The Redei–Berge symmetric function of a directed graph [talk slides]

Darij Grinberg
joint work with Richard P. Stanley

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In 1934, Laszlo Redei observed a peculiar property of tournaments (directed graphs that have an arc between every pair of distinct vertices): Each tournament has an odd number of Hamiltonian paths. In 1996, Chow introduced the "path-cycle symmetric function" of a directed graph, a symmetric function in two sets of arguments, which was later used in rook theory. We study Chow's symmetric function in the case when the y-variables are 0. In this case, we give new nontrivial expansions of the function in terms of the power-sum basis; in particular, we find that it is p-positive as long as the directed graph has no 2-cycles. We use our expansions to reprove Redei's theorem and refine it to a mod-4 congruence.

This is joint work with Richard P. Stanley.

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Preprint:

- Darij Grinberg and Richard P. Stanley, *The Redei–Berge symmetric function of a directed graph*, arXiv:2307.05569.

Slides of this talk:

- `https://www.cip.ifi.lmu.de/~grinberg/algebra/ipac2023a.pdf`

1. Digraphs and tournaments

- **Definition.** A **digraph** (short for "directed graph") means a pair (V, A) of a finite set V and a subset $A \subseteq V \times V$.

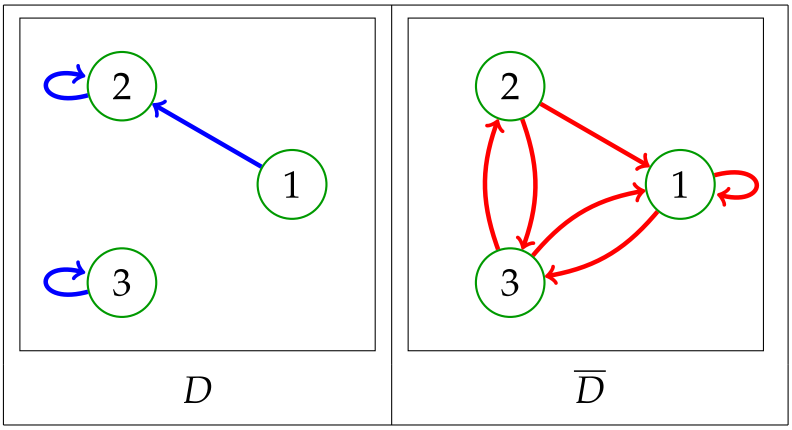
  The elements $(u, v) \in A$ are called **arcs** of this digraph, and are drawn accordingly.

  We allow loops ($(u, u) \in A$) and antiparallel arcs ($(u, v) \in A$ and $(v, u) \in A$) but not parallel arcs (A is not a multiset).

- **Definition.** Let $D = (V, A)$ be a digraph. Then, $\overline{D}$ denotes the **complement** of D; this is the digraph $(V, (V \times V) \setminus A)$. Its arcs are the **non-arcs** of D.

- **Example.**

$$\text{If } D = (\{1,2,3\}, \{(1,2), (2,2), (3,3)\}),$$
$$\text{then } \overline{D} = (\{1,2,3\}, \{(1,1), (1,3), (2,1), (2,3), (3,1), (3,2)\}).$$

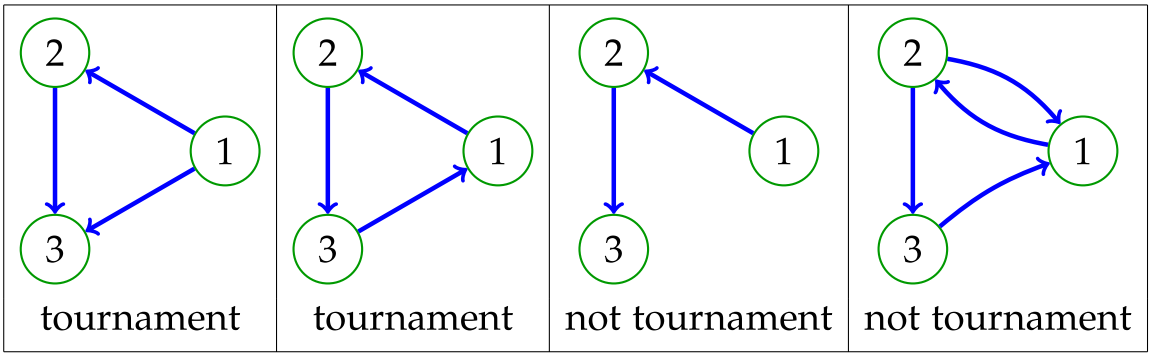


- **Definition.** A digraph $D = (V, A)$ is **loopless** if it has no loops (i.e., no arcs (u, u)).

- **Definition.** A loopless digraph $D = (V, A)$ is a **tournament** if it has the following property: For any distinct $u, v \in V$, exactly one of the two pairs (u, v) and (v, u) is an arc of D.

  In other words, a tournament is an orientation of the complete undirected graph K_V.

- **Examples.**


2
1
3
tournament
2
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tournament
2
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2
1
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not tournament

2. Hamiltonian paths and Rédei's and Berge's theorems

- **Definition.** Let V be a finite set. A V**-listing** will mean a list of elements of V that contains each element of V exactly once.

- **Definition.** Let $D = (V, A)$ be a digraph. A **Hamiltonian path** (short: **hamp**) of D means a V-listing $(v_1, v_2, \ldots, v_n)$ such that

$$(v_i, v_{i+1}) \in A \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}.$$

  In other words (for $V \neq \varnothing$), it means a path of D that contains each vertex.

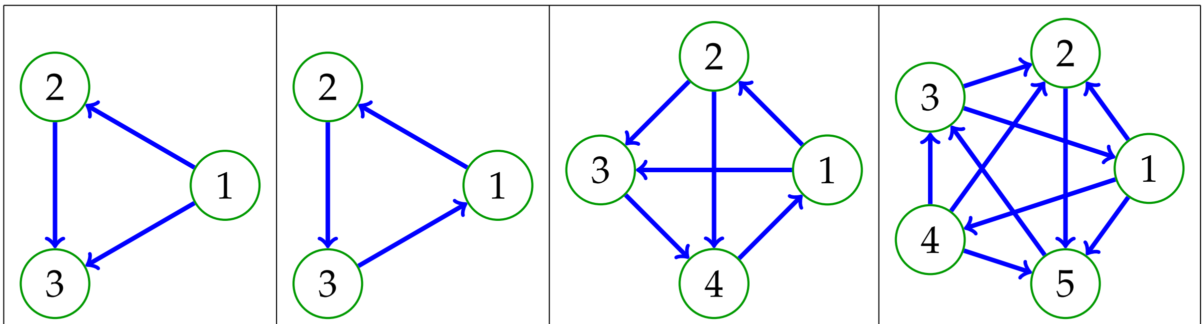
- **Easy proposition (Rédei 1933):** Any tournament has a hamp.

- This is an easy exercise in graph theory. But Rédei proved a lot more:

- **Theorem (Rédei 1933):** Let D be a tournament. Then,

$$(\text{\# of hamps of } D) \text{ is odd.}$$

- **Example.** Here are some tournaments:



has 1 hamp: $(1, 2, 3)$	has 3 hamps: e.g., $(2, 3, 1)$	has 5 hamps: e.g., $(1, 2, 3, 4)$	has 9 hamps: e.g., $(1, 4, 2, 5, 3)$

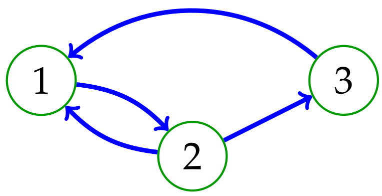
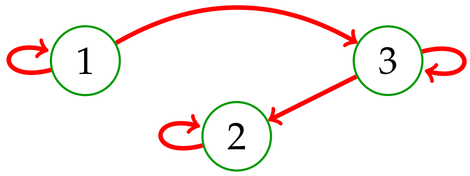
- Rédei's proof is complicated and intransparent (see Moon, *Topics on Tournaments* for an English version).

  To give a more conceptual proof, Berge discovered the following:

- **Theorem (Berge 1976):** Let D be a digraph. Then,

$$(\text{\# of hamps of } \overline{D}) \equiv (\text{\# of hamps of } D) \bmod 2.$$

- **Example.**

 D	 $\overline{D}$
has 3 hamps	has 1 hamp

(loops don't actually matter, but I draw them to be fully correct).

- Berge proves his theorem (in his *Graphs* textbook) using an elegant inclusion-exclusion argument.

  Then he uses his theorem to prove Rédei's theorem via induction on the number of "inversions" (arcs directed the "wrong way").

  This proof is much cleaner than Rédei's, but still far from simple.

  For a detailed exposition, see `https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf` .

- **Remark.** Can we improve on Rédei's theorem even further?

  MathOverflow question #232751 asks for the possible values of (# of hamps of D) for a tournament D.

  Among the numbers between 1 and 80555, the answer is "all odd numbers except for 7 and 21" (proved by bof and Gordon Royle).

  Question: Are these the only exceptions?

3. The Rédei–Berge symmetric function

- Independently, Chow (*The Path-Cycle Symmetric Function of a Digraph*, 1996) introduced a symmetric function assigned to each digraph D.

  (This was inspired by Chung/Graham's cover polynomial in rook theory.)

- We only discuss a coarsening of his construction (Chow has two families of variables, and we set the second family to 0).

  Question: Which of the results below can be generalized to the full version?

- **Definition.** Let $n \in \mathbb{N}$, and let I be a subset of $\{1, 2, \ldots, n-1\}$. Then, we define the power series

  $$L_{I,n} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$$

  (where the summation indices $i_1, i_2, \ldots, i_n$ range over $\{1, 2, 3, \ldots\}$).

  Remark: This is a **(Gessel's) fundamental quasisymmetric function**.

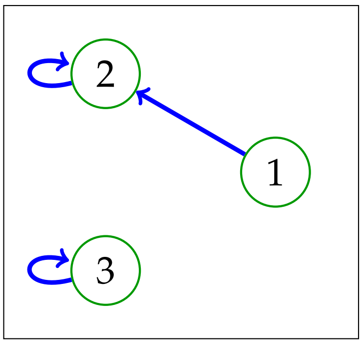
- **Definition.** Let $n \in \mathbb{N}$. Let $D = (V, A)$ be a digraph with n vertices. We define the **Redei–Berge symmetric function**

  $$U_D := \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]],$$

  where

  $$\operatorname{Des}(w, D) := \{i \in \{1, 2, \ldots, n-1\} \mid (w_i, w_{i+1}) \in A\} \quad \text{for each } V\text{-listing } w = (w_1, w_2, \ldots, w_n).$$

- **Example:** Let

  $D =$ 

  .

Then,

$$\begin{aligned}
U_D &= \sum_{w \text{ is a } V\text{-listing}} L_{\operatorname{Des}(w,D),\ 3} \\
&= L_{\operatorname{Des}((1,2,3),D),\ 3} + L_{\operatorname{Des}((1,3,2),D),\ 3} + L_{\operatorname{Des}((2,1,3),D),\ 3} \\
&\qquad + L_{\operatorname{Des}((2,3,1),D),\ 3} + L_{\operatorname{Des}((3,1,2),D),\ 3} + L_{\operatorname{Des}((3,2,1),D),\ 3} \\
&= L_{\{1\},\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\{2\},\ 3} + L_{\varnothing,\ 3} \\
&= 4 \cdot \underbrace{L_{\varnothing,\ 3}}_{= \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{1\},\ 3}}_{= \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3}} + \underbrace{L_{\{2\},\ 3}}_{= \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}} \\
&= 4 \cdot \sum_{i_1 \leq i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 < i_2 \leq i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 \leq i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}.
\end{aligned}$$

- We can restate the definition of U_D directly as follows:

- **Proposition.** Let $D = (V, A)$ be a digraph with n vertices. Then,

$$U_D = \sum_{f : V \to \{1,2,3,\ldots\}} a_{D,f} \prod_{v \in V} x_{f(v)},$$

  where $a_{D,f}$ is the # of all V-listings $w = (w_1, w_2, \ldots, w_n)$ such that

  - we have $f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n)$;
  - we have $f(w_i) < f(w_{i+1})$ if $(w_i, w_{i+1}) \in A$.

- This is similar (though not directly related) to P-partition enumerators and chromatic symmetric functions.

- **Remark.** We can restate the definition of $a_{D,f}$ in nicer terms. Namely, fix a digraph $D = (V, A)$ and a map $f : V \to \{1,2,3,\ldots\}$. For any $j \in f(V)$, let $\overline{D_j}$ denote the induced subdigraph of the complement $\overline{D}$ on the vertex set $f^{-1}(j) = \{v \in V \mid f(v) = j\}$. Then,

$$a_{D,f} = \prod_{j \in f(V)} \left(\# \text{ of hamps of } \overline{D_j} \right).$$

  (Think of f as assigning a "level" to each vertex of D; then $f^{-1}(j)$ are the level sets.)

- Note that U_D is $\Xi_{\overline{D}}(x, 0)$ in the notations of Chow's 1996 paper.

- What is U_D good for? Counting hamps, for one:

- **Proposition.** Let D be a digraph. Then,

$$U_D(1,0,0,0,\ldots) = \left(\#\text{ of hamps of } \overline{D}\right).$$

- Thus, any results about U_D might give us information about the # of hamps!

- Formulas for U_D in some specific cases (D acyclic, D poset, D path) can be found in Additional Problem 120 to Chapter 7 of Stanley's EC2.

4. p-expansions: the main theorems

- I called U_D the "Rédei–Berge symmetric function", but is it actually symmetric? Yes, and in fact something better holds:
- **Definition.** For each $k \geq 1$, let

  $$p_k := x_1^k + x_2^k + x_3^k + \cdots$$

  be the k**-th power-sum symmetric function**.
- **Theorem.** For any digraph D, we have

  $$U_D \in \mathbb{Z}\left[p_1, p_2, p_3, \ldots\right].$$

  That is, U_D can be written as a polynomial in $p_1, p_2, p_3, \ldots$ over $\mathbb{Z}$.
- Which polynomial, though?
- **Definition.** Fix a digraph $D = (V, A)$.

  Let $\mathfrak{S}_V$ be the symmetric group on the set V.

  For any $\sigma \in \mathfrak{S}_V$, we let $\operatorname{Cycs}\sigma$ be the set of all cycles of σ, and we let

  $$p_{\operatorname{type}\sigma} := \prod_{\gamma \in \operatorname{Cycs}\sigma} p_{\ell(\gamma)},$$

  where $\ell(\gamma)$ denotes the length of γ. In other words, if σ has cycles of lengths $a, b, \ldots, k$ (including 1-cycles), then $p_{\operatorname{type}\sigma} = p_a p_b \cdots p_k$.

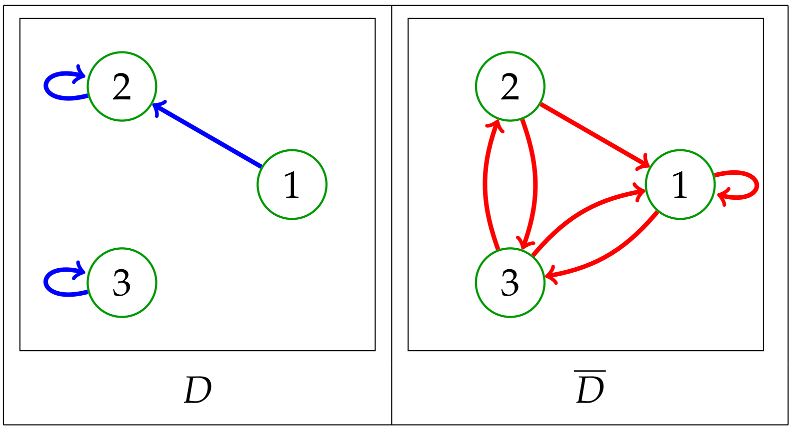
  We say that a cycle γ of σ is **a** D**-cycle** if all the pairs $(i, \sigma(i))$ for $i \in \gamma$ are arcs of D.
- **Main Theorem I.** Let $D = (V, A)$ be a digraph. Set

  $$\varphi(\sigma) := \sum_{\substack{\gamma \in \operatorname{Cycs}\sigma; \\ \gamma \text{ is a } D\text{-cycle}}} (\ell(\gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

  Then,

  $$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} (-1)^{\varphi(\sigma)} p_{\operatorname{type}\sigma}.$$
- This yields the $U_D \in \mathbb{Z}\left[p_1, p_2, p_3, \ldots\right]$ theorem, of course.

- **Example.** Recall our favorite example:



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The cycles of D are $(2)_\sim$ and $(3)_\sim$, whereas the cycles of $\overline{D}$ are $(1)_\sim$, $(2,3)_\sim$, $(3,1)_\sim$ and $(1,3,2)_\sim$ (the "$\sim$" means "rotation-equivalence class").

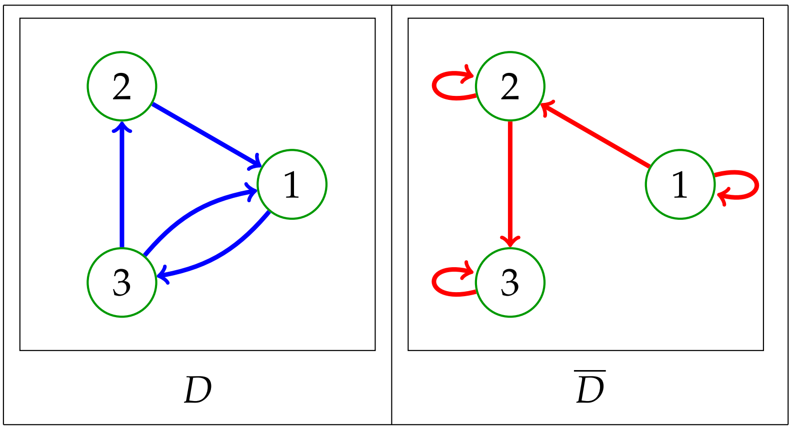
Thus, the $\sum\limits_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has four addends, corresponding to (σ written in one-line notation)

$\sigma =$	$[1,2,3]$	$[3,1,2]$	$[1,3,2]$	$[3,2,1]$
$(-1)^{\varphi(\sigma)} =$	1	1	1	1
$p_{\operatorname{type}\sigma} =$	p_1^3	p_3	p_2p_1	p_2p_1

Hence, Main Theorem I yields

$$U_D = p_1^3 + p_3 + p_2p_1 + p_2p_1 = p_1^3 + 2p_1p_2 + p_3.$$

- **Example.** Let



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Thus, the $\sum\limits_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has three addends, with

$\sigma =$	$[1,2,3]$	$[3,1,2]$	$[3,2,1]$
$(-1)^{\varphi(\sigma)} =$	1	1	-1
$p_{\operatorname{type}\sigma} =$	p_1^3	p_3	p_2p_1

Hence, Main Theorem I yields

$$U_D = p_1^3 + p_3 - p_2p_1.$$

- Main Theorem I yields Berge's theorem, since the sum for D and the sum for $\overline{D}$ range over the same σ's, and the addends only differ in sign.

- **Corollary.** Let $D = (V, A)$ be a digraph. Assume that every D-cycle has odd length. Then,

$$U_D = \sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} p_{\operatorname{type}\sigma} \in \mathbb{N}\left[p_1, p_2, p_3, \ldots\right].$$

- **Main Theorem II.** Let $D = (V, A)$ be a tournament. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. (A cycle is called **nontrivial** if it has length > 1.) Then,

$$U_D = \sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is a } D\text{-cycle;}\\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type}\sigma}$$
$$\in \mathbb{N}\left[p_1, 2p_3, 2p_5, 2p_7, \ldots\right] = \mathbb{N}\left[p_1,\ 2p_i \ \mid\ i > 1 \text{ is odd}\right].$$

- Main Theorem II easily yields Rédei's theorem, as the only addend with $2^{\psi(\sigma)}$ odd is the $\sigma = \operatorname{id}$ addend.

- The above corollary yields that U_D is p-positive when D has no even-length cycles. But this holds even more generally:

- **Main Theorem III.** Let $D = (V, A)$ be a digraph that has no cycles of length 2. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle;} \\ \text{no even-length cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a reversed } D\text{-cycle}}} p_{\operatorname{type} \sigma}.$$

- **Remark.** Even this does not cover all p-positive U_D's; there are more.

5. Proof ideas

- The proof of Main Theorem I is long and intricate. It might be simplifiable. Here are the main ideas.

- **Pólya-style lemma.** Let V be a finite set. Let $\sigma \in \mathfrak{S}_V$ be a permutation of V. Then,

$$\sum_{\substack{f:V\to\{1,2,3,\ldots\};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

Proof. Easy exercise.

- Using this lemma (and the above formula for $a_{D,f}$), we can easily reduce Main Theorem I to the following lemma:

- **Main combinatorial lemma.** Let $D = (V, A)$ be a digraph with n vertices. Let $f : V \to \{1, 2, 3, \ldots\}$ be any map. Then,

$$\prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right) = \sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle;}\\ f\circ\sigma=f}} (-1)^{\varphi(\sigma)},$$

where $\overline{D_j}$ is the induced subdigraph of $\overline{D}$ on the vertex set $f^{-1}(j)$.

- Work on each level:

Main combinatorial lemma (simplified). Let $D = (V, A)$ be a digraph with n vertices. Then,

$$\left(\text{\# of hamps of } \overline{D}\right) = \sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} (-1)^{\varphi(\sigma)}.$$

- This can be proved using a nontrivial exclusion-inclusion.

- Main Theorems II and III follow from Main Theorem I by combining σ's into equivalence classes by reversing certain cycles.

6. A surprise

- **Theorem.** Let D be a tournament. Then,

$$(\text{\# of hamps of } D) \\ \equiv 1 + 2\,(\text{\# of nontrivial odd-length } D\text{-cycles}) \bmod 4.$$

  Here, "nontrivial" means "having length > 1".

- We can prove this using Main Theorem II. We have not seen this anywhere in the literature.

7. I thank

- **Richard P. Stanley** for the obvious reasons.
- **Mike Zabrocki** for helpful comments.
- **Anna Pun** for the invitation and the implied compliment.
- **you** for your patience.