A GRAPH THEORETIC PROOF OF THE FUNDAMENTAL TRACE IDENTITY

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Abstract. To each circuit decomposition of a directed multigraph a permutation of the edge set is assigned, and a lemma on the signs of the resulting permutations is proved yielding the fundamental trace identity for matrices over a commutative unitary ring as a corollary.

Let Γ be a directed multigraph. A *circuit decomposition* of Γ is a set D of circuits[1] of Γ such that each edge of Γ occurs in exactly one element of D. Each circuit $[\gamma_1, \ldots, \gamma_t] \in D$ determines a permutation of $\{\gamma_1, \ldots, \gamma_t\}$, viz.

$$(\gamma_1, \ldots, \gamma_t) = \begin{pmatrix} \gamma_1, \ldots, \gamma_{t-1}, \gamma_t \\ \gamma_2, \ldots, \quad \gamma_t, \gamma_1 \end{pmatrix},$$

and

$$\sigma_D := \prod_{[\gamma_1,\ldots,\gamma_t] \in D} (\gamma_1, \ldots, \gamma_t)$$

is a permutation of the edge set of Γ. Let

$$\mathscr{C}(\Gamma) := \{\sigma \mid \sigma = \sigma_D \text{ for some circuit decomposition } D \text{ of } \Gamma\}.$$

Lemma 1. *If Γ has more edges than vertices, then the number of even permutations in $\mathscr{C}(\Gamma)$ is equal to the number of odd permutations in $\mathscr{C}(\Gamma)$.*

[1] *Comment by DG:* A *circuit* means a closed walk with no repeated edges (but repeated vertices are allowed) that has at least one edge. Circuits are viewed as lists of edges up to cyclic rotation (so $[\gamma_1, \ldots, \gamma_t]$ and $[\gamma_2, \ldots, \gamma_t, \gamma_1]$ are considered to be the same circuit).

Proof. By our hypothesis, there exist two distinct edges α, β which begin at the same vertex. Let Δ be the set of all circuit decompositions D of Γ with the property that α, β belong to the same circuit in D, and let Δ^* consist of the other circuit decompositions of Γ. Let $D \in \Delta$, and $[\gamma_1, \ldots, \gamma_k, \alpha, \gamma_{k+1}, \ldots, \gamma_l, \beta, \gamma_{l+1}, \ldots, \gamma_m] \in D$. Replacing this single circuit by $[\alpha, \gamma_{k+1}, \ldots, \gamma_l]$ and $[\beta, \gamma_{l+1}, \ldots, \gamma_m, \gamma_1, \ldots, \gamma_k]$ yields a circuit decomposition $D^* \in \Delta^*$, and $\operatorname{sgn}(\sigma_{D^*}) = -\operatorname{sgn}(\sigma_D)$. It is easy to see that $\sigma_D \mapsto \sigma_{D^*}$ is a bijection of $\{\sigma_D \mid D \in \Delta\}$ onto $\{\sigma_{D^*} \mid D^* \in \Delta^*\}$. The lemma follows. □

Corollary 1 (cf. [2], [3, Theorem 4.3.(b)])**.** *Let R be a commutative ring with identity* 1*, and let $\mathscr{S}_k$ be the symmetric group on $\{1, \ldots, k\}$. If $k > n$, then for all $n \times n$ matrices $M_1, \ldots, M_k$ over R,*

$$\sum_{\sigma \in \mathscr{S}_k} \operatorname{sgn}(\sigma) \cdot \prod_{(i_1, \ldots, i_t)} \operatorname{Tr}(M_{i_1} \cdots M_{i_t}) = 0,$$

where $(i_1, \ldots, i_t)$ ranges over the set of all cycles of the cycle decomposition of the permutation σ.

Proof. We put

$$\Phi_\sigma(M_1, \ldots, M_k) := \prod_{(i_1, \ldots, i_t)} \operatorname{Tr}(M_{i_1} \cdots M_{i_t}).$$

Then Φ_σ is a k-fold R-linear mapping. Therefore we may assume that each M_j is a matrix whose only non-zero entry is a 1, say, in the (r_j, s_j)-place. We define a directed multigraph Γ with vertex set $\{1, \ldots, n\}$ and with k edges $\gamma_1, \ldots, \gamma_k$ such that γ_j begins at r_j and ends at s_j. Obviously, $\Phi_\sigma(M_1, \ldots, M_k) \neq 0$ if and only if $\operatorname{Tr}(M_{i_1} \cdots M_{i_t}) \neq 0$, i.e., $s_{i_j} = r_{i_{j+1}}$ for $1 \leq j < t$ and $s_{i_t} = r_{i_1}$, for all cycles $(i_1, \ldots, i_t)$ occurring in the cycle decomposition of σ. This is equivalent to $\sigma = \sigma_D$ for some circuit decomposition D of Γ. Therefore,

$$\sum_{\sigma \in \mathscr{S}_k} \operatorname{sgn}(\sigma) \Phi_\sigma(M_1, \ldots, M_k) = \sum_{\sigma \in \mathscr{C}(\Gamma)} \operatorname{sgn}(\sigma) = 0,$$

by the lemma. □

The assertion of our Corollary plays a fundamental role in the theory of trace identities (cf. [3, 4]). The graph theoretic approach in this note is closely related to Swan's proof of the Amitsur–Levitzky identity [1, I, Theorem 14].

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