A short proof of the Littlewood-Richardson rule

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`gasharov - lrrule.ps` (preprint available at `http://www.math.cornell.edu/~vesko/papers/lrrule.ps`)

Errata (collected by Darij Grinberg)

The following list of errata refers to the preprint version of Vesselin Gasharov's article "A short proof of the Littlewood-Richardson rule" available from his website (`http://www.math.cornell.edu/~vesko/papers/lrrule.ps`). The same errors appear in the published version (European Journal of Combinatorics, Volume 19, Issue 4, May 1998, Pages 451–453), although the page numbers in the published version are different.

I will refer to the results appearing in the preprint by the numbers under which they appear in it.

- **Page 1, Definition 1.1:** Here, the author defines $N(i, w_{\leq r})$ to mean "the number of occurrences of the symbol i in $w_{\leq r}$". There is nothing wrong about this definition, but later in the text the notation $N(i, v)$ is used for various words v which aren't always given in the form $w_{\leq r}$ for some w and r. So a more general definition would be good, such as the following one: "For any $i \geq 1$ and any word v, let $N(i, v)$ denote the number of occurrences of the symbol i in v."

- **Page 1, Definition 1.1:** While $w_{\leq r}$ is defined in this definition, $w_{>r}$ (a notation used in the proof of Proposition 2.1) is not defined. It should be defined, for example as follows: "For $0 \leq r \leq n$, let $w_{>r}$ denote the word $w_{r+1}...w_n$."

- **Page 1:** I am not sure what "Theorem 1.2 is not more general" means. If it means to say that Theorem 1.2 follows easily from the classical formulation (which requires $\theta = 0$), then I don't see how it follows from it. If it merely means that the classical formulation is enough to compute all $\langle s_{\lambda/\mu}, s_{\nu/\theta} \rangle$, that is true, but I would word it differently to avoid confusion.

- **Page 2:** Replace "For a partition π" by "For a permutation π".

- **Page 3, proof of Proposition 2.1:** Here it is claimed that "The fact that all $(i+1)$'s in R which are in $w_{>r}$ are free implies" (4). This is a slightly incomplete argument, because the fact that all $(i+1)$'s in R which are in $w_{>r}$ are free does not guarantee that there are no non-free $(i+1)$'s in the row directly under R. Fortunately, this gap is easy to fill; here is the precise argument:

  There are no $(i+1)$'s in $w_{>r}$ in columns weakly to the right of C (because any such $(i+1)$'s would lie weakly to the right and strictly below w_r, so they would have to be $> w_r$ (because the tableau P is column-strict), which is absurd because $w_r = i+1$). In particular, there are no non-free $(i+1)$'s in $w_{>r}$ in these columns. This (combined with the fact that all $(i+1)$'s in R which are in $w_{>r}$ are free) yields that for each non-free $(i+1)$ in $w_{>r}$, the i directly above it also belongs to $w_{>r}$. Hence, the non-free $(i+1)$'s in $w_{>r}$ are in 1-to-1 correspondence with the non-free i's in $w_{>r}$, so that their contributions to $N(i, w_{>r})$ and to $N(i+1, w_{>r})$ are the same.

- **Page 4, proof of Theorem 1.2:** The "*Tab*" should be a roman "Tab".
- **Page 4, proof of Theorem 1.2:** The equality

$$\sum_{\pi \in S_l} \operatorname{sgn}(\pi) \left\langle s_{\lambda/\mu}, h_{\pi(\nu)-\theta} \right\rangle = \sum_{\pi \in S_l} \operatorname{sgn}(\pi) \left| \operatorname{Tab}(\lambda/\mu, \pi(\nu) - \theta) \right|$$

might need a couple more explanations. The proof of this equality goes as follows:

It is clearly enough to show that $\left\langle s_{\lambda/\mu}, h_{\pi(\nu)-\theta} \right\rangle = \left| \operatorname{Tab}(\lambda/\mu, \pi(\nu) - \theta) \right|$ for every $\pi \in S_l$. So let $\pi \in S_l$. When the l-tuple $\pi(\nu) - \theta$ has a negative entry, both $h_{\pi(\nu)-\theta}$ and $\left| \operatorname{Tab}(\lambda/\mu, \pi(\nu) - \theta) \right|$ are 0, so that the equality $\left\langle s_{\lambda/\mu}, h_{\pi(\nu)-\theta} \right\rangle = \left| \operatorname{Tab}(\lambda/\mu, \pi(\nu) - \theta) \right|$ is trivial in this case. Hence, we can WLOG assume that we are not in this case. Assume this. Then, the l-tuple $\pi(\nu) - \theta$ consists of nonnegative integers only. Let κ denote the partition obtained by removing all zero entries from this l-tuple $\pi(\nu) - \theta$ and reordering all the remaining entries in nonincreasing order.

Recall that $s_{\lambda/\mu} = \sum_{\substack{P \text{ is a tableau} \\ \text{of shape } \lambda/\mu}} x^P$. Hence, if η is any l-tuple of nonnegative integers, then

$$\left(\text{the coefficient of } s_{\lambda/\mu} \text{ before } x^{\eta}\right) = \left| \operatorname{Tab}(\lambda/\mu, \eta) \right|. \tag{1}$$

Now, it is known that $(h_\lambda)_{\lambda \text{ is a partition}}$ and $(m_\lambda)_{\lambda \text{ is a partition}}$ are orthogonal bases of the vector space of symmetric functions (where m_λ denotes the λ-th monomial symmetric function). Hence, for every symmetric function f and every partition τ, we have

$$\begin{aligned} \langle f, h_\tau \rangle &= \left(\text{the } m_\tau\text{-coordinate of } f \text{ with respect to the basis } (m_\lambda)_{\lambda \text{ is a partition}}\right) \\ &= \left(\text{the coefficient of } f \text{ before } x^\tau\right). \end{aligned}$$

Hence, if f is a symmetric function, and ϕ is a tuple of nonnegative integers, and if τ is the partition obtained by removing all zero entries from this tuple ϕ and reordering all the remaining entries in nonincreasing order, then we have

$$\begin{aligned}\left\langle f, h_\phi \right\rangle &= \left\langle f, h_\tau \right\rangle \qquad \left(\begin{array}{c}\text{since } h_\phi = h_\tau \text{ (because the product } h_\phi \\ \text{depends neither on the order of its factors} \\ \text{nor on the appearance of } h_0 = 1 \text{ factors)}\end{array}\right) \\ &= \left(\text{the coefficient of } f \text{ before } x^\tau\right) \\ &= \left(\text{the coefficient of } f \text{ before } x^\phi\right) \\ &\qquad \left(\begin{array}{c}\text{since the function } f \text{ is symmetric, and thus its} \\ \text{coefficients before any two monomials with the} \\ \text{same multisets of positive exponents are equal}\end{array}\right).\end{aligned}$$

Applying this to $f = s_{\lambda/\mu}$, $\tau = \pi(\nu) - \theta$ and $\phi = \kappa$, we obtain

$$\begin{aligned}\left\langle s_{\lambda/\mu}, h_\kappa \right\rangle &= \left(\text{the coefficient of } s_{\lambda/\mu} \text{ before } x^{\pi(\nu)-\theta}\right) \\ &= \left|\operatorname{Tab}\left(\lambda/\mu, \pi(\nu) - \theta\right)\right| \qquad \left(\text{by (1), applied to } \eta = \pi(\nu) - \theta\right),\end{aligned}$$

qed.

- **Page 4, proof of Theorem 1.2:** Replace "w" by "$w := w(P)$" in "implies that w is a θ-lattice permutation".
- **Page 4, proof of Theorem 1.2:** After "which implies that π is the identity permutation", maybe add an explanation why this is true. For example, one such explanation would be "(because $\pi(\nu)_{i+1} \leq \pi(\nu)_i$ rewrites as $\nu_{\pi(i+1)} - \pi(i+1) < \nu_{\pi(i)} - \pi(i)$, which can hold for all $i \geq 1$ only when $\pi = \mathrm{id}$)".