

The Robinson-Schensted and Schützenberger algorithms, an elementary approach

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Errata by Darij Grinberg - I

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The following text is an annotation to Marc A. A. van Leeuwen’s paper “The Robinson-Schensted and Schützenberger algorithms, an elementary approach” in its version of 30 January 2013.

This annotation contains corrections of mistakes (or what I believe to be mistakes) and additional comments (in particular, elaborations of some arguments that I found insufficiently detailed in the paper). The latter are printed in [blue](#).

Different comments are separated by horizontal lines, like this:

Page 3: Replace “complementary the approach” by “complementary to the approach”.

Page 5, §1.1: Replace “a ‘differential poset’ as defined in [Stan3]” by “a ‘1-differential poset’ as defined in [Stan3]”; and even this is not completely correct, since a 1-differential poset as defined in [Stan3] is also required to have a global minimum and to be locally finite.

Also, replace “by any differential poset” by “by any 1-differential poset”.

The same kind of inaccuracy appears in §3.1 on page 16.

Page 5, §1.2: I think it should be explained that a “saturated decreasing chain in \mathcal{P} ” means a sequence $(\lambda_{[0]}, \lambda_{[1]}, \dots, \lambda_{[k]})$ of partitions such that $\lambda_{[k]} = (0)$ and such that every $i \in \{0, 1, \dots, k-1\}$ satisfies $\lambda_{[i+1]} \in (\lambda_{[i]})^-$ (or, equivalently, $\lambda_{[i]} \in (\lambda_{[i+1]})^+$). (So, unlike in [Stan2], you are requiring the chain to end at (0) .) At least, this is what you mean by “saturated decreasing chain in \mathcal{P} ” up to §4. (In §5, you do consider saturated decreasing chains which do not end at (0) .)

Similarly, if \mathbf{P} is a poset which has a unique minimal element $\widehat{0}$, then a “saturated decreasing chain” in \mathbf{P} means a sequence (p_0, p_1, \dots, p_k) of elements of \mathbf{P} such that $p_k = \widehat{0}$ and such that, for every $i \in \{0, 1, \dots, k-1\}$, the element p_{i+1} is covered by p_i in \mathbf{P} (this means that $p_{i+1} < p_i$ and that there exists no $q \in \mathbf{P}$ satisfying $p_{i+1} < q < p_i$).

Page 6, proof of Lemma 1.3.1: Here is a more detailed version of the compu-

tation at the end of the proof:

$$\begin{aligned}
(n+1) \# \mathcal{T}_\lambda &= \# \mathcal{T}_\lambda + n \underbrace{\# \mathcal{T}_\lambda}_{=\sum_{\mu \in \lambda^-} \# \mathcal{T}_\mu \text{ (by (3))}} = \# \mathcal{T}_\lambda + n \underbrace{\sum_{\mu \in \lambda^-} \# \mathcal{T}_\mu}_{=\sum_{\mu \in \lambda^-} n \# \mathcal{T}_\mu} = \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \underbrace{n}_{=|\mu|+1} \# \mathcal{T}_\mu \\
&= \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \underbrace{(|\mu|+1) \# \mathcal{T}_\mu}_{=\sum_{\lambda' \in \mu^+} \# \mathcal{T}_{\lambda'}} \quad \text{(by the induction hypothesis)} = \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \underbrace{(|\mu|+1) \# \mathcal{T}_\mu}_{=\sum_{\lambda' \in \mu^+} \# \mathcal{T}_{\lambda'}} \quad \text{(by the induction hypothesis)} \\
&= \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \underbrace{\sum_{\substack{\lambda' \in \mu^+ \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'}}_{=\# \mathcal{T}_\lambda + \sum_{\substack{\lambda' \in \mu^+ \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'}} = \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \underbrace{\left(\# \mathcal{T}_\lambda + \sum_{\substack{\lambda' \in \mu^+ \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'} \right)}_{=\# \lambda^- \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^-} \sum_{\substack{\lambda' \in \mu^+ \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'}} \\
&= \underbrace{\# \mathcal{T}_\lambda + \# \lambda^- \# \mathcal{T}_\lambda}_{=(1+\# \lambda^-) \# \mathcal{T}_\lambda} + \underbrace{\sum_{\mu \in \lambda^-} \sum_{\substack{\lambda' \in \mu^+ \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'}}_{=\sum_{\lambda' \neq \lambda} \# (\lambda^- \cap (\lambda')^-) \# \mathcal{T}_{\lambda'}} = \underbrace{(1 + \# \lambda^-)}_{=\# \lambda^- + 1 = \# \lambda^+ \text{ (by (1))}} \# \mathcal{T}_\lambda + \sum_{\lambda' \neq \lambda} \underbrace{\# (\lambda^- \cap (\lambda')^-)}_{=\# (\lambda^+ \cap (\lambda')^+) \text{ (by (2), applied to } \mu = \lambda')} \# \mathcal{T}_{\lambda'} \\
&= \underbrace{\# \lambda^+ \# \mathcal{T}_\lambda}_{=\sum_{\mu \in \lambda^+} \# \mathcal{T}_\lambda} + \underbrace{\sum_{\lambda' \neq \lambda} \# (\lambda^+ \cap (\lambda')^+) \# \mathcal{T}_{\lambda'}}_{=\sum_{\mu \in \lambda^+} \sum_{\substack{\lambda' \in \mu^- \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'}} = \sum_{\mu \in \lambda^+} \# \mathcal{T}_\lambda + \sum_{\mu \in \lambda^+} \sum_{\substack{\lambda' \in \mu^- \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'} \\
&= \sum_{\mu \in \lambda^+} \underbrace{\left(\# \mathcal{T}_\lambda + \sum_{\substack{\lambda' \in \mu^- \\ \lambda' \neq \lambda}} \# \mathcal{T}_{\lambda'} \right)}_{=\sum_{\lambda' \in \mu^-} \# \mathcal{T}_{\lambda'}} = \sum_{\mu \in \lambda^+} \underbrace{\sum_{\lambda' \in \mu^-} \# \mathcal{T}_{\lambda'}}_{=\# \mathcal{T}_\mu \text{ (by (3))}} = \sum_{\mu \in \lambda^+} \# \mathcal{T}_\mu.
\end{aligned}$$

Page 6: It might be helpful to point out that you define the symmetric group \mathbf{S}_n as the group of all permutations of $\{1, 2, \dots, n\}$. (This is in contrast to some other papers you wrote, where you defined it as the group of all permutations of $\{0, 1, \dots, n-1\}$.)

Page 8, §2.1: Remove the “this” from “which we shall call this”.

Page 8, §2.1: Replace “the proceeds” by “then proceeds”.

Page 8, §2.1: In the description of the deflation procedure, replace “Given a tableau” by “Given a nonempty tableau”.

Page 9: For the sake of clarity, after “there are no more than 2 intermediate partitions” I would suggest adding “between $\text{sh } P^\downarrow$ and $\text{sh } P$ (and from (4), we can immediately see that both $\text{sh } P^\downarrow$ and $\text{sh } P^-$ have to be such intermediate partitions)”.

Page 9: Replace “with a minimal element” by “with a unique minimal element”. (At least I believe that you mean it this way. You could, however, work without the “unique minimal element” requirement as well, if you set $c^\downarrow = c_1$ for every two-element chain $c = (c_0, c_1)$. But then you do not even have to require the existence of minimal elements: If they don’t exist, then there are no saturated decreasing chains to begin with, and so the procedures are vacuously well-defined.)

Page 10, §2.1: You never define what “the algorithm S ” is! I assume you just want to define it as the algorithm sending every tableau P to the tableau P^* defined on the first paragraph of page 10 (so that $S(P) = P^*$).

Page 10, proof of Theorem 2.2.1: Replace “for $i + j \leq n$ ” by “for nonnegative integers i and j satisfying $i + j \leq n$ ”.

Page 10, proof of Theorem 2.2.1: Replace “for $P^{[0,0]} = P$ ” by “ $P^{[0,0]} = P$ ”.

Page 10, proof of Theorem 2.2.1: The definition of the $P^{[i,j]}$ here is a tad confusing: It is not completely obvious that it is well-defined. Let me instead suggest an equivalent definition whose well-definedness is clear:

- For every Young tableau T and every nonnegative integer $k \leq |\text{sh } T|$, we define a Young tableau $T^{-[k]}$ of size $|\text{sh } T| - k$. Indeed, we define it recursively by setting $T^{-[0]} = T$, and by setting $T^{-[k]} = (T^{-[k-1]})^-$ for every $k \in \{1, 2, \dots, |\text{sh } T|\}$.
- For every Young tableau T and every nonnegative integer $\ell \leq |\text{sh } T|$, we define a Young tableau $T^{\downarrow[k]}$ of size $|\text{sh } T| - k$. Indeed, we define it recursively by setting $T^{\downarrow[0]} = T$, and by setting $T^{\downarrow[k]} = (T^{\downarrow[k-1]})^\downarrow$ for every $k \in \{1, 2, \dots, |\text{sh } T|\}$.
- For every Young tableau T and every nonnegative integer $k < |\text{sh } T|$, we have

$$(T^{-[k]})^\downarrow = (T^\downarrow)^{-[k]}. \quad (1)$$

(This is proven straightforwardly by induction over k , using (4) in the induction step.)

- For every Young tableau T and every two nonnegative integers k and ℓ satisfying $k + \ell \leq |\text{sh } T|$, we define a tableau $T^{\{k,\ell\}}$ by

$$T^{\{k,\ell\}} = (T^{-[\ell]})^{\downarrow[k]}. \quad (2)$$

- For every Young tableau T and every two nonnegative integers k and ℓ satisfying $k + \ell \leq |\text{sh } T|$, we have

$$T^{\{k,\ell\}} = (T^{\downarrow[k]})^{-[\ell]}. \quad (3)$$

(This is proven straightforwardly by induction over k , using (1) in the induction step.)

- Now, consider our Young tableau P with $|\text{sh } P| = n$. For every two nonnegative integers i and j satisfying $i + j \leq n$, we define a Young tableau $P^{[i,j]}$ by $P^{[i,j]} = P^{\downarrow\{i,j\}}$. To see that this definition is equivalent to your definition, we need to show that it satisfies the properties

$$P^{[0,0]} = P, \tag{4}$$

and that every two nonnegative integers i and j satisfying $i + j + 1 \leq n$ satisfy

$$P^{[i,j+1]} = (P^{[i,j]})^- \quad \text{and} \quad P^{[i+1,j]} = (P^{[i,j]})^\downarrow. \tag{5}$$

But this is easy (using (2) and (3)). Thus, my definition of $P^{[i,j]}$ is equivalent to yours.

Page 11, §2.2: In the picture, the 8-th column should be labelled “8”.

Page 11, §2.2: Replace “path is that ends” by “path that ends”.

Page 12, §2.3: Replace “ $D^{-1}(T_{i-1}, -m_i, s)$ ” by “ $D^{-1}(T_{i-1}, s, -m_i)$ ”.

Page 12, §2.4: Replace “ $Y(\lambda) \setminus \{(1, 1)\}$ ” by “ $Y(\lambda) \setminus \{(0, 0)\}$ ”.

Page 12, §2.4: After “the shape λ admits only domino tableaux of rank r ”, add “(and it does indeed admit at least one such tableau)”.

Page 13: Replace “the number of domino tableaux” by “the number of normalized domino tableaux”.

Page 13: Replace “apply to algorithm” by “apply the algorithm”.

Page 13: Before “the context of thin interval posets”, add “in”.

Page 13: You write: “In this way domino tableaux may be considered to represent equivalence classes of Young tableaux that are “as self-dual as possible””. But is it obvious that no two different domino tableaux can lead to equivalent “almost self-dual” Young tableaux?

Page 13: Replace “ $S(X)$ differs from S ” by “ $S(X)$ differs from X ”.

Page 14: Replace “In the is section” by “In this section”.

Page 14: Replace “and the $\text{sh } P = \text{sh } T + s$ ” by “and we have $\text{sh } P = \text{sh } T + s$ ”.

Page 15: When you write “since the final position of h lies outside $(T \leftarrow m)^-$ ”, it would be helpful to remind the reader that you are still assuming that $m \not\prec T$. (You only drop this assumption after (8).)

Page 15: Replace “presevers” by “preserves”.

Page 15: Replace “the conditions $m > T$ ” by “the condition $m > T$ ”.

Page 15: Replace “do determine” by “to determine”.

Page 16, §3.1: Replace “tableaux of equal shape” by “normalized tableaux of equal shape $\in \mathcal{P}_n$ ”.

Page 16, §3.1: In “another bijection $RS^t : \mathbf{S}_n \rightarrow \bigcup_{\lambda \in \mathcal{P}} \mathcal{T}_\lambda \times \mathcal{T}_\lambda$ ”, replace “ \mathcal{P} ” by “ \mathcal{P}_n ”.

Page 17, proof of Theorem 3.2.1: At the beginning of this proof, add the following sentence: “Set $(P, Q) = RS(\sigma)$; we then need to show that $RS(\sigma^{-1}) = (Q, P)$.” (Indeed, the way you formulate Theorem 3.2.1, it is not clear a-priori what P and Q are.)

Page 17, proof of Theorem 3.2.1: Replace “the the set” by “the set”.

Page 17, proof of Theorem 3.2.1: Here you write that “Any configuration $\begin{pmatrix} \lambda^{[i-1, j-1]} & \lambda^{[i-1, j]} \\ \lambda^{[i, j-1]} & \lambda^{[i, j]} \end{pmatrix}$ is of type $RS1$ if $j = \sigma_i$, and otherwise of type $RS0$, $RS2$, or $RS3$ ”. This claim (which I will call the *growth-cell claim*) is correct, but in my opinion not trivial enough to deserve no justification. Here is a sketch of how this is proven:

If λ is a partition, then we treat any Young tableau of shape λ as a map from $Y(\lambda)$ to \mathbf{Z} . Thus, it makes sense to speak (for example) of the restriction of a Young tableau to a subset of $Y(\lambda)$ (because it makes sense to speak of the restriction of a map to a subset of its domain). For every Young tableau Z and every integer j , we let $Z|_{\leq j}$ denote the restriction of the tableau Z to the set of its squares whose entry does not exceed j . (Clearly, this restriction is again a Young tableau.)

Now, we can state a general fact:

Proposition 3.2.1a. Let T be a Young tableau. Let m be an integer which does not appear in T . Let j be an integer. Then, the configuration $\begin{pmatrix} \text{sh}(T|_{\leq j-1}) & \text{sh}(T|_{\leq j}) \\ \text{sh}((T \leftarrow m)|_{\leq j-1}) & \text{sh}((T \leftarrow m)|_{\leq j}) \end{pmatrix}$ is of type $RS1$ if $j = m$, and otherwise is of type $RS0$, $RS2$, or $RS3$.

The growth-cell claim follows from Proposition 3.2.1a (applied to $T = P_{i-1}$ and $m = \sigma_i$), because we have $P^{[i, j]} = P_i|_{\leq j}$ and $P_i = P_{i-1} \leftarrow \sigma_i$. Thus, it remains to prove Proposition 3.2.1a.

The proof of Proposition 3.2.1a mainly rests on the following two results:

Proposition 3.2.1b. Let T be a Young tableau. Let m be an integer which does not appear in T . Let j be an integer.

(a) If $m \leq j$, then $(T \leftarrow m)|_{\leq j} = (T|_{\leq j}) \leftarrow m$.

(b) If $m > j$, then $(T \leftarrow m)|_{\leq j} = T|_{\leq j}$.

(c) Let α and β be two partitions such that $\alpha = \text{sh}((T \leftarrow m) \mid_{\leq j})$ and $\beta = \text{sh}(T \mid_{\leq j})$. Then, $\alpha \in \{\beta\} \cup \beta^+$.

Proposition 3.2.1c. Let T be a nonempty Young tableau. Let m be an integer which does not appear in T . Assume that $m \not\asymp T$. Then, the configuration $\begin{pmatrix} \text{sh}(T^-) & \text{sh} T \\ \text{sh}((T \leftarrow m)^-) & \text{sh}(T \leftarrow m) \end{pmatrix}$ is of type $RS2$, or $RS3$.

While I still would not call these results trivial, they are the kinds of results I would feel comfortable leaving to the reader.¹

Finally, Proposition 3.2.1a can easily be derived from Proposition 3.2.1c using Proposition 3.2.1b and (7):

Proof of Proposition 3.2.1a. We need to prove the following two statements:

Statement P3.2.1.a.1: If $j = m$, then the configuration $\begin{pmatrix} \text{sh}(T \mid_{\leq j-1}) & \text{sh}(T \mid_{\leq j}) \\ \text{sh}((T \leftarrow m) \mid_{\leq j-1}) & \text{sh}((T \leftarrow m) \mid_{\leq j}) \end{pmatrix}$ is of type $RS1$.

Statement P3.2.1.a.2: If $j \neq m$, then the configuration $\begin{pmatrix} \text{sh}(T \mid_{\leq j-1}) & \text{sh}(T \mid_{\leq j}) \\ \text{sh}((T \leftarrow m) \mid_{\leq j-1}) & \text{sh}((T \leftarrow m) \mid_{\leq j}) \end{pmatrix}$ is of type $RS0$, $RS2$, or $RS3$.

Proof of Statement P3.2.1.a.1: Assume that $j = m$. Thus, Proposition 3.2.1b (a) yields that $(T \leftarrow m) \mid_{\leq j} = (T \mid_{\leq j}) \leftarrow m$.

Recall that the integer m does not appear in T . Since $j = m$, this rewrites as follows: The integer j does not appear in T . Hence, $T \mid_{\leq j-1} = T \mid_{\leq j}$.

On the other hand, $m = j > j - 1$. Hence, Proposition 3.2.1b (b) (applied to $j - 1$ instead of j) yields that $(T \leftarrow m) \mid_{\leq j-1} = T \mid_{\leq j-1}$.

Furthermore, $m = j > T \mid_{\leq j-1} = T \mid_{\leq j}$. Hence, (6) (applied to $T \mid_{\leq j}$ instead of T) shows that $\text{ch}((T \mid_{\leq j}) \leftarrow m) = \rho^+(\text{sh}(T \mid_{\leq j})) : \text{ch}(T \mid_{\leq j})$. Hence, $\text{sh}((T \mid_{\leq j}) \leftarrow m) = \rho^+(\text{sh}(T \mid_{\leq j}))$. Hence,

$$\text{sh} \left(\underbrace{(T \leftarrow m) \mid_{\leq j}}_{=(T \mid_{\leq j}) \leftarrow m} \right) = \text{sh}((T \mid_{\leq j}) \leftarrow m) = \rho^+(\text{sh}(T \mid_{\leq j})).$$

Now, set $\kappa = \text{sh}(T \mid_{\leq j-1})$, $\lambda = \text{sh}(T \mid_{\leq j})$, $\mu = \text{sh}((T \leftarrow m) \mid_{\leq j-1})$ and $\nu = \text{sh}((T \leftarrow m) \mid_{\leq j})$. Then,

$$\kappa = \text{sh} \left(\underbrace{T \mid_{\leq j-1}}_{=T \mid_{\leq j}} \right) = \text{sh}(T \mid_{\leq j}) = \lambda$$

¹The main idea of the proof of Proposition 3.2.1b (a) is the following: The entries being displaced during the insertion procedure form an increasing sequence; hence, the entries $\leq j$ are getting displaced before the entries $> j$. Thus, the insertion procedure for $I(T, m)$ first emulates the insertion procedure for $I(T \mid_{\leq j}, m)$, and after that only displaces entries that are $> j$ (whence the entries $\leq j$ remain unmoved). This proves Proposition 3.2.1b (a). Proposition 3.2.1b (b) is proven the same way, except that the entries $\leq j$ are never touched to begin with (since we are inserting an entry $> j$ right away). Part (c) of Proposition 3.2.1b follows from parts (a) and (b).

Proposition 3.2.1c is just a restatement of the analysis of the case $m \not\asymp T$ that you have made between Definition 3.1.1 and Definition 3.1.2.

and

$$\mu = \text{sh} \left(\underbrace{(T \leftarrow m) |_{\leq j-1}}_{=T|_{\leq j-1}} \right) = \text{sh}(T |_{\leq j-1}) = \kappa = \lambda$$

From $\lambda = \kappa$, we obtain $\lambda \in \{\kappa\} \subseteq \{\kappa\} \cup \kappa^+$. From $\mu = \kappa$, we obtain $\mu \in \{\kappa\} \subseteq \{\kappa\} \cup \kappa^+$.

Furthermore, $\nu = \text{sh}((T \leftarrow m) |_{\leq j}) = \rho^+ \left(\underbrace{\text{sh}(T |_{\leq j})}_{=\lambda} \right) = \rho^+(\lambda) \in \lambda^+$, so that $\lambda \in \nu^- \subseteq \{\nu\} \cup \nu^-$. Hence, $\mu = \lambda \in \{\nu\} \cup \nu^-$ and $\nu = \lambda \in \{\nu\} \cup \nu^-$.

Combining $\lambda, \mu \in \{\kappa\} \cup \kappa^+$ with $\lambda, \mu \in \{\nu\} \cup \nu^-$, we obtain $\lambda, \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \nu^-)$. Combined with $\kappa = \lambda = \mu$ and $\nu = \rho^+(\lambda)$, this shows that $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is a configuration of type *RS1*. Since $\kappa = \text{sh}(T |_{\leq j-1})$, $\lambda = \text{sh}(T |_{\leq j})$, $\mu = \text{sh}((T \leftarrow m) |_{\leq j-1})$ and $\nu = \text{sh}((T \leftarrow m) |_{\leq j})$, this is precisely Statement P3.2.1.a.1. Thus, Statement P3.2.1.a.1 is proven.

Proof of Statement P3.2.1.a.2: Assume that $j \neq m$. Let $Q = T |_{\leq j}$. The integer m does not appear in Q (since m does not appear in T).

Set $\kappa = \text{sh}(T |_{\leq j-1})$, $\lambda = \text{sh}(T |_{\leq j})$, $\mu = \text{sh}((T \leftarrow m) |_{\leq j-1})$ and $\nu = \text{sh}((T \leftarrow m) |_{\leq j})$. Then, it is easy to see that $\lambda \in \{\kappa\} \cup \kappa^+$ and $\mu \in \{\kappa\} \cup \kappa^+$ (by Proposition 3.2.1b (c), applied to $j-1$, μ and κ instead of j , α and β). Also, $\nu \in \{\mu\} \cup \mu^+$, and thus $\mu \in \{\nu\} \cup \nu^-$. Furthermore, $\nu \in \{\lambda\} \cup \lambda^+$ (by Proposition 3.2.1b (c), applied to ν and λ instead of α and β). Thus, $\lambda \in \{\nu\} \cup \nu^-$.

Combining $\lambda, \mu \in \{\kappa\} \cup \kappa^+$ with $\lambda, \mu \in \{\nu\} \cup \nu^-$, we obtain $\lambda, \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \nu^-)$.

Now, we shall show that

$$\text{the configuration } \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix} \text{ is of type } RS0, RS2, \text{ or } RS3. \quad (6)$$

Proof of (6): We are in one of the following two cases:

Case 1: The integer j appears in T .

Case 2: The integer j does not appear in T .

Let us first consider Case 1. In this case, the integer j appears in T . Thus, j is

the highest entry of $T |_{\leq j}$. Hence, $T |_{\leq j-1} = \left(\underbrace{T |_{\leq j}}_{=Q} \right)^- = Q^-$. In particular, Q^- is

well-defined. Hence, the tableau Q is nonempty.

Also, the integer j appears in $T \leftarrow m$ (since j appears in T). Thus, j is the highest entry of $(T \leftarrow m) |_{\leq j}$. Therefore, $(T \leftarrow m) |_{\leq j-1} = ((T \leftarrow m) |_{\leq j})^-$.

Now, we must be in one of the following two subcases:

Subcase 1.1: We have $m \leq j$.

Subcase 1.2: We have $m > j$.

Let us first consider Subcase 1.1. In this Subcase, we have $m \leq j$. Combined with $j \neq m$, this shows that $m < j$, so that $m \leq j-1$ (since m and j are integers).

Now, Proposition 3.2.1b (a) shows that $(T \leftarrow m) \mid_{\leq j} = \underbrace{(T \mid_{\leq j})}_{=Q} \leftarrow m = Q \leftarrow m$.

Hence, $(T \leftarrow m) \mid_{\leq j-1} = \left(\underbrace{(T \leftarrow m) \mid_{\leq j}}_{=Q \leftarrow m} \right)^- = (Q \leftarrow m)^-$.

Now, $\kappa = \text{sh} \left(\underbrace{T \mid_{\leq j-1}}_{=Q^-} \right) = \text{sh}(Q^-)$, $\lambda = \text{sh} \left(\underbrace{T \mid_{\leq j}}_{=Q} \right) = \text{sh} Q$, $\mu = \text{sh} \left(\underbrace{(T \leftarrow m) \mid_{\leq j-1}}_{=(Q \leftarrow m)^-} \right) = \text{sh}((Q \leftarrow m)^-)$ and $\nu = \text{sh} \left(\underbrace{(T \leftarrow m) \mid_{\leq j}}_{=Q \leftarrow m} \right) = \text{sh}(Q \leftarrow m)$.

But j is the highest entry of $T \mid_{\leq j} = Q$; thus, j appears in Q . Hence, $m \not\leq j$ (since $m \leq j$). Therefore, Proposition 3.2.1c (applied to Q instead of T) shows that the configuration $\begin{pmatrix} \text{sh}(Q^-) & \text{sh} Q \\ \text{sh}((Q \leftarrow m)^-) & \text{sh}(Q \leftarrow m) \end{pmatrix}$ is of type $RS2$, or $RS3$. Since $\text{sh}(Q^-) = \kappa$, $\text{sh} Q = \lambda$, $\text{sh}((Q \leftarrow m)^-) = \mu$ and $\text{sh}(Q \leftarrow m) = \nu$, this rewrites as follows: The configuration $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is of type $RS2$, or $RS3$. Hence, (6) is proven in Subcase 1.1.

Let us now consider Subcase 1.2. In this Subcase, we have $m > j$. Thus, $m > j > j - 1$.

Now, Proposition 3.2.1b (b) shows that $(T \leftarrow m) \mid_{\leq j} = T \mid_{\leq j}$. Hence, $\nu = \text{sh} \left(\underbrace{(T \leftarrow m) \mid_{\leq j}}_{=T \mid_{\leq j}} \right) = \text{sh}(T \mid_{\leq j}) = \lambda$. In other words, $\lambda = \nu$. Also, Proposition 3.2.1b (b) (applied to $j - 1$ instead of j) shows that $(T \leftarrow m) \mid_{\leq j-1} = T \mid_{\leq j-1}$ (since $m > j - 1$). Hence, $\mu = \text{sh} \left(\underbrace{(T \leftarrow m) \mid_{\leq j-1}}_{=T \mid_{\leq j-1}} \right) = \text{sh}(T \mid_{\leq j-1}) = \kappa$, so that $\kappa = \mu$.

Now, we have $\kappa = \mu \wedge \lambda = \nu$. Hence, $(\kappa = \lambda \wedge \mu = \nu$ or $\kappa = \mu \wedge \lambda = \nu)$.

Thus, we know that $\lambda, \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \nu^-)$ and $(\kappa = \lambda \wedge \mu = \nu$ or $\kappa = \mu \wedge \lambda = \nu)$.

In other words, the configuration $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is of type $RS0$. Hence, (6) is proven in Subcase 1.2.

We have thus proven (6) in each of the two Subcases 1.1 and 1.2. Therefore, (6) always holds in Case 1.

Let us now consider Case 2. In this Case, the integer j does not appear in T . Thus,

$T \mid_{\leq j-1} = T \mid_{\leq j}$. Hence, $\kappa = \text{sh} \left(\underbrace{T \mid_{\leq j-1}}_{=T \mid_{\leq j}} \right) = \text{sh}(T \mid_{\leq j}) = \lambda$.

But the integer j does not appear in $T \leftarrow m$ (since j neither appears in T , nor

equals m). Thus, $(T \leftarrow m) \upharpoonright_{\leq j-1} = (T \leftarrow m) \upharpoonright_{\leq j}$, so that $\mu = \text{sh} \left(\underbrace{(T \leftarrow m) \upharpoonright_{\leq j-1}}_{=(T \leftarrow m) \upharpoonright_{\leq j}} \right) = \text{sh}((T \leftarrow m) \upharpoonright_{\leq j}) = \nu$.

Now, we have $\kappa = \lambda \wedge \mu = \nu$. Hence, $(\kappa = \lambda \wedge \mu = \nu$ or $\kappa = \mu \wedge \lambda = \nu)$.

Thus, we know that $\lambda, \mu \in (\{\kappa\} \cup \kappa^+) \cap (\{\nu\} \cup \nu^-)$ and $(\kappa = \lambda \wedge \mu = \nu$ or $\kappa = \mu \wedge \lambda = \nu)$.

In other words, the configuration $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ is of type $RS0$. Hence, (6) is proven in

Case 2.

We have thus proven (6) in each of the two Cases 1 and 2. Thus, (6) is proven.

Now, recall that $\kappa = \text{sh}(T \upharpoonright_{\leq j-1})$, $\lambda = \text{sh}(T \upharpoonright_{\leq j})$, $\mu = \text{sh}((T \leftarrow m) \upharpoonright_{\leq j-1})$ and $\nu = \text{sh}((T \leftarrow m) \upharpoonright_{\leq j})$. Hence, (6) rewrites as follows: The configuration $\begin{pmatrix} \text{sh}(T \upharpoonright_{\leq j-1}) & \text{sh}(T \upharpoonright_{\leq j}) \\ \text{sh}((T \leftarrow m) \upharpoonright_{\leq j-1}) & \text{sh}((T \leftarrow m) \upharpoonright_{\leq j}) \end{pmatrix}$ is of type $RS0$, $RS2$, or $RS3$. This proves Statement P3.2.1.a.2.

Now, we have proven both Statement P3.2.1.a.1 and Statement P3.2.1.a.2. Combining these two statements, we obtain Proposition 3.2.1a.

Page 17, proof of Theorem 3.2.1: Replace “the final case” by “the case $RS0$ ”.

Page 17, proof of Theorem 3.2.1: Here is the reason why “ $\lambda^{[i,j]} = \lambda^{[j,i]}$ ” holds:

We can prove $\lambda^{[i,j]} = \lambda^{[j,i]}$ (for every $(i, j) \in \{0, 1, \dots, n\}^2$) by strong induction over $i + j$:

- In the case when one of i and j is 0, it follows from $\lambda^{[i,j]} = (0) = \lambda^{[j,i]}$.
- In the other case, it can be verified as follows: Neither i nor j is 0; hence, i and j both belong to $\{1, 2, \dots, n\}$. Thus, the growth-cell claim² (applied to j and i instead of i and j) shows that the configuration $\begin{pmatrix} \lambda^{[j-1,i-1]} & \lambda^{[j-1,i]} \\ \lambda^{[j,i-1]} & \lambda^{[j,i]} \end{pmatrix}$ is of type $RS1$ if $i = \sigma_j$, and otherwise of type $RS0$, $RS2$, or $RS3$. In other words,

$$\left(\begin{array}{l} \text{the configuration } \begin{pmatrix} \lambda^{[j-1,i-1]} & \lambda^{[j,i-1]} \\ \lambda^{[j,i-1]} & \lambda^{[j,i]} \end{pmatrix} \text{ is of type } RS1 \\ \text{if } i = \sigma_j, \text{ and otherwise of type } RS0, RS2, \text{ or } RS3 \end{array} \right) \quad (7)$$

(because the statement that a configuration $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ has any of the types $RS1$, $RS0$, $RS2$, and $RS3$ is symmetric in λ and μ). But by the inductive hypothesis, we have $\lambda^{[i-1,j]} = \lambda^{[j,i-1]}$, $\lambda^{[i,j-1]} = \lambda^{[j-1,i]}$ and $\lambda^{[i-1,j-1]} = \lambda^{[j-1,i-1]}$. On the other hand, the growth-cell claim (applied to σ^{-1} and $\lambda^{[u,v]}$ instead of σ and $\lambda^{[u,v]}$) shows that the configuration $\begin{pmatrix} \lambda^{[i-1,j-1]} & \lambda^{[i-1,j]} \\ \lambda^{[i,j-1]} & \lambda^{[i,j]} \end{pmatrix}$ is of type $RS1$ if $j = (\sigma^{-1})_i$, and otherwise of type $RS0$, $RS2$, or $RS3$. In other words, the configuration $\begin{pmatrix} \lambda^{[i-1,j-1]} & \lambda^{[i-1,j]} \\ \lambda^{[i,j-1]} & \lambda^{[i,j]} \end{pmatrix}$ is of type $RS1$ if $i = \sigma_j$, and otherwise of type $RS0$,

²This is the claim that “Any configuration $\begin{pmatrix} \lambda^{[i-1,j-1]} & \lambda^{[i-1,j]} \\ \lambda^{[i,j-1]} & \lambda^{[i,j]} \end{pmatrix}$ is of type $RS1$ if $j = \sigma_i$, and otherwise of type $RS0$, $RS2$, or $RS3$ ” made in the proof of Theorem 3.2.1

RS2, or *RS3* (because $j = (\sigma^{-1})_i$ is equivalent to $i = \sigma_j$). Since $\lambda^{[i-1,j]} = \lambda^{[j,i-1]}$, $\lambda^{[i,j-1]} = \lambda^{[j-1,i]}$ and $\lambda^{[i-1,j-1]} = \lambda^{[j-1,i-1]}$, this rewrites as follows:

$$\left(\begin{array}{l} \text{the configuration } \left(\begin{array}{cc} \lambda^{[j-1,i-1]} & \lambda^{[j,i-1]} \\ \lambda^{[j-1,i]} & \lambda^{[i,j]} \end{array} \right) \text{ is of type } RS1 \\ \text{if } i = \sigma_j, \text{ and otherwise of type } RS0, RS2, \text{ or } RS3 \end{array} \right). \quad (8)$$

Comparing this with (7), we conclude that $\lambda^{[i,j]} = \lambda^{[j,i]}$, because of the observation that if a configuration $\begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$ has any of the types *RS1*, *RS2*, *RS3*, and *RS0*, then ν is uniquely determined by κ , λ and μ and the information whether *RS1* applies.

Page 17: You write: “For a rectangular region we get the algorithm of [SaSt].”. It would be better to replace “[SaSt]” by “[SaSt, Theorem 5.1]” here, since only the case of partial permutations (not matrix words) is obtained directly from your growth approach.

Page 17: You write: “The moves occurring in the original formulation of the insertion procedure are related to configurations of types *RS1* and *RS2*; for a description in terms of chains of partitions, type *RS3* has to be considered as well”. I have not been able to find a correct way to make this statement precise. In my opinion, *RS3* “belongs” to the Insertion procedure no less than *RS1* and *RS2* do, and appears when a bumping-out is happening “deep inside the tableau” (i.e., far away from the corners). What was your intended interpretation?

Page 18, §3.2: Replace “then its own number” by “than its own number”.

Page 18, §3.2: Replace “the poset associated to” by “the partition associated to”.

Page 19, §4.1: Replace “[Schü1] not only” by “[Schü1] is not only”.

Page 20: Replace “configurations type” by “configurations of type”.

Page 20: Replace “*RS^t2* and *RS^t3* are identical to *RS2* and *RS3*, respectively” by “*RS^t0* and *RS^t3* are identical to *RS0* and *RS3*, respectively”.

Page 20, proof of Theorem 4.1.1: You write: “For $i + k = n + 1$ we either have $j = \sigma_i = \sigma_{n+1-k}$, in which case $\lambda^{[i,j-1,k]} = (0)$ and $\lambda^{[i,j,k]} = (1)$, or $j \neq \sigma_i = \sigma_{n+1-k}$, in which case $\lambda^{[i,j-1,k]} = \lambda^{[i,j,k]}$ ”. This raises the question of how to construct such a family of $\lambda^{[i,j,k]}$. The answer is simple, but (I think) is worth writing out explicitly: We set $\lambda^{[i,j,k]} = \begin{cases} (0), & \text{if } j < \sigma_i; \\ (1), & \text{if } j \geq \sigma_i \end{cases}$ for every (i, j, k) satisfying $i + k = n + 1$.

Page 20, proof of Theorem 4.1.1: Replace “such such” by “such”.

Page 20, proof of Theorem 4.1.1: Replace “by indiction” by “by induction”, unless you wish to imply that the jury is still out on the claim about the configurations.

Page 25: Replace “the the different” by “to the different”.

Page 27, §5: Replace “to to” by “to”.

Page 27, Definition 5.1: Remove the “a” in “be a skew tableaux”.

Page 28: I think it is worth pointing out that the notion of “one-step glissement” is equivalent to what is commonly called a “(jeu-de-taquin) slide” (e.g., in Fulton’s “Young tableaux”, with the only difference that Fulton carries around the partitions λ and μ defining the skew shape λ/μ and not just the set $Y(\lambda) \setminus Y(\mu)$). (That said, the proof of this equivalence is not completely obvious.)

Page 28, Lemma 5.3: Typo “their their”.

Page 28, proof of Lemma 5.3: In “From the definition of R ”, replace “ R ” by “ RS^{-1} ”.

Page 29, proof of Theorem 5.4: “Then existence” should be “The existence”.

Page 29: “the square s it the corner” should be “the square s is the corner”.

Page 29: Typo: “be be”.

Page 31, reference [Garf]: “Algebra’s” should be “Algebras”.
