

Noncommutative birational rowmotion on a rectangle

A case study in noncommutative dynamics

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joint work with Tom Roby (UConn)

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slides: [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/ipac2023b.pdf)

[//www.cip.ifi.lmu.de/~grinberg/algebra/ipac2023b.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/ipac2023b.pdf)

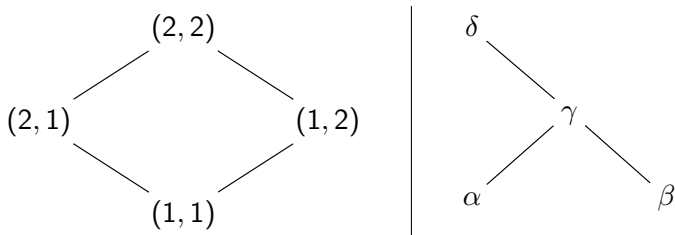
paper: <https://arxiv.org/abs/2208.11156>

FPSAC abstract: [https:](https://www.cip.ifi.lmu.de/~grinberg/algebra/fps2023.pdf)

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Introduction: Posets

- A **poset** (= partially ordered set) is a set P with a reflexive, transitive and antisymmetric relation.
- We use the symbols $<$, $\leq$, $>$ and $\geq$ accordingly.
- We draw posets as Hasse diagrams:

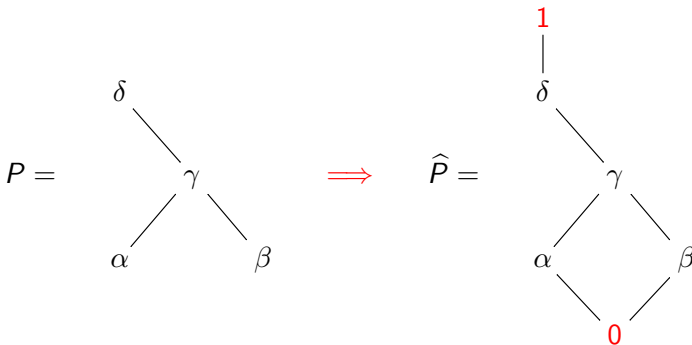


- We only care about finite posets here.
- We say that $u \in P$ **is covered by** $v \in P$ (written $u \triangleleft v$) if we have $u < v$ and there is no $w \in P$ satisfying $u < w < v$.
- We say that $u \in P$ **covers** $v \in P$ (written $u \triangleright v$) if we have $u > v$ and there is no $w \in P$ satisfying $u > w > v$.

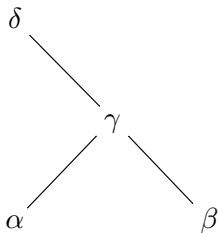
More poset basics: $\hat{P}$

- Let P be a finite poset. We define $\hat{P}$ to be the poset obtained by adjoining two new elements 0 and 1 to P and forcing
 - 0 to be less than every other element, and
 - 1 to be greater than every other element.

Example:



- A **linear extension** of P means a list $(v_1, v_2, \dots, v_n)$ of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- For instance,

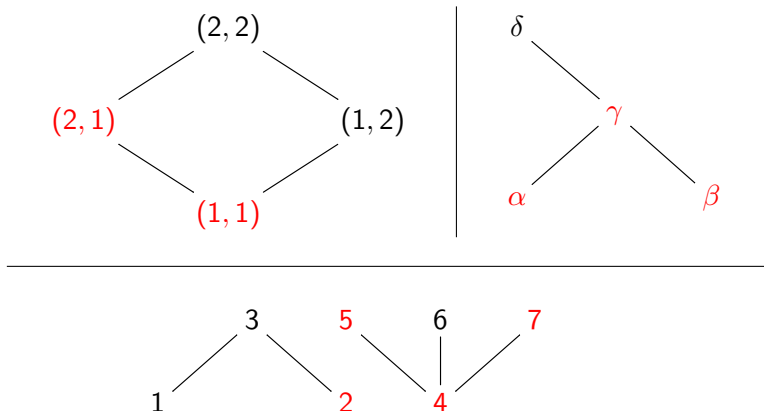


has two linear extensions $(\alpha, \beta, \gamma, \delta)$ and $(\beta, \alpha, \gamma, \delta)$.

- Every finite poset has at least one linear extension.

More poset basics: order ideals

- An **order ideal** of a poset P is a subset S of P such that if $v \in S$ and $w \leq v$, then $w \in S$.
- Examples (the elements of the order ideal are marked in red):



- We let $J(P)$ denote the set of all order ideals of P .

- **Classical rowmotion** is the rowmotion studied by Striker/Williams ([arXiv:1108.1172](#)). It has appeared many times before, under different guises:
 - Brouwer/Schrijver (1974) (as a permutation of the antichains),
 - Fon-der-Flaass (1993) (as a permutation of the antichains),
 - Cameron/Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
 - Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or “nonnesting partitions”, with relations to Lie theory).

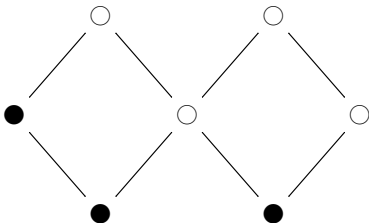
Classical rowmotion: the standard definition

- Let P be a finite poset. **Classical rowmotion** is the map $\mathbf{r} : J(P) \rightarrow J(P)$ which sends **every order ideal** S to a new order ideal $\mathbf{r}(S)$ defined as follows:
 - Invert colors** (i.e., take the complement $P \setminus S$).
 - Boil down to generators** (i.e., take the set M of minimal elements of this complement).
 - Complete downwards** (i.e., take the set J of all $w \in P$ such that there exists an $m \in M$ such that $w \leq m$).

Then, $\mathbf{r}(S) = J$.

Example:

Let S be the following order ideal ($\bullet$ = inside order ideal):



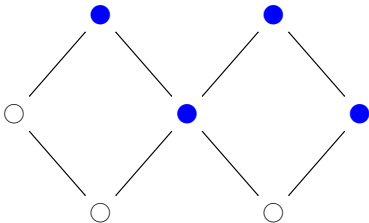
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Example:

Mark the elements of the complement **blue**.



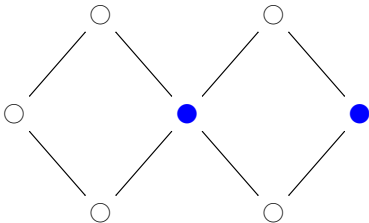
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Example:

Leave only the minimal elements:



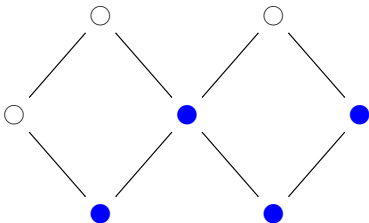
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Then, $\mathbf{r}(S) = J$.

Example:

$\mathbf{r}(S)$ is the order ideal generated by M (“everything below M ”):



Classical rowmotion: properties

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

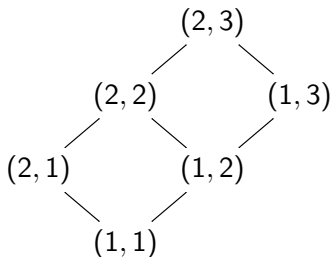
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However, **for some types of P** , the order can be explicitly computed or bounded from above.

See Striker/Williams ([arXiv:1108.1172](#)) for the first generation of results.

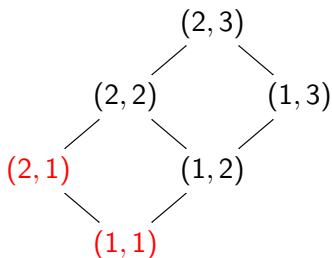
- If P is a $p \times q$ -rectangle:



(shown here for $p = 2$ and $q = 3$), then $\text{ord}(\mathbf{r}) = p + q$.

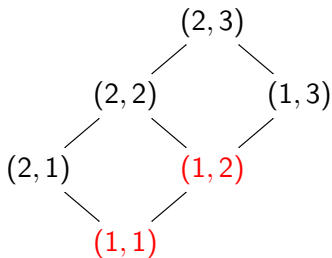
Example:

Let S be the order ideal of the 2×3 -rectangle given by:



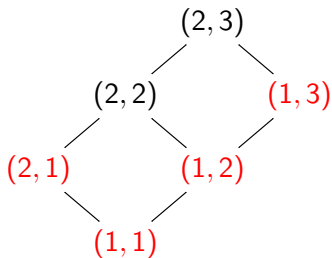
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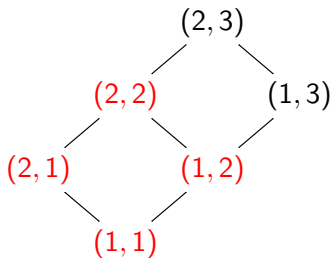
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$r^2(S)$ is



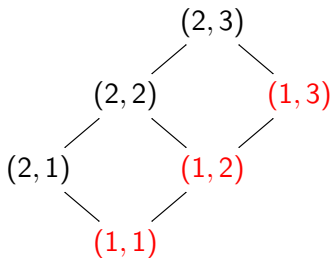
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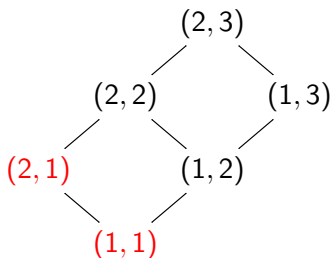
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Example:

$\mathbf{r}^5(S)$ is

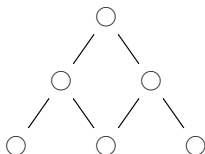


which is precisely the S we started with.

$$\text{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$$

Further posets for which classical rowmotion has small order:

- If P is a Δ -shaped triangle with sidelength $p - 1$:

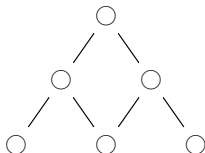


(shown here for $p = 4$), then $\text{ord}(\mathbf{r}) = 2p$ (if $p > 2$).

- In this case, $\mathbf{r}^p$ is “reflection in the y -axis” (i.e., the central vertical axis).

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- More general examples come from finite Weyl groups (Armstrong/Stump/Thomas, [arXiv:1101.1277](#)) and from minuscule weights of classical groups (Rush/Shi, [arXiv:1108.5245](#); Okada, [arXiv:2004.05364](#)).

There is an alternative definition of classical rowmotion, which splits it into many little steps.

- If P is a poset and $v \in P$, then the v -**toggle** is the map $\mathbf{t}_v : J(P) \rightarrow J(P)$ which takes every order ideal S to:
 - $S \cup \{v\}$, if v is not in S but all elements of P covered by v are in S already;
 - $S \setminus \{v\}$, if v is in S but none of the elements of P covering v is in S ;
 - S otherwise.
- Simpler way to state this: $\mathbf{t}_v(S)$ is:
 - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
 - S otherwise.

(“Try to add or remove v from S ; if this breaks the order ideal axiom, leave S fixed.”)

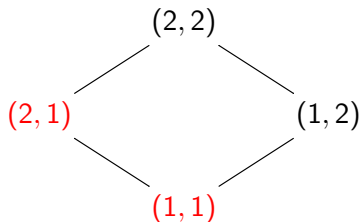
Classical rowmotion: the toggling definition

- Let $(v_1, v_2, \dots, v_n)$ be a **linear extension** of P ; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \dots \circ \mathbf{t}_{v_n}.$$

Example:

Start with this order ideal S :



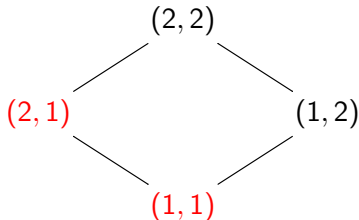
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Example:

First apply $\mathbf{t}_{(2,2)}$, which changes nothing:



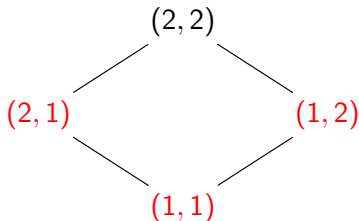
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Then apply $\mathbf{t}_{(1,2)}$, which adds $(1,2)$ to the order ideal:



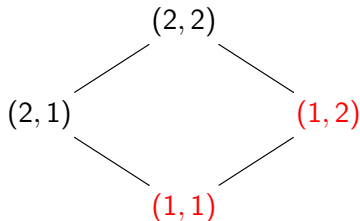
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Then apply $\mathbf{t}_{(2,1)}$, which removes $(2, 1)$ from the order ideal:



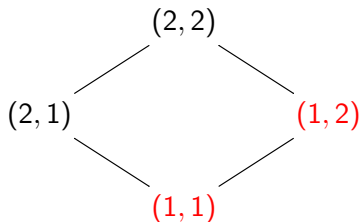
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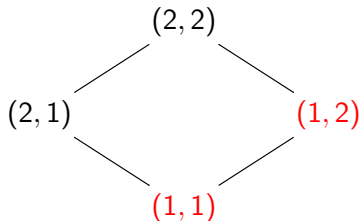
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So this is $\mathbf{r}(S)$:

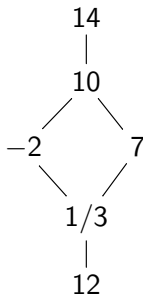


- define **noncommutative birational rowmotion**: a generalization of classical rowmotion on several levels, due to David Einstein, James Propp, Tom Roby and myself, based on ideas of Anatol Kirillov and Arkady Berenstein.
- extend the “order $p + q$ ” theorem for rectangles to this generalization.
- ask some questions.

Noncommutative birational rowmotion: definition

- Let $\mathbb{K}$ be a ring (not necessarily commutative).
- A $\mathbb{K}$ -labelling of P will mean a function $\hat{P} \rightarrow \mathbb{K}$.
- The values of such a function will be called the **labels** of the labelling.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\hat{P}$.

Example: This is a $\mathbb{Q}$ -labelling of the 2×2 -rectangle:



- For any $v \in P$, define the **birational v -toggle** as the partial map $T_v : \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}}$ defined by

$$(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v; \\ \left(\sum_{\substack{u \in \hat{P}; \\ u < v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \hat{P}; \\ u > v}} \overline{f(u)}}, & \text{if } w = v \end{cases}$$

for all $w \in \hat{P}$.

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- Notice that this is a **local change** to the label at v ; all other labels stay the same.
- If $\mathbb{K}$ is commutative, then $T_v^2 = \text{id}$ (on the range of T_v).

- We define **(noncommutative) birational rowmotion** as the partial map

$$R := T_{v_1} \circ T_{v_2} \circ \cdots \circ T_{v_n} : \mathbb{K}^{\hat{P}} \dashrightarrow \mathbb{K}^{\hat{P}},$$

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- This is indeed independent on the linear extension, because:

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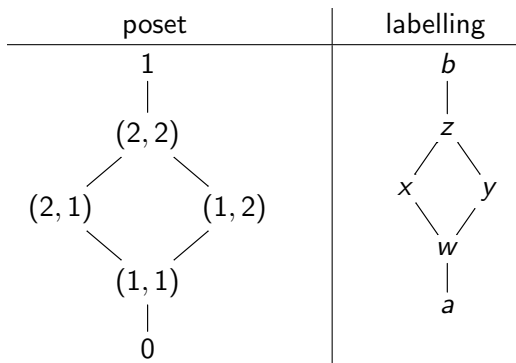
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- This is indeed independent on the linear extension, because:
 - T_v and T_w commute whenever v and w are incomparable (or just don't cover each other);
 - we can get from any linear extension to any other by switching incomparable adjacent elements.

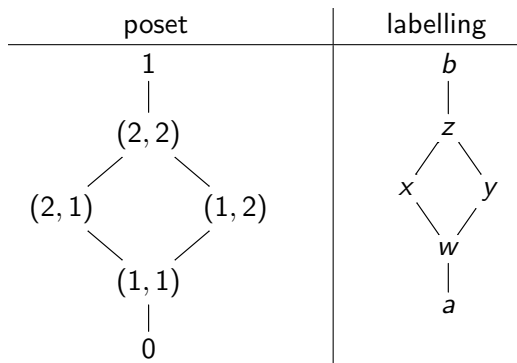
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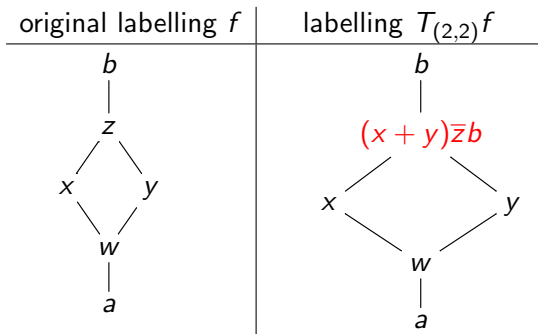


We have $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ (using the linear extension $((1,1), (1,2), (2,1), (2,2))$).

That is, toggle in the order “top, left, right, bottom”.

Example:

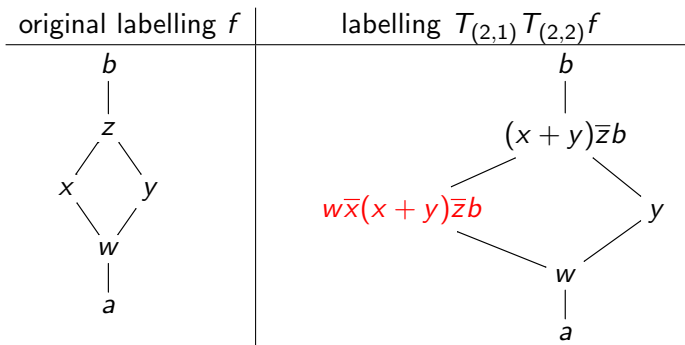
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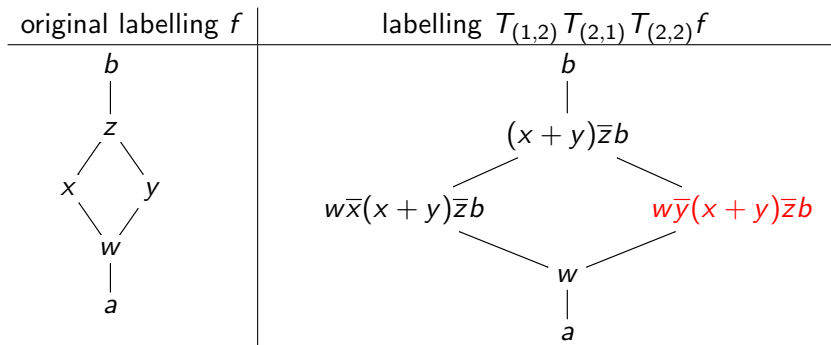
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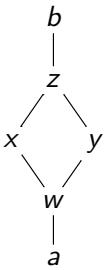
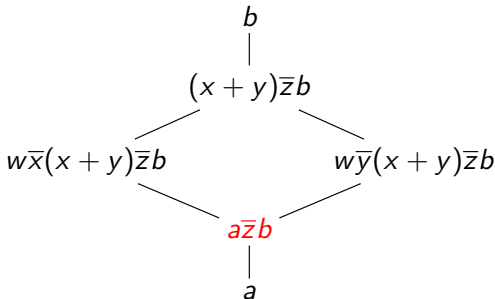
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original labelling f	labelling $T_{(1,1)} T_{(1,2)} T_{(2,1)} T_{(2,2)} f = Rf$
$ \begin{array}{c} b \\ \\ z \\ \swarrow \quad \searrow \\ x \qquad y \\ \swarrow \quad \searrow \\ w \\ \\ a \end{array} $	$ \begin{array}{c} b \\ \\ (x+y)\bar{z}b \\ \swarrow \qquad \searrow \\ w\bar{x}(x+y)\bar{z}b \qquad w\bar{y}(x+y)\bar{z}b \\ \swarrow \qquad \searrow \\ \underline{\underline{a\bar{w} \cdot w\bar{x}(x+y)\bar{z}b + w\bar{y}(x+y)\bar{z}b}} \\ \\ a \end{array} $

We are using $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$.

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We have used $R = T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$ and simplified the result.

- Why is this called birational rowmotion?
- Indeed, it generalizes classical rowmotion of order ideals:
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- Don't like semifields? Use $\mathbb{Q}$ and take the “tropical limit”.

- If $\mathbb{K}$ is commutative, then birational rowmotion R has nice orders for nice posets (mostly Grinberg/Roby 2014):
 - If P is a rectangle $[p] \times [q]$, then $R^{p+q} = \text{id}$.

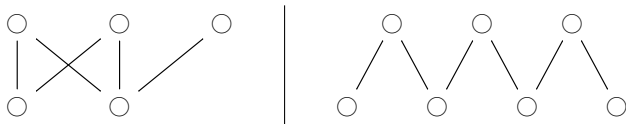
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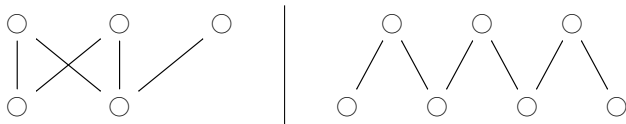
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 - More generally, if P is the minuscule poset associated to a minuscule weight λ of a finite-dimensional simple Lie algebra $\mathfrak{g}$, then $R^h = \text{id}$, where h is the Coxeter number of $\mathfrak{g}$. ([Soichi Okada, doi:10.37236/9557](#) .)

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 - If P is an “ n -graded forest” (a forest with all leaves having rank n), then $R^\ell = \text{id}$ for $\ell = \text{lcm}(1, 2, \dots, n+1)$.

- In general, even if $\mathbb{K}$ is commutative, R can have infinite order
 - e.g., for the following two posets:



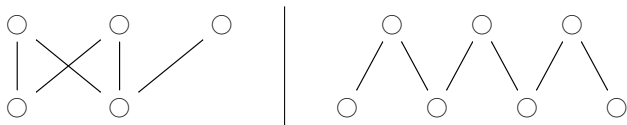
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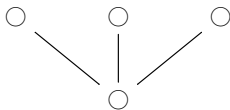
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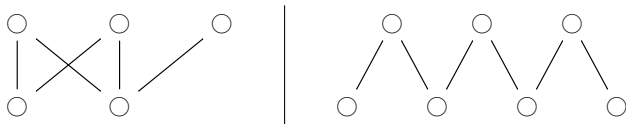
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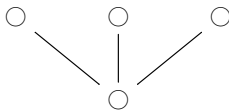
This satisfies $R^6 = \text{id}$ if $\mathbb{K}$ is commutative, but nothing like that in general.

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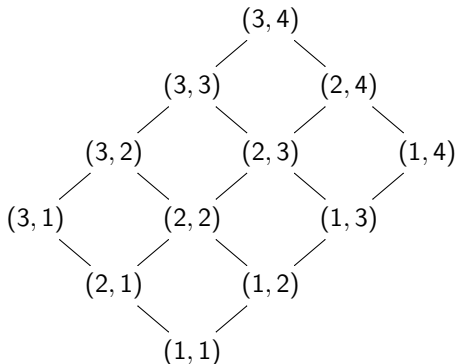
- However, not all is lost!

- Let p and q be two positive integers. Let $\mathbb{K}$ be a ring. Let P be the $p \times q$ -rectangle poset: i.e.,

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Example: For $p = 3$ and $q = 4$, this is



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Periodicity theorem (* 2015, † 2022 G & Roby):

If a and b are invertible and $R^{p+q}f$ is well-defined, then

$$(R^{p+q}f)(x) = a\bar{b} \cdot f(x) \cdot \bar{a}b \quad \text{for each } x \in \hat{P}.$$

Note that $a\bar{b} \cdot f(x) \cdot \bar{a}b$ is **not** generally conjugate to $f(x)$.

Birational rowmotion: the rectangle case

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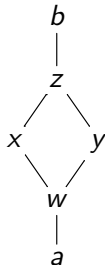
Reciprocity theorem (* 2015, † 2022 G & Roby):

Let $\ell \in \mathbb{N}$. Let $(i, j) \in P$. If $R^\ell f$ is well-defined and $\ell \geq i + j - 1$, then

$$(R^\ell f)(i, j) = a \cdot \underbrace{(R^{\ell-i-j+1}f)(p+1-i, q+1-j)}_{=\text{antipode of } (i,j) \text{ in } P} \cdot b.$$

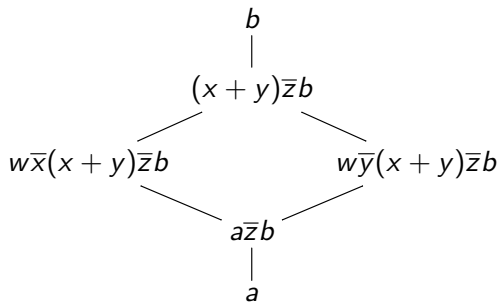
- **Example:** Iteratively apply R to a labelling of the 2×2 -rectangle.

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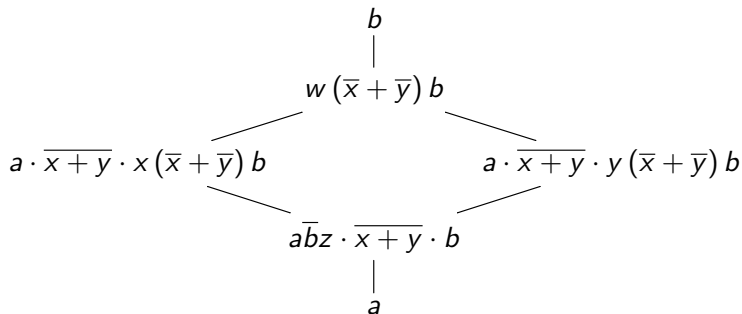
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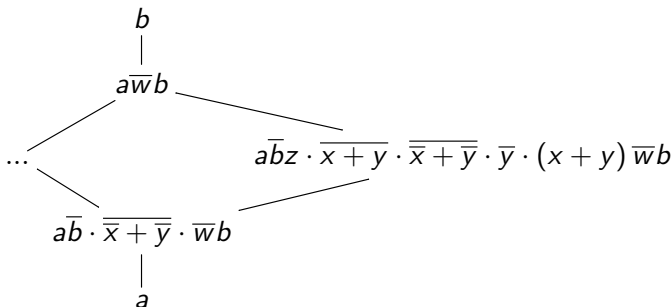
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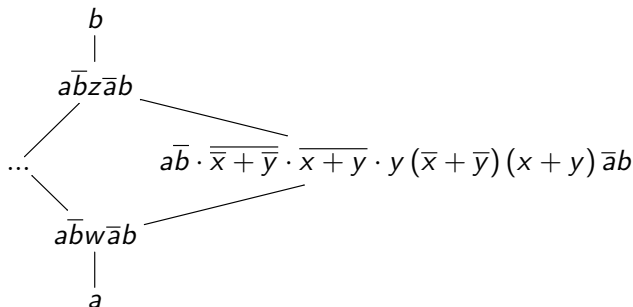
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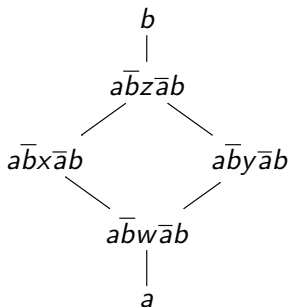
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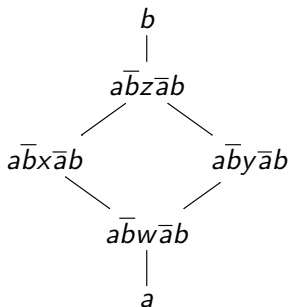
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(after nontrivial simplifications).

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This confirms the periodicity theorem for $p = q = 2$.

- Note that this is similar to Kontsevich's periodicity conjecture, proved by Iyudu/Shkarin ([arXiv:1305.1965](#)).

Birational rowmotion: the rectangle case, example

- Here are $R^0 f, R^1 f, \dots, R^4 f$ for a generic $f \in \widehat{\mathbb{K}^{[2] \times [2]}}$ again, this time fully simplified and with the $f(0) = a$ and $f(1) = b$ labels removed:

$$R^0 f = \begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array} ; \quad R^1 f = \begin{array}{ccccc} & & (x+y)\bar{z}b & & \\ & & / & \backslash & \\ w\bar{x}(x+y)\bar{z}b & & & & w\bar{y}(x+y)\bar{z}b \\ & & \backslash & / & \\ & & a\bar{z}b & & \end{array}$$

$$R^2 f = \begin{array}{ccccc} & & w(\bar{x} + \bar{y})b & & \\ & & / & \backslash & \\ a\bar{y}b & & & & a\bar{x}b \\ & & \backslash & / & \\ & & a\bar{b}z\bar{x} + \bar{y}b & & \end{array} ; \quad R^3 f = \begin{array}{ccccc} & & a\bar{w}b & & \\ & & / & \backslash & \\ a\bar{b}z\bar{x} + \bar{y}y\bar{w}b & & & & a\bar{b}z\bar{x} + \bar{y}x\bar{w}b \\ & & \backslash & / & \\ & & a\bar{b} \cdot \bar{x} + \bar{y} \cdot \bar{w}b & & \end{array}$$

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Equally colored labels are related by reciprocity. Can you spot some more?

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Here are some more instances of reciprocity. (There are more.)

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Explicitly, if $A \in \mathbb{K}^{p \times (p+q)}$ is any matrix, then
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for all $(i, j) \in P$, where the $A[a : b \mid c : d]$ s are certain submatrices of A . (Note that this map Grasp_0 actually factors through the Grassmannian.)

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- Reciprocity also easy using Grasp_0 .

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 - Can we WLOG assume that $\mathbb{K}$ is a skew field?
No: e.g., the identity $x\overline{xy}y = 1$ holds in all skew fields but not in all rings.

First attempts at general proof

- This looks easy; the devil is in the details (particularly the “almost all” part: not just Zariski density but also some rescaling required).
 - Can this be generalized to arbitrary $\mathbb{K}$?
 - **In some sense, yes:** Replace determinants by quasideterminants (Gelfand/Retakh, [arXiv:q-alg/9705026](#); see also [arXiv:math/0208146](#)).
- Specifically, redefine Grasp_0 by

$$(\text{Grasp}_0 A)(i, j) = (-1)^i q_{0, i+j-1}^{\{1:i|i+j:p+j\}}(A).$$

The “algebra” works!

- Unfortunately, the technical parts no longer work:
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No: e.g., the identity $x\overline{xy}y = 1$ holds in all skew fields but not in all rings.
- We now believe this approach is a dead end.

- New proofs of periodicity and reciprocity in the commutative- $\mathbb{K}$ case were found by Gregg Musiker and Tom Roby in [arXiv:1801.03877](https://arxiv.org/abs/1801.03877).

They proceed by giving an explicit formula for $(R^k f)(i, j)$.
For instance, $(R^3 f)(3, 2)$

$$= \frac{1}{A_{02} + A_{11} + A_{20}} (A_{01}A_{02}A_{11}A_{12} + A_{01}A_{02}A_{12}A_{20} + A_{01}A_{02}A_{20}A_{21} \\ + A_{02}A_{10}A_{12}A_{20} + A_{02}A_{10}A_{20}A_{21} + A_{10}A_{11}A_{20}A_{21}),$$

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- General formula for $(R^k f)(i, j)$ involves sums over NILPs (non-intersecting lattice path families) in numerator and denominator, as well as index shifting and a case split (“small” k and “large” k behave differently).
- Lattice paths can be generalized to noncommutative $\mathbb{K}$, but NILPs? Unclear in what order to multiply different paths.

- We are back at square 1: no known theory available.

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- Let's play around with the setting.
Step 1: Introduce notations...

A new beginning

- Fix p , q , P and f . Assume that $R^\ell f$ is well-defined for all necessary ℓ . Let $a = f(0)$ and $b = f(1)$.

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- For any $x \in \widehat{P}$ and $\ell \in \mathbb{N}$, write

$$x_\ell := \left(R^\ell f\right)(x).$$

Thus, $x_0 = f(x)$ and $0_\ell = a$ and $1_\ell = b$.

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- The definition of R yields

$$(Rf)(v) = \left(\sum_{u < v} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{u > v} \overline{(Rf)(u)}} \quad \text{for each } v \in P.$$

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- In other words,

$$v_1 = \left(\sum_{u < v} u_0 \right) \cdot \overline{v_0} \cdot \overline{\sum_{u > v} u_1} \quad \text{for each } v \in P.$$

- We have just shown that

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- So far, we have just rewritten our setup using the (more convenient) $x_\ell := (R^\ell f)(x)$ notation.

Simplifying the goal

- We must prove:

periodicity: $x_{p+q} = \bar{a}\bar{b} \cdot x_0 \cdot \bar{a}b;$

reciprocity: $x_\ell = a \cdot \overline{y_{\ell-i-j+1}} \cdot b$

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- Periodicity follows from reciprocity: Indeed, if $x = (i, j)$ and $x' = (p + 1 - i, q + 1 - j)$, then

$$\begin{aligned}x_{p+q} &= a \cdot \overline{x'_{p+q-i-j+1}} \cdot b && \text{(by reciprocity)} \\&= a \cdot \overline{a \cdot \overline{x_0} \cdot b} \cdot b && \text{(by reciprocity again)} \\&= \overline{ab} \cdot x_0 \cdot \overline{ab}.\end{aligned}$$

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- Moreover, reciprocity in general follows from reciprocity for $\ell = i + j - 1$ (just apply it to $R^k f$ instead of f otherwise).

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$$A_\ell^v := v_\ell \cdot \overline{\sum_{u \prec v} u_\ell} \quad \text{and} \quad \forall_\ell^v := \overline{\sum_{u \succ v} u_\ell} \cdot \overline{v_\ell}.$$

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- If u and v are elements of $\widehat{P}$, set

$$A_\ell^{u \rightarrow v} := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A_\ell^{\mathbf{p}} \quad \text{and}$$

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- **Path formulas:**

(a) We have

$$u_\ell = \overline{\forall_\ell^{1 \rightarrow u}} \cdot b \quad \text{for each } u \in P.$$

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Prove this by downwards induction on u .

Induction step: Given $v \in P$ such that $\mathcal{V}^{1 \rightarrow u} = b\overline{u_\ell}$ for all $u \succ v$. Since any path $1 \rightarrow v$ passes through a unique $u \succ v$, we have

$$\begin{aligned} \mathcal{V}^{1 \rightarrow v} &= \sum_{u \succ v} \mathcal{V}^{1 \rightarrow u} \mathcal{V}^v = \sum_{u \succ v} b\overline{u_\ell} \mathcal{V}^v && \text{(by induction hypothesis)} \\ &= b\overline{v_\ell} && \text{(by definition of } \mathcal{V}^v), \quad \text{qed.} \end{aligned}$$

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- (b) Analogous, but use upwards induction instead.

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$$u_\ell = A_\ell^{u \rightarrow (1,1)} \cdot a \quad \text{for each } u \in P.$$

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- *Proof idea:* Each path $1 \rightarrow u$ begins with the step $1 \succ (p, q)$. Thus, $V_\ell^{1 \rightarrow u} = V_\ell^{(p,q) \rightarrow u}$ (since $V_\ell^1 = 1$). Hence, (c) follows from (a).

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Similarly, (d) follows from (b).

- **Transition equation in $A\forall$ -form:**

$$\forall_{\ell+1}^v = A_\ell^v \quad \text{for each } v \in \hat{P} \text{ and } \ell \in \mathbb{N}.$$

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$$v_{\ell+1} = \left(\sum_{u \leq v} u_\ell \right) \cdot \overline{v_\ell} \cdot \sum_{u \geq v} \overline{u_{\ell+1}}.$$

Take reciprocals on both sides, multiply by $\sum_{u \geq v} \overline{u_{\ell+1}}$ and rewrite using $\forall_{\ell+1}^v$ and A_ℓ^v .

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- **Corollary of the transition equation:**

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Hence, $\forall_{\ell+1}^{u \rightarrow v} = A_\ell^{u \rightarrow v}$ for any $u, v \in \widehat{P}$.

- Now, for the bottommost element $(1, 1)$ of P , we have

$$\begin{aligned}(1, 1)_1 &= \overline{V_1^{(p,q) \rightarrow (1,1)}} \cdot b && \text{(by path formula \textbf{(c)})} \\ &= \overline{A_0^{(p,q) \rightarrow (1,1)}} \cdot b && \text{(since } V_{\ell+1}^{u \rightarrow v} = A_\ell^{u \rightarrow v} \text{)} \\ &= a \cdot \overline{(p, q)_0} \cdot b && \text{(by path formula \textbf{(d)})} .\end{aligned}$$

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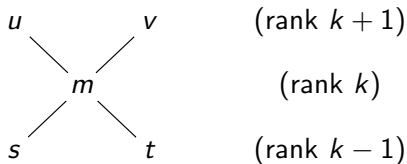
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- What now?

The case $j = 1$ suffices: part 1

- We can simplify our goal one bit further. Consider the “neighborhood” of an element of our rectangle P :



(where the **rank** of an $(i, j) \in P$ is defined to be $i + j - 1$).

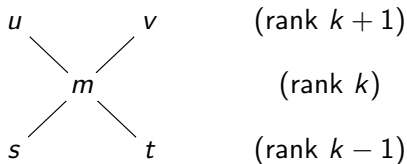
Say we have shown (our “induction hypotheses”) that reciprocity holds for each of s, t, m, u ; that is, we have

$$\begin{aligned}
 s_\ell &= a \cdot \overline{s'_{\ell-(k-1)}} \cdot b, & t_\ell &= a \cdot \overline{t'_{\ell-(k-1)}} \cdot b, \\
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for all sufficiently high ℓ , where x' denotes the antipode of x (that is, if $x = (i, j)$, then $x' = (p + 1 - i, q + 1 - j)$).

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Claim: Then, reciprocity also holds for v ; that is, we have $v_\ell = a \cdot \overline{v'_{\ell-(k+1)}} \cdot b$ for all $\ell \geq k + 1$.

The case $j = 1$ suffices: part 2

- *Proof idea.* Fix $\ell \geq k + 1$, and compare the transition equations

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{u_\ell + v_\ell} \quad \text{and}$$

$$m'_{\ell-k} = (u'_{\ell-k-1} + v'_{\ell-k-1}) \cdot \overline{m'_{\ell-k-1}} \cdot \overline{s'_{\ell-k} + t'_{\ell-k}}$$

using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b$,

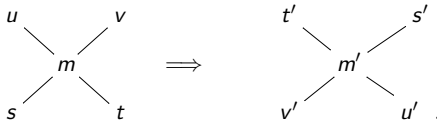
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The case $j = 1$ suffices: part 2

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using the induction hypotheses $m_\ell = a \cdot \overline{m'_{\ell-k}} \cdot b$,

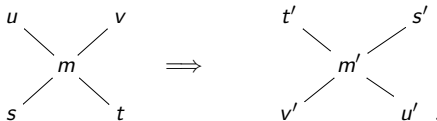
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noting that



After subtracting $u_\ell = a \cdot \overline{u'_{\ell-(k+1)}} \cdot b$, out comes

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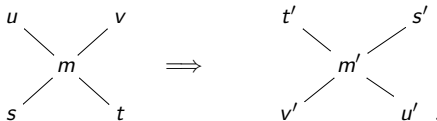
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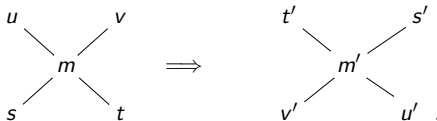
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- This argument still works if s , t or u does not exist.
- Thus, in order to prove reciprocity for all (i, j) , it suffices (by induction) to prove it in the case when $j = 1$.

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Note the lack of rowmotion in this formula! The ℓ here is constantly 1, so it is a property of a single labeling. Thus, we drop the subscripts.

- **Our new goal:** Prove that

$$A^{(p,q) \rightarrow (2,1)} = V^{(p-1,q) \rightarrow (1,1)}.$$

The conversion lemma

- More generally:

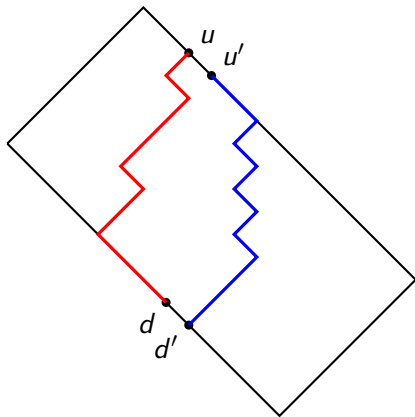
- **Conversion lemma:**

Let u and u' be two adjacent elements on the top-right edge of P (that is, $u = (k, q)$ and $u' = (k - 1, q)$). Let d and d' be two adjacent elements on the bottom-left edge of P (that is, $d = (i, 1)$ and $d' = (i - 1, 1)$). Then,

$$A_\ell^{u \rightarrow d} = \mathcal{V}_\ell^{u' \rightarrow d'} \quad \text{for each } \ell \in \mathbb{N}.$$

In short:

$$A^{u \rightarrow d} = \mathcal{V}^{u' \rightarrow d'}.$$



- If we can prove the conversion lemma, we will obtain reciprocity not only for $(i, j) = (2, 1)$, but also for all (i, j) on the bottom-left edge of P (that is, for the entire case $j = 1$), because we can argue as follows:

$$\begin{aligned}
 (i, 1)_i &= \overline{\mathcal{V}_i^{(p,q) \rightarrow (i,1)}} \cdot b \\
 &= \overline{A_{i-1}^{(p,q) \rightarrow (i,1)}} \cdot b \\
 &= \overline{\mathcal{V}_{i-1}^{(p-1,q) \rightarrow (i-1,1)}} \cdot b \\
 &= \overline{A_{i-2}^{(p-1,q) \rightarrow (i-1,1)}} \cdot b \\
 &= \overline{\mathcal{V}_{i-2}^{(p-2,q) \rightarrow (i-2,1)}} \cdot b \\
 &= \dots \\
 &= \overline{\mathcal{V}_1^{(p-i+1,q) \rightarrow (1,1)}} \cdot b \\
 &= \overline{A_0^{(p-i+1,q) \rightarrow (1,1)}} \cdot b \\
 &= a \cdot \overline{(p-i+1, q)_0} \cdot b
 \end{aligned}$$

(by path formula **(c)**)

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$$(i, 1)_\ell = a \cdot \overline{(p - i + 1, q)_{\ell-i}} \cdot b$$

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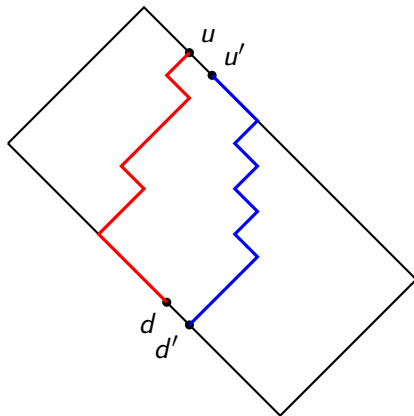
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- Thus, we only need to prove the conversion lemma. We can now drop all subscripts forever!

Proving the conversion lemma: the intuition

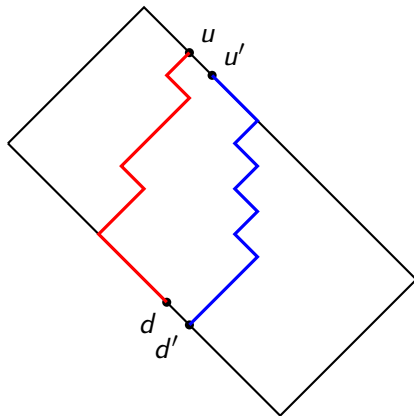
- Let us again look at the picture:



We must prove $A^{u \rightarrow d} = \forall^{u' \rightarrow d'}$.

Proving the conversion lemma: the intuition

- Let us again look at the picture:



We must prove $A^{u \rightarrow d} = \forall u' \rightarrow d'$.

- How do we interpolate between paths $u \rightarrow d$ and paths $u' \rightarrow d'$?

- We define a **path-jump-path** to be a sequence

$$\mathbf{p} = (v_0 \succ v_1 \succ \cdots \succ v_i \blacktriangleright v_{i+1} \succ v_{i+2} \succ \cdots \succ v_k)$$

of elements of P , where the relation $x \blacktriangleright y$ means “ y is one step down and some steps to the right of x ” (that is, if $x = (r, s)$, then $y = (r - k, s + k - 1)$ for some $k > 0$).

We say that this path-jump-path $\mathbf{p}$ has **jump at i** .

Proving the conversion lemma: path-jump-paths

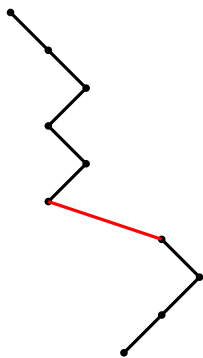
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Example of a path-jump-path:



(The red edge is the jump.)

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For any such path-jump-path $\mathbf{p}$, we set

$$E_{\mathbf{p}} := A^{v_0} A^{v_1} \cdots A^{v_{i-1}} v_i \overline{v_{i+1}} \mathcal{V}^{v_{i+2}} \mathcal{V}^{v_{i+3}} \cdots \mathcal{V}^{v_k}.$$

(Here, we are omitting the ℓ subscripts – so v_i means $(v_i)_{\ell}$ and v_{i+1} means $(v_{i+1})_{\ell}$.)

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- Now, if $k = \text{rank } u - \text{rank } (d')$, then

$$A^{u \rightarrow d} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \rightarrow d' \\ \text{with jump at } k-1}} E_{\mathbf{p}},$$

since $A^d = d \overline{d'}$, and similarly

$$\forall^{u' \rightarrow d'} = \sum_{\substack{\mathbf{p} \text{ is a path-jump-path } u \rightarrow d' \\ \text{with jump at } 0}} E_{\mathbf{p}}.$$

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- So we need to show that

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- This is indeed true and can be proved by a “local” argument (rewriting two consecutive steps of the path).
- This is similar to the “zipper argument” in lattice models. (Is there a Yang–Baxter equation lurking?)

Proving the conversion lemma: the civilized version, part 1

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- Modulo the details omitted, this finishes the proof of the reciprocity theorem.
- However, the path-jump-path argument is somewhat messy. We can make it slicker by rewriting it in matrix notation:
- Define three $P \times P$ -matrices $\mathbf{A}$, $\mathbf{V}$ and $\mathbf{U}$ by

$$\begin{aligned}\mathbf{A}_{x,y} &:= A^x [x \succ y], & \mathbf{V}_{x,y} &:= V^y [x \succ y], \\ \mathbf{U}_{x,y} &:= x\bar{y} [x \blacktriangleright y] & & \text{for all } x, y \in P.\end{aligned}$$

Here, $[\mathcal{A}]$ is the Iverson bracket (i.e., truth value) of a statement $\mathcal{A}$; the relation $x \blacktriangleright y$ means “ y is one step down and some steps to the right of x ” as before. And again, we are omitting the ℓ subscripts, so $x\bar{y}$ actually means $x_\ell \bar{y}_\ell$.

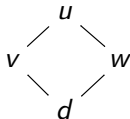
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Indeed, this follows easily from the following neat lemma: If



are four adjacent elements of P , then

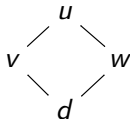
$$\overline{w} \cdot \forall^d \cdot d = \overline{u} \cdot A^u \cdot v \quad \text{and} \quad \overline{v} \cdot \forall^d \cdot d = \overline{u} \cdot A^u \cdot w.$$

(The u and d here are unrelated to the u and d from the conversion lemma!)

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- From $\mathbf{AU} = \mathbf{UV}$, we easily obtain

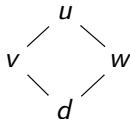
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- Setting $k = \text{rank } u - \text{rank } d$ and comparing the (u, d') -entries of both sides, we quickly obtain $A^{u \rightarrow d} = \forall^{u' \rightarrow d'}$ (since $x \blacktriangleright d'$ holds only for $x = d$, and since $u \blacktriangleright x$ holds only for $x = u'$). This proves the conversion lemma again.

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In the **commutative** case, the theorems hold for semifields (and, more generally, commutative semirings) because they hold for fields and because they are “essentially” polynomial identities (once you clear denominators).

Is that all? Part 1: Semirings

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This **fails** for noncommutative $\mathbb{K}$!

- **Scary example** ([David Speyer, MathOverflow #401273](#)): If x and y are two elements of a ring such that $x + y$ is invertible, then

$$x \cdot \overline{x + y} \cdot y = y \cdot \overline{x + y} \cdot x.$$

But this is not true if “ring” is replaced by “semiring”!

- Thus, we are left with a

Question:

Are the periodicity and reciprocity theorems still true if “ring” is replaced by “semiring”? (I.e., we no longer require $\mathbb{K}$ to have a subtraction.)

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- Note that the main hurdle is the argument that reduced the general case to the $j = 1$ case. That argument used subtraction!
- We have partial results, e.g., for $p = q = 3$ and for $p = 2$.

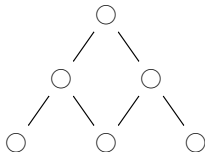
- Other posets remain to be studied.

Conjecture:

Let P be the triangle-shaped poset $\Delta(p)$ or its reflection $\nabla(p)$. Let $f \in \mathbb{K}^{\hat{P}}$ be a labelling such that $R^P f$ exists. Let $a = f(0)$ and $b = f(1)$. Then, for each $x \in \hat{P}$, we have

$$(R^P f)(x) = \bar{a}\bar{b} \cdot f(x') \cdot \bar{a}b,$$

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Is that all? Part 3: Other posets

- Other posets remain to be studied.

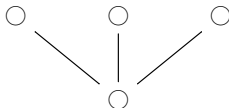
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- We have a similar conjecture for other kinds of triangles and (still unproved even in the commutative case!) for trapezoids.
- As already mentioned, other simple posets such as



do not have periodic behavior for noncommutative $\mathbb{K}$.

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Question:

What other results like ours are known in the noncommutative case?

- **Tom Roby**: collaboration
- **Mathematisches Forschungsinstitut Oberwolfach**: hospitality in July/August 2021
- **Banff International Research Station**: 2021 conference where this was first presented
- **Anna Pun**: invitation
- **Michael Joseph, Tim Champion, Max Glick, Maxim Kontsevich, Gregg Musiker, Pace Nielsen, James Propp, Pasha Pylyavskyy, Bruce Sagan, Roland Speicher, David Speyer, Hugh Thomas, and Jurij Volcic**: discussions
- **Sage and Sage-combinat**: computations
- **the birational combinatorics community**: keeping the subject exciting since 2013
- **you**: your patience

Some references

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- **Zamolodchikov periodicity conjecture in type AA** (proved by A. Yu. Volkov, [arXiv:hep-th/0606094v1](https://arxiv.org/abs/hep-th/0606094v1)): Let r and s be positive integers. Let $Y_{i,j,k}$ be elements of a commutative ring for $i \in [r]$ and $j \in [s]$ and $k \in \mathbb{Z}$. Assume that

$$Y_{i,j,k+1} Y_{i,j,k-1} = \frac{(1 + Y_{i+1,j,k})(1 + Y_{i-1,j,k})}{(1 + 1/Y_{i,j+1,k})(1 + 1/Y_{i,j-1,k})}$$

for all i, j, k , where sums involving “off-grid” points (e.g., $1 + Y_{0,j,k}$) are understood as 1.

Then, $Y_{i,j,k+2(r+s+2)} = Y_{i,j,k}$ for all i, j, k .

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Then, $Y_{i,j,k+2(r+s+2)} = Y_{i,j,k}$ for all i, j, k .

- **Observation (Max Glick and others, ca. 2015?):** This is equivalent to periodicity of birational rowmotion ($R^{p+q} = 1$) for $[p] \times [q]$, where $p = r + 1$ and $q = s + 1$, when the ring is commutative. Explicitly,

$$Y_{i,j,i+j-2k} = (R^k f)(i, j+1) / (R^k f)(i+1, j).$$

(Fine points omitted.)

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- **Disappointment:** Zamolodchikov periodicity does not generalize to noncommutative rings (no matter how we order the five factors).

- A recent preprint by Joseph Johnson and Ricky Ini Liu (*Birational rowmotion and the octahedron recurrence*, [arXiv:2204.04255](https://arxiv.org/abs/2204.04255)) reproves the “order $p + q$ ” theorem for commutative $\mathbb{K}$ in a simpler way (besides doing a number of other interesting things).

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- Lemma 4.1 in the Johnson-Liu preprint generalizes our conversion lemma in the commutative case from single paths to k -tuples of nonintersecting paths. We don't know how this could be done in the noncommutative case; it is unclear in what order to multiply labels from different paths.

Proposition (2022, G & Roby):

Let P be any finite poset. Let $f \in \mathbb{K}^{\hat{P}}$. Then,

$$f(1) \cdot \sum_{\substack{u \in \hat{P}; \\ u > 0}} \overline{(Rf)(u)} \cdot f(0) = \sum_{\substack{u \in \hat{P}; \\ u < 1}} f(u),$$

assuming that the inverses $\overline{(Rf)(u)}$ are well-defined.

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Corollary (2022+, G & Roby):

Let P be any finite poset. Let $f \in \mathbb{K}^{\hat{P}}$ with $f(0) = f(1) = 1$. Then, the quantity

$$\sum_{\substack{u, v \in \hat{P}; \\ u < v}} f(u) \cdot \overline{f(v)}$$

is unchanged under birational rowmotion (i.e., when we replace f by Rf).