)$ is therefore also injective.
The diagram

$$\begin{array}{ccccc} C & & & & \\ & \searrow^{\Delta} & & & \\ & & C \otimes C & \xrightarrow{\pi_{X \wedge Y} \otimes \pi_Z} & (C \diagup (X \wedge Y)) \otimes (C \diagup Z) \\ & & \downarrow{\scriptstyle \Delta \otimes \operatorname{id}} & & \downarrow{\scriptstyle \overline{\Delta} \otimes \operatorname{id}} \\ & & C \otimes C \otimes C & \xrightarrow{\pi_X \otimes \pi_Y \otimes \pi_Z} & (C \diagup X) \otimes (C \diagup Y) \otimes (C \diagup Z) \end{array}$$

[2] *Proof.* Let $\varphi \in C \diagup (X \wedge Y)$ be such that $\overline{\Delta}\varphi = 0$. Then, $\varphi = \pi_{X \wedge Y}(c)$ for some $c \in C$ (since $\pi_{X \wedge Y}$ is surjective). Hence,

$$\overline{\Delta}\varphi = \overline{\Delta}(\pi_{X \wedge Y}(c)) = \underbrace{\left(\overline{\Delta} \circ \pi_{X \wedge Y}\right)}_{=(\pi_X \otimes \pi_Y) \circ \Delta}(c) = ((\pi_X \otimes \pi_Y) \circ \Delta)(c).$$

Thus, $\overline{\Delta}\varphi = 0$ becomes $((\pi_X \otimes \pi_Y) \circ \Delta)(c) = 0$ and hence $c \in \operatorname{Ker}((\pi_X \otimes \pi_Y) \circ \Delta) = X \wedge Y = \operatorname{Ker} \pi_{X \wedge Y}$, so that $\pi_{X \wedge Y}(c) = 0$. Thus, $\varphi = \pi_{X \wedge Y}(c) = 0$. We have therefore shown that every $\varphi \in C \diagup (X \wedge Y)$ such that $\overline{\Delta}\varphi = 0$ satisfies $\varphi = 0$. Thus, $\overline{\Delta}$ is injective.

commutes. Hence,

$$\begin{aligned}
&\operatorname{Ker}\left(\left(\pi_X \otimes \pi_Y \otimes \pi_Z\right) \circ (\Delta \otimes \operatorname{id}) \circ \Delta\right) \\
&= \operatorname{Ker}\left(\left(\overline{\Delta} \otimes \operatorname{id}\right) \circ \left(\pi_{X\wedge Y} \otimes \pi_Z\right) \circ \Delta\right) = \operatorname{Ker}\left(\left(\pi_{X\wedge Y} \otimes \pi_Z\right) \circ \Delta\right) \\
&\qquad \left(\begin{array}{c}\text{since the map } \overline{\Delta} \otimes \operatorname{id} \text{ is injective, and thus composing with it} \\ \text{does not change the kernel}\end{array}\right).
\end{aligned}$$

Now, applying $X \wedge Y = \operatorname{Ker}\left(\left(\pi_X \otimes \pi_Y\right) \circ \Delta\right)$ to $X \wedge Y$ and Z instead of X and Y, we get $(X \wedge Y) \wedge Z = \operatorname{Ker}\left(\left(\pi_{X\wedge Y} \otimes \pi_Z\right) \circ \Delta\right)$. We conclude that

$$\operatorname{Ker}\left(\left(\pi_X \otimes \pi_Y \otimes \pi_Z\right) \circ (\Delta \otimes \operatorname{id}) \circ \Delta\right) = \operatorname{Ker}\left(\left(\pi_{X\wedge Y} \otimes \pi_Z\right) \circ \Delta\right) = (X \wedge Y) \wedge Z.$$

Similarly,

$$\operatorname{Ker}\left(\left(\pi_X \otimes \pi_Y \otimes \pi_Z\right) \circ (\operatorname{id} \otimes \Delta) \circ \Delta\right) = X \wedge (Y \wedge Z).$$

Comparing these two equalities, we get $(X \wedge Y) \wedge Z = X \wedge (Y \otimes Z)$ since $(\Delta \otimes \operatorname{id}) \circ \Delta = (\operatorname{id} \otimes \Delta) \circ \Delta$. This proves Proposition I.3.12 1).

- **I.3.4:** In the definition of $N \wedge X$ (shortly before Proposition I.3.13), you write:

  $$N \wedge X = \{x \in M,\ \Phi x \in N \otimes C + C \otimes X\}.$$

  The $C \otimes X$ should be $M \otimes X$ here.

- **Proposition I.3.13:** Replace $\wedge^n M$ by $\wedge^n R$ here.

- **I.5:** In the definition of a bialgebra, you need one more condition: the condition that $\varepsilon(1) = 1$ [3]. (Normally this would follow from the condition that ε is an algebra morphism, but apparently in your text "algebra morphism" does not mean "unital algebra morphism", and then it does not follow from this condition. Also it does not follow from any of the four commutative diagrams on page 21.)

- **I.6.1:** In the first paragraph of I.6.1, replace "map from $kG \times kG$ into kG" by "map from $kG \times kG$ to kG" (since "into" sounds like a claim that the map is injective).

- **I.6.2:** Add a point after "$\varepsilon\mid_V = 0$".

- **Lemma I.6.2:** First, replace $S(\mathcal{H}) \subset \mathcal{H}$ by $S(J) \subset J$. Besides, the usual definition of a "Hopf ideal" involves a third condition: $\varepsilon(J) = 0$. However, this condition is redundant in almost every case - the only exception is when $J = \mathcal{H}$, which brings us back to the question whether 0 should be considered a Hopf algebra.

- **End of proof of Proposition I.7.1:** On page 25, replace $\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes H)$ by $\mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathcal{H})$ (the third H should be calligraphic).

[3] Theoretically you could also want to exclude the $\varepsilon(1) = 1$ condition from the definition of a bialgebra. The only difference it would make is that it would cause the zero space 0 to be a bialgebra (and a Hopf algebra). But I think it is not common to consider 0 as a bialgebra (and you actually use the condition $\varepsilon(1) = 1$ in the proof of Proposition I.7.1).

- **End of proof of Proposition I.7.1:** On page 25, in the computation

$$\begin{aligned}\left(\left(\Delta\circ S\right)\widetilde{*}\Delta\right)(x) &= [\text{... several lines of computation ...}]\\ &= u\circ\varepsilon(x)\otimes u\circ\varepsilon(x),\end{aligned}$$

  the term $u\circ\varepsilon(x)\otimes u\circ\varepsilon(x)$ should be replaced by $u\circ\varepsilon(x_1)\otimes u\circ\varepsilon(x_2)$ (otherwise it wouldn't be linear in x).

- **End of proof of Proposition I.7.1:** On page 26, in the computation

$$\begin{aligned}\left(\Delta\widetilde{*}\left(\tau\circ\left(S\otimes S\right)\circ\Delta\right)\right)(x) &= [\text{... several lines of computation ...}]\\ &= \left(\sum_{(x)} x_1 S x_2\right)\otimes\left(u\circ\varepsilon\right)(x)\\ &= u\circ\varepsilon(x)\otimes u\circ\varepsilon(x),\end{aligned}$$

  the last two lines of this computation are incorrect (the terms should be linear in x). I would replace them by

$$\begin{aligned}&= \sum_{(x)} x_1\varepsilon(x_2)\, S x_3\otimes 1 = \left(\sum_{(x)} x_1 S x_2\right)\otimes 1 = u\circ\varepsilon(x)\otimes 1\\ &= u\circ\varepsilon(x_1)\otimes u\circ\varepsilon(x_2),\end{aligned}$$

- **Proof of Proposition I.7.3:** Replace $S(x)-x$ by $S(x)+x$ here.

- **Proof of Proposition I.7.3:** Replace $\mathbf{1}\otimes(xy+yx)$ by $\mathbf{1}\otimes(xy-yx)$ here.

- **II.1:** On page 27, I think it is worth a mention that every graded bialgebra automatically satisfies $1\in\mathcal{H}_0$ and $\varepsilon(\mathcal{H}_n)=0$ for every $n>0$. (More strongly, every graded counital coalgebra C automatically satisfies $\varepsilon(C_n)=0$ for every $n>0$, and every graded unital algebra A automatically satisfies $1\in A_0$.) This is used when you say that $\operatorname{Ker}\varepsilon=\bigoplus_{n\geq 1}\mathcal{H}_n$ for any connected graded bialgebra $\mathcal{H}$.

- **Proposition II.1.1:** Replace $\mathcal{H}^n$ by $\mathcal{H}_n$.

- **Proof of Proposition II.1.1:** At the beginning of this proof, you write: "Thanks to connectedness we clearly can write :

$$\Delta x = a\left(x\otimes 1\right)+b\left(1\otimes x\right)+\widetilde{\Delta}x$$

  with $a,b\in k$ and $\widetilde{\Delta}x\in\operatorname{Ker}\varepsilon\otimes\operatorname{Ker}\varepsilon$. The co-unity property then tells us that, with $k\otimes\mathcal{H}$ and $\mathcal{H}\otimes k$ canonically identified with $\mathcal{H}$:

$$\begin{aligned}x &= \left(\varepsilon\otimes I\right)(\Delta x) = bx\\ x &= \left(I\otimes\varepsilon\right)(\Delta x) = ax,\end{aligned}$$

  hence $a=b=1$."
  This whole paragraph is slightly flawed. There is a typo ("co-unity" should be

"co-unit"), but I am talking about something more serious: Connectedness does not *directly* give us $\Delta x = a(x \otimes 1) + b(1 \otimes x) + \widetilde{\Delta} x$ for some $a, b \in k$, but rather gives us $\Delta x = u \otimes 1 + 1 \otimes v + \widetilde{\Delta} x$ for some $u, v \in \mathcal{H}_n$. We do not yet know that u and v are multiples of x; to see that, we need the counit property. Hence I would rewrite the above paragraph as follows:
"Thanks to connectedness we clearly can write :

$$\Delta x = u \otimes 1 + 1 \otimes v + \widetilde{\Delta} x$$

with $u, v \in \mathcal{H}_n$ and $\widetilde{\Delta} x \in \operatorname{Ker} \varepsilon \otimes \operatorname{Ker} \varepsilon$. The co-unit property then tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$:

$$x = (\varepsilon \otimes I)(\Delta x) = v$$
$$x = (I \otimes \varepsilon)(\Delta x) = u,$$

hence $\Delta x = x \otimes 1 + 1 \otimes x + \widetilde{\Delta} x$."

- **Proof of Proposition II.1.1:** Add a point between "$|x'| + |x''| = n$" and "We easily compute :".

- **II.2:** On page 29, in the definition of a "filtered Hopf algebra", I don't understand the meaning of the word "characteristic" in "a unit characteristic $u : k \to \mathcal{H}$".

- **II.2:** On page 29, replace "if x is an homogeneous element" by "if x is a nonzero homogeneous element".

- **Proposition II.2.1:** Replace $x \in \mathcal{H}^n$ by $x \in \mathcal{H}^n \cap \operatorname{Ker} \varepsilon$. (Otherwise, $x = 1$ and $n = 1$ is a counterexample.)

- **Proof of Proposition II.2.1:** I am not sure whether this proof is really a "Straightforward adaptation of proof of proposition II.1.1". For example, you cannot apply the co-unit property as easily as you did in the proof of Proposition II.1.1, since you don't have $\varepsilon(\mathcal{H}^n) = 0$ for $n \geq 1$.
  Here is what I think is a correct proof of Proposition II.2.1:
  First, it is easy to see that any grouplike element of a filtered coalgebra must lie in the 0-th part of the filtration.[4] Applied to the grouplike element 1 of the filtered coalgebra $\mathcal{H}$, we get $1 \in \mathcal{H}^0$. Note that we have not used the connectedness of $\mathcal{H}$

[4] *Proof.* Let C be a filtered coalgebra with filtration $(C_n)_{n \geq 0}$. Let $g \in C$ be a grouplike element of C. We must prove that $g \in C_0$.

Since $g \in C$, there exists some $n \geq 0$ such that $g \in C_n$. Let m be the smallest such n. Then, $g \in C_m$, but $g \notin C_{m-1}$, where we set $C_{-1} = 0$.

If $m = 0$, then we are done, so let us assume that $m > 0$. Since $g \in C_m$, we have

$$\Delta(g) \in \Delta(C_m) \subseteq \sum_{i=0}^{m} C_i \otimes C_{m-i} = \sum_{i=0}^{m-1} \underbrace{C_i}_{\subseteq C_{m-1}} \otimes \underbrace{C_{m-i}}_{\subseteq C_m} + C_m \otimes \underbrace{C_0}_{\subseteq C_{m-1}}$$
$$\subseteq \sum_{i=0}^{m-1} C_{m-1} \otimes C_m + C_m \otimes C_{m-1} \subseteq C_{m-1} \otimes C_m + C_m \otimes C_{m-1}$$

(since $C_{m-1} \otimes C_m$ is a k-vector space). If π denotes the canonical projection $C_m \to C_m / C_{m-1}$, we yet.

Next, define a subspace $\mathcal{H}^{+i}$ of $\mathcal{H}^i$ by $\mathcal{H}^{+i} = \mathcal{H}^i \cap \operatorname{Ker}\varepsilon$ for every $i > 0$. It is easy to see that $\mathcal{H}^i = \mathcal{H}^0 + \mathcal{H}^{+i}$ for every $i > 0$ (because every $x \in \mathcal{H}^i$ satisfies $x = \underbrace{\varepsilon(x) \cdot 1}_{\in \mathcal{H}^0} + \underbrace{(x - \varepsilon(x) \cdot 1)}_{\in \mathcal{H}^i \cap \operatorname{Ker}\varepsilon = \mathcal{H}^{+i}} \in \mathcal{H}^0 + \mathcal{H}^{+i}$). Now every $n \geq 1$ satisfies

$$\Delta(\mathcal{H}^n) \subseteq \sum_{p+q=n} \mathcal{H}^p \otimes \mathcal{H}^q = \mathcal{H}^0 \otimes \mathcal{H}^n + \sum_{\substack{p+q=n;\\ p\neq 0;\ q\neq 0}} \mathcal{H}^p \otimes \mathcal{H}^q + \mathcal{H}^n \otimes \mathcal{H}^0.$$

Since every $p \neq 0$ and $q \neq 0$ with $p + q = n$ satisfy

$$\begin{aligned}
\underbrace{\mathcal{H}^p}_{=\mathcal{H}^0+\mathcal{H}^{+p}} \otimes \underbrace{\mathcal{H}^q}_{=\mathcal{H}^0+\mathcal{H}^{+q}} &= \left(\mathcal{H}^0 + \mathcal{H}^{+p}\right) \otimes \left(\mathcal{H}^0 + \mathcal{H}^{+q}\right) \\
&= \mathcal{H}^0 \otimes \underbrace{\mathcal{H}^0}_{\subseteq \mathcal{H}^n} + \mathcal{H}^0 \otimes \underbrace{\mathcal{H}^{+q}}_{\subseteq \mathcal{H}^q \subseteq \mathcal{H}^n} + \underbrace{\mathcal{H}^{+p}}_{\subseteq \mathcal{H}^p \subseteq \mathcal{H}^n} \otimes \mathcal{H}^0 + \mathcal{H}^{+p} \otimes \mathcal{H}^{+q} \\
&\subseteq \mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^n \otimes \mathcal{H}^0 + \mathcal{H}^{+p} \otimes \mathcal{H}^{+q},
\end{aligned}$$

this becomes

$$\begin{aligned}
\Delta(\mathcal{H}^n) &\subseteq \mathcal{H}^0 \otimes \mathcal{H}^n + \sum_{\substack{p+q=n;\\ p\neq 0;\ q\neq 0}} \left(\mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^0 \otimes \mathcal{H}^n + \mathcal{H}^n \otimes \mathcal{H}^0 + \mathcal{H}^{+p} \otimes \mathcal{H}^{+q}\right) + \mathcal{H}^n \otimes \mathcal{H}^0 \\
&\subseteq \mathcal{H}^n \otimes \mathcal{H}^0 + \mathcal{H}^0 \otimes \mathcal{H}^n + \sum_{\substack{p+q=n;\\ p\neq 0;\ q\neq 0}} \mathcal{H}^{+p} \otimes \mathcal{H}^{+q}.
\end{aligned}$$

Thus, for every $n \geq 1$ and every $x \in \mathcal{H}^n \cap \operatorname{Ker}\varepsilon$, we can write

$$\Delta x = u \otimes 1 + 1 \otimes v + \widetilde{\Delta} x$$

for some $u, v \in \mathcal{H}^n$ and $\widetilde{\Delta} x \in \sum\limits_{\substack{p+q=n;\\ p\neq 0;\ q\neq 0}} \mathcal{H}^{+p} \otimes \mathcal{H}^{+q}$ (here we are using, for the first time, that $\mathcal{H}$ is connected). The co-unit property now tells us that, with $k \otimes \mathcal{H}$ and $\mathcal{H} \otimes k$ canonically identified with $\mathcal{H}$, we have

$$x = (\varepsilon \otimes I)(\Delta x) = v + \varepsilon(u) \qquad \text{and}$$
$$x = (I \otimes \varepsilon)(\Delta x) = u + \varepsilon(v)$$

(here we are using that $\widetilde{\Delta} x \in \sum\limits_{\substack{p+q=n;\\ p\neq 0;\ q\neq 0}} \mathcal{H}^{+p} \otimes \mathcal{H}^{+q}$, so that $(\varepsilon \otimes I)\left(\widetilde{\Delta} x\right) = 0$ and $(I \otimes \varepsilon)\left(\widetilde{\Delta} x\right) = 0$), and therefore $u = x - \varepsilon(v)$ and $v = x - \varepsilon(u)$. Hence,

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thus have

$$(\pi \otimes \pi)(\Delta(g)) \in (\pi \otimes \pi)(C_{m-1} \otimes C_m + C_m \otimes C_{m-1}) \subseteq \underbrace{\pi(C_{m-1})}_{=0} \otimes \pi(C_m) + \pi(C_m) \otimes \underbrace{\pi(C_{m-1})}_{=0} = 0,$$

so that $(\pi \otimes \pi)(\Delta(g)) = 0$. But since $\Delta(g) = g \otimes g$ (since g is grouplike), we have $(\pi \otimes \pi)(\Delta(g)) = (\pi \otimes \pi)(g \otimes g) = \pi(g) \otimes \pi(g) \neq 0$ (since $\pi(g) \neq 0$, which is because $g \notin C_{m-1}$), and thus we get a contradiction to $(\pi \otimes \pi)(\Delta(g)) = 0$. This contradiction shows that the case $m > 0$ cannot occur. Thus, $m = 0$, so that $g \in C_m = C_0$, qed.

$\Delta x = u \otimes 1 + 1 \otimes v + \widetilde{\Delta} x$ rewrites as $\Delta x = x \otimes 1 + 1 \otimes x - (\varepsilon(u) + \varepsilon(v)) 1 \otimes 1 + \widetilde{\Delta} x$. Applying $\varepsilon \otimes I$ to this equation, we get $x = \varepsilon(x) 1 + x - (\varepsilon(u) + \varepsilon(v)) 1$ (since $(\varepsilon \otimes I)\left(\widetilde{\Delta} x\right) = 0$), which rewrites as $\varepsilon(x) = \varepsilon(u) + \varepsilon(v)$. Hence, if $x \in \mathcal{H}^{+n}$, we have $\varepsilon(u) + \varepsilon(v) = \varepsilon(x) = 0$, so that $\Delta x = x \otimes 1 + 1 \otimes x - (\varepsilon(u) + \varepsilon(v)) 1 \otimes 1 + \widetilde{\Delta} x$ simplifies to $\Delta x = x \otimes 1 + 1 \otimes x + \widetilde{\Delta} x$.
We thus have proven that every $x \in \mathcal{H}^{+n}$ for every $n > 0$ satisfies $\Delta x = x \otimes 1 + 1 \otimes x + \widetilde{\Delta} x$ with $\widetilde{\Delta} x \in \sum\limits_{\substack{p+q=n; \\ p \neq 0;\ q \neq 0}} \mathcal{H}^{+p} \otimes \mathcal{H}^{+q}$. The rest of the proof now indeed proceeds analogously to the proof of Proposition II.1.1 (except that we don't use homogeneity).

- **Proof of Theorem II.2.2:** In the first line of this proof, replace $S(H^n) \subset H^n$ by $S(\mathcal{H}^n) \subset \mathcal{H}^n$ (with calligraphic $\mathcal{H}$).

- **Proof of Theorem II.2.2:** In the second paragraph of this proof, replace "inclusion $S\mathcal{H}_0 \subset \mathcal{H}_0$" by "inclusion $S\mathcal{H}^0 \subset \mathcal{H}^0$".

- **Proof of Theorem II.2.2:** In the second paragraph of this proof, you write

$$\mathcal{H}^n = \mathcal{H}^0 \wedge \mathcal{H}^{n-1} = \mathcal{H}^{n-1} \wedge H^0.$$

  The last H should be calligraphic here.

- **Proof of Theorem II.2.2:** In the second paragraph of this proof, replace the formula

$$Sx = \sum_{(x)} Sx_2 \otimes Sx_1$$

  by

$$\Delta(Sx) = \sum_{(x)} Sx_2 \otimes Sx_1.$$

- **Proof of Theorem II.2.2:** In the second paragraph of this proof, replace "its is obviously" by "it is obviously".

- **Remark 2 after the proof of Theorem II.2.2:** Replace "subcoagebra" by "subcoalgebra".

- **Proof of Proposition II.3.1:** You write:

$$(e - \varphi)^{*k}(x) = m_{\mathcal{A},k-1}(\varphi \otimes \cdots \otimes \varphi) \widetilde{\Delta}_{k-1}(x).$$

  This should be

$$\begin{aligned}(e - \varphi)^{*k}(x) &= m_{\mathcal{A},k-1}((e - \varphi) \otimes \cdots \otimes (e - \varphi)) \Delta_{k-1}(x) \\ &= m_{\mathcal{A},k-1}((-\varphi) \otimes \cdots \otimes (-\varphi)) \widetilde{\Delta}_{k-1}(x).\end{aligned}$$

- **II.3:** Between the proof of Corollary II.3.2 and Proposition II.3.3, you write: "For any $x \in \mathcal{H}^n$ the exponential :

$$e^{*\alpha}(x) = \sum_{k \geq 0} \frac{\alpha^{*k}(x)}{k!}$$

is a finite sum (ending up at $k = n$)." You could add "and any $\alpha \in \mathfrak{g}$" after "For any $x \in \mathcal{H}^n$" here, in order to make it clear what α is.

- **Proposition II.3.3:** In part 2), replace "then" by "the".

- **II.3:** On page 32, in the definition of $\mathcal{L}_n$, you falsely write $\mathcal{L}^n$ instead of $\mathcal{L}_n$. The same mistake is repeated on page 33, in the last line of II.3 ("the Lie algebras $\mathfrak{g}/\mathcal{L}^n$"). Also the same mistake, this time is repeated in Proposition II.3 (this time the $\mathcal{L}^p$, $\mathcal{L}^q$ and $\mathcal{L}^{p+q}$ should be $\mathcal{L}_p$, $\mathcal{L}_q$ and $\mathcal{L}_{p+q}$).

- **II.4:** When you define "characters", replace "algebra morphisms" by "unital algebra morphisms" (otherwise, 0 would be a character, contradicting Proposition II.4.1 3)).

- **Proof of Proposition II.4.1:** In the commutative diagram on page 34, replace $\otimes\mathcal{A}$ by $\mathcal{A} \otimes \mathcal{A}$.

- **Proof of Proposition II.4.1:** This proof ends with the equation

$$\tau^{-1}(x) = \sum_{k \geq 0} (e - \tau)^{*k}(x).$$

To keep notations consistent, I believe you should replace τ^{-1} by τ^{*-1} here.

- A remark about Proposition II.4.1: The following statement generalizes Proposition II.4.1 2):
If $\mathcal{H}$ is a Hopf algebra over a field k (not necessarily of characteristic 0), and $\mathcal{A}$ is a k-algebra, and if $\xi : \mathcal{H} \to \mathcal{A}$ is a cocycle which has an inverse with respect to the convolution, then this inverse ξ^{*-1} is a cocycle as well.
Proof of this statement: Define a k-linear map $\Phi_1 : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A}$ by

$$\Phi_1(x \otimes y) = \xi(xy) \qquad \text{for every } x \in \mathcal{H} \text{ and } y \in \mathcal{H}.$$

Define a k-linear map $\Phi_2 : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A}$ by

$$\Phi_2(x \otimes y) = \xi^{*-1}(xy) \qquad \text{for every } x \in \mathcal{H} \text{ and } y \in \mathcal{H}.$$

Define a k-linear map $\Phi_3 : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A}$ by

$$\Phi_3(x \otimes y) = \xi^{*-1}(yx) \qquad \text{for every } x \in \mathcal{H} \text{ and } y \in \mathcal{H}.$$

Then, every $x \in \mathcal{H}$ and $y \in \mathcal{H}$ satisfy

$$\begin{aligned}
(\Phi_1 * \Phi_2)(x \otimes y) &= \sum_{(x\otimes y)} \Phi_1\left((x\otimes y)_1\right) \Phi_2\left((x \otimes y)_2\right) = \sum_{(x)(y)} \underbrace{\Phi_1(x_1 \otimes y_1)}_{=\xi(x_1y_1)} \underbrace{\Phi_2(x_2 \otimes y_2)}_{=\xi^{*-1}(x_2y_2)} \\
&= \sum_{(x)(y)} \xi(x_1y_1)\, \xi^{*-1}(x_2y_2) = \sum_{(xy)} \xi\left((xy)_1\right) \xi^{*-1}\left((xy)_2\right) \\
&= \underbrace{\left(\xi * \xi^{*-1}\right)}_{=e}(xy) = e(xy) = e(x \otimes y),
\end{aligned}$$

so that $\Phi_1 * \Phi_2 = e$. Also, every $x \in \mathcal{H}$ and $y \in \mathcal{H}$ satisfy

$$\begin{aligned}
(\Phi_3 * \Phi_1)(x \otimes y) &= \sum_{(x\otimes y)} \Phi_3\left((x\otimes y)_1\right) \Phi_1\left((x \otimes y)_2\right) = \sum_{(x)(y)} \underbrace{\Phi_3(x_1 \otimes y_1)}_{=\xi^{*-1}(y_1x_1)} \underbrace{\Phi_1(x_2 \otimes y_2)}_{\substack{=\xi(x_2y_2)=\xi(y_2x_2) \\ \text{(since } \xi \text{ is a cocycle)}}} \\
&= \sum_{(x)(y)} \xi^{*-1}(y_1x_1)\, \xi(y_2x_2) = \sum_{(yx)} \xi^{*-1}\left((yx)_1\right) \xi\left((yx)_2\right) \\
&= \underbrace{\left(\xi^{*-1} * \xi\right)}_{=e}(yx) = e(yx) = e(y)\, e(x) = e(x \otimes y),
\end{aligned}$$

so that $\Phi_3 * \Phi_1 = e$. Thus, $\Phi_2 = \underbrace{e}_{=\Phi_3*\Phi_1} * \Phi_2 = \Phi_3 * \underbrace{\Phi_1 * \Phi_2}_{=e} = \Phi_3 * e = \Phi_3$. Thus, every $x \in \mathcal{H}$ and $y \in \mathcal{H}$ satisfy $\xi^{*-1}(xy) = \underbrace{\Phi_2}_{=\Phi_3}(x \otimes y) = \Phi_3(x \otimes y) = \xi^{*-1}(yx)$.

In other words, ξ^{*-1} is a cocycle, qed.

- **Proof of Proposition II.4.2:** Replace $\sum\limits_{(x)(y)} \alpha(x_1x_2)\, \beta(y_1y_2)$ by $\sum\limits_{(x)(y)} \alpha(x_1y_1)\, \beta(x_2y_2)$.
  Also, one line further below, replace $e(x_2)\, \alpha(y_2)$ by $e(x_2)\, \beta(y_2)$.

- **Proof of Proposition II.4.2:** I fear you don't really prove that the exponential restricts to a bijection from $\mathfrak{g}_1$ onto G_1; instead you only show that it maps $\mathfrak{g}_1$ into G_1 (but not necessarily surjectively). Do you have an easy proof for the fact that it restricts to a bijection from $\mathfrak{g}_1$ onto G_1 ? Here is the only proof I have:
  Proof. You have shown that the exponential maps $\mathfrak{g}_1$ into G_1. Now it remains to show that any $\alpha \in \mathfrak{g}$ satisfying $e^{*\alpha} \in G_1$ must lie in $\mathfrak{g}_1$.
  So consider some $\alpha \in \mathfrak{g}$ satisfying $e^{*\alpha} \in G_1$.
  Any two elements β and γ of $\mathfrak{g}$ which commute satisfy $e^{*(\beta+\gamma)} = e^{*\beta} \cdot e^{*\gamma}$. [5]
  Using this fact and induction over n, we can prove the following: Every $m \in \mathbb{N}$ satisfies $e^{*m\alpha} = \left(e^{*\alpha}\right)^m$. Thus, $e^{*m\alpha} \in G_1$ for every $m \in \mathbb{N}$ (since $e^{*\alpha} \in G_1$ and since G_1 is a group).
  Now let $x \in \mathcal{H}$ and $y \in \mathcal{H}$ be arbitrary. We will prove that $\alpha(xy) = e(x)\, \alpha(y) + \alpha(x)\, e(y)$.
  Since $x \in \mathcal{H}$, there exists some $i \in \mathbb{N}$ such that $x \in \mathcal{H}^i$. Consider this i. Then, the power series $e^{*m\alpha}(x) = \sum\limits_{k\geq 0} \dfrac{(m\alpha)^{*k}(x)}{k!}$ ends up at $k = i$.

[5] This can be proven by the same argument as the classical one used to prove that $\exp(x+y) = \exp x \cdot \exp y$ for two reals x and y (where exp is defined by the power series).

Since $y \in \mathcal{H}$, there exists some $j \in \mathbb{N}$ such that $y \in \mathcal{H}^j$. Consider this j. Then, the power series $e^{*m\alpha}(y) = \sum_{k\geq 0} \frac{(m\alpha)^{*k}(y)}{k!}$ ends up at $k = j$.

Also, $\underbrace{x}_{\in \mathcal{H}^i} \underbrace{y}_{\in \mathcal{H}^j} \in \mathcal{H}^i \mathcal{H}^j \subseteq \mathcal{H}^{i+j}$. Thus, the power series $e^{*m\alpha}(y) = \sum_{k\geq 0} \frac{(m\alpha)^{*k}(y)}{k!}$ ends up at $k = i + j$.

Every $m \in \mathbb{N}$ satisfies

$$\begin{aligned}
e^{*m\alpha}(xy) &= e^{*m\alpha}(x)\, e^{*m\alpha}(y) \qquad (\text{since } e^{*m\alpha} \in G_1, \text{ so that } e^{*m\alpha} \text{ is a character}) \\
&= \sum_{k\geq 0} \frac{(m\alpha)^{*k}(x)}{k!} \sum_{k\geq 0} \frac{(m\alpha)^{*k}(y)}{k!} \qquad (\text{by the definition of } e^{*m\alpha}) \\
&= \sum_{k\geq 0,\ \ell\geq 0} \underbrace{\frac{(m\alpha)^{*k}(x) \cdot (m\alpha)^{*\ell}(y)}{k!\ell!}}_{= m^{k+\ell} \frac{\alpha^{*k}(x)\cdot \alpha^{*\ell}(y)}{k!\ell!}} = \sum_{k\geq 0,\ \ell\geq 0} m^{k+\ell} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!} \\
&= \sum_{n\geq 0} m^n \sum_{k+\ell=n} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!}
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{n\geq 0} m^n \sum_{k+\ell=n} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!} &= e^{*m\alpha}(xy) = \sum_{k\geq 0} \underbrace{\frac{(m\alpha)^{*k}(xy)}{k!}}_{= m^k \frac{\alpha^{*k}(xy)}{k!}} \qquad (\text{by the definition of } e^{*m\alpha}) \\
&= \sum_{k\geq 0} m^k \frac{\alpha^{*k}(xy)}{k!} = \sum_{n\geq 0} m^n \frac{\alpha^{*n}(xy)}{n!}. \qquad (1)
\end{aligned}$$

Thus, the identity (1) holds for infinitely many distinct values of $m \in \mathcal{A}$ (because it holds for every $m \in \mathbb{N}$, and because $\mathbb{N}$ injects into $\mathcal{A}$ [6]). But (1) is a polynomial identity in m, since both sums $\sum_{n\geq 0} m^n \sum_{k+\ell=n} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!}$ and $\sum_{n\geq 0} m^n \frac{\alpha^{*n}(xy)}{n!}$ end up at $n = i+j$ (since every $n > i+j$ satisfies $\sum_{k+\ell=n} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!} = 0$ [7] and $\frac{\alpha^{*n}(xy)}{n!} = 0$ (since $xy \in \mathcal{H}^{i+j}$)). Since this polynomial identity (1) holds for infinitely many distinct values of $m \in \mathcal{A}$, it must therefore hold as a

[6] Here we have used the condition that $\operatorname{char} k = 0$.

[7] *Proof.* Let $n > i + j$ be arbitrary. Any k and ℓ with $k + \ell = n$ satisfy at least one of the two inequalities $k > i$ and $\ell > j$ (since otherwise, we would have $k \leq i$ and $\ell \leq j$, so that $k + \ell \leq i + j < n$, contradicting $k + \ell = n$). But in each of these two cases we have $\alpha^{*k}(x) \cdot \alpha^{*\ell}(y) = 0$ (in fact, in the case $k > i$ we have $\alpha^{*k}(x) = 0$ (because $x \in \mathcal{H}^i$), whereas in the case $\ell > j$ we have $\alpha^{*\ell}(y) = 0$ (because $y \in \mathcal{H}^j$)). Thus, any k and ℓ with $k + \ell = n$ satisfy $\alpha^{*k}(x) \cdot \alpha^{*\ell}(y) = 0$. Hence, $\sum_{k+\ell=n} \frac{\alpha^{*k}(x) \cdot \alpha^{*\ell}(y)}{k!\ell!} = \sum_{k+\ell=n} \frac{0}{k!\ell!} = 0$.

formal polynomial identity, i. e., we must have

$$\sum_{n\geq 0} X^n \sum_{k+\ell=n} \frac{\alpha^{*k}(x)\cdot\alpha^{*\ell}(y)}{k!\ell!} = \sum_{n\geq 0} X^n \frac{\alpha^{*n}(xy)}{n!} \qquad (2)$$

as an identity between elements of the polynomial ring $\mathcal{A}[X]$. But if two polynomials are equal as elements of the polynomial ring $\mathcal{A}[X]$, their corresponding coefficients must be equal to each other; therefore, we can compare coefficients in (2) and conclude that every $n \geq 0$ satisfies $\sum\limits_{k+\ell=n} \frac{\alpha^{*k}(x)\cdot\alpha^{*\ell}(y)}{k!\ell!} = \frac{\alpha^{*n}(xy)}{n!}$. Applying this to $n = 1$, we get $\sum\limits_{k+\ell=1} \frac{\alpha^{*k}(x)\cdot\alpha^{*\ell}(y)}{k!\ell!} = \frac{\alpha^{*1}(xy)}{1!}$. Since

$$\sum_{k+\ell=1} \frac{\alpha^{*k}(x)\cdot\alpha^{*\ell}(y)}{k!\ell!} = \frac{\alpha^{*0}(x)\cdot\alpha^{*1}(y)}{0!1!} + \frac{\alpha^{*1}(x)\cdot\alpha^{*0}(y)}{1!0!} = \frac{e(x)\cdot\alpha(y)}{1} + \frac{\alpha(x)\cdot e(y)}{1}$$
$$= e(x)\,\alpha(y) + \alpha(x)\,e(y)$$

and $\frac{\alpha^{*1}(xy)}{1!} = \alpha(xy)$, this rewrites as $e(x)\,\alpha(y) + \alpha(x)\,e(y) = \alpha(xy)$. Since this holds for any $x \in \mathcal{H}$ and $y \in \mathcal{H}$, we thus conclude that α is a derivation. In other words, $\alpha \in \mathfrak{g}$, qed.

- **Theorem II.5.1:** Replace "$\operatorname{Ker}\varepsilon$ into $\mathcal{A}_-$" by "$\operatorname{Ker}\varepsilon$ to $\mathcal{A}_-$". Also, replace "$\mathcal{H}$ into A_+" by "$\mathcal{H}$ to $\mathcal{A}_+$" (this includes replacing the A by a calligraphic $\mathcal{A}$).

- **Proof of Theorem II.5.1, part 1):** In the "easy computation" you make in order to check $\varphi_+ = \varphi_- * \varphi$, remove the point after $(I-\pi)\left(\varphi(x) + \sum\limits_{(x)} \varphi_-(x')\,\varphi(x'')\right)$.

- **Proof of Theorem II.5.1:** The proof of part 1) of this theorem is not complete: it is not clear whether the recurrence equation

$$\varphi_-(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\,\varphi(x'')\right) \qquad (3)$$

defining the function φ_- is "stable" in the sense that if we have take some $x \in \mathcal{H}^n$, then we get one and the same value of $\varphi_-(x)$ no matter whether we treat x as an element of $\mathcal{H}^n$ and apply (3) n times or we treat x as an element of $\mathcal{H}^{n+1}$ and apply (3) $n+1$ times.
Here is how I would fix this proof:
Proof of part 1) of Theorem II.5.1: For every $n \in \mathbb{N}$, let $\mathcal{H}^{+n}$ be the subspace $\mathcal{H}^n \cap \operatorname{Ker}\varepsilon$ of $\operatorname{Ker}\varepsilon$. It is easy to see that $\operatorname{Ker}\varepsilon = \bigcup\limits_{n\geq 0} \mathcal{H}^{+n}$ and $\mathcal{H}^{+0} = 0$.
For every $n \in \mathbb{N}$ we will now define two maps $\varphi_{n-} : \mathcal{H}^{+n} \to \mathcal{A}$ and $\varphi_{n+} : \mathcal{H}^{+n} \to \mathcal{A}$. We do this by induction over n:
For $n = 0$, define both maps φ_{n-} and φ_{n+} to be the zero map (this is the only choice anyway, since $\mathcal{H}^{+0} = 0$).

Let $m \in \mathbb{N}$. Assume that we already have defined two maps $\varphi_{m-} : \mathcal{H}^{+m} \to \mathcal{A}$ and $\varphi_{m+} : \mathcal{H}^{+m} \to \mathcal{A}$. Then we define two maps $\varphi_{(m+1)-} : \mathcal{H}^{+(m+1)} \to \mathcal{A}$ and $\varphi_{(m+1)+} : \mathcal{H}^{+(m+1)} \to \mathcal{A}$ by

$$\varphi_{(m+1)-}(x) = -\pi \left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x') \varphi(x'') \right) \qquad \text{for every } x \in \mathcal{H}^{+(m+1)} \tag{4}$$

and

$$\varphi_{(m+1)+}(x) = (I - \pi) \left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x') \varphi(x'') \right) \qquad \text{for every } x \in \mathcal{H}^{+(m+1)}, \tag{5}$$

respectively.[8]

Now we will prove that every $n \in \mathbb{N}$ satisfies the three equations

$$\varphi_{(n+1)-} \mid_{\mathcal{H}^{+n}} = \varphi_{n-}, \tag{6}$$

$$\varphi_{(n+1)+} \mid_{\mathcal{H}^{+n}} = \varphi_{n+}, \tag{7}$$

$$\varphi_{n+} = \varphi_{n-} * \varphi. \tag{8}$$

(The last of these equations is supposed to mean that every $x \in \mathcal{H}^{+n}$ satisfies $\varphi_{n+}(x) = \sum_{(x)} \varphi_{n-}(x_1) \varphi(x_2)$.)

We will do this by induction over n: The induction base (the case $n = 0$) is trivial since $\mathcal{H}^0 = 0$. Now to the induction step: Consider some positive $m \in \mathbb{N}$. Assume that the equations (6), (7), (8) are all proven for $n = m - 1$. Now let us prove (6), (7), (8) for $n = m$.

Since the equations (6), (7), (8) are all proven for $n = m-1$, we have $\varphi_{m-} \mid_{\mathcal{H}^{+(m-1)}} = \varphi_{(m-1)-}$, $\varphi_{m+} \mid_{\mathcal{H}^{+(m-1)}} = \varphi_{(m-1)+}$ and $\varphi_{(m-1)+} = \varphi_{(m-1)-} * \varphi$.

Let $x \in \mathcal{H}^m$ be arbitrary. Then,

$$\left(\varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}}\right)(x) = \varphi_{(m+1)-}(x) = -\pi \left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x') \varphi(x'') \right).$$

Since $x \in \mathcal{H}^m$ yields $\sum_{(x)} x' \otimes x'' \in \mathcal{H}^{+(m-1)} \otimes \mathcal{H}$ [9], we can assume that $x' \in$

[8] This definition rests on the fact that every $x \in \mathcal{H}^{+(m+1)}$ satisfies $\sum_{(x)} x' \otimes x'' = \widetilde{\Delta} x \in \sum_{\substack{p+q=m+1; \\ p \neq 0;\ q \neq 0}} \underbrace{\mathcal{H}^{+p}}_{\subseteq \mathcal{H}^{+m}} \otimes \underbrace{\mathcal{H}^{+q}}_{\subseteq \mathcal{H}} \subseteq \sum_{\substack{p+q=m+1; \\ p \neq 0;\ q \neq 0}} \mathcal{H}^{+m} \otimes \mathcal{H} \subseteq \mathcal{H}^{+m} \otimes \mathcal{H}$.

[9] In fact, $\sum_{(x)} x' \otimes x'' = \widetilde{\Delta} x \in \sum_{\substack{p+q=m; \\ p \neq 0;\ q \neq 0}} \underbrace{\mathcal{H}^{+p}}_{\subseteq \mathcal{H}^{+(m-1)}} \otimes \underbrace{\mathcal{H}^{+q}}_{\subseteq \mathcal{H}} \subseteq \sum_{\substack{p+q=m; \\ p \neq 0;\ q \neq 0}} \mathcal{H}^{+(m-1)} \otimes \mathcal{H} \subseteq \mathcal{H}^{+(m-1)} \otimes \mathcal{H}$.

$\mathcal{H}^{+(m-1)}$ in this equation, and thus we get

$$\left(\varphi_{(m+1)-} \mid_{\mathcal{H}^m}\right)(x) = -\pi\left(\varphi(x) + \sum_{(x)} \underbrace{\varphi_{m-}(x')}_{\substack{=\left(\varphi_{m-}\mid_{\mathcal{H}^{+(m-1)}}\right)(x') \\ (\text{since } x' \in \mathcal{H}^{+(m-1)})}} \varphi(x'')\right)$$

$$= -\pi\left(\varphi(x) + \sum_{(x)} \underbrace{\left(\varphi_{m-} \mid_{\mathcal{H}^{+(m-1)}}\right)}_{=\varphi_{(m-1)-}}(x')\,\varphi(x'')\right)$$

$$= -\pi\left(\varphi(x) + \sum_{(x)} \varphi_{(m-1)-}(x')\,\varphi(x'')\right).$$

But comparing this to

$$\varphi_{m-}(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_{(m-1)-}(x')\,\varphi(x'')\right)$$
$$(\text{by } (4), \text{ applied to } m-1 \text{ instead of } m),$$

we obtain $\left(\varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}}\right)(x) = \varphi_{m-}(x)$. Since this holds for every $x \in \mathcal{H}^m$, we thus have showed that $\varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}} = \varphi_{m-}$. In other words, we proved (6) for $n = m$. Similarly to our proof of $\varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}} = \varphi_{m-}$, we can show that $\varphi_{(m+1)+} \mid_{\mathcal{H}^{+m}} = \varphi_{m+}$. Thus, we proved (7) for $n = m$. To complete the induction step, we now need to verify (8) for $n = m$.
Let $x \in \mathcal{H}^m$. Since $\varphi_{m-} = \varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}}$, we get

$$\varphi_{m-}(x) = \left(\varphi_{(m+1)-} \mid_{\mathcal{H}^{+m}}\right)(x) = \varphi_{(m+1)-}(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'')\right).$$

But since $\varphi_{m+} = \varphi_{(m+1)+} \mid_{\mathcal{H}^{+m}}$, we have

$$\varphi_{m+}(x) = \left(\varphi_{(m+1)+} \mid_{\mathcal{H}^{+m}}\right)(x) = \varphi_{(m+1)+}(x) = (I-\pi)\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'')\right)$$

$$= \varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'') - \pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'')\right)$$

$$= \varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'') + \underbrace{\left(-\pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'')\right)\right)}_{=\varphi_{m-}(x)}$$

$$= \varphi(x) + \sum_{(x)} \varphi_{m-}(x')\,\varphi(x'') + \varphi_{m-}(x) = \sum_{(x)} \varphi_{m-}(x_1)\,\varphi(x_2) = (\varphi_{m-} * \varphi)(x).$$

Since this is proven for every $x \in \mathcal{H}^m$, we conclude that $\varphi_{m+} = \varphi_{m-} * \varphi$. Thus, (8) is verified for $n = m$.
We have thus proven the three equations (6), (7), (8) for $n = m$. This completes the induction step, and thus we have proven (6), (7), (8) for all $n \in \mathbb{N}$.
Note that $I - \pi$ is a projection onto $\mathcal{A}_+$ (since π is a projection parallel to $\mathcal{A}_+$).
For every $x \in \operatorname{Ker} \varepsilon$, let $N(x)$ be the smallest $m \in \mathbb{N}$ satisfying $x \in \mathcal{H}^m$.
Now, let us define a map $\varphi_{\infty-} : \operatorname{Ker} \varepsilon \to \mathcal{A}$ by setting

$$\varphi_{\infty-}(x) = \varphi_{(N(x))-}(x) \qquad \text{for every } x \in \operatorname{Ker} \varepsilon.$$

Then, (6) shows that

$$\varphi_{\infty-}(x) = \varphi_{m-}(x) \qquad \text{for every } x \in \operatorname{Ker} \varepsilon \text{ and every } m \in \mathbb{N} \text{ satisfying } x \in \mathcal{H}^m.$$

Similarly, let us define a map $\varphi_{\infty+} : \operatorname{Ker} \varepsilon \to \mathcal{A}$ by setting

$$\varphi_{\infty+}(x) = \varphi_{(N(x))+}(x) \qquad \text{for every } x \in \operatorname{Ker} \varepsilon.$$

Then, (7) shows that

$$\varphi_{\infty+}(x) = \varphi_{m+}(x) \qquad \text{for every } x \in \operatorname{Ker} \varepsilon \text{ and every } m \in \mathbb{N} \text{ satisfying } x \in \mathcal{H}^m.$$

Now (8) proves that $\varphi_{\infty+} = \varphi_{\infty-} * \varphi$.
We now extend the map $\varphi_{\infty-} : \operatorname{Ker} \varepsilon \to \mathcal{A}$ to a map $\varphi_- : \mathcal{H} \to \mathcal{A}$ by setting $\varphi_-(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$. Similarly, we extend the map $\varphi_{\infty+} : \operatorname{Ker} \varepsilon \to \mathcal{A}$ to a map $\varphi_+ : \mathcal{H} \to \mathcal{A}$ by setting $\varphi_+(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$. It is easy to see that $\varphi_+ = \varphi_- * \varphi$ (since $\varphi_{\infty+} = \varphi_{\infty-} * \varphi$ and since $\varphi_+(\mathbf{1}) = \mathbf{1}_{\mathcal{A}} = (\varphi_- * \varphi)(\mathbf{1})$), so that $\varphi = \varphi_-^{*-1} * \varphi_+$.
Also, it is clear that φ_- sends $\mathbf{1}$ to $\mathbf{1}_{\mathcal{A}}$ and $\operatorname{Ker} \varepsilon$ to $\mathcal{A}_-$ (the latter is because of (4) and because π is a projection onto $\mathcal{A}_-$), and that φ_+ sends $\mathcal{H}$ to $\mathcal{A}_+$ (this is because of (5) and because $I - \pi$ is a projection onto $\mathcal{A}_+$).
We have now proven the existence of the Birkhoff decomposition. To complete the proof of Theorem II.5.1 part 1), we must now show that it is unique. To show this, we assume that we have some elements ψ_- and ψ_+ of G satisfying $\varphi = \psi_-^{-1} * \psi_+$ such that ψ_- sends $\mathbf{1}$ to $\mathbf{1}_{\mathcal{A}}$ and $\operatorname{Ker} \varepsilon$ to $\mathcal{A}_-$, and such that ψ_+ sends $\mathcal{H}$ to $\mathcal{A}_+$. Now let us prove that $\psi_- = \varphi_-$ and $\psi_+ = \varphi_+$; this will clearly prove the uniqueness of the Birkhoff decomposition.
To prove that $\psi_- = \varphi_-$, we will show by induction over n that $\psi_- \mid_{\mathcal{H}^{+n}} = \varphi_{n-}$ for every $n \in \mathbb{N}$: The induction base (the case $n = 0$) is clear (again due to $\mathcal{H}^{+0} = 0$). Now to the induction step: Let $m \in \mathbb{N}$ be arbitrary. Assume that $\psi_- \mid_{\mathcal{H}^{+m}} = \varphi_{m-}$. Let us now show that $\psi_- \mid_{\mathcal{H}^{+(m+1)}} = \varphi_{(m+1)-}$.
Let $x \in \mathcal{H}^{+(m+1)}$. Then,

$$\sum_{(x)} x' \otimes x'' = \widetilde{\Delta} x \in \sum_{\substack{p+q=m+1;\\ p \neq 0;\ q \neq 0}} \underbrace{\mathcal{H}^{+p}}_{\subseteq \mathcal{H}^{+m}} \otimes \underbrace{\mathcal{H}^{+q}}_{\subseteq \mathcal{H}} \subseteq \sum_{\substack{p+q=m+1;\\ p \neq 0;\ q \neq 0}} \mathcal{H}^{+m} \otimes \mathcal{H} \subseteq \mathcal{H}^{+m} \otimes \mathcal{H}.$$

We can thus WLOG assume that $x' \in \mathcal{H}^{+m}$. On the other hand, $\psi_-(x) \in \mathcal{A}_-$ (since $x \in \mathcal{H}^{+(m+1)} \subseteq \operatorname{Ker} \varepsilon$ and $\psi_-(\operatorname{Ker} \varepsilon) \subseteq \mathcal{A}_-$), so that $\pi(\psi_-(x)) = \psi_-(x)$

(because π is a projection onto $\mathcal{A}_-$). Besides, $\psi_+(x) \in \mathcal{A}_+$, and thus $\pi(\psi_+(x)) = 0$ (since π is a projection parallel to $\mathcal{A}_+$). But since $\psi_+ = \psi_- * \varphi$ (because $\varphi = \psi_-^{-1} * \psi_+$), we have

$$\psi_+(x) = (\psi_- * \varphi)(x) = \sum_{(x)} \psi_-(x_1)\varphi(x_2) = \psi_-(x) + \varphi(x) + \sum_{(x)} \underbrace{\psi_-(x')}_{\substack{=(\psi_- |_{\mathcal{H}^{+m}})(x') \\ (\text{since } x' \in \mathcal{H}^{+m})}} \varphi(x'')$$

$$= \psi_-(x) + \varphi(x) + \sum_{(x)} \underbrace{(\psi_- |_{\mathcal{H}^{+m}})}_{=\varphi_{m-}}(x')\varphi(x'') = \psi_-(x) + \varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'').$$

Thus

$$\pi(\psi_+(x)) = \pi\left(\psi_-(x) + \varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'')\right)$$
$$= \underbrace{\pi(\psi_-(x))}_{=\psi_-(x)} + \pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'')\right)$$
$$= \psi_-(x) + \pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'')\right).$$

Since $\pi(\psi_+(x)) = 0$, this becomes $0 = \psi_-(x) + \pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'')\right)$, so that

$$\psi_-(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_{m-}(x')\varphi(x'')\right) = \varphi_{(m+1)-}(x).$$

Since this holds for every $x \in \mathcal{H}^{+(m+1)}$, we thus conclude that $\psi_- |_{\mathcal{H}^{+(m+1)}} = \varphi_{(m+1)-}$. This completes the induction.
We thus have shown that $\psi_- |_{\mathcal{H}^{+n}} = \varphi_{n-}$ for every $n \in \mathbb{N}$. By the construction of $\varphi_{\infty-}$, this yields that $\psi_- |_{\operatorname{Ker}\varepsilon} = \varphi_{*-}$. This means that the maps ψ_- and φ_- coincide on $\operatorname{Ker}\varepsilon$. Since they also coincide on $k\cdot\mathbf{1}$, this yields $\psi_- = \varphi_-$. Analogous arguments show that $\psi_+ = \varphi_+$. This completes the proof of the uniqueness of the Birkhoff decomposition. Thus, part 1) of Theorem II.5.1 is finally proven.

- **Proof of Theorem II.5.1:** On page 36, you write: "The same property for τ_+ comes then from proposition II.3.1." Maybe you mean Proposition II.4.1 instead of II.3.1 here?

- **Proof of Theorem II.5.1:** On page 36, you prove $\tau_-(xy) = \tau_-(yx)$ by decomposing $\Delta(xy)$ using Sweedler's notation. This decomposition is only correct when x and y lie in $\operatorname{Ker}\varepsilon$. (Fortunately, it is enough to prove $\tau_-(xy) = \tau_-(yx)$ for x and y lying in $\operatorname{Ker}\varepsilon$, because it is trivially true when x or y lies in $k \cdot \mathbf{1}$.)

- **Proof of Theorem II.5.1:** On page 37, you write "with $X = \chi(x) - \sum_{(x)} \chi_-(x') \chi(x'')$ and $Y = \chi(y) - \sum_{(y)} \chi_-(y') \chi(y'')$" (during the proof of assertion 3)). This should be "with $X = \chi(x) + \sum_{(x)} \chi_-(x') \chi(x'')$ and $Y = \chi(y) + \sum_{(y)} \chi_-(y') \chi(y'')$".

- **Remark at the end of II.5:** Remove the word "recursively": the current definition of b is not recursive at all. Besides, $\varphi_- = -\pi \circ b(\varphi)$ should probably be replaced by $\varphi_- = e - \pi \circ b(\varphi)$, unless you want it to hold on $\operatorname{Ker} \varepsilon$ only (since $\varphi_-(\mathbf{1}) = \mathbf{1}_{\mathcal{A}} \notin \pi(\mathcal{A})$).

- **II.6:** At the very end of page 39, you write: "K. Ebrahimi-Fard, L. Guo and D. Kreimer derive in [EGK2] two identities involving the Bogoliubov character :

$$e^{*-R(\chi_R(X))} = R\left(b\left(e^{*X}\right)\right), \qquad e^{*\widetilde{R}(\chi_R(X))} = -\widetilde{R}\left(b\left(e^{*X}\right)\right).$$

  " I have not read [EGK2], but I think that by applying the uniqueness of the Birkhoff decomposition and the (corrected!) version of the Remark at the end of II.5 ("corrected" in the sense that $\varphi_- = -\pi \circ b(\varphi)$ is replaced by $\varphi_- = e - \pi \circ b(\varphi)$), I get two slightly different identities:

$$e^{*-R(\chi_R(X))} = e - R\left(b\left(e^{*X}\right)\right), \qquad e^{*\widetilde{R}(\chi_R(X))} = \widetilde{R}\left(b\left(e^{*X}\right)\right).$$

- **Remark at the end of II.6:** What do you mean by "Rota-Baxter identity for R just guarantees that equation (**) gives a Birkhoff decomposition"? How do you define a Birkhoff decomposition when there is no projection π but just an operator R satisfying Rota-Baxter?

- **II.7:** Replace "fuction" by "function".

- **II.8:** On the first line of II.8, you speak of "biderivation". What does "biderivation" mean? I used to understand this word as "derivation and coderivation at the same time", but the meaning of "derivation" that makes Y a derivation (and makes $\varphi \mapsto \varphi \circ Y$ a derivation of $(\mathcal{L}(\mathcal{H}, \mathcal{A}), *)$ in Lemma II.8.1) is different from the meaning of "derivation" in II.4, so you might want to add a remark about these two meanings (the first one defines a derivation as a map f satisfying $f(xy) = xf(y) + f(x)y$, whereas the second one defines a derivation as a map f satisfying $f(xy) = e(x) f(y) + f(x) e(y)$) and which of them is used at what place. Also, you have never defined what a coderivation is.

- **II.8:** Before Lemma II.8.1, you define a map θ_t. Maybe you should add that t is assumed to be an element for which e^{nt} makes sense, e. g., a complex number, or an element of a maximal ideal in a complete local ring.

- **Lemma II.8.2:** This lemma is not completely proved in the text preceding it. What remains to be proven is that if α is a derivation of $\mathcal{H}$ with values in $\mathcal{A}$, then $\alpha \circ Y^{-1}$ is a derivation of $\mathcal{H}$ with values in $\mathcal{A}$ as well.
  Here is a *proof* for this: Assume that α is a derivation of $\mathcal{H}$ with values in $\mathcal{A}$. Now, in order to prove that $\alpha \circ Y^{-1}$ is a derivation of $\mathcal{H}$ with values in $\mathcal{A}$, we let x and y be two elements of $\mathcal{H}$. We must then prove the equation $(\alpha \circ Y^{-1})(xy) = e(x)(\alpha \circ Y^{-1})(y) + (\alpha \circ Y^{-1})(x) e(y)$.

Since this equation is linear in x and y, we can WLOG assume that x and y are homogeneous. We now distinguish between four cases:
Case 1: We have $|x| > 0$ and $|y| > 0$.
Case 2: We have $|x| = 0$ and $|y| > 0$.
Case 3: We have $|y| = 0$ and $|x| > 0$.
Case 4: We have $|x| = 0$ and $|y| = 0$.
Let us consider Case 1 first. In this case,

$$\left(\alpha \circ Y^{-1}\right)(xy) = \alpha\left(\underbrace{Y^{-1}(xy)}_{=\frac{1}{|xy|}xy}\right) = \frac{1}{|xy|}\underbrace{\alpha(xy)}_{\substack{=e(x)\alpha(y)+\alpha(x)e(y) \\ \text{(since } \alpha \text{ is a derivation)}}}$$
$$= \frac{1}{|xy|}\left(\underbrace{e(x)}_{=0 \text{ (since } |x|>0)}\alpha(y) + \alpha(x)\underbrace{e(y)}_{=0 \text{ (since } |y|>0)}\right)$$
$$= \frac{1}{|xy|}\left(0\alpha(y) + \alpha(x)0\right) = 0$$

and

$$\underbrace{e(x)}_{=0 \text{ (since } |x|>0)}\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)\underbrace{e(y)}_{=0 \text{ (since } |y|>0)} = 0\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)0 = 0,$$

so that $\left(\alpha \circ Y^{-1}\right)(xy) = e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)e(y)$ is proven in Case 1.
Let us now consider Case 2. In this case,

$$\left(\alpha \circ Y^{-1}\right)(xy) = \alpha\left(\underbrace{Y^{-1}(xy)}_{=\frac{1}{|xy|}xy}\right) = \frac{1}{|xy|}\underbrace{\alpha(xy)}_{\substack{=e(x)\alpha(y)+\alpha(x)e(y) \\ \text{(since } \alpha \text{ is a derivation)}}} = \frac{1}{|xy|}\left(e(x)\alpha(y) + \alpha(x)\underbrace{e(y)}_{=0 \text{ (since } |y|>0)}\right)$$
$$= \frac{1}{|xy|}\left(e(x)\alpha(y) + \alpha(x)0\right) = \frac{1}{|xy|}e(x)\alpha(y) = \frac{1}{|y|}e(x)\alpha(y)$$
$$\left(\text{since } |xy| = \underbrace{|x|}_{=0} + |y| = |y|\right)$$

and

$$e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)\underbrace{e(y)}_{=0 \text{ (since } |y|>0)}$$
$$= e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)0 = e(x)\underbrace{\left(\alpha \circ Y^{-1}\right)(y)}_{=\alpha(Y^{-1}(y))=\alpha\left(\frac{1}{|y|}y\right)=\frac{1}{|y|}\alpha(y)} = \frac{1}{|y|}e(x)\alpha(y),$$

so that $\left(\alpha \circ Y^{-1}\right)(xy) = e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)\, e(y)$ is proven in Case 2.
Similarly we can prove $\left(\alpha \circ Y^{-1}\right)(xy) = e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)\, e(y)$ in Case 3.
In Case 4, the proof is left to the reader.
Thus, in all four cases, we have shown that $\left(\alpha \circ Y^{-1}\right)(xy) = e(x)\left(\alpha \circ Y^{-1}\right)(y) + \left(\alpha \circ Y^{-1}\right)(x)\, e(y)$, so that we can conclude that $\alpha \circ Y^{-1}$ is a derivation, qed.

- **Remark at the end of II.8:** The comment that "any other value of $Y^{-1}(\mathbf{1})$ would give the same result" is correct only as long as this value of $Y^{-1}(\mathbf{1})$ is still supposed to be a scalar multiple of $\mathbf{1}$.

- **II.9.1:** You write: "and we extend Δ to an algebra isomorphism". You mean "homomorphism", not "isomorphism".

- **II.9.1:** It would be useful to add that this Hopf algebra $\mathcal{N}$ is isomorphic to the symmetric Hopf algebra on the k-module $k^{\mathbb{P}}$ (where $\mathbb{P}$ is the set of all primes) by the isomorphism

$$\mathcal{N} \to k^{\mathbb{P}};$$
$$e_n \mapsto \sum_{p \text{ prime divisor of } n} (\text{multiplicity of } p \text{ in } n) \cdot \left(\text{basis vector of } k^{\mathbb{P}} \text{ corresponding to } p\right).$$

- **II.9.3:** The definition you give for the notion of a "planar rooted tree" confused me. If it is taken literally, two planar graphs are considered different if they differ in the position of their vertices on the plane; for instance, two planar rooted trees consisting of one vertex each are different if these vertices are located at different points in the plane. I *believe* this is not what you want to achieve. After having looked up some definitions of "planar rooted tree" on the internet, I would say I prefer the purely combinatorial definition: A planar rooted tree is an oriented tree (this means an oriented connected graph with no oriented cycles such that only one vertex - the root - has only outgoing edges, whereas every other vertex has exactly one incoming edge) along with, for every vertex v of the tree, a linear order on the set of children of v (here, a "child" means the target of an outgoing edge from v). When we draw a planar rooted tree on plane, we have to start with the root, and at each step choose a vertex already drawn and add all of its children; we should do it in such a way that the lowest child (with respect to the linear order) is drawn on the very left, the second lowest child is the second from the left, and so on, and the highest child is drawn on the very right.
Also, when you say "Let $\mathcal{T}$ be the set of planar rooted trees", you mean "Let $\mathcal{T}$ be the set of planar rooted trees up to isomorphism".
Also, it should be mentioned that a "planar rooted forest" means a finite (possibly empty) sequence of planar rooted trees. This means that changing the order of the trees in a planar rooted forest changes the forest.

- **II.9.3:** You write: "The trunk of a tree is a tree". This is only true if the cut is not the empty cut (since there is no tree with 0 vertices).

- **II.9.3:** You write: "and let $\operatorname{Adm}^*(F)$ the set of elementary cuts disregarding the empty cut and the total cut". You mean "admissible" rather than "elementary" here.

- **II.9.3:** I would prefer to have a formal definition of a notion of a "cut", particularly because there is a very simple one:
  - A *cut* of a forest F denotes a pair (A, B) where A and B are two subsets of $\operatorname{V}(F)$ (the set of the vertices of F) such that $A \cap B = \varnothing$ and $A \cup B = \operatorname{V}(F)$. The *trunk* of F with respect to this cut is then defined as the forest $\operatorname{ind}_A(F)$, and the *crown* of F with respect to this cut is defined as the forest $\operatorname{ind}_B(F)$.
  Here, whenever S is a subset of $\operatorname{V}(F)$, we denote by $\operatorname{ind}_S(F)$ the induced subgraph of F on the vertex set S.
  With this definition, we can reformulate the definitions of admissible and elementary cuts, as well as of bi-admissible couples, in a more formal fashion:
  - An *admissible cut* of a forest F denotes a cut (A, B) such that every path on F (viewed as a sequence of vertices) has the form (some vertices in A, some vertices in B) (where "some" might also mean "none"). Equivalently, an *admissible cut* of a forest F denotes a cut (A, B) such that the set A is closed under taking ancestors (i. e., every ancestor of an element of A lies in A) and the set B is closed under taking descendants (i. e., every descendant of an element of B lies in B).
  - An *elementary cut* of a forest F denotes a cut (A, B) such that $B =$ (the set of all descendants of v) for some $v \in \operatorname{V}(F)$. Here, v is considered its own descendant.
  - A *bi-admissible couple* means a triple (A_1, A_2, A_3) where A_1, A_2 and A_3 are three subsets of $\operatorname{V}(F)$ (the set of the vertices of F) such that

  $$A_i \cap A_j = \varnothing \text{ for all } i, j \in \{1, 2, 3\} \text{ with } i \neq j$$

  and $A_1 \cup A_2 \cup A_3 = \operatorname{V}(F)$, and such that every $i, j \in \{1, 2, 3\}$, every vertex $v \in A_i$ and every descendant $w \in A_j$ of v satisfy $i \leq j$. The *trunk* of F with respect to this bi-admissible couple is defined as the forest $\operatorname{ind}_A(F)$, the *middle* of F with respect to this bi-admissible couple is defined as the forest $\operatorname{ind}_B(F)$, and the *crown* of F with respect to this bi-admissible couple is defined as the forest $\operatorname{ind}_C(F)$.
  It is then easy to see that choosing a bi-admissible couple on a forest F is equivalent to choosing an admissible cut on F and then choosing an admissible cut on the crown of F (indeed, a triple (A_1, A_2, A_3) of subsets of $\operatorname{V}(F)$ is a bi-admissible couple if and only if $(A_1, A_2 \cup A_3)$ is an admissible cut of F and (A_2, A_3) is an admissible cut of $\operatorname{ind}_{A_2 \cup A_3}(F)$), and also equivalent to choosing an admissible cut on F and then choosing an admissible cut on the trunk of F (indeed, a triple (A_1, A_2, A_3) of subsets of $\operatorname{V}(F)$ is a bi-admissible couple if and only if $(A_1 \cup A_2, A_3)$ is an admissible cut of F and (A_1, A_2) is an admissible cut of $\operatorname{ind}_{A_1 \cup A_2}(F)$). This proves the coassociativity of Δ.

- **II.9.3:** On page 44, replace "n-uples" by "n-uples" (the "n" should be math, not text).

- **II.9.3:** On page 44, "By corollary II.2.2" should be "By corollary II.3.2".

- **III.3:** On the fourth line of page 47 (not counting the title of III.3), you write: "the free commutative algebra generated by V". Is it possible that you mean $V_{\mathcal{T}}$ when you say V ?

- **III.3:** On the sixth line of page 47, replace "co-unity" by "co-unit".

- **III.3:** On page 47, you write that "any nonempty subgraph has a non-vanishing loop number". How can this be true? For example, assume that our original graph is a 4-cycle, and we choose the subgraph formed by two adjacent edges of this cycle. This subgraph has 2 internal edges, 2 external edges and 3 vertices, so the loop number is $2 - 3 + 1 = 0$. Maybe you want the I in the definition of the loop number to be the number of *all* edges?

- **III.3:** In the fourth line from bottom of page 48, you write: "The ideal J is the a bi-ideal". Here, the "the" should be removed.

- **III.3:** On page 49, you write: "We can identify the quotient with $S(V'_T)$, where V'_T stands for [...]". Both of the T's here should be calligraphic $\mathcal{T}$'s.

- **Chapter IV.** On page 50, you write: "We denote by Y (resp. θ_t) the biderivation (resp. the one-parameter group of automorphisms) of the Hopf algebra $\mathcal{H}$ induced by the graduation (cf. § II.6)." I think you mean II.8, not II.6 here.

- **Proof of Proposition IV.1.2:** You mention a "derivation property

  $$\left(e^{*t\alpha} * e^{*s\alpha}\right) \circ Y = \left(e^{*t\alpha} \circ Y\right) * e^{*s\alpha} + e^{*t\alpha} * \left(e^{*s\alpha} \circ Y\right)$$

  " here. It is correct, but it will probably become a bit clearer if you refer to Lemma II.8.1 for this property (otherwise, it looks like you are applying the fact that Y is a derivation).

- **Proof of Corollary IV.1.3:** I don't understand how exactly you show the first assertion here.
  Anyway, I don't like the proof as it uses the analytic Proposition IV.1.2, whereas there is a much more straightforward proof using pure algebra (and therefore works for any commutative k-algebra $\mathcal{A}$ over any field k, not necessarily $\mathbb{C}$):
  Alternative proof of Corollary IV.1.3: Let us prove the first assertion first:
  Let α be a derivation of $\mathcal{H}$ with values in $\mathcal{A}$. We must prove that $R(\alpha)$ is a derivation of $\mathcal{H}$ with values in $\mathcal{A}$ as well.
  Let $\varphi = e^{*\alpha}$. Then, Proposition II.4.2 yields that $\varphi \in G_1$ (since α is a derivation and thus $\alpha \in \mathfrak{g}_1$), so that φ is an algebra homomorphism. Thus, φ^{*-1} is an algebra homomorphism as well (since G_1 is a group). Now, we are going to prove that any $x \in \mathcal{H}$ and $y \in \mathcal{H}$ satisfy

  $$\left(\varphi^{*-1} * (\varphi \circ Y)\right)(xy) = e(x)\left(\varphi^{*-1} * (\varphi \circ Y)\right)(y) + \left(\varphi^{*-1} * (\varphi \circ Y)\right)(x)\, e(y). \tag{9}$$

  In order to prove (9), we can WLOG assume that x and y are homogeneous.

Then, we can write $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ with homogeneous x_1 and x_2, and we can write $\Delta(y) = \sum_{(y)} y_1 \otimes y_2$ with homogeneous y_1 and y_2. Clearly,

$$\left(\varphi^{*-1} * (\varphi \circ Y)\right)(x) = \sum_{(x)} \varphi^{*-1}(x_1) \underbrace{(\varphi \circ Y)(x_2)}_{\substack{=\varphi(Y(x_2))=\varphi(|x_2|x_2) \\ (\text{since } Y(x_2)=|x_2|x_2)}} = \sum_{(x)} \varphi^{*-1}(x_1)\varphi(|x_2| x_2)$$
$$= \sum_{(x)} |x_2| \varphi^{*-1}(x_1)\varphi(x_2).$$

Similarly, $(\varphi^{*-1} * (\varphi \circ Y))(y) = \sum_{(y)} |y_2| \varphi^{*-1}(y_1)\varphi(y_2)$. But

$$\left(\varphi^{*-1} * (\varphi \circ Y)\right)(xy)$$
$$= \sum_{(xy)} \varphi^{*-1}\left((xy)_1\right)(\varphi \circ Y)\left((xy)_2\right) = \sum_{(x)(y)} \underbrace{\varphi^{*-1}(x_1 y_1)}_{\substack{=\varphi^{*-1}(x_1)\varphi^{*-1}(y_1) \\ (\text{since } \varphi^{*-1} \text{ is an algebra} \\ \text{homomorphism})}} \underbrace{(\varphi \circ Y)(x_2 y_2)}_{=\varphi(Y(x_2 y_2))}$$
$$= \sum_{(x)(y)} \varphi^{*-1}(x_1)\varphi^{*-1}(y_1)\varphi\left(\underbrace{Y(x_2 y_2)}_{=(|x_2|+|y_2|)x_2 y_2}\right) = \sum_{(x)(y)} (|x_2| + |y_2|)\varphi^{*-1}(x_1)\varphi^{*-1}(y_1) \underbrace{\varphi(x_2 y_2)}_{\substack{=\varphi(x_2)\varphi(y_2) \\ (\text{since } \varphi \text{ is an algebra} \\ \text{homomorphism})}}$$
$$= \sum_{(x)(y)} \underbrace{(|x_2| + |y_2|)\varphi^{*-1}(x_1)\varphi^{*-1}(y_1)\varphi(x_2)\varphi(y_2)}_{=|x_2|\varphi^{*-1}(x_1)\varphi(x_2)\varphi^{*-1}(y_1)\varphi(y_2)+|y_2|\varphi^{*-1}(x_1)\varphi(x_2)\varphi^{*-1}(y_1)\varphi(y_2)}$$
$$= \sum_{(x)(y)} \left(|x_2| \varphi^{*-1}(x_1)\varphi(x_2)\varphi^{*-1}(y_1)\varphi(y_2) + |y_2| \varphi^{*-1}(x_1)\varphi(x_2)\varphi^{*-1}(y_1)\varphi(y_2)\right)$$
$$= \underbrace{\sum_{(x)} |x_2| \varphi^{*-1}(x_1)\varphi(x_2)}_{=(\varphi^{*-1}*(\varphi\circ Y))(x)} \underbrace{\sum_{(y)} \varphi^{*-1}(y_1)\varphi(y_2)}_{=(\varphi^{*-1}*\varphi)(y)=e(y)} + \underbrace{\sum_{(x)} \varphi^{*-1}(x_1)\varphi(x_2)}_{=(\varphi^{*-1}*\varphi)(x)=e(x)} \underbrace{\sum_{(y)} |y_2| \varphi^{*-1}(y_1)\varphi(y_2)}_{=(\varphi^{*-1}*(\varphi\circ Y))(y)}$$
$$= \left(\varphi^{*-1} * (\varphi \circ Y)\right)(x)\, e(y) + e(x) \left(\varphi^{*-1} * (\varphi \circ Y)\right)(y).$$

This proves (9). Thus, $\varphi^{*-1} * (\varphi \circ Y)$ is a derivation from $\mathcal{H}$ with values in $\mathcal{A}$. But since $\varphi = e^{*\alpha}$ yields $\varphi^{*-1} * (\varphi \circ Y) = e^{*-\alpha} * (e^{*\alpha} \circ Y) = R(\alpha)$, this means that $R(\alpha)$ is a derivation from $\mathcal{H}$ with values in $\mathcal{A}$. We thus have shown the first assertion of Corollary IV.1.3.

Let us now prove the second assertion of Corollary IV.1.3:

Let β be a cocycle from $\mathcal{H}$ to $\mathcal{A}$. We must prove that $R(\beta)$ is a cocycle from $\mathcal{H}$ to $\mathcal{A}$ as well.

For this we need a *lemma:* If η is a cocycle from $\mathcal{H}$ to $\mathcal{A}$, then $\eta \circ Y$ is a cocycle from $\mathcal{H}$ to $\mathcal{A}$ as well.

Proof of the lemma: Let x and y be two homogeneous elements of $\mathcal{H}$. Then,

$Y(xy) = \underbrace{|xy|}_{=|x|+|y|} xy = (|x|+|y|)\,xy$ and

$$(\eta \circ Y)(xy) = \eta\left(\underbrace{Y(xy)}_{=(|x|+|y|)xy}\right) = \eta\left((|x|+|y|)\,xy\right) = (|x|+|y|)\,\eta(xy).$$

Similarly, $(\eta \circ Y)(yx) = (|y|+|x|)\,\eta(yx)$. Comparing these two equalities (and recalling that $\eta(xy) = \eta(yx)$ (since η is a cocycle) and $|x|+|y| = |y|+|x|$), we conclude that $(\eta \circ Y)(xy) = (\eta \circ Y)(yx)$. We have thus shown the identity $(\eta \circ Y)(xy) = (\eta \circ Y)(yx)$ for any two homogeneous elements x and y of $\mathcal{H}$. Since this identity is linear in x and y, we can thus conclude that this identity holds for *any two* (not necessarily homogeneous) elements x and y of $\mathcal{H}$. In other words, $\eta \circ Y$ is a cocycle. The lemma is proven.

Now, Proposition II.4.2 yields $\exp(\mathfrak{g}_2) = G_2$. Thus, $\beta \in \mathfrak{g}_2$ (since β is a cocycle) yields $e^{*\beta} \in G_2$. Since G_2 is a group, this yields $e^{*-\beta} \in G_2$. On the other hand, the lemma (applied to $Y = e^{*\beta}$) yields that $e^{*\beta} \circ Y$ is a cocycle. Now, Proposition II.4.1 1) shows that the convolution $e^{*-\beta} * \left(e^{*\beta} \circ Y\right)$ is a cocycle. Since $R(\beta) = e^{*-\beta} * \left(e^{*\beta} \circ Y\right)$, this means that $R(\beta)$ is a cocycle. We haved thus proven the second assertion of Corollary IV.1.3, all without using analysis.

- **Remark at the end of IV.1:** Apparently you use Proposition IV.1.2 to show that $R(\alpha) = \alpha \circ Y$ in the case when $\mathcal{H}$ is cocommutative. Here is an alternative proof of this statement:

  Alternative proof of $R(\alpha) = \alpha \circ Y$ in the case of a cocommutative Hopf algebra $\mathcal{H}$:

  From Lemma II.8.1 we know that $\varphi \mapsto \varphi \circ Y$ is a derivation of the algebra $(\mathcal{L}(\mathcal{H},\mathcal{A}), *)$. In other words, any $\varphi \in \mathcal{L}(\mathcal{H},\mathcal{A})$ and $\psi \in \mathcal{L}(\mathcal{H},\mathcal{A})$ satisfy $(\varphi * \psi) \circ Y = \varphi * (\psi \circ Y) + (\varphi \circ Y) * \psi$. Applying this property and using induction over n, we can show that every $\alpha \in \mathcal{L}(\mathcal{H},\mathcal{A})$ and every integer $n > 0$ satisfy $\alpha^{*n} \circ Y = n(\alpha \circ Y) * \alpha^{*(n-1)}$ (here we use that $\mathcal{L}(\mathcal{H},\mathcal{A})$ is commutative, which follows from the cocommutativity of $\mathcal{H}$ and the commutativity of $\mathcal{A}$). Now, if $\alpha \in \mathfrak{g}$, then $e^{*\alpha} = \sum\limits_{n \geq 0} \dfrac{\alpha^{*n}}{n!}$, so that

$$\begin{aligned} e^{*\alpha} \circ Y &= \sum_{n\geq 0} \frac{\alpha^{*n}}{n!} \circ Y = \underbrace{\frac{\alpha^{*0}}{0!}}_{=e} \circ Y + \sum_{n\geq 1} \frac{\alpha^{*n}}{n!} \circ Y = \underbrace{e \circ Y}_{=0} + \sum_{n\geq 1} \frac{\alpha^{*n}}{n!} \circ Y = \sum_{n\geq 1} \frac{\alpha^{*n}}{n!} \circ Y \\ &= \sum_{n\geq 1} \frac{1}{n!} \underbrace{\alpha^{*n} \circ Y}_{=n(\alpha\circ Y)*\alpha^{*(n-1)}} = \sum_{n\geq 1} \underbrace{\frac{1}{n!} n}_{=\frac{1}{(n-1)!}} (\alpha \circ Y) * \alpha^{*(n-1)} \\ &= \sum_{n\geq 1} \frac{1}{(n-1)!} (\alpha \circ Y) * \alpha^{*(n-1)} = (\alpha \circ Y) * \sum_{n\geq 1} \frac{\alpha^{*(n-1)}}{(n-1)!} = (\alpha \circ Y) * \underbrace{\sum_{n\geq 0} \frac{\alpha^{*n}}{n!}}_{=e^{*\alpha}} \\ &\qquad \text{(here, we substituted } n \text{ for } n-1 \text{ in the sum)} \\ &= (\alpha \circ Y) * e^{*\alpha} = e^{*\alpha} * (\alpha \circ Y) \qquad \text{(by the commutativity of } \mathcal{L}(\mathcal{H},\mathcal{A})). \end{aligned}$$

But sinec $R(\alpha)$ is defined as the $\gamma \in \mathfrak{g}$ satisfying $e^{*\alpha} \circ Y = e^{*\alpha} * \gamma$, this yields $R(\alpha) = \alpha \circ Y$, qed.
This proof works for any field k of characteristic 0 (not necessarily $\mathbb{C}$) and for any commutative k-algebra $\mathcal{A}$.

- **IV.2:** On page 52, you write: "we define the Lie algebra :

$$\widetilde{g} = \mathfrak{g} \rtimes \mathbb{C}.Z_0,$$

". I think the $\widetilde{g}$ should be a $\widetilde{\mathfrak{g}}$ (since you call it $\widetilde{\mathfrak{g}}$ later).

- **IV.2:** On page 52, you write: "(see lemma II.7.1)". You probably mean Lemma II.8.1 here.

- **IV.2:** On page 52, you write: "We shall not dig out a Lie group structure for $\widetilde{G}$ here" - but haven't you already dug it out? You have defined the product in $\widetilde{G}$ in the sentence before, and I assume the differentiable-manifold structure isn't much of a problem - or is it?

- **IV.2:** At the very end of page 52, I don't understand Araki's formula:

$$\exp(tZ_0 + \gamma) = \sum_{n=0}^{\infty} \int_{\sum_{j=0}^{n} u_j = 1,\ u_j \geq 0} \exp(u_0 t Z_0)\,\gamma \exp(u_1 t Z_0)\,\gamma \cdots \gamma \exp(u_n t Z_0)\, du_1 \cdots du_n.$$

The $\exp(u_i t Z_0)$ terms lie in $\widetilde{G}$, while γ lies in $\mathfrak{g}$. How can we multiply them with each other?
(I assume you define such a product by extending the law $(\varphi, t)(\psi, s) = (\varphi * (\psi \circ \theta_t), t + s)$ to arbitrary $\varphi, \psi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$, rather than only $\varphi, \psi \in G$. Can you confirm this?)

- **IV.2:** On page 53, replace $x = \mathcal{H}^{n_0}$ by $x \in \mathcal{H}^{n_0}$.

- **IV.2:** On page 53, in the long calculation (the one that proves the one-parameter group property), there are some typos:
  - The second $\sum_{p=0}^{\infty}$ should be a $\sum_{q=0}^{\infty}$.
  - The very last integral should be $\int_{0 \leq u_n \leq \cdots \leq u_1 \leq t+s}$ rather than $\int_{s \leq u_p \leq \cdots \leq u_1 \leq t+s}$.

- **Proof of Theorem IV.2.1:** I don't really understand the proof of part 4) of this theorem at the place where you write "and thus replace the group G by any of the groups G_1, G_2 in assertions 1), 2) and 3)". This does seem to work for G_2, but I am not convinced that this works for G_1: in fact, it is not clear how the argument that "The right-hand side belongs manifestly to G" (this argument is on page 54, in the proof of part 1)) is supposed to work for G_1 (the sum of elements of G_1 needs not be in G_1).
Anyway, I have an alternative proof of part 4), which is devoid of any analysis and therefore works in the case of an arbitrary commutative algebra $\mathcal{A}$ over any field k of characteristic 0 (not necessarily $k = \mathbb{C}$):
Alternative proof of Theorem IV.2.1 part 4): Let $\gamma \in \mathfrak{g}$. According to the proof of part 3) of Theorem IV.2.1, the element $\widetilde{R}^{-1}(\gamma)$ is the limit of the sequence

(φ_n) defined by $\varphi_0 = e$ and $\varphi_{n+1} = e + T(\varphi_n)$ for all $n \geq 0$, where $T(\psi)$ means $(\psi * \gamma) \circ Y^{-1}$.

Now, let $\gamma \in \mathfrak{g}_2$. Then, we are going to prove that $\varphi_n \in G_2$ for all $n \in \mathbb{N}$.

We prove this by induction over n: The induction base (the case $n = 0$) is obvious ($\varphi_0 = e$ is an algebra homomorphism and thus lies in G_1).

For the induction step, let $m \in \mathbb{N}$ be arbitrary. Assume that $\varphi_m \in G_2$. We now must prove that $\varphi_{m+1} = e + T(\varphi_m)$.

Since the recursive definition of the sequence (φ_n) yields $\varphi_{m+1} = e + T(\varphi_m)$.

Since $\varphi_m \in G_2$, we know that φ_m is a cocycle. Since γ is a cocycle as well (because $\gamma \in \mathfrak{g}_2$), Proposition II.4.1 part 1) yields that $\varphi_m * \gamma$ is a cocycle as well. Thus, $(\varphi_m * \gamma) \circ Y^{-1}$ is a cocycle. (This is because of the following *lemma:* If η is a cocycle from $\mathcal{H}$ to $\mathcal{A}$, then $\eta \circ Y^{-1}$ is a cocycle from $\mathcal{H}$ to $\mathcal{A}$ as well.[10])

By the definition of T, we have $T(\varphi_m) = (\varphi_m * \gamma) \circ Y^{-1}$. Since $(\varphi_m * \gamma) \circ Y^{-1}$ is a cocycle, this shows that $T(\varphi_m)$ is a cocycle. Since e is a cocycle, this yields that $e + T(\varphi_m)$ is a cocycle (being the sum of two cocycles). Since $(e + T(\varphi_m))(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$, this entails $e + T(\varphi_m) \in G_2$. We thus have $\varphi_{m+1} = e + T(\varphi_m) \in G_2$.

This completes the induction, and thus we have proven that $\varphi_n \in G_2$ for all $n \in \mathbb{N}$. Thus, the limit of the sequence (φ_n) is in G_2 as well (since the limit of a convergent sequence of elements of G_2 must always be an element of G_2). In other words, $\widetilde{R}^{-1}(\gamma)$ is in G_2 (since $\widetilde{R}^{-1}(\gamma)$ is the limit of the sequence (φ_n)).

We have thus shown that $\widetilde{R}^{-1}(\gamma) \in G_2$ for every $\gamma \in \mathfrak{g}_2$. In other words, we have shown that $\widetilde{R}^{-1}(\mathfrak{g}_2) \subseteq G_2$.

Now let us prove that $\widetilde{R}^{-1}(\mathfrak{g}_1) \subseteq G_1$:

Forget about the $\gamma \in \mathfrak{g}_2$ we took above. Now let us take some $\gamma \in \mathfrak{g}_1$. Then, γ is a derivation from $\mathcal{H}$ with values in $\mathcal{A}$ and satisfies $\gamma(\mathbf{1}) = 0$.

Let $\varphi = \widetilde{R}^{-1}(\gamma)$. By the definition of $\widetilde{R}$, this means that $\varphi \in G$ and $\varphi \circ Y = \varphi * \gamma$.

From $\varphi \in G$, we conclude that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$.

---

[10] *Proof of the lemma:* Let x and y be two homogeneous elements of $\mathcal{H}$ such that $|x| + |y| > 0$. Then, $Y^{-1}(xy) = \frac{1}{|xy|} xy = \frac{1}{|x| + |y|} xy$ (since $|xy| = |x| + |y|$) and

$$\left(\eta \circ Y^{-1}\right)(xy) = \eta\left(\underbrace{Y^{-1}(xy)}_{= \frac{1}{|x|+|y|} xy}\right) = \eta\left(\frac{1}{|x| + |y|} xy\right) = \frac{1}{|x| + |y|} \eta(xy).$$

Similarly, $\left(\eta \circ Y^{-1}\right)(yx) = \frac{1}{|y| + |x|} \eta(yx)$. Comparing these two equalities (and recalling that $\eta(xy) = \eta(yx)$ (since η is a cocyle) and $|x| + |y| = |y| + |x|$), we conclude that $\left(\eta \circ Y^{-1}\right)(xy) = \left(\eta \circ Y^{-1}\right)(yx)$. We have thus shown the identity $\left(\eta \circ Y^{-1}\right)(xy) = \left(\eta \circ Y^{-1}\right)(yx)$ for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x| + |y| > 0$. Since the identity $\left(\eta \circ Y^{-1}\right)(xy) = \left(\eta \circ Y^{-1}\right)(yx)$ also trivially holds for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x| + |y| = 0$ (because $|x| = |y| = 0$ yields $x, y \in k \cdot \mathbf{1}$), this yields that the identity $\left(\eta \circ Y^{-1}\right)(xy) = \left(\eta \circ Y^{-1}\right)(yx)$ holds for any two homogeneous elements x and y of $\mathcal{H}$. Since this identity is linear in x and y, we can thus conclude that this identity holds for *any two* (not necessarily homogeneous) elements x and y of $\mathcal{H}$. In other words, $\eta \circ Y$ is a cocyle. The lemma is proven.

Every homogeneous $x \in \mathcal{H}$ satisfies

$$(\varphi \circ Y)(x) = \varphi\left(\underbrace{Y(x)}_{=|x|x}\right) = \varphi(|x|\, x) = |x|\, \varphi(x),$$

so that

$$|x|\, \varphi(x) = \underbrace{(\varphi \circ Y)}_{=\varphi * \gamma}(x) = (\varphi * \gamma)(x) = \sum_{(x)} \varphi(x_1)\, \gamma(x_2). \tag{10}$$

We are now going to prove that

$$\varphi(xy) = \varphi(x)\, \varphi(y) \qquad \text{for any two homogeneous elements } x \text{ and } y \text{ of } \mathcal{H}. \tag{11}$$

Proof of (11). We will prove (11) by strong induction over $|x| + |y|$:
The induction base (the case $|x| + |y| = 0$) is clear (since $\mathcal{H}_0 = k \cdot \mathbf{1}$ and $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$).
Now to the induction step: Let $d \in \mathbb{N}$ be positive. Assume that (11) is proven for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x| + |y| < d$. Now let us prove (11) for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x|+|y| = d$.
Let x and y be two homogeneous elements of $\mathcal{H}$ satisfying $|x|+|y| = d$. Then, we can WLOG assume that $x = \sum_{(x)} x_1 \otimes x_2$ with homogeneous x_1 and x_2 satisfying $|x_1| + |x_2| = |x|$, and that $y = \sum_{(y)} y_1 \otimes y_2$ with homogeneous y_1 and y_2 satisfying $|y_1| + |y_2| = y$.
Any three homogeneous elements u, v and w of $\mathcal{H}$ satisfying $|u| + |v| + |w| = d$ satisfy

$$\varphi(uv)\, \gamma(w) = \varphi(u)\, \varphi(v)\, \gamma(w). \tag{12}$$

[11] But (10) (applied to xy instead of x) yields

$$|xy|\varphi(xy)=\sum_{(xy)}\varphi((xy)_1)\gamma((xy)_2)=\sum_{(x)(y)}\underbrace{\varphi(x_1y_1)\gamma(x_2y_2)}_{\substack{=\varphi(x_1)\varphi(y_1)\gamma(x_2y_2)\\ \text{(by (12), applied to } u=x_1,\\ v=y_1 \text{ and } w=x_2y_2)}}=\sum_{(x)(y)}\varphi(x_1)\varphi(y_1)\underbrace{\gamma(x_2y_2)}_{\substack{=e(x_2)\gamma(y_2)+\gamma(x_2)e(y_2)\\ \text{(since }\gamma\text{ is a derivation)}}}$$

$$=\sum_{(x)(y)}\varphi(x_1)\varphi(y_1)(e(x_2)\gamma(y_2)+\gamma(x_2)e(y_2))$$

$$=\underbrace{\sum_{(x)(y)}\varphi(x_1)\varphi(y_1)e(x_2)\gamma(y_2)}_{\substack{=\sum_{(x)}\varphi(x_1)e(x_2)\sum_{(y)}\varphi(y_1)\gamma(y_2)\\ \text{(since }\mathcal{A}\text{ is commutative)}}}+\underbrace{\sum_{(x)(y)}\varphi(x_1)\varphi(y_1)\gamma(x_2)e(y_2)}_{\substack{=\sum_{(x)}\varphi(x_1)\gamma(x_2)\sum_{(y)}\varphi(y_1)e(y_2)\\ \text{(since }\mathcal{A}\text{ is commutative)}}}$$

$$=\underbrace{\sum_{(x)}\varphi(x_1)e(x_2)}_{=(\varphi*e)(x)}\underbrace{\sum_{(y)}\varphi(y_1)\gamma(y_2)}_{=(\varphi*\gamma)(y)}+\underbrace{\sum_{(x)}\varphi(x_1)\gamma(x_2)}_{=(\varphi*\gamma)(x)}\underbrace{\sum_{(y)}\varphi(y_1)e(y_2)}_{=(\varphi*e)(x)}$$

$$=\underbrace{(\varphi*e)}_{=\varphi}(x)\cdot\underbrace{(\varphi*\gamma)}_{=\varphi\circ Y}(y)+\underbrace{(\varphi*\gamma)}_{=\varphi\circ Y}(x)\cdot\underbrace{(\varphi*e)}_{=\varphi}(y)$$

$$=\varphi(x)\cdot\underbrace{(\varphi\circ Y)(y)}_{=\varphi(Y(y))}+\underbrace{(\varphi\circ Y)(x)}_{=\varphi(Y(x))}\cdot\varphi(y)$$

$$=\varphi(x)\cdot\varphi\left(\underbrace{Y(y)}_{=|y|y}\right)+\varphi\left(\underbrace{Y(x)}_{=|x|x}\right)\cdot\varphi(y)$$

$$=\varphi(x)\cdot\underbrace{\varphi(|y|y)}_{=|y|\varphi(y)}+\underbrace{\varphi(|x|x)}_{=|x|\varphi(x)}\cdot\varphi(y)=|y|\varphi(x)\cdot\varphi(y)+|x|\varphi(x)\cdot\varphi(y)$$

$$=\underbrace{(|y|+|x|)}_{=|xy|}\varphi(x)\varphi(y)=|xy|\varphi(x)\varphi(y).$$

Since $|xy|=|x|+|y|=d>0$, we can divide this equation by $|xy|$ and obtain $\varphi(xy)=\varphi(x)\varphi(y)$. In other words, we have proven (11) for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x|+|y|=d$. This completes the induction step.

Thus, (11) is proven for any two homogeneous elements x and y of $\mathcal{H}$. Since this equation (11) is linear in x and y, we thus conclude that (11) holds for any two

[11] *Proof of (12).* We distinguish between two cases:

Case 1: We have $|w|=0$.

Case 2: We have $|w|>0$.

In Case 1, we have $\gamma(w)=0$ (since $|w|=0$ yields $w\in\mathcal{H}^0=k\cdot\mathbf{1}$, and since we know that $\gamma(\mathbf{1})=0$). Therefore, (12) trivially holds in Case 1.

Let us now consider Case 2. In this case, $|u|+|v|+|w|=d$ rewrites as $|u|+|v|=d-\underbrace{|w|}_{>0}<d$. Thus, in Case 2, we can apply (11) to u and v instead of x and y (since we assumed that (11) is proven for any two homogeneous elements x and y of $\mathcal{H}$ satisfying $|x|+|y|<d$), and we obtain $\varphi(uv)=\varphi(u)\varphi(v)$. This shows that (12) holds in Case 2.

We have now shown that (12) holds in each of the two cases 1 and 2. This completes the proof of (12).

elements x and y of $\mathcal{H}$ (not necessarily homogeneous). In other words, φ is an algebra morphism (since we also know that $\varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$). In other words, $\varphi \in G_1$. Thus, $\widetilde{R}^{-1}(\gamma) = \varphi \in G_1$.
We have thus shown that $\widetilde{R}^{-1}(\gamma) \in G_1$ for every $\gamma \in \mathfrak{g}_1$. Hence, $\widetilde{R}^{-1}(\mathfrak{g}_1) \subseteq G_1$. Combined with $\widetilde{R}^{-1}(\mathfrak{g}_2) \subseteq G_2$, this completes our proof of part 4) of Proposition IV.2.1.

- **Proof of Lemma IV.4.2:** In the first line of the computation that you do in this proof, replace $\psi \circ Y^n$ by $z^n \psi \circ Y^n$.

- **Proof of Theorem IV.4.1:** On page 57, you write: "We have still to fix the convergence of the exponential just above in the case when $z\widetilde{R}(\psi)$ belongs to $L(\mathcal{H}, \mathcal{A}_+)$." The L here should be a calligraphic $\mathcal{L}$.

- **Proof of Theorem IV.4.1:** On page 57, you write:

$$\mathcal{L}_+^n = \bigcup_{p+q=n} \mathcal{L}_+^{p,q}.$$

  I assume the $\bigcup$ sign should be a $\sum$.

- **Proof of Theorem IV.4.1:** On page 58, you write:

$$\frac{d}{dt} \mid_{t=0} \psi_t = z(\psi \circ Y) = \psi \dot{h}_t \mid_{t=0} .$$

  The ψ_t should be a ψ^t here.

- **Theorem IV.4.4:** When you say $\mathbb{C}$ in this theorem, you mean $\mathbb{C}$ as a subspace of $\mathcal{A}$. Maybe it would be good to state this explicitly (I got confused by this $\mathbb{C}$).

- **Proof of Lemma IV.4.5:** The reference to "theorem II.4.1" should refer to Theorem II.5.1 instead.

- **Proof of Theorem IV.4.4:** On page 60, you write: "it is easily seen by induction on $|x|$ that the right-hand side evaluated at z has a limit when z tends to infinity. Thus $\psi(x) \in \mathcal{A}_-$". I am not sure about the "Thus" here: the function $\dfrac{1}{z-1}$ also has a limit when z tends to infinity, but doesn't lie in $\mathcal{A}_-$ for $z_0 = 0$. on the other hand, I think you don't need the detour through limits: you can just say that "it is thus easily seen by induction on $|x|$ that $\psi(x) \in \mathcal{A}_-$". Or am I missing something here?