# Noncommutative birational rowmotion on a rectangle

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*joint work with Tom Roby (UConn)*

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BIRS, Kelowna
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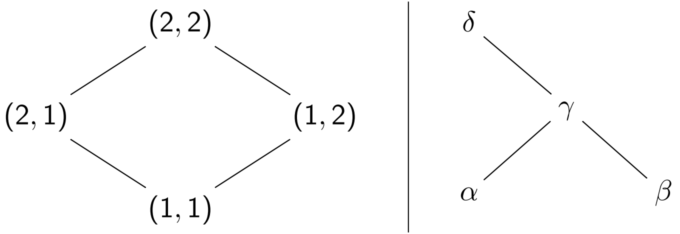
**slides:** http://www.cip.ifi.lmu.de/~grinberg/algebra/kelowna2021.pdf

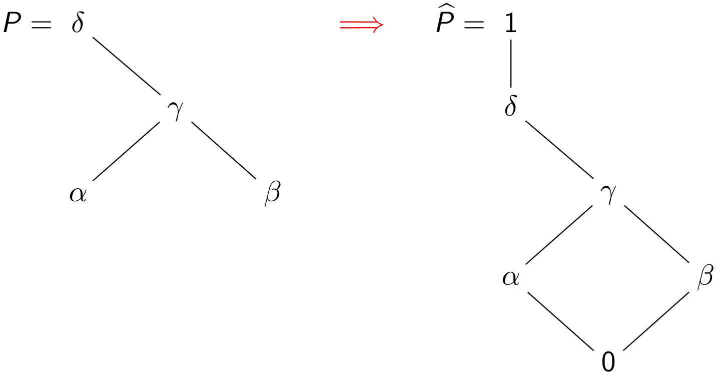
- A **poset** (= partially ordered set) is a set $P$ with a reflexive, transitive and antisymmetric relation.
- We use the symbols $<$, $\leq$, $>$ and $\geq$ accordingly.
- We draw posets as Hasse diagrams:



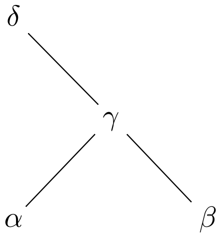

- We only care about finite posets here.
- We say that $u \in P$ **is covered by** $v \in P$ (written $u \lessdot v$) if we have $u < v$ and there is no $w \in P$ satisfying $u < w < v$.
- We say that $u \in P$ **covers** $v \in P$ (written $u \gtrdot v$) if we have $u > v$ and there is no $w \in P$ satisfying $u > w > v$.

- Let $P$ be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to $P$ and forcing
  - 0 to be less than every other element, and
  - 1 to be greater than every other element.

**Example:**



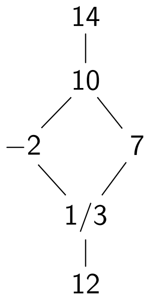
- A **linear extension** of $P$ means a list $(v_1, v_2, \ldots, v_n)$ of all elements of $P$ (each only once) such that $i < j$ whenever $v_i < v_j$.
- For instance,




has two linear extensions $(\alpha, \beta, \gamma, \delta)$ and $(\beta, \alpha, \gamma, \delta)$.
- Every finite poset has at least one linear extension.

- Let $\mathbb{K}$ be a ring (not necessarily commutative).
- A $\mathbb{K}$-**labelling of** $P$ will mean a function $\widehat{P} \to \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.

**Example:** This is a $\mathbb{Q}$-labelling of the $2 \times 2$-rectangle:



- For any $v \in P$, define the **birational $v$-toggle** as the partial map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left( \sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \widehat{P}$.

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# Birational rowmotion: definition

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- This is a **partial** map. If any of the inverses does not exist in $\mathbb{K}$, then $T_v f$ is undefined!
- Notice that this is a **local change** to the label at $v$; all other labels stay the same.
- If $\mathbb{K}$ is commutative, then $T_v^2 = \text{id}$ (on the range of $T_v$).

- We define **(noncommutative) birational rowmotion** as the partial map

$$R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of $P$.

- This is indeed independent on the linear extension, because:

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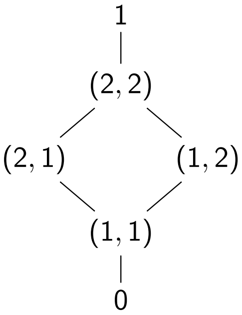
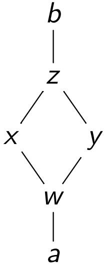
where $(v_1, v_2, \ldots, v_n)$ is a linear extension of $P$.

- This is indeed independent on the linear extension, because:
  - $T_v$ and $T_w$ commute whenever $v$ and $w$ are incomparable (or just don't cover each other);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.

**Example:**

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the $2 \times 2$-rectangle:

| poset | labelling |
|---|---|


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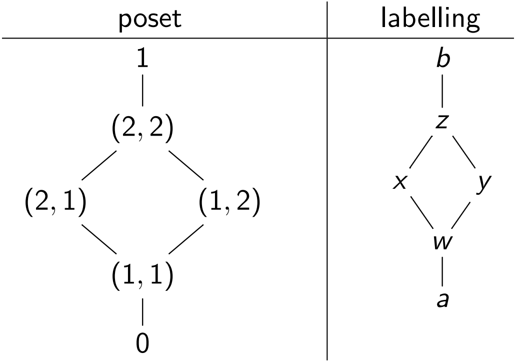


We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1),(1,2),(2,1),(2,2))$).

That is, toggle in the order "top, left, right, bottom".

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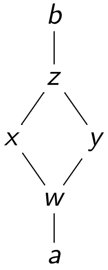
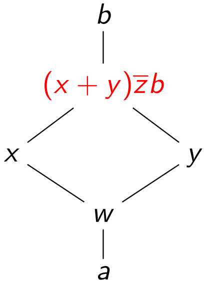
|   |   |

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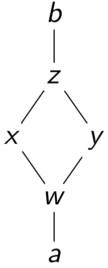


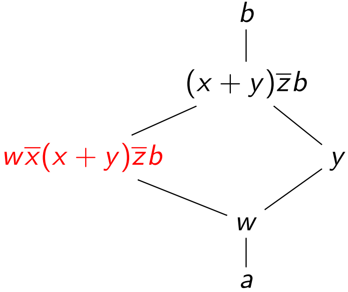


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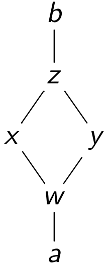
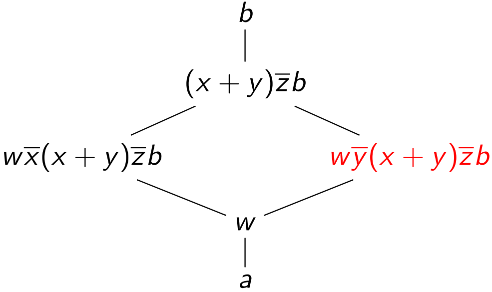
|  |  |





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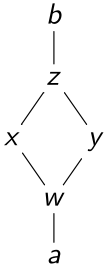
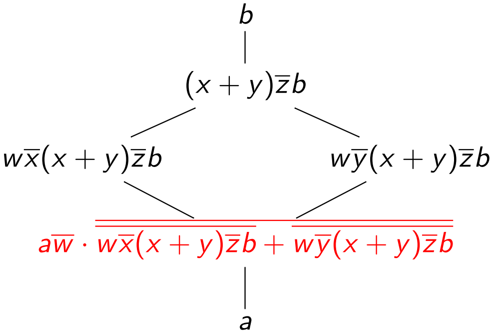
|  |  |





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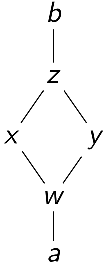
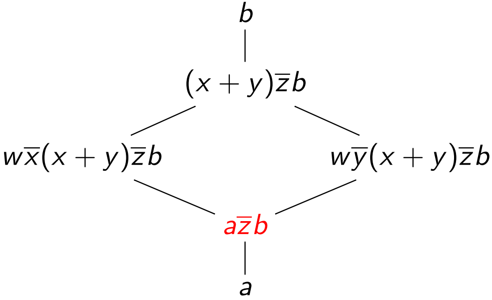
|   |   |

We have used $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ and simplified the result.

- Why is this called birational rowmotion?
- Indeed, it generalizes classical rowmotion of order ideals:
  - Let $\operatorname{Trop}\mathbb{Z}$ be the **tropical semiring** over $\mathbb{Z}$. This is the set $\mathbb{Z} \cup \{-\infty\}$ with "addition" $(a, b) \mapsto \max\{a, b\}$ and "multiplication" $(a, b) \mapsto a + b$. This is a semifield.

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  - To every order ideal $S \in J(P)$, assign a $\operatorname{Trop}\mathbb{Z}$-labelling $\operatorname{tlab} S$ defined by

    $$(\operatorname{tlab} S)(v) = \begin{cases} 1, & \text{if } v \notin S \cup \{0\}; \\ 0, & \text{if } v \in S \cup \{0\}. \end{cases}$$

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  - Don't like semifields? Use $\mathbb{Q}$ and take the "tropical limit".

- If $\mathbb{K}$ is commutative, then birational rowmotion $R$ has nice orders for nice posets (mostly Grinberg/Roby 2014):
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  - More generally, if $P$ is the minuscule poset associated to a minuscule weight $\lambda$ of a finite-dimensional simple Lie algebra $\mathfrak{g}$, then $R^h = \mathrm{id}$, where $h$ is the Coxeter number of $\mathfrak{g}$. (Soichi Okada, doi:10.37236/9557 .)

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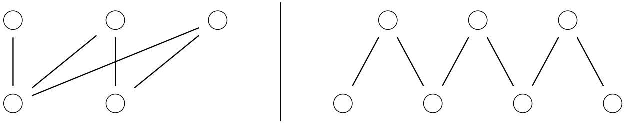
  - If $P$ is an "$n$-graded forest" (a forest with all leaves having rank $n$), then $R^{\ell} = \mathrm{id}$ for $\ell = \mathrm{lcm}\,(1, 2, \ldots, n+1)$.

- In general, $R$ can have infinite order – e.g., for the following two posets:



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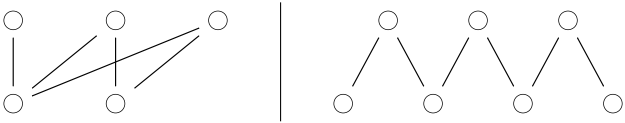

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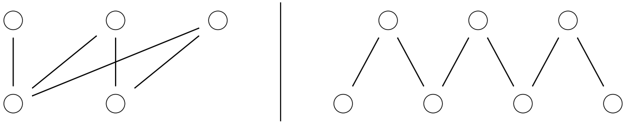

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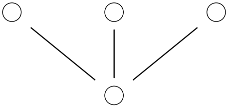

This satisfies $R^6 = \mathrm{id}$ if $\mathbb{K}$ is commutative, but nothing like that in general (apparently).

# Birational rowmotion: some chaos

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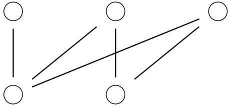

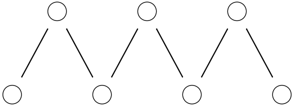

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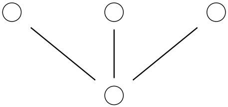


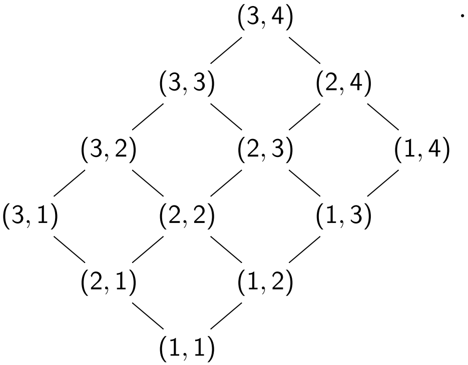
This satisfies $R^6 = \mathrm{id}$ if $\mathbb{K}$ is commutative, but nothing like that in general (apparently).

- However, not all is lost!

- Let $p$ and $q$ be two positive integers. Let $\mathbb{K}$ be a ring. Let $P$ be the $p \times q$-rectangle poset: i.e.,

$$P := [p] \times [q], \qquad \text{where } [m] := \{1, 2, \ldots, m\}.$$

(The order on $P$ is entrywise.)
**Example:** For $p = 3$ and $q = 4$, this is



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**Periodicity theorem (* 2015, † 2021+ G & Roby):**

If $a$ and $b$ are invertible and $R^{p+q} f$ is well-defined, then

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a} b \qquad \text{for each } x \in \widehat{P}.$$

Note that $a\overline{b} \cdot f(x) \cdot \overline{a} b$ is **not** generally conjugate to $f(x)$.

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$$P := [p] \times [q], \qquad \text{where } [m] := \{1, 2, \ldots, m\}.$$

(The order on $P$ is entrywise.)

- Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Let $a = f(0)$ and $b = f(1)$.

**Periodicity theorem (* 2015, † 2021+ G & Roby):**

If $a$ and $b$ are invertible and $R^{p+q} f$ is well-defined, then

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a} b \qquad \text{for each } x \in \widehat{P}.$$

**Reciprocity theorem (* 2015, † 2021+ G & Roby):**

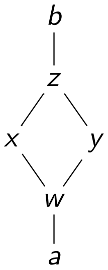
Let $\ell \in \mathbb{N}$. If $R^\ell f$ is well-defined and $\ell \geq i + j - 1$, then

$$\left(R^\ell f\right)(i,j) = a \cdot \overline{\left(R^{\ell-i-j+1} f\right) \underbrace{(p+1-i, q+1-j)}_{=\text{antipode of } (i,j) \text{ in } P}} \cdot b$$

for each $(i, j) \in P$.

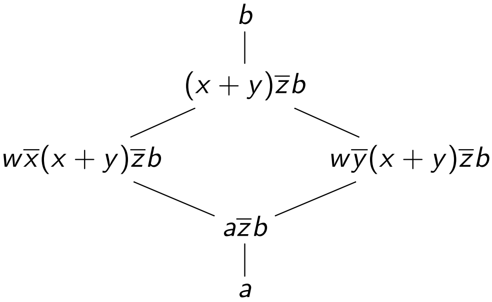
- **Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

  $R^0 f =$



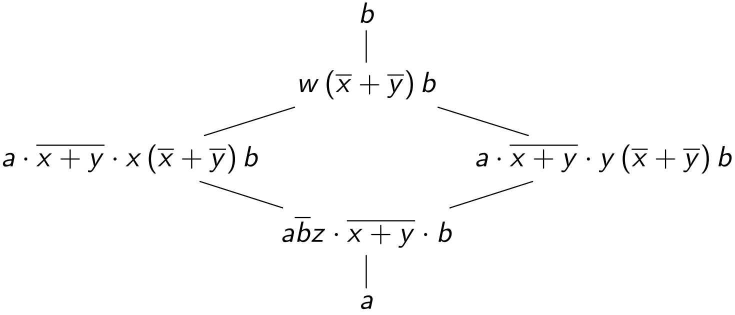
- **Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

$R^1 f =$



- **Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

  $R^2 f =$



- **Example:** Iteratively apply $R$ to a labelling of the $2 \times 2$-rectangle.

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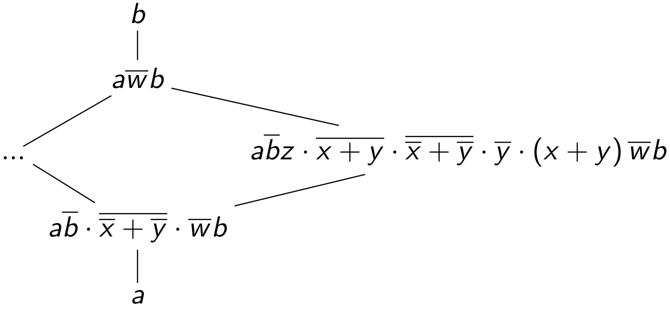

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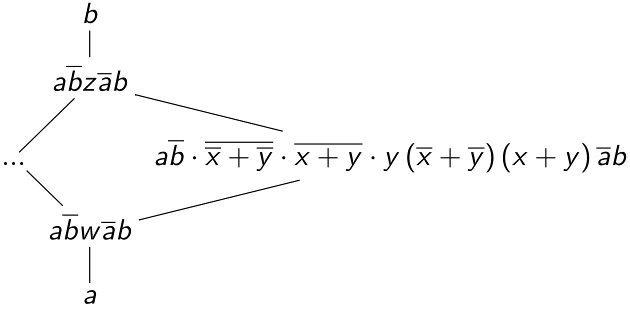

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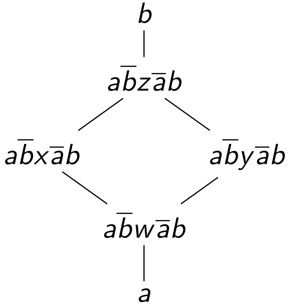


(after nontrivial simplifications).

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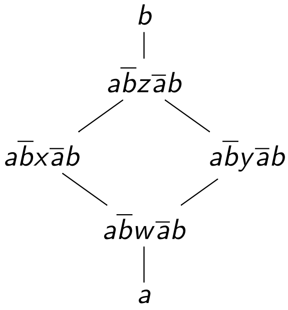


  This confirms the periodicity theorem for $p = q = 2$.

- Note that this is similar to Kontsevich's periodicity conjecture, proved by Iyudu/Shkarin (arXiv:1305.1965).

## The commutative case

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    Explicitly, if $A \in \mathbb{K}^{p \times (p+q)}$ is any matrix, then $(\operatorname{Grasp}_0 A)(0) = (\operatorname{Grasp}_0 A)(1) = 1$ and

    $$(\operatorname{Grasp}_0 A)(i,j) = \frac{\det(A[1:i \mid i+j-1:p+j])}{\det(A[0:i \mid i+j:p+j])}$$

    for all $(i,j) \in P$, where the $A[a:b \mid c:d]$s are certain submatrices of $A$. (Note that this map $\operatorname{Grasp}_0$ actually factors through the Grassmannian.)

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  - Reciprocity also easy using $\mathrm{Grasp}_0$.

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    No: e.g., the identity $x\overline{y}\overline{x}y = 1$ holds in all skew fields but not in all rings.
- We now believe this approach is a dead end.

- New proofs of periodicity and reciprocity in the commutative-$\mathbb{K}$ case were found by Gregg Musiker and Tom Roby in arXiv:1801.03877.
  They proceed by giving an explicit formula for $\left(R^k f\right)(i,j)$.
  For instance, $\left(R^3 f\right)(3,2)$

  $$= \frac{1}{A_{02}+A_{11}+A_{20}} (A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}),$$

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- Lattice paths can be generalized to noncommutative $\mathbb{K}$, but NILPs? Unclear in what order to multiply different paths.

# What now?

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- Let's play around with the setting. Step 1: Introduce notations...

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- The definition of $R$ yields

$$(Rf)(v) = \left(\sum_{u \lessdot v} f(u)\right) \cdot \overline{f(v)} \cdot \overline{\sum_{u \gtrdot v} \overline{(Rf)(u)}} \qquad \text{for each } v \in P.$$

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- In other words,

$$v_1 = \left(\sum_{u \lessdot v} u_0\right) \cdot \overline{v_0} \cdot \overline{\sum_{u \gtrdot v} \overline{u_1}} \qquad \text{for each } v \in P.$$

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- Similarly,

$$v_{\ell+1} = \left( \sum_{u \lessdot v} u_\ell \right) \cdot \overline{v_\ell} \cdot \overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}} \qquad \text{for each } v \in P \text{ and } \ell \in \mathbb{N}.$$

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- We haven't done anything serious yet, just rewritten the setup using the (more convenient) $x_\ell := \left(R^\ell f\right)(x)$ notation.

# Simplifying the goal

- We must prove:

**periodicity**: $x_{p+q} = a\overline{b} \cdot x_0 \cdot \overline{a}b$;

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- Periodicity follows from reciprocity: Indeed, if $x = (i,j)$ and $x' = (p+1-i, q+1-j)$, then

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- Moreover, reciprocity in general follows from reciprocity for $\ell = i + j - 1$ (just apply it to $R^k f$ instead of $f$ otherwise).

## Paths, Deltas and Nablas

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- If $u$ and $v$ are elements of $\widehat{P}$, set

$$\Delta_\ell^{u \to v} := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} \Delta_\ell^{\mathbf{p}} \qquad \text{and}$$

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- **Path formulas:**
  **(a)** We have

$$u_\ell = \overline{\nabla_\ell^{1\to u}} \cdot b \qquad \text{for each } u \in P.$$

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  Induction step: Given $v \in P$ such that $\nabla^{1 \to u} = b\overline{u_\ell}$ for all $u \gtrdot v$. Since any path $1 \to v$ passes through a unique $u \gtrdot v$, we have

$$\nabla^{1 \to v} = \sum_{u \gtrdot v} \nabla^{1 \to u} \nabla^v = \sum_{u \gtrdot v} b\overline{u_\ell} \nabla^v \qquad \text{(by induction hypothesis)}$$

$$= b\overline{v_\ell} \qquad \text{(by definition of } \nabla^v\text{)}, \qquad \text{qed.}$$

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  **(b)** Analogous, but use upwards induction instead.

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  **(c)** We have
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  Similarly, **(d)** follows from **(b)**.

- **Transition equation in $\Delta$-$\nabla$-form:**

$$\nabla_{\ell+1}^{v} = \Delta_{\ell}^{v} \qquad \text{for each } v \in \widehat{P} \text{ and } \ell \in \mathbb{N}.$$

- **Transition equation in Δ-∇-form:**

$$\nabla^v_{\ell+1} = \Delta^v_\ell \qquad \text{for each } v \in \widehat{P} \text{ and } \ell \in \mathbb{N}.$$

- *Proof idea:* Above we showed that

$$v_{\ell+1} = \left(\sum_{u \lessdot v} u_\ell\right) \cdot \overline{v_\ell} \cdot \overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}}.$$

  Take reciprocals on both sides, multiply by $\overline{\sum_{u \gtrdot v} \overline{u_{\ell+1}}}$ and rewrite using $\nabla^v_{\ell+1}$ and $\Delta^v_\ell$.

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- As a consequence of $\nabla^{v}_{\ell+1} = \Delta^{v}_{\ell}$, we have

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- As a consequence of $\nabla_{\ell+1}^{v} = \Delta_{\ell}^{v}$, we have

$$\nabla_{\ell+1}^{\mathbf{p}} = \Delta_{\ell}^{\mathbf{p}} \qquad \text{for each path } \mathbf{p} \text{ and each } \ell \in \mathbb{N}.$$

  Hence, $\nabla_{\ell+1}^{u \to v} = \Delta_{\ell}^{u \to v}$ for any $u, v \in \widehat{P}$.

- Now, for the bottommost element $(1,1)$ of $P$, we have

$$\begin{aligned} (1,1)_1 &= \overline{\nabla_1^{(p,q)\to(1,1)}} \cdot b && \text{(by path formula } \mathbf{(c)}\text{)} \\ &= \overline{\Delta_0^{(p,q)\to(1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\ &= a \cdot \overline{(p,q)_0} \cdot b && \text{(by path formula } \mathbf{(d)}\text{)}. \end{aligned}$$

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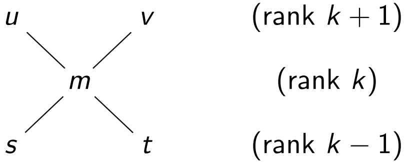
- What now?

## The case $j = 1$ suffices: part 1

- We can simplify our goal one bit further. Consider the "neighborhood" of an element of our rectangle $P$:




(where the **rank** of an $(i,j) \in P$ is defined to be $i + j - 1$). Say we have shown (our "induction hypotheses") that reciprocity holds for each of $s, t, m, u$; that is, we have

$$s_\ell = a \cdot \overline{s'_{\ell-(k-1)}} \cdot b, \qquad t_\ell = a \cdot \overline{t'_{\ell-(k-1)}} \cdot b,$$
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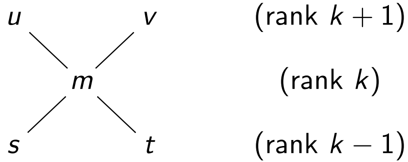


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**Claim:** Then, reciprocity also holds for $v$; that is, we have $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$ for all $\ell \geq k + 1$.

## The case $j = 1$ suffices: part 2

- *Proof idea.* Fix $\ell \geq k+1$, and compare the transition equations

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{\overline{u_\ell} + \overline{v_\ell}} \qquad \text{and}$$

$$m'_{\ell-k} = (u'_{\ell-k-1} + v'_{\ell-k-1}) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{\overline{s'_{\ell-k}} + \overline{t'_{\ell-k}}}$$

using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b$,

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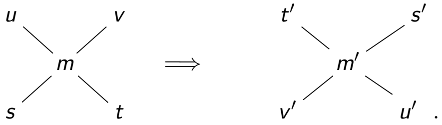

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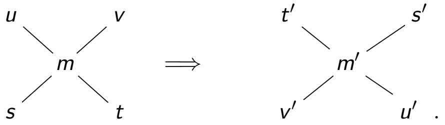


After subtracting $u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$, out comes $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$.

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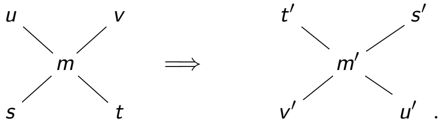


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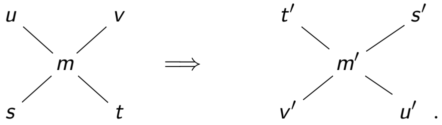


- This argument still works if $s$, $t$ or $u$ does not exist.
- Thus, in order to prove reciprocity for all $(i, j)$, it suffices (by induction) to prove it in the case when $j = 1$.

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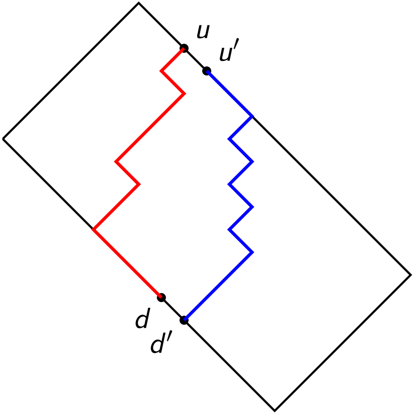
  Note the lack of rowmotion in this formula! The $\ell$ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.
- **Our new goal:** Prove that

$$\Delta^{(p,q)\to(2,1)} = \nabla^{(p-1,q)\to(1,1)}.$$

- More generally:
- **Conversion lemma:** Let $u$ and $u'$ be two adjacent elements on the top-right edge of $P$ (that is, $u = (k, q)$ and $u' = (k-1, q)$). Let $d$ and $d'$ be two adjacent elements on the bottom-left edge of $P$ (that is, $d = (i, 1)$ and $d' = (i-1, 1)$). Then,




$$\Delta_\ell^{u \to d} = \nabla_\ell^{u' \to d'} \qquad \text{for each } \ell \in \mathbb{N}.$$

In short:

$$\Delta^{u \to d} = \nabla^{u' \to d'}.$$

- If we can prove the conversion lemma, we will obtain reciprocity not only for $(i, j) = (2, 1)$, but also for all $(i, j)$ on the bottom-left edge of $P$ (that is, for the entire case $j = 1$), because we can argue as follows:

$$\begin{aligned}
(i,1)_i &= \overline{\nabla_i^{(p,q)\to(i,1)}} \cdot b && \text{(by path formula } \textbf{(c)}\text{)} \\
&= \overline{\Delta_{i-1}^{(p,q)\to(i,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= \overline{\nabla_{i-1}^{(p-1,q)\to(i-1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{\Delta_{i-2}^{(p-1,q)\to(i-1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= \overline{\nabla_{i-2}^{(p-2,q)\to(i-2,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \cdots \\
&= \overline{\nabla_1^{(p-i+1,q)\to(1,1)}} \cdot b && \text{(by the conversion lemma)} \\
&= \overline{\Delta_0^{(p-i+1,q)\to(1,1)}} \cdot b && \left(\text{since } \nabla_{\ell+1}^{u\to v} = \Delta_\ell^{u\to v}\right) \\
&= a \cdot \overline{(p-i+1,q)_0} \cdot b && \text{(by path formula } \textbf{(d)}\text{)}\,.
\end{aligned}$$

- This proves reciprocity

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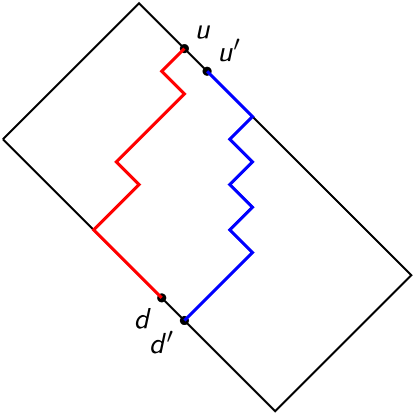
  The case $\ell > i$ follows by applying this to $R^{\ell - i} f$ instead of $f$.
- Thus, we only need to prove the conversion lemma. We can now drop all subscripts forever!

# Proving the conversion lemma: the intuition

- Let us again look at the picture:




We must prove $\Delta^{u \to d} = \nabla^{u' \to d'}$.

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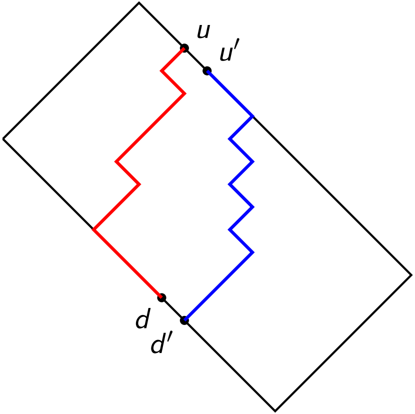


We must prove $\Delta^{u \to d} = \nabla^{u' \to d'}$.

- How do we interpolate between paths $u \to d$ and paths $u' \to d'$ ?

- We define a **path-jump-path** to be a sequence

$$\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$$

  of elements of $P$, where the relation $x \blacktriangleright y$ means "$y$ is one step down and some steps to the right of $x$" (that is, if $x = (r, s)$, then $y = (r - k, s + k - 1)$ for some $k > 0$). We say that this path-jump-path $\mathbf{p}$ has **jump at** $i$.

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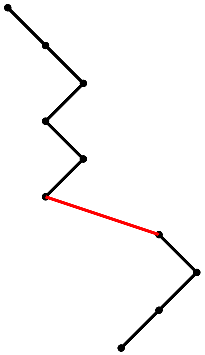
**Example** of a path-jump-path:



(The red edge is the jump.)

## Proving the conversion lemma: path-jump-paths

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We say that this path-jump-path $\mathbf{p}$ has **jump at** $i$.
For any such path-jump-path $\mathbf{p}$, we set

$$E_{\mathbf{p}} := \Delta^{v_0} \Delta^{v_1} \cdots \Delta^{v_{i-1}} v_i \overline{v_{i+1}} \nabla^{v_{i+2}} \nabla^{v_{i+3}} \cdots \nabla^{v_k}.$$

(Here, we are omitting the $\ell$ subscripts – so $v_i$ means $(v_i)_\ell$ and $v_{i+1}$ means $(v_{i+1})_\ell$.)

- We define a **path-jump-path** to be a sequence

$$\mathbf{p} = (v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_i \blacktriangleright v_{i+1} \gtrdot v_{i+2} \gtrdot \cdots \gtrdot v_k)$$

  of elements of $P$, where the relation $x \blacktriangleright y$ means "$y$ is one step down and some steps to the right of $x$" (that is, if $x = (r, s)$, then $y = (r-k, s+k-1)$ for some $k > 0$).
  We say that this path-jump-path $\mathbf{p}$ has **jump at** $i$.
  For any such path-jump-path $\mathbf{p}$, we set

$$E_{\mathbf{p}} := \Delta^{v_0} \Delta^{v_1} \cdots \Delta^{v_{i-1}} v_i \overline{v_{i+1}} \nabla^{v_{i+2}} \nabla^{v_{i+3}} \cdots \nabla^{v_k}.$$

- Now, if $k = \operatorname{rank} u - \operatorname{rank}(d')$, then

$$\Delta^{u \to d} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } k-1}} E_{\mathbf{p}},$$

  since $\Delta^d = d\overline{d'}$, and similarly

$$\nabla^{u' \to d'} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } 0}} E_{\mathbf{p}}.$$

## Proving the conversion lemma: moving the jump

- So we need to show that

$$\sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } k-1}} E_{\mathbf{p}} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \to d' \\ \text{with jump at } 0}} E_{\mathbf{p}}.$$

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for each $0 \leq i < k - 1$.

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  for each $0 \leq i < k - 1$.
- And yes, this is true and can be proved by a "local" argument (rewriting two consecutive steps of the path).
- This is similar to the "zipper argument" in lattice models. (Is there a Yang–Baxter equation lurking?)

## Proving the conversion lemma: the civilized version, part 1

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- However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:
- Define three $P \times P$-matrices $\Delta$, $\nabla$ and $U$ by

$$\Delta_{x,y} := \Delta^x [x \gtrdot y], \qquad \nabla_{x,y} := \nabla^y [x \gtrdot y],$$
$$U_{x,y} := x\overline{y} [x \blacktriangleright y] \qquad \text{for all } x, y \in P.$$

Here, $[\mathcal{A}]$ is the Iverson bracket (i.e., truth value) of a statement $\mathcal{A}$; the relation $x \blacktriangleright y$ means "$y$ is one step down and some steps to the right of $x$" as before. And again, we are omitting the $\ell$ subscripts, so $x\overline{y}$ actually means $x_\ell \overline{y_\ell}$.
- Now, we claim that

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# Proving the conversion lemma: the civilized version, part 2

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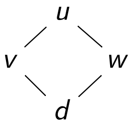
  Indeed, this follows easily from the following neat lemma: If




are four adjacent elements of $P$, then

$\overline{w} \cdot \nabla^d \cdot d = \overline{u} \cdot \Delta^u \cdot v$ and $\overline{v} \cdot \nabla^d \cdot d = \overline{u} \cdot \Delta^u \cdot w$.

(The $u$ and $d$ here are unrelated to the $u$ and $d$ from the conversion lemma!)

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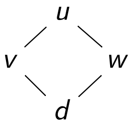


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- From $\Delta U = U \nabla$, we easily obtain

  $$\Delta^{\circ k} U = U \nabla^{\circ k} \qquad \text{for any } k \in \mathbb{N},$$

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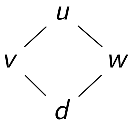


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  where $A^{\circ k}$ means the $k$-th power of a matrix $A$.
- Setting $k = \operatorname{rank} u - \operatorname{rank} d$ and comparing the $(u, d')$-entries of both sides, we quickly obtain $\Delta^{u \to d} = \nabla^{u' \to d'}$ (since $x \blacktriangleright d'$ holds only for $x = d$, and since $u \blacktriangleright x$ holds only for $x = u'$). This proves the conversion lemma again.

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  In the **commutative** case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are "essentially" polynomial identities (once you clear denominators).
  This **fails** for noncommutative $\mathbb{K}$ !
- **Scary example** (David Speyer, MathOverflow #401273): If $x$ and $y$ are two elements of a ring such that $x + y$ is invertible, then

$$x \cdot \overline{x+y} \cdot y = y \cdot \overline{x+y} \cdot x.$$

  But this is not true if "ring" is replaced by "semiring"!

## Is that all? Part 2: Questions

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**Question:**

Are the periodicity and reciprocity theorems still true if "ring" is replaced by "semiring"? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)

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**Question:**

Are any other results like ours known in the noncommutative case?

# Acknowledgments


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- **Mathematisches Forschungsinstitut Oberwolfach**: hospitality in July/August 2021
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- **Sage and Sage-combinat**: computations
- **the birational combinatorics community**: keeping the subject interesting since 2013
- **you**: your patience

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