Partition Algebras

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Darij's list of errata and comments

- **Page 2:** Typo: "partiton".
- **Page 2:** Replace "the algebras $A_k(n)$" by "the algebras $\mathbb{C}A_k(n)$".
- **Page 3:** In the definition of A_k and $A_{k+\frac{1}{2}}$, replace "$\mathbb{Z}_{>0}$" by "$\mathbb{Z}_{\geq 0}$". Similarly, in many other places throughout the article (but not everywhere), "$\mathbb{Z}_{>0}$" can and should be replaced by "$\mathbb{Z}_{\geq 0}$". (While the first two monoids A_0 and $A_{\frac{1}{2}}$ are not very interesting, you do use them – e.g., they appear in the graph on page 14.)
- **Page 4:** The set I_k is not a submonoid of A_k, but a **nonunital** submonoid[1] of A_k (unlike S_k, P_k, B_k and T_k, all of which are unital monoids). I don't think that you want to use the word "monoid" (without qualification) for nonunital monoids, because if you do, then you would have to include the element 1 in the presentation in Theorem 1.11 (a).
- **Page 5, (1.8):** Replace "$\sum\limits_{\ell\geq 0} C(\ell-1)z^\ell$" by "$\sum\limits_{\ell\geq 0} C(\ell)z^\ell$".
- **Page 5, (1.8):** Replace "$\sum\limits_{\ell\geq 0}(2(\ell-1))!!\frac{z^\ell}{\ell!}$" by "$\sum\limits_{\ell\geq 0}(2\ell)!!\frac{z^\ell}{(\ell+1)!}$".
- **Page 6:** Here you introduce the notation $d_1d_2 = d_1 \circ d_2$, which is perfectly fine, but it would have been better to introduce it before, since it was already used on page 5 (when you wrote "$d = \sigma_1 t \sigma_2$").
- **Page 6, Theorem 1.11:** I think the generator $p_{\frac{1}{2}}$ occuring in parts (b) and (d) of Theorem 1.11 doesn't actually exist (at least you have never defined it!) and is not needed. I have not checked the proof, but I assume it can just be removed.
- **Page 8:** On the first line of page 8, replace "the the" by "the".
- **Page 9, §2:** Replace "$\mathbb{C}$span-" by "$\mathbb{C}$-span".

[1] i.e., a subsemigroup

- **Page 9, (2.2):** Please explain that whenever $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, you are abbreviating $\mathbb{C}A_k(n)$ by $\mathbb{C}A_k$.
- **Page 10:** The sentence "The map $\varepsilon_{\frac{1}{2}}$ is the composition $\mathbb{C}A_{k-\frac{1}{2}} \hookrightarrow \mathbb{C}A_k \xrightarrow{\varepsilon_1} \mathbb{C}A_{k-1}$" should be moved to below (2.4) (because it uses the map ε_1 which is only defined in (2.4)).
- **Page 10:** The "tr" in (2.7) and the "*tr*" on the line above should appear in the same font.
- **Page 12, (2.19):** Replace "$\lambda \vdash n$" by "$\lambda \vdash k$".
- **Page 12, (2.20):** This equation should end with a comma, rather than with a period.
- **Page 13:** In the picture showing the first few levels of $\widehat{S}$, the "k = 2" should be in mathmode.
- **Page 13:** "Young tableaux of shape λ" should be "Young tableaux of shape μ".
- **Page 13:** "the box of λ" should be "the box of μ".
- **Page 14:** In the picture showing the first few levels of $\widehat{A}$, the "k = 2" should be in mathmode.
- **Page 16:** Replace "for some constant p" by "for some constant $k \in \mathbb{C}$".
- **Page 16:** Replace "so that there are A-submodules" by "so that there are nonzero A-submodules".
- **Page 16:** It would be useful to replace "If p is an idempotent in A and Ap is a simple A-module" by "If p is an idempotent in a $\mathbb{C}$-algebra A and Ap is a simple A-module" to remind the reader that A is a $\mathbb{C}$-algebra (this becomes particularly important here, because the $pAp = \mathbb{C}p$ claim requires the base ring to be algebraically closed).
- I have never figured out whether you require algebras to be unital in your paper or not. Sometimes it seems that you do (for example, on page 16, you write "$\mathbb{C}(p \cdot 1 \cdot p)$", which seems to assume there exists a 1, although you could just as well avoid this by writing "$\mathbb{C}(p \cdot p \cdot p)$" instead), and sometimes you definitely do (e.g., in (4.20a) you use the 1 of A), but sometimes you definitely don't (e.g., when defining the basic construction you don't assume algebras to be unital, since the basic construction for A and B could be non-unital even when A and B are unital).

- **Page 17:** On the line just above (2.38), replace "define the $\mathbb{Z}[x]$-algebra by" by "define the $\mathbb{Z}[x]$-algebra $A_{k,\mathbb{Z}}$ by".

- **Page 17, (2.39):** Replace "$\mathbb{Z}$-module homomorphism" by "$\mathbb{Z}$-algebra homomorphism". (A $\mathbb{Z}$-module homomorphism $\mathbb{Z}[x] \to \mathbb{C}$ would not be uniquely determined by where it takes x.)

- **Page 18, Proposition 2.43:** Replace "$\mathbb{Z}$-module homomorphism" by "$\mathbb{Z}$-algebra homomorphism".

- **Page 18, (3.2):** Replace the summation index "$1 \leq i_{1'}, ...i_{k'} \leq n$" by "$1 \leq i_{1'}, ..., i_{k'} \leq n$".

- **Page 19, proof of Theorem 3.6 (a):** Here it would be helpful to introduce the following notation you are using:

  The family $\left(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}\right)_{(i_1,i_2,\ldots,i_k) \in \{1,2,\ldots,n\}^k}$ is a basis of the $\mathbb{C}$-vector space $V^{\otimes k}$. For every $b \in \operatorname{End}\left(V^{\otimes k}\right)$ and every $(u_1, u_2, \ldots, u_k) \in \{1, 2, \ldots, n\}^k$ and every $(j_1, j_2, \ldots, j_k) \in \{1, 2, \ldots, n\}^k$, we denote by $b_{j_1,j_2,\ldots,j_k}^{u_1,u_2,\ldots,u_k}$ the $\left(v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_k}\right)$-coordinate of $b\left(v_{u_1} \otimes v_{u_2} \otimes \cdots \otimes v_{u_k}\right)$ (with respect to the basis $\left(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}\right)_{(i_1,i_2,\ldots,i_k) \in \{1,2,\ldots,n\}^k}$ of $V^{\otimes k}$). This coordinate $b_{j_1,j_2,\ldots,j_k}^{u_1,u_2,\ldots,u_k}$ is called the *matrix entry* of b at the *matrix coordinates* $((u_1, u_2, \ldots, u_k), (j_1, j_2, \ldots, j_k))$.

  This notation has the consequence that

  $$b\left(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}\right) = \sum_{1 \leq i_{1'}, i_{2'}, \ldots, i_{k'} \leq n} b_{i_{1'},i_{2'},\ldots,i_{k'}}^{i_1,i_2,\ldots,i_k} v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{k'}}$$

  for every $b \in \operatorname{End}\left(V^{\otimes k}\right)$ and every $(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k$. Comparing this with (3.2), we conclude that every $d \in A_k$, every $(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, n\}^k$ and every $(i_{1'}, i_{2'}, \ldots, i_{k'}) \in \{1, 2, \ldots, n\}^k$ satisfy

  $$(d)_{i_{1'},i_{2'},\ldots,i_{k'}}^{i_1,i_2,\ldots,i_k} = \left(\Phi_k(d)\right)_{i_{1'},i_{2'},\ldots,i_{k'}}^{i_1,i_2,\ldots,i_k}.$$

- **Page 20, proof of Theorem 3.6 (b):** Replace "vertices i_{k+1} and $i_{(k+1)'}$ must be in the same block of d" by "vertices $k+1$ and $(k+1)'$ must be in the same block of d".

- **Page 25, proof of Theorem 3.27:** Replace "is cannot be" by "cannot be".

- **Page 25, proof of Theorem 3.27:** I suppose "Theorem Theorem 2.26(c)" should be "Theorem 2.26(c)".

- **Page 26, (3.32):** There seems to be one closing parenthesis too much here.

- **Page 31:** Replace "statment" by "statement".
- **Page 31:** Remove the comma at the end of (4.3).
- **Page 32:** Replace "$a^{\mu}_{PQ} \leftarrow E^{\mu}_{PQ}$" by "$a^{\mu}_{PQ} \longleftarrow\!\!\dashv E^{\mu}_{PQ}$".
- **Page 32:** At the very end of (4.13), replace "$\varepsilon^{\mu}_{XY} a^{\mu}_{ST}$" by "$\delta_{\lambda\mu} \varepsilon^{\mu}_{XY} a^{\mu}_{ST}$".
- **Page 33, (4.16):** Replace "$\overrightarrow{a}^{\mu}_{P} \otimes \overleftarrow{a}^{\mu}_{P}$" by "$\overleftarrow{a}^{\mu}_{P} \otimes \overrightarrow{a}^{\mu}_{P}$".
- **Page 33, (4.17):** Replace "$\overrightarrow{a}^{\lambda}_{W} \otimes \overleftarrow{a}^{\mu}_{Z}$" by "$\overleftarrow{a}^{\lambda}_{W} \otimes \overrightarrow{a}^{\mu}_{Z}$" on the left-hand side of (4.17). Make similar replacements on the other sides (every time, the second tensorand should have an $\overleftarrow{a}$ and the third tensorand an $\overrightarrow{a}$).
- **Page 33:** Here you claim that "$\left\{ \overline{m}^{\mu}_{XY} \mid \mu \in \widehat{A},\ X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu} \right\}$ is a basis of $\overline{R} \otimes_{\overline{A}} \overline{L}$". It took me a while to understand why this holds. Here is my proof for it: Recall that $\overline{A} \cong \bigoplus_{\mu \in \widehat{A}} M_{d_\mu}(\mathbb{F}) = \bigoplus_{\nu \in \widehat{A}} M_{d_\nu}(\mathbb{F})$ as $\mathbb{F}$-algebras. Use this isomorphism to identify $\overline{A}$ with $\bigoplus_{\nu \in \widehat{A}} M_{d_\nu}(\mathbb{F})$. Fix $\mu \in \widehat{A}$. Then, $\overleftarrow{A}_{\mu}$ is isomorphic to the right $\overline{A}$-module of length-d_μ row vectors over $\mathbb{F}$ on which the $M_{d_\mu}(\mathbb{F})$ addend of the direct sum $\bigoplus_{\nu \in \widehat{A}} M_{d_\nu}(\mathbb{F})$ acts by vector-matrix multiplication, whereas all other addends act as 0. Similarly, $\overrightarrow{A}_{\mu}$ is isomorphic to the left $\overline{A}$-module of length-d_μ column vectors over $\mathbb{F}$ on which the $M_{d_\mu}(\mathbb{F})$ addend of the direct sum $\bigoplus_{\nu \in \widehat{A}} M_{d_\nu}(\mathbb{F})$ acts by matrix-vector multiplication, whereas all other addends act as 0. From these descriptions of $\overleftarrow{A}_{\mu}$ and $\overrightarrow{A}_{\mu}$, it is easy to see that $\overleftarrow{A}_{\mu} \otimes_{\overline{A}} \overrightarrow{A}_{\mu} \cong \mathbb{F}$ (as $\mathbb{F}$-vector spaces), and more precisely, that the one-element family $\left(\overleftarrow{a}^{\mu}_{P} \otimes \overrightarrow{a}^{\mu}_{P} \right)$ is an $\mathbb{F}$-vector space basis of $\overleftarrow{A}_{\mu} \otimes_{\overline{A}} \overrightarrow{A}_{\mu}$ for every $P \in \widehat{A}^{\mu}$. Now, if we fix some $P \in \widehat{A}^{\mu}$, then the $\mathbb{F}$-vector space

$$\underbrace{R^{\mu}}_{\substack{\text{this } \mathbb{F}\text{-vector space} \\ \text{has basis } \left(r^{\mu}_{Y}\right)_{Y \in \widehat{R}^{\mu}}}} \otimes \underbrace{\overleftarrow{A}^{\mu} \otimes_{\overline{A}} \overrightarrow{A}^{\mu}}_{\substack{\text{this } \mathbb{F}\text{-vector space} \\ \text{has basis } \left(\overleftarrow{a}^{\mu}_{P} \otimes \overrightarrow{a}^{\mu}_{P}\right)}} \otimes \underbrace{L^{\mu}}_{\substack{\text{this } \mathbb{F}\text{-vector space} \\ \text{has basis } \left(\ell^{\mu}_{X}\right)_{X \in \widehat{L}^{\mu}}}}$$

clearly has basis

$$\left(r_Y^\mu \otimes \overleftarrow{a}_P^\mu \otimes \overrightarrow{a}_P^\mu \otimes \ell_X^\mu\right)_{Y \in \widehat{R}^\mu,\ X \in \widehat{L}^\mu}$$
$$= \left(\underbrace{r_X^\mu \otimes \overleftarrow{a}_P^\mu \otimes \overrightarrow{a}_P^\mu \otimes \ell_Y^\mu}_{= \overline{m}_{XY}^\mu}\right)_{X \in \widehat{R}^\mu,\ Y \in \widehat{L}^\mu}$$
$$\text{(here, we have renamed the indices } Y \text{ and } X \text{ as } X \text{ and } Y\text{)}$$
$$= \left(\overline{m}_{XY}^\mu\right)_{X \in \widehat{R}^\mu,\ Y \in \widehat{L}^\mu}.$$

Now, let us forget that we fixed μ. We thus see that for every $\mu \in \widehat{A}$, the $\mathbb{F}$-vector space $R^\mu \otimes \overleftarrow{A}^\mu \otimes_{\overline{A}} \overrightarrow{A}^\mu \otimes L^\mu$ has basis $\left(\overline{m}_{XY}^\mu\right)_{X \in \widehat{R}^\mu,\ Y \in \widehat{L}^\mu}$. Now,

$$\underbrace{\overline{R}}_{= \bigoplus_{\mu \in \widehat{A}} R^\mu \otimes \overleftarrow{A}^\mu} \otimes_{\overline{A}} \underbrace{\overline{L}}_{= \bigoplus_{\mu \in \widehat{A}} \overrightarrow{A}^\mu \otimes L^\mu}$$
$$= \left(\bigoplus_{\mu \in \widehat{A}} R^\mu \otimes \overleftarrow{A}^\mu\right) \otimes_{\overline{A}} \left(\bigoplus_{\mu \in \widehat{A}} \overrightarrow{A}^\mu \otimes L^\mu\right) \cong \bigoplus_{\mu \in \widehat{A},\ \nu \in \widehat{A}} R^\mu \otimes \overleftarrow{A}^\mu \otimes_{\overline{A}} \overrightarrow{A}^\nu \otimes L^\nu$$
$$= \bigoplus_{\mu \in \widehat{A}} \underbrace{R^\mu \otimes \overleftarrow{A}^\mu \otimes_{\overline{A}} \overrightarrow{A}^\mu \otimes L^\mu}_{\substack{\text{this } \mathbb{F}\text{-vector space has basis} \\ \left(\overline{m}_{XY}^\mu\right)_{X \in \widehat{R}^\mu,\ Y \in \widehat{L}^\mu}}} \qquad \left(\text{since } \overleftarrow{A}^\mu \otimes_{\overline{A}} \overrightarrow{A}^\nu = 0 \text{ whenever } \mu \neq \nu\right).$$

If we regard the isomorphisms in this equality as identities, we thus conclude that the $\mathbb{F}$-vector space $\overline{R} \otimes_{\overline{A}} \overline{L}$ has basis $\left(\overline{m}_{XY}^\mu\right)_{\mu \in \widehat{A},\ X \in \widehat{R}^\mu,\ Y \in \widehat{L}^\mu}$, qed.

- **Page 34:** In the first displayed equation on this page, replace "$\overline{n}_{XY}$" by "$\overline{n}_{XY}^\mu$", and replace "$\overline{m}_{Q_1 Q_2}$" by "$\overline{m}_{Q_1 Q_2}^\mu$".
- **Page 34:** Replace "using (4.10) and (4.12)" by "using (4.10) and (4.13)".
- **Page 34:** Replace "$\overrightarrow{a}_W^\lambda \otimes \overleftarrow{a}_W^\lambda$" by "$\overleftarrow{a}_W^\lambda \otimes \overrightarrow{a}_W^\lambda$" in the chain of equalities below the words "By direct computations". Make similar replacements throughout this chain of equalities.
- **Page 34:** Replace "$\overline{a}_{WZ}^\lambda$" by "$a_{WZ}^\lambda$".
- **Page 34:** In "$\frac{1}{\varepsilon_T^\lambda} \frac{1}{\varepsilon_V^\mu} n_{YT}^\lambda n_{UV}^\mu = \delta_{\lambda\mu} \delta_{TU} \frac{1}{\varepsilon_T^\lambda \varepsilon_V^\lambda} \varepsilon_T^\lambda n_{YV}^\lambda$", replace the "$=$" sign by an "$\equiv$" sign.
- **Page 34:** You claim that "the images of the elements e_{YT}^λ in (4.7) form a set of matrix units in the algebra $(R \otimes_A L) / I$". First, I think you should remove the words "in (4.7)" here, because they are confusing (they sounds

as if you mean the images under π, but instead you actually mean the images under the projection $R \otimes_A L \to (R \otimes_A L) / I$). Second, this might need some further explanation. You have proven that the images of the elements e^{λ}_{YT} under the projection $R \otimes_A L \to (R \otimes_A L) / I$ multiply like matrix units, but it remains to show that these images form a basis of the $\mathbb{F}$-vector space $(R \otimes_A L) / I$ (in fact, a family of 0's also multiplies like matrix units, but does not constitute matrix units unless it is empty). However, this is not hard to show: We already know that $\left\{ \overline{m}^{\mu}_{XY} \mid \mu \in \widehat{A},\ X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu} \right\}$ is a basis of $\overline{R} \otimes_{\overline{A}} \overline{L}$. Consequently, $\left\{ \overline{n}^{\mu}_{XY} \mid \mu \in \widehat{A},\ X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu} \right\}$ is a basis of $\overline{R} \otimes_{\overline{A}} \overline{L}$ as well (because the definition of $\overline{n}^{\mu}_{XY}$ shows that for every $\mu \in \widehat{A}$, we have the matrix equality

$$\left(\overline{n}^{\mu}_{XY}\right)_{X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu}} = \underbrace{\left(C^{\mu}_{ZW}\right)_{W \in \widehat{R}^{\mu},\ Z \in \widehat{R}^{\mu}}}_{\substack{\text{this is an invertible matrix} \\ \text{(being the transpose of the invertible matrix } C^{\mu}\text{)}}} \cdot \left(\overline{m}^{\mu}_{XY}\right)_{X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu}} \cdot \underbrace{\left(D^{\mu}_{ST}\right)_{T \in \widehat{L}^{\mu},\ S \in \widehat{L}^{\mu}}}_{\substack{\text{this is an invertible matrix} \\ \text{(being the transpose of the invertible matrix } D^{\mu}\text{)}}}$$

). In other words, $\left\{ \pi\left(n^{\mu}_{XY}\right) \mid \mu \in \widehat{A},\ X \in \widehat{R}^{\mu},\ Y \in \widehat{L}^{\mu} \right\}$ is a basis of $\pi(R \otimes_A L)$ (since $\overline{n}^{\mu}_{XY} = \pi\left(n^{\mu}_{XY}\right)$ and $\overline{R} \otimes_{\overline{A}} \overline{L} = \pi(R \otimes_A L)$). In other words, $\left\{ \pi\left(n^{\mu}_{YT}\right) \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \right\}$ is a basis of $\pi(R \otimes_A L)$ (here, we renamed the indices X and Y as Y and T). Therefore, the family

$$\mathfrak{F} := \left\{ k_i,\ n^{\mu}_{YT} \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \right\}$$

is a basis of $R \otimes_A L$ (because $\{k_i\}$ is a basis of $\ker \pi$). But the subfamily

$$\mathfrak{G} := \left\{ k_i,\ n^{\mu}_{YT} \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu},\ \left(\varepsilon^{\mu}_{Y} = 0 \text{ or } \varepsilon^{\mu}_{T} = 0\right) \right\}$$

of this latter family is a basis of I (because I was defined as the $\mathbb{F}$-span of $\mathfrak{G}$). Hence, the images of the elements of $\mathfrak{F} \setminus \mathfrak{G}$ under the projection $R \otimes_A L \to (R \otimes_A L) / I$ form a basis of $(R \otimes_A L) / I$. Since

$$\begin{aligned}
&\mathfrak{F} \setminus \mathfrak{G} \\
&= \left\{ k_i,\ n^{\mu}_{YT} \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \right\} \\
&\qquad \setminus \left\{ k_i,\ n^{\mu}_{YT} \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu},\ \left(\varepsilon^{\mu}_{Y} = 0 \text{ or } \varepsilon^{\mu}_{T} = 0\right) \right\} \\
&= \left\{ n^{\mu}_{YT} \mid \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu},\ \left(\text{neither } \varepsilon^{\mu}_{Y} = 0 \text{ nor } \varepsilon^{\mu}_{T} = 0\right) \right\},
\end{aligned}$$

this rewrites as follows: The images of the elements

$$n^{\mu}_{YT} \text{ for } \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \text{ satisfying } \left(\text{neither } \varepsilon^{\mu}_{Y} = 0 \text{ nor } \varepsilon^{\mu}_{T} = 0\right)$$

under the projection $R \otimes_A L \to (R \otimes_A L) / I$ form a basis of $(R \otimes_A L) / I$. But recall that we need to prove that the images of the elements

$$e^{\mu}_{YT} \text{ for } \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \text{ satisfying } \left(\text{neither } \varepsilon^{\mu}_{Y} = 0 \text{ nor } \varepsilon^{\mu}_{T} = 0\right)$$

under the projection $R \otimes_A L \to (R \otimes_A L) / I$ form a basis of $(R \otimes_A L) / I$. This immediately follows from the fact that the images of the elements

$$n^{\mu}_{YT} \text{ for } \mu \in \widehat{A},\ Y \in \widehat{R}^{\mu},\ T \in \widehat{L}^{\mu} \text{ satisfying } \left(\text{neither } \varepsilon^{\mu}_{Y} = 0 \text{ nor } \varepsilon^{\mu}_{T} = 0\right)$$

under the projection $R \otimes_A L \to (R \otimes_A L) / I$ form a basis of $(R \otimes_A L) / I$ (because $e^{\mu}_{YT} = \dfrac{1}{\varepsilon^{\mu}_{T}} n^{\mu}_{YT}$ differs from n^{μ}_{YT} only in a nonzero multiplicative factor). This completes the proof of your claim that "the images of the elements e^{λ}_{YT} in (4.7) form a set of matrix units in the algebra $(R \otimes_A L) / I$".

- **Page 35:** You write: "Let $A \subseteq B$ be an inclusion of algebras". I think this is one of the places where you want A and B (or B at least) to be unital, or else (4.20a) and (4.20c) don't make sense.
- **Page 35, (4.20c):** After "$pAp = \mathbb{F}p$", add "and p is an idempotent".
- **Page 35, (4.22):** It would help to explain that your notation $P \to \mu \to \lambda$ is shorthand for a pair $(P \to \mu, \mu \to \lambda)$ of an element $P \to \mu$ of $\widehat{A}^{\mu}$ and an edge $\mu \to \lambda$ of Γ.

  (Anyway, I am wondering why you don't define an extended graph $\widehat{\Gamma}$ which consists of Γ and an additional vertex $\mathbb{F}$, and which has the same edges as Γ and, additionally, $\left|\widehat{A}^{\mu}\right|$ edges from $\mathbb{F}$ to μ for every $\mu \in \widehat{A}$. Then, you could identify $\widehat{B}^{\lambda}$ with the set of edges from $\mathbb{F}$ to λ in this graph $\widehat{\Gamma}$ for every $\lambda \in \widehat{B}$.)
- **Page 36, (4.24):** Replace "$\delta_{\lambda\sigma}\delta_{QS}\delta_{\gamma\tau}b_{\substack{PT\\ \mu\nu\\ \lambda}}$" by "$\delta_{\lambda\sigma}\delta_{Q\to\gamma,S\to\tau}\delta_{\gamma\tau}\delta_{\gamma\to\lambda,\ \tau\to\sigma}b_{\substack{PT\\ \mu\nu\\ \lambda}}$".

  (The $\delta_{\gamma\to\lambda,\ \tau\to\sigma}$ factor is important; there might be several edges from γ to λ, and they give rise to different matrix elements.)
- **Page 38:** Replace "The rese" by "The rest".
- **Page 39, §5:** In the definition of "trace", replace "linear" by "$\overline{\mathbb{F}}$-linear".
- **Page 39, §5:** In the definition of "nondegenerate", replace "for each $b \in A$" by "for each nonzero $b \in A$".
- **Page 39, Lemma 5.1:** The notations here conflict with the notations introduced just a few moments earlier. For example, you want the trace $\overrightarrow{t}$ in Lemma 5.1 to be an $\mathbb{F}$-linear map $A \to \mathbb{F}$, whereas you previously defined

a trace as an $\overline{\mathbb{F}}$-linear map $\overline{A} \to \overline{\mathbb{F}}$. It would probably best to define the notions of "trace" and "nondegenerate" over arbitrary fields first, and only then apply them to the case of $\overline{\mathbb{F}}$.

- **Page 39, proof of Lemma 5.1:** Replace "$\overline{\mathbb{F}}$" by "$\mathbb{F}$".
- **Page 39, proof of Lemma 5.1:** Replace "the columns of G are linearly dependent" by "the rows of G are linearly dependent".
- **Page 40, Proposition 5.2:** In part (a), replace "$\operatorname{Hom}_{\overline{\mathbb{F}}}$" by "$\operatorname{Hom}_{\mathbb{F}}$".
- **Page 43, proof of Theorem 5.8:** I would replace "vacuously true" by "obviously true". ("Vacuously true" means that the conditions can never be satisfied; this is probably not what you meant.)
- **Page 43, proof of Theorem 5.8:** Replace "a proper submodule N" by "a proper nonzero submodule N".
- **Page 44, proof of Theorem 5.8:** Replace "complementary to M" by "complementary to N in M".
- **Page 44, Theorem 5.10:** Remove the comma in "$\mathbb{F}$, the field of fractions".
- **Page 44, Theorem 5.10:** Remove the comma in "and $\overline{R}$, the integral closure".
- **Page 44, Theorem 5.10:** Replace "$t_1 \overrightarrow{A}(b_1) + \cdots t_d \overrightarrow{A}(b_d)$" by "$t_1 \overrightarrow{A}(b_1) + \cdots + t_d \overrightarrow{A}(b_d)$".
- **Page 45, Theorem 5.10 (a):** Replace the "$\longmapsto$" arrow by a "$\longrightarrow$" arrow in "$A_{\overline{\mathbb{F}}} \longmapsto \overline{\mathbb{F}}$".
- **Page 45, Theorem 5.10 (a):** Replace "be the extension" by "be an extension".
- **Page 45, Theorem 5.10 (b):** Replace the "$\longmapsto$" arrow by a "$\longrightarrow$" arrow in "$A_{\overline{\mathbb{K}}} \longmapsto \overline{\mathbb{K}}$".