Schur functions and alternating sums
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version of 10 November 2006
Errata and comments by Darij Grinberg

The following text is an annotation to Marc A. A. van Leeuwen's paper "Schur functions and alternating sums" in its version of 10 November 2006.

This annotation contains corrections of mistakes (or what I believe to be mistakes) and additional comments (in particular, elaborations of some arguments that I found insufficiently detailed in the paper). The latter are printed in blue.

Different comments are separated by horizontal lines, like this:

Page 1, Abstract: "Our our goal" should be "our goal".

Page 1, §0: Lemma 2.2 actually has appeared in the literature (although in a far less readable form than in your paper). It appears (in an equivalent form) in [DeLeTh, second equation on p. 29], where it is attributed to Muir (the reference unfortunately being the whole book "A treatise on the theory of determinants").

Page 1, §0: "an equally symmetrical doubly alternating expressions" should be either "an equally symmetrical doubly alternating expression" or "equally symmetrical doubly alternating expressions".

Page 4, §1.2: "to transpose semistandard tableau" should be "to transpose semistandard tableaux".

Page 5, Definition 1.2.3: Replace "$\operatorname{Tabl}(\lambda/\nu)$" by "$\operatorname{Tabl}(\lambda/\mu)$".

Page 5, §1.3: I would replace "and for which $\deg f_n$ is bounded" by "and for which the sequence $(\deg f_n)_{n\in\mathbf{N}}$ is bounded", to make the wording unambiguous.

Page 6, §1.4: Replace "be the sign" by "by the sign".

Page 6, §1.4: It would be useful to define the notations $\mathbf{S}_n$ and $\varepsilon(\sigma)$ here (both of which are not really standard):

- Given $n \in \mathbf{N}$, we let $\mathbf{S}_n$ denote the group of all permutations of the set $[n] = \{0, 1, \ldots, n-1\}$.
- Given $n \in \mathbf{N}$ and $\sigma \in \mathbf{S}_n$, we let $\varepsilon(\sigma)$ denote the sign of σ.

Page 6, §1.4: In the formula "$a_\alpha\left[X_{[n]}\right] = \sum_{\sigma\in\mathbf{S}_n} \varepsilon(\sigma) X^{\sigma\cdot\alpha}$", you are using the notation "$\sigma\cdot\alpha$" for $\sigma(\alpha)$. This notation should be defined.

Page 6, §1.4: In (13), replace "$X_j^{\alpha_j}$" by "$X_i^{\alpha_j}$".

Page 7: In the formula for $s_{(3,1)}\left[X_{[3]}\right]$, replace "$(X_0 - X_1)(X_0 - X_1)(X_1 - X_2)$" by "$(X_0 - X_1)(X_0 - X_2)(X_1 - X_2)$".

Page 7, proof of Proposition 1.4.1: The sentence "We may assume that $\alpha \in \mathbf{N}^{n+1}$ holds, since otherwise both $s_\alpha\left[X_{[n+1]}\right]$ and $s_\alpha\left[X_{[n]}\right]$ are zero by definition." should be moved to the very beginning of this proof, because already in the very first sentence this assumption is being used.

Page 8, §1.4: The claim that "The set $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ forms a $\mathbf{Z}$-basis of Λ" is not completely obvious (in my opinion) and could use a proof. Here is the simplest proof that I am aware of:

Proof of the fact that the set $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ forms a $\mathbf{Z}$-basis of Λ:

- For every $n \in \mathbf{N}$, the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0}$ is a $\mathbf{Z}$-basis of $\Lambda_{[n]}$. In particular, this family is linearly independent. Hence, the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ must be linearly independent (because if this family would satisfy a nontrivial linear dependency relation, then, for some sufficiently high $n \in \mathbf{N}$, we could apply the projection $\Lambda \to \Lambda_{[n]}$ to this relation, and obtain a nontrivial linear dependency relation for the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0}$; but this would contradict the fact that the latter family is linearly independent).

- Now, let us show that the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ. Indeed, here are two ways to prove this:

  First proof of the fact that the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ: We shall use the notation $\varepsilon(\alpha, \lambda)$ defined in (19). However, we cannot yet use the equality (19) as its definition, because we have not yet proven that the set $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ forms a $\mathbf{Z}$-basis of Λ (and thus we have not yet proven that the scalar product $\langle\cdot,\cdot\rangle$ is well-defined). We shall instead use a different (but equivalent) definition of this notation: If $\alpha \in \mathcal{C}$ and $\lambda \in \mathcal{P}$, then $\varepsilon(\alpha, \lambda)$ shall be defined as follows:

  - If the sequence $\alpha[]$ (defined as in §1.5) has two equal entries, then we set $\varepsilon(\alpha, \lambda) = 0$.
  - If the sequence $\alpha[]$ has no two equal entries, then there exists a unique permutation $\sigma \in \mathbf{S}_\infty$ such that $\sigma(\alpha[])$ is strictly decreasing. We consider this σ, and we set $\varepsilon(\alpha, \lambda) = \begin{cases} \varepsilon(\sigma), & \text{if } \sigma(\alpha[]) = \lambda[]; \\ 0, & \text{if } \sigma(\alpha[]) \neq \lambda[] \end{cases}$.

  It is rather clear that, with this definition of $\varepsilon(\alpha, \lambda)$, the equality (20) holds for every $\alpha \in \mathcal{C}$. Proposition 2.1 and Lemma 2.2 hold. (Their proofs did not make use of the fact that the set $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ forms a $\mathbf{Z}$-basis of Λ; they only used the notation $\varepsilon(\alpha, \lambda)$.)

  Let $f \in \Lambda$. Then, we can write $f[X_\mathbf{N}]$ in the form $f[X_\mathbf{N}] = \sum_{\alpha\in\mathcal{C}} c_\alpha X^\alpha$ for some family $(c_\alpha)_{\alpha\in\mathcal{C}} \in \mathbf{Z}^\mathcal{C}$ of integers. Consider this family $(c_\alpha)_{\alpha\in\mathcal{C}}$. Applying Lemma 2.2 to $\beta = (0)$, we obtain $f s_{(0)} = \sum_{\alpha\in\mathcal{C}} c_\alpha s_{\alpha+(0)}$ (and the sum on the right hand side of this equality is effectively finite – i.e., all but finitely many of its addends are zero). Compared with $f \underbrace{s_{(0)}}_{=1} = f$, this yields

$$f = \sum_{\alpha\in\mathcal{C}} c_\alpha \underbrace{s_{\alpha+(0)}}_{=s_\alpha} = \sum_{\alpha\in\mathcal{C}} c_\alpha \underbrace{s_\alpha}_{=\sum_{\lambda\in\mathcal{P}}\varepsilon(\alpha,\lambda)s_\lambda} = \sum_{\alpha\in\mathcal{C}} c_\alpha \left(\sum_{\lambda\in\mathcal{P}} \varepsilon(\alpha,\lambda) s_\lambda\right) = \sum_{\alpha\in\mathcal{C}}\sum_{\lambda\in\mathcal{P}} c_\alpha \varepsilon(\alpha,\lambda) s_\lambda.$$

Thus, f belongs to the $\mathbf{Z}$-linear span of the family $(s_\lambda)_{\lambda\in\mathcal{P}}$.

Now, let us forget that we fixed f. We thus have shown that every $f\in\Lambda$ belongs to the $\mathbf{Z}$-linear span of the family $(s_\lambda)_{\lambda\in\mathcal{P}}$. In other words, the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ.

Second proof of the fact that the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ: For every $m\in\mathbf{N}$ and every graded ring A, we let $A_{(m)}$ denote the m-th graded component of the ring A. Then, if $n\in\mathbf{N}$ and $m\in\mathbf{N}$ are arbitrary, then

$$\text{the family } \left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0;\ |\lambda|=m} \text{ is a } \mathbf{Z}\text{-basis of } \left(\Lambda_{[n]}\right)_m \tag{1}$$

[1].

For every $n\in\mathbf{N}$, let π_n be the projection $\Lambda_{[n+1]}\to\Lambda_{[n]}$ defined by the substitution $X_n := 0$. Then, $\pi_n(f) = f[X_n := 0]$ for every $n\in\mathbf{N}$ and $f\in\Lambda_{[n+1]}$. Hence, every $n\in\mathbf{N}$ and $\lambda\in\mathcal{P}$ satisfy

$$\pi_n\left(s_\lambda\left[X_{[n+1]}\right]\right) = s_\lambda\left[X_{[n+1]}\right][X_n := 0] = s_\lambda\left[X_{[n]}\right] \tag{2}$$

(by Proposition 1.4.1, applied to $\alpha=\lambda$). For every $n\in\mathbf{N}$ and $m\in\mathbf{N}$ satisfying $n\geq m$, it is easy to see that

$$\text{the map } \pi_n\mid_{\left(\Lambda_{[n+1]}\right)_m} : \left(\Lambda_{[n+1]}\right)_m \to \Lambda_{[n]} \text{ is injective} \tag{3}$$

[2].

[1] *Proof of (1):* Let $n\in\mathbf{N}$ and $m\in\mathbf{N}$. We know that the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0}$ is a $\mathbf{Z}$-basis of $\Lambda_{[n]}$. Therefore, for reasons of gradedness, we see that the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0;\ |\lambda|=m}$ is a $\mathbf{Z}$-basis of $\left(\Lambda_{[n]}\right)_m$. Qed.

[2] *Proof of (3):* Let $n\in\mathbf{N}$ and $m\in\mathbf{N}$ be such that $n\geq m$. The family $\left(s_\lambda\left[X_{[n+1]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_{n+1}=0;\ |\lambda|=m}$ is a $\mathbf{Z}$-basis of $\left(\Lambda_{[n+1]}\right)_m$ (because of (1), applied to $n+1$ instead of n).

Recall that $n\geq m$. Thus, it is easy to see that any $\lambda\in\mathcal{P}$ satisfying $|\lambda|=m$ must also satisfy $\lambda_n=0$. Hence, the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_{n+1}=0;\ |\lambda|=m}$ is a subfamily of the $\mathbf{Z}$-basis $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_n=0}$ of $\Lambda_{[n]}$. Therefore, this family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_{n+1}=0;\ |\lambda|=m}$ is linearly independent.

Now, for every $\lambda\in\mathcal{P}$, we have $\left(\pi_n\mid_{\left(\Lambda_{[n+1]}\right)_m}\right)\left(s_\lambda\left[X_{[n+1]}\right]\right) = \pi_n\left(s_\lambda\left[X_{[n+1]}\right]\right) = s_\lambda\left[X_{[n]}\right]$ (by (2)). Therefore, the $\mathbf{Z}$-linear map $\pi_n\mid_{\left(\Lambda_{[n+1]}\right)_m}$ sends the $\mathbf{Z}$-basis $\left(s_\lambda\left[X_{[n+1]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_{n+1}=0;\ |\lambda|=m}$ of $\left(\Lambda_{[n+1]}\right)_m$ to the family $\left(s_\lambda\left[X_{[n]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_{n+1}=0;\ |\lambda|=m}$. Since the latter family is linearly independent, this shows that the map $\pi_n\mid_{\left(\Lambda_{[n+1]}\right)_m}$ is injective (since it sends a $\mathbf{Z}$-basis of its domain to a linearly independent family). This proves (3).

Now, let $f \in \Lambda$ be homogeneous. Set $m = \deg f$. We know that the family $\left(s_\lambda\left[X_{[m]}\right]\right)_{\lambda\in\mathcal{P};\ \lambda_m=0;\ |\lambda|=m}$ is a $\mathbf{Z}$-basis of $\left(\Lambda_{[m]}\right)_m$ (by (1), applied to $n = m$). Thus, there exists a family $(w_\lambda)_{\lambda\in\mathcal{P};\ \lambda_m=0}$ of integers such that

$$f\left[X_{[m]}\right] = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[m]}\right] \tag{4}$$

(since $f\left[X_{[m]}\right] \in \left(\Lambda_{[m]}\right)_m$). Consider this family. We claim that

$$f\left[X_{[n]}\right] = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n]}\right] \qquad \text{for every } n\in\mathbf{N}. \tag{5}$$

Indeed, we can prove (5) as follows: For $n < m$, the equality (5) follows by applying the canonical projection $\Lambda_{[m]} \to \Lambda_{[n]}$ (defined by the substitutions $X_i := 0$ for all $i \in \{n, n+1, \ldots, m-1\}$) to (4). For $n \geq m$, we can prove (5) by induction over n: The induction base ($n = m$) follows from (4). For the induction step, we fix an $n \in \mathbf{N}$ satisfying $n \geq m$, and we assume (as the induction hypothesis) that

$$f\left[X_{[n]}\right] = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n]}\right]; \tag{6}$$

our goal is then to prove that

$$f\left[X_{[n+1]}\right] = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n+1]}\right]. \tag{7}$$

But both $f\left[X_{[n+1]}\right]$ and $\sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n+1]}\right]$ are elements of $\left(\Lambda_{[n+1]}\right)_m$. We have

$$\begin{aligned}
\pi_n\left(f\left[X_{[n+1]}\right]\right) &= f\left[X_{[n]}\right] \qquad \text{(by the definition of } \pi_n\text{)}\\
&= \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda \underbrace{s_\lambda\left[X_{[n]}\right]}_{\substack{=\pi_n\left(s_\lambda\left[X_{[n+1]}\right]\right)\\ \text{(by (2))}}} \qquad \text{(by (6))}\\
&= \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda \pi_n\left(s_\lambda\left[X_{[n+1]}\right]\right) = \pi_n\left(\sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n+1]}\right]\right).
\end{aligned}$$

Since the map $\pi_n \mid_{\left(\Lambda_{[n+1]}\right)_m}$ is injective (by (3)), this shows that $f\left[X_{[n+1]}\right] = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n+1]}\right]$ (because both $f\left[X_{[n+1]}\right]$ and $\sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda\left[X_{[n+1]}\right]$ live in $\left(\Lambda_{[n+1]}\right)_m$). In other words, (7) holds. This completes the induction step, and thus (5) is proven. Now, from (5), it follows that $f = \sum_{\lambda\in\mathcal{P};\ \lambda_m=0} w_\lambda s_\lambda$. Therefore, f belongs to the $\mathbf{Z}$-linear span of the family $(s_\lambda)_{\lambda\in\mathcal{P}}$.

Now, let us forget that we fixed f. We thus have shown that every homogeneous $f \in \Lambda$ belongs to the $\mathbf{Z}$-linear span of the family $(s_\lambda)_{\lambda\in\mathcal{P}}$. Therefore, every $f \in \Lambda$ belongs to the $\mathbf{Z}$-linear span of the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ (since every $f \in \Lambda$ is a sum of homogeneous elements of Λ). In other words, the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ.

- The family $(s_\lambda)_{\lambda\in\mathcal{P}}$ spans Λ and is linearly independent. Hence, the family $(s_\lambda)_{\lambda\in\mathcal{P}}$ is a $\mathbf{Z}$-basis of Λ, qed.

Page 8, §1.5: Remove the comma before "that transforms α [] into λ []" on the last line of page 8.

Page 10, Proposition 1.5.1: Replace "$\mathbf{Z}$" by "$\mathbf{N}$".

Page 12: Replace "save intervals" by "safe intervals".

Page 13: On the second line of page 13, replace "$K'_{\lambda/\lambda,\alpha} = K'_{\lambda/\lambda,\alpha} = [\alpha = (0)]$" by "$K'_{\lambda/\lambda,\alpha} = K_{\lambda/\lambda,\alpha} = [\alpha = (0)]$".

Page 13, Corollary 2.6: Replace "transposed" by "transpose".

Page 13, §2: Two lines above (25), replace "transposed" by "transpose".

Page 14, proof of Proposition 3.2: After "of equal shape", add "(or, for (30), transposed shape)".

Page 14: You prove Proposition 3.2 using a reference to Knuth's paper [Knu]. Let me give an alternative proof of Proposition 3.2, which relies only on results that you prove in your paper.

Second proof of Proposition 3.2: The following proof of Proposition 3.2 uses only results you prove in your paper. Thus, it would make the paper self-contained.

We first notice some basic properties of minimal symmetric functions (whose proofs are very easy):

- Every $\nu \in \mathcal{C}$ satisfies
$$m_\nu [X_{\mathbf{N}}] = \sum_{\beta \in \mathcal{C}} \left[\nu^+ = \beta^+\right] X^\beta. \tag{8}$$

- Every $\nu \in \mathcal{P}$ satisfies
$$m_\nu [X_{\mathbf{N}}] = \sum_{\alpha \in \mathcal{C}} \left[\nu = \alpha^+\right] X^\alpha. \tag{9}$$

- For every $\nu \in \mathcal{P}$ and $\alpha \in \mathcal{C}$, we have
$$(\text{the } X^\alpha\text{-coefficient of } m_\nu [X_{\mathbf{N}}]) = \left[\nu = \alpha^+\right]. \tag{10}$$

Now, every $\lambda \in \mathcal{P}$ satisfies

$$s_\lambda [X_{\mathbf{N}}] = \sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)} \tag{11}$$

[3] and therefore

$$((\text{the } X^\alpha\text{-coefficient of } s_\lambda [X_{\mathbf{N}}]) = K_{\lambda,\alpha} \qquad \text{for every } \alpha \in \mathcal{C}) \tag{13}$$

[3] *Proof of (11):* Let $\lambda \in \mathcal{P}$. Define a power series $f [X_{\mathbf{N}}] \in \mathbf{Z} [[X_{\mathbf{N}}]]$ by $f [X_{\mathbf{N}}] = \sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)}$. In §5, you show that the power series $f [X_{\mathbf{N}}]$ is symmetric. Since this power series is also homogeneous,

[4]. Moreoer, every $\lambda \in \mathcal{P}$ satisfies

$$s_\lambda = \sum_{\nu \in \mathcal{P}} K_{\lambda,\nu} m_\nu \tag{14}$$

(by (13), since s_λ is symmetric). Finally, every $\lambda \in \mathcal{P}$ and $\alpha \in \mathcal{C}$ satisfy

$$K_{\lambda,\alpha^+} = K_{\lambda,\alpha} \tag{15}$$

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this shows that $f[X_\mathbf{N}] \in \Lambda$. Now,

$$f[X_\mathbf{N}] = \sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)} = \sum_{\alpha \in \mathcal{C}} |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}| X^\alpha.$$

Thus, Lemma 2.2 (applied to $\beta = (0)$ and $c_\alpha = |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}|$) yields

$$\begin{aligned} f s_{(0)} &= \sum_{\alpha \in \mathcal{C}} |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}| \underbrace{s_{\alpha+(0)}}_{=s_\alpha} \\ &= \sum_{\alpha \in \mathcal{C}} |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}| s_\alpha. \end{aligned}$$

Compared with $f \underbrace{s_{(0)}}_{=1} = f$, this yields

$$f = \sum_{\alpha \in \mathcal{C}} |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}| s_\alpha. \tag{12}$$

But in (48), you have proven that $\sum_{T \in \mathrm{SST}(\lambda)} s_{\mathrm{wt}(T)} = s_\lambda$, so that

$$s_\lambda = \sum_{T \in \mathrm{SST}(\lambda)} s_{\mathrm{wt}(T)} = \sum_{\alpha \in \mathcal{C}} |\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\}| s_\alpha = f \qquad \text{(by (12))}.$$

Hence, $s_\lambda[X_\mathbf{N}] = f[X_\mathbf{N}] = \sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)}$, qed.

[4]*Proof of (13):* Let $\lambda \in \mathcal{P}$. Let $\alpha \in \mathcal{C}$. We have

$$\begin{aligned} &\left(\text{the } X^\alpha\text{-coefficient of } \underbrace{s_\lambda[X_\mathbf{N}]}_{\substack{=\sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)} \\ \text{(by (11))}}}\right) \\ &= \left(\text{the } X^\alpha\text{-coefficient of } \sum_{T \in \mathrm{SST}(\lambda)} X^{\mathrm{wt}(T)}\right) = \sum_{T \in \mathrm{SST}(\lambda)} \underbrace{\left(\text{the } X^\alpha\text{-coefficient of } X^{\mathrm{wt}(T)}\right)}_{=[\alpha = \mathrm{wt}(T)]} \\ &= \sum_{T \in \mathrm{SST}(\lambda)} [\alpha = \mathrm{wt}(T)] = \sum_{\substack{T \in \mathrm{SST}(\lambda); \\ \alpha = \mathrm{wt}(T)}} \underbrace{[\alpha = \mathrm{wt}(T)]}_{\substack{=1 \\ \text{(since } \alpha = \mathrm{wt}(T))}} + \sum_{\substack{T \in \mathrm{SST}(\lambda); \\ \alpha \neq \mathrm{wt}(T)}} \underbrace{[\alpha = \mathrm{wt}(T)]}_{\substack{=0 \\ \text{(since } \alpha \neq \mathrm{wt}(T))}} \\ &= \sum_{\substack{T \in \mathrm{SST}(\lambda); \\ \alpha = \mathrm{wt}(T)}} 1 + \underbrace{\sum_{\substack{T \in \mathrm{SST}(\lambda); \\ \alpha \neq \mathrm{wt}(T)}} 0}_{=0} = \sum_{\substack{T \in \mathrm{SST}(\lambda); \\ \alpha = \mathrm{wt}(T)}} 1 = |\{T \in \mathrm{SST}(\lambda) \mid \alpha = \mathrm{wt}(T)\}| \cdot 1 \\ &= \left| \underbrace{\{T \in \mathrm{SST}(\lambda) \mid \alpha = \mathrm{wt}(T)\}}_{\substack{=\{T \in \mathrm{SST}(\lambda) \mid \mathrm{wt}(T) = \alpha\} \\ = \mathrm{SST}(\lambda, \alpha)}} \right| = |\mathrm{SST}(\lambda, \alpha)| = \# \mathrm{SST}(\lambda, \alpha) = K_{\lambda,\alpha}, \end{aligned}$$

(by (13), since s_λ is symmetric).

Now, we have

$$f = \sum_{\lambda \in \mathcal{P}} \langle h_\lambda \mid f \rangle m_\lambda \qquad \text{for every } f \in \Lambda \tag{16}$$

[5]. Hence, the two bases $(h_\lambda)_{\lambda \in \mathcal{P}}$ and $(m_\lambda)_{\lambda \in \mathcal{P}}$ of Λ are dual to each other with respect to the bilinear form $\langle \cdot \mid \cdot \rangle$.

Now, as you show right before Proposition 3.2, every $\alpha \in \mathcal{C}$ and $\beta \in \mathcal{C}$ satisfy

$$\langle h_\alpha \mid e_\beta \rangle = \sum_{\lambda \in \mathcal{P}} K_{\lambda,\alpha} K'_{\lambda,\beta} \tag{19}$$

and

$$\langle h_\alpha \mid h_\beta \rangle = \sum_{\lambda \in \mathcal{P}} K_{\lambda,\alpha} K_{\lambda,\beta}. \tag{20}$$

Now, fix $\beta \in \mathcal{C}$. Then, (16) (applied to $f = h_\beta$) yields

$$\begin{aligned} h_\beta &= \sum_{\lambda \in \mathcal{P}} \langle h_\lambda \mid h_\beta \rangle m_\lambda = \sum_{\nu \in \mathcal{P}} \underbrace{\langle h_\nu \mid h_\beta \rangle}_{\substack{= \sum_{\lambda \in \mathcal{P}} K_{\lambda,\nu} K_{\lambda,\beta} \\ \text{(by (20), applied to } \alpha = \nu)}} m_\nu \\ &\qquad \text{(here, we renamed the summation index } \lambda \text{ as } \nu) \\ &= \sum_{\nu \in \mathcal{P}} \left(\sum_{\lambda \in \mathcal{P}} K_{\lambda,\nu} K_{\lambda,\beta} \right) m_\nu , \end{aligned}$$

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and thus (13) is proven.

[5] *Proof of (16):* Let $f \in \Lambda$. We need to prove the equality (16). This equality is $\mathbf{Z}$-linear in f. Hence, we can WLOG assume that f belongs to the basis $(s_\lambda)_{\lambda \in \mathcal{P}}$ of the $\mathbf{Z}$-module Λ. Assume this. Thus, $f = s_\mu$ for some $\mu \in \mathcal{P}$. Consider this μ.

For every $\nu \in \mathcal{P}$, we have $K_{\mu/(0),\nu} = \langle h_\nu \mid s_{\mu/(0)} \rangle$ (by the definition of $K_{\mu/(0),\nu}$). Since $\mu/(0) = \mu$, this equality rewrites as

$$K_{\mu,\nu} = \langle h_\nu \mid s_\mu \rangle . \tag{17}$$

Now,

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} \langle h_\lambda \mid f \rangle m_\lambda &= \sum_{\nu \in \mathcal{P}} \left\langle h_\nu \mid \underbrace{f}_{= s_\mu} \right\rangle m_\nu \qquad \text{(here, we renamed the summation index } \lambda \text{ as } \nu) \\ &= \sum_{\nu \in \mathcal{P}} \underbrace{\langle h_\nu \mid s_\mu \rangle}_{\substack{= K_{\mu,\nu} \\ \text{(by (17))}}} m_\nu = \sum_{\nu \in \mathcal{P}} K_{\mu,\nu} m_\nu . \end{aligned} \tag{18}$$

On the other hand, $f = s_\mu = \sum_{\nu \in \mathcal{P}} K_{\mu,\nu} m_\nu$ (by (14), applied to $\lambda = \mu$). Comparing this with (18), we obtain $f = \sum_{\lambda \in \mathcal{P}} \langle h_\lambda \mid f \rangle m_\lambda$. This proves (16).

so that

$$h_\beta[X_\mathbf{N}] = \sum_{\nu\in\mathcal{P}} \left(\sum_{\lambda\in\mathcal{P}} K_{\lambda,\nu}K_{\lambda,\beta}\right) \underbrace{m_\nu[X_\mathbf{N}]}_{\substack{=\sum_{\alpha\in\mathcal{C}}[\nu=\alpha^+]X^\alpha \\ \text{(by (9))}}} = \sum_{\nu\in\mathcal{P}} \left(\sum_{\lambda\in\mathcal{P}} K_{\lambda,\nu}K_{\lambda,\beta}\right)\left(\sum_{\alpha\in\mathcal{C}}[\nu=\alpha^+]X^\alpha\right)$$

$$= \sum_{\alpha\in\mathcal{C}} \underbrace{\sum_{\nu\in\mathcal{P}} \left(\sum_{\lambda\in\mathcal{P}} K_{\lambda,\nu}K_{\lambda,\beta}\right)[\nu=\alpha^+]}_{=\sum_{\lambda\in\mathcal{P}} K_{\lambda,\alpha^+}K_{\lambda,\beta}} X^\alpha = \sum_{\alpha\in\mathcal{C}} \left(\sum_{\lambda\in\mathcal{P}} \underbrace{K_{\lambda,\alpha^+}}_{\substack{=K_{\lambda,\alpha} \\ \text{(by (15))}}} K_{\lambda,\beta}\right) X^\alpha$$

$$= \sum_{\alpha\in\mathcal{C}} \left(\sum_{\lambda\in\mathcal{P}} K_{\lambda,\alpha}K_{\lambda,\beta}\right) X^\alpha.$$

Hence,

$$\sum_{\alpha\in\mathcal{C}} \left(\sum_{\lambda\in\mathcal{P}} K_{\lambda,\alpha}K_{\lambda,\beta}\right) X^\alpha = h_\beta[X_\mathbf{N}] = \sum_{\alpha\in\mathcal{C}} \sharp\mathcal{M}_{\alpha,\beta}X^\alpha \qquad \text{(by (10))}.$$

Comparing coefficients in this equality, we conclude that

$$\sum_{\lambda\in\mathcal{P}} K_{\lambda,\alpha}K_{\lambda,\beta} = \sharp\mathcal{M}_{\alpha,\beta}.$$

This proves (29). Similarly, we can prove (30) (by applying (16) to $f = e_\beta$ instead of $f = h_\beta$, and by using (19) instead of (20)). The proof of Proposition 3.2 is complete.

(Alternatively, it is also possible to prove Proposition 3.2 by deriving it from the identities (33) and (34), which are more widespread than Proposition 3.2 and which can be proven in various ways, not only using the RSK algorithm.)

Page 15: "the the image" should be "the image".

Page 15: You write: "From the proof of proposition 2.1 we see that if $\lambda \in \mathbf{N}^l$, then one needs to consider only permutations $\sigma \in \mathbf{S}_l$". By this, you mean that if $\lambda \in \mathbf{N}^l$ and if $\sigma \in \mathbf{S}_\infty$ is a permutation satisfying $\sigma(\lambda[]) \geq \mu[]$, then

$$\sigma \in \mathbf{S}_l. \qquad (21)$$

I do not see how (21) is supposed to follow from the proof of Proposition 2.1. Here is, instead, my proof of (21):

Proof of (21): Let $\lambda \in \mathbf{N}^l$, and let $\sigma \in \mathbf{S}_\infty$ be a permutation satisfying $\sigma(\lambda[]) \geq \mu[]$. We need to prove (21). Indeed, assume the contrary (for the sake of contradiction). Thus, $\sigma \notin \mathbf{S}_l$. Thus, there exists some $i \in \{l, l+1, l+2, \ldots\}$ such that $\sigma(i) \neq i$. Let j be the **largest** such i. Thus, j is an element of $\{l, l+1, l+2, \ldots\}$ and satisfies $\sigma(j) \neq j$. From $j \in \{l, l+1, l+2, \ldots\}$, we obtain $j \geq l$.

Now, we have $\sigma(j) < j$ [6]. Now, set $i = \sigma(j)$. Thus, $\sigma^{-1}(i) = j$.

We have $\lambda \in \mathbf{N}^l$, and thus $\lambda_w = 0$ for every $w \in \mathbf{N}$ satisfying $w \geq l$. Applying this to $w = j$, we obtain $\lambda_j = 0$. The definition of $\lambda[j]$ now yields $\lambda[j] = \underbrace{\lambda_j}_{=0} - 1 - j = -1 - j$, so that

$$\begin{aligned}
-1 - j &= \lambda\left[\underbrace{j}_{=\sigma^{-1}(i)}\right] = \lambda\left[\sigma^{-1}(i)\right] = (\lambda[])_{\sigma^{-1}(i)} = (\sigma(\lambda[]))_i \\
&\qquad \text{(by the definition of } \sigma(\lambda[])\text{)} \\
&\geq (\mu[])_i \qquad \text{(since } \sigma(\lambda[]) \geq \mu[]\text{)} \\
&= \mu[i] = \underbrace{\mu_i}_{\geq 0} - 1 - \underbrace{i}_{=\sigma(j)<j} \qquad \text{(by the definition of } \mu[i]\text{)} \\
&> -1 - j.
\end{aligned}$$

This is absurd. This contradiction shows that our assumption was wrong. Hence, (21) is proven.

Page 16, §3: You claim that "we may apply the automorphism ω to equation (37)" to obtain the equality (40). This is correct, but in my opinion could use a bit more justification. You are tacitly using the fact that

$$\omega\left(s_{\lambda/\mu}\right) = s_{\lambda^{\mathbf{t}}/\mu^{\mathbf{t}}} \qquad \text{for any } \lambda \in \mathcal{P} \text{ and } \mu \in \mathcal{P}. \tag{22}$$

The proof of (22) is simple, but not something I would leave to the reader:

Proof of (22): The basis $(s_\lambda)_{\lambda \in \mathcal{P}}$ of the $\mathbf{Z}$-module Λ is orthonormal with respect to the scalar product $\langle \cdot \mid \cdot \rangle$ (because of the definition of this scalar product). In other words, we have

$$\langle s_\lambda \mid s_\mu \rangle = [\lambda = \mu] \qquad \text{for any } \lambda \in \mathcal{P} \text{ and } \mu \in \mathcal{P}. \tag{23}$$

Furthermore,

$$\langle \omega(f) \mid \omega(g) \rangle = \langle f \mid g \rangle \qquad \text{for any } f \in \Lambda \text{ and } g \in \Lambda \tag{24}$$

[7].

Recall that ω is a ring morphism (since ω coincides with the ring morphism $\Lambda \to \Lambda$ that sends $h_i \mapsto e_i$ for all $i > 0$). Thus,

$$s_{\mu^{\mathbf{t}}} s_{\nu^{\mathbf{t}}} = \omega(s_\mu s_\nu) \qquad \text{for any } \mu \in \mathcal{P} \text{ and } \nu \in \mathcal{P} \tag{25}$$

[6] *Proof.* Assume the contrary. Thus, $\sigma(j) \geq j$. Combined with $\sigma(j) \neq j$, this yields $\sigma(j) > j$. Now, σ is injective (since $\sigma \in \mathbf{S}_\infty$). Thus, from $\sigma(j) \neq j$, we obtain $\sigma(\sigma(j)) \neq \sigma(j)$. Also, $\sigma(j) \geq j \geq l$, so that $\sigma(j) \in \{l, l+1, l+2, \ldots\}$.

But recall that j is the **largest** $i \in \{l, l+1, l+2, \ldots\}$ such that $\sigma(i) \neq i$. Hence, every $i \in \{l, l+1, l+2, \ldots\}$ such that $\sigma(i) \neq i$ satisfies $i \leq j$. Applying this to $i = \sigma(j)$, we obtain $\sigma(j) \leq j$ (since $\sigma(j) \in \{l, l+1, l+2, \ldots\}$ and $\sigma(\sigma(j)) \neq \sigma(j)$). This contradicts $\sigma(j) > j$. This contradiction proves that our assumption was wrong, qed.

[7] *Proof of (24):* The equality $\langle \omega(f) \mid \omega(g) \rangle = \langle f \mid g \rangle$ is $\mathbf{Z}$-linear in each of f and g. Thus, it suffices to prove it in the case when f and g belong to the basis $(s_\lambda)_{\lambda \in \mathcal{P}}$ of the $\mathbf{Z}$-module Λ. But in this case, it boils down to the identity $\left[\alpha^{\mathbf{t}} = \beta^{\mathbf{t}}\right] = [\alpha = \beta]$ for any $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$ (because of (23)), which identity is obvious.

[8].

Every $f \in \Lambda$ satisfies

$$f = \sum_{\lambda \in \mathcal{P}} \underbrace{\langle f \mid s_\lambda \rangle}_{= \langle s_\lambda \mid f \rangle} s_\lambda = \sum_{\lambda \in \mathcal{P}} \langle s_\lambda \mid f \rangle s_\lambda = \sum_{\nu \in \mathcal{P}} \langle s_\nu \mid f \rangle s_\nu \tag{26}$$

(here, we have renamed the summation index λ as ν).

Let $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{P}$. The definition of ω yields $\omega(s_\lambda) = s_{\lambda^{\mathbf{t}}}$. Applying (26) to $f = s_{\lambda/\mu}$, we obtain

$$s_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}} \underbrace{\left\langle s_\nu \mid s_{\lambda/\mu} \right\rangle}_{\substack{= \langle s_\mu s_\nu \mid s_\lambda \rangle \\ \text{(since } \langle s_\mu s_\nu \mid s_\lambda \rangle = \left\langle s_\nu \mid s_{\lambda/\mu} \right\rangle \\ \text{(by (17), applied to } f = s_\nu\text{))}}} s_\nu = \sum_{\nu \in \mathcal{P}} \langle s_\mu s_\nu \mid s_\lambda \rangle s_\nu . \tag{27}$$

The same argument (applied to $\lambda^{\mathbf{t}}$ and $\mu^{\mathbf{t}}$ instead of λ and μ) yields

$$\begin{aligned} s_{\lambda^{\mathbf{t}}/\mu^{\mathbf{t}}} &= \sum_{\nu \in \mathcal{P}} \langle s_{\mu^{\mathbf{t}}} s_\nu \mid s_{\lambda^{\mathbf{t}}} \rangle s_\nu = \sum_{\nu \in \mathcal{P}} \left\langle \underbrace{s_{\mu^{\mathbf{t}}} s_{\nu^{\mathbf{t}}}}_{\substack{= \omega(s_\mu s_\nu) \\ \text{(by (25))}}} \mid \underbrace{s_{\lambda^{\mathbf{t}}}}_{= \omega(s_\lambda)} \right\rangle \underbrace{s_{\nu^{\mathbf{t}}}}_{\substack{= \omega(s_\nu) \\ \text{(since } \omega(s_\nu) = s_{\nu^{\mathbf{t}}} \\ \text{(by the definition of } \omega\text{))}}} \\ &\qquad \left(\begin{array}{c} \text{here, we have substituted } \nu^{\mathbf{t}} \text{ for } \nu \text{ in the sum,} \\ \text{because the map } \mathcal{P} \to \mathcal{P},\ \nu \mapsto \nu^{\mathbf{t}} \text{ is a bijection} \end{array} \right) \\ &= \sum_{\nu \in \mathcal{P}} \underbrace{\langle \omega(s_\mu s_\nu) \mid \omega(s_\lambda) \rangle}_{\substack{= \langle s_\mu s_\nu \mid s_\lambda \rangle \\ \text{(by (24))}}} \omega(s_\nu) \\ &= \sum_{\nu \in \mathcal{P}} \langle s_\mu s_\nu \mid s_\lambda \rangle \omega(s_\nu) = \omega \left(\underbrace{\sum_{\nu \in \mathcal{P}} \langle s_\mu s_\nu \mid s_\lambda \rangle s_\nu}_{\substack{= s_{\lambda/\mu} \\ \text{(by (27))}}} \right) \qquad \text{(since the map } \omega \text{ is } \mathbf{Z}\text{-linear)} \\ &= \omega \left(s_{\lambda/\mu} \right). \end{aligned}$$

This proves (22).

Page 18: Replace "by a single factor h_j" by "by a single factor h_{β_j}" (or something like this, but not h_j, since j already means something different here).

Page 19: In the paragraph that begins with "Now suppose to the contrary" and ends with "of the sequence $(\mu^{\mathbf{t}} + \operatorname{col}(M))\,[]$", every appearance of "$\beta^{(i)}$" should be replaced by "$\beta^{(i+1)}$", and every appearance of "$\beta^{(i-1)}$" should be replaced by "$\beta^{(i)}$". Also, the "α_{i-1}" should be replaced by "α_i". (The reason for this is that the claim

[8] *Proof of (25):* Let $\mu \in \mathcal{P}$ and $\nu \in \mathcal{P}$. Since ω is a ring morphism, we have $\omega(s_\mu s_\nu) = \underbrace{\omega(s_\mu)}_{\substack{= s_{\mu^{\mathbf{t}}} \\ \text{(by the definition of } \omega\text{)}}} \underbrace{\omega(s_\nu)}_{\substack{= s_{\nu^{\mathbf{t}}} \\ \text{(by the definition of } \omega\text{)}}} = s_{\mu^{\mathbf{t}}} s_{\nu^{\mathbf{t}}}$. This proves (25).

that "$\beta^{(i)}$ was obtained by adding the binary composition M_i to the partition $\beta^{(i-1)}$" is false; the addition of M_i rather turns $\beta^{(i)}$ into $\beta^{(i+1)}$.)

Page 25: You claim that "Conversely any entry $m+1$ for which this condition in terms of α' holds, and for which k is minimal, cannot have an entry m in the same column". This is true, but I find this rather nontrivial.

References

[DeLeTh] J. Désarménien, B. Leclerc, J.-Y. Thibon, *Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory*, Séminaire Lotharingien de Combinatoire [electronic only] (1994), Volume: 32.