Why should the Littlewood–Richardson rule be true?
Roger Howe and Soo Teck Lee
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "**Why should the Littlewood–Richardson rule be true?**" by the numbers under which they appear in this article (specifically, in its published version). **The page numbers are relative to the article** (i.e., "page 5" means "the 5-th page of the article", not "the 5-th page of the issue").

10. Errata

My familiarity with this paper is fleeting. Thus, I will not be surprised if some of the following corrections are actually wrong; and even if not, I am fairly sure they are far from complete.

- **Page 5:** Replace "any collection J" by "any strictly increasing sequence J".
- **Page 5:** Replace "to be the span the basis vectors" by "to be the span of the basis vectors".
- **Page 6:** The words "for $1 \leq a \leq d$" which are directly after (2.4) should actually be inside the displayed equation (2.4).
- **Page 6:** Add a whitespace in "$U_i^{\mathrm{opp}}$is the span".
- **Page 7:** "A partition α is specified by a weakly decreasing sequence of non-negative integers," should be replaced by "A partition α is specified by a weakly decreasing sequence of non-negative integers such that all but finitely many of its entries are 0.".
- **Page 7:** In (2.14), replace "$n - d - j_d - d)$" by "$n - d - (j_d - d)$".
- **Page 10:** You write: "It is not hard to argue that we can find an orthonormal basis $\{ \overrightarrow{\mathbf{y}}_b \}$ for V, such that $\overrightarrow{\mathbf{y}}_b$ belongs to U_{A,j_b}". It might be helpful to point out that this follows from Gram-Schmidt orthonormalization.
- **Page 10:** You write: "if $J_V^{\mathrm{opp}} = J^{\mathrm{opp}}$ is the jump sequence of V with respect to the opposite flag $\mathcal{F}_A^{\mathrm{opp}}$ defined by the spaces $U_{A,n-j}^{\perp}$". This notation conflicts with the definition of J^{opp} on page 6, unless you mean to say that the J^{opp} defined on page 6 actually **is** the jump sequence of V with respect to the opposite flag $\mathcal{F}_A^{\mathrm{opp}}$ – but this, I believe, is false.

Due to this confusion, I do not understand how you get $\operatorname{tr}(P_V A) \leq \sum_{c=1}^{d} \lambda_{n-j_c^{\text{opp}}+1}(A)$ and prove Theorem 2.1. I also think there are further typos in these arguments: for example, I believe "the intersection of Schubert varieties $\mathcal{S}_{\mathcal{F}_{(A+B),K}} \cap \mathcal{S}_{\mathcal{F}_{A,I}^{\text{opp}}} \cap \mathcal{S}_{\mathcal{F}_{B,J}^{\text{opp}}}$" should be "the intersection of Schubert varieties $\Omega_{\mathcal{F}_{A+B,K}} \cap \Omega_{\mathcal{F}_A^{\text{opp}},I} \cap \Omega_{\mathcal{F}_B^{\text{opp}},J}$" on page 10, and I am also wondering if the "$c_{\alpha_J,\alpha_K}^{\alpha_I^{\text{opp}}}$" in Theorem 2.1 shouldn't rather be something like "$c_{\alpha_{I^{\text{opp}}},\alpha_{J^{\text{opp}}}}^{\alpha_K}$".

- **Page 11, Theorem 2.1:** Replace "the the" by "the".
- **Page 12, §3:** Replace "for any vector $\mathbf{v}$ in V" by "for any vector v in V".
- **Page 12, §3:** On the same line, replace "the function on G" by "the function on GL_n". (Or maybe define $G = \mathrm{GL}_n$, if you call it G again later.)
- **Page 13:** "repsentations" $\to$ "representations".
- **Page 15:** In condition (ii'), replace "for $j, a \geq 1$" by "for $j \geq 1$ and $a \geq 0$".
- **Page 16:** Replace "skew-row" by "skew row" (twice).
- **Page 17:** In "by a nested sequence $D = D_0 \subset D_1' \subset D_2' \subset \cdots \subset D_r'$ of Young diagrams", replace "D_0" by "D_0'".
- **Page 19:** You write: "Suppose that the row lengths a_j are weakly decreasing". I find it unmotivated that you refer to the a_j as "row lengths" here, since so far they are just nonnegative integers, and I don't think you have declared your intention to consider them as row lengths of a Young diagram.
- **Page 20:** "nubmers" $\to$ "numbers".
- **Page 22:** Replace "among all the possible tableau" by "among all the possible tableaux".
- **Page 22, Lemma 6.1:** It would be good to add the sentence "Let a and b be nonnegative integers satisfying $a \geq b$" at the beginning of this lemma. This would remind the reader of the standing assumption that $a \geq b$ (which was briefly mentioned on page 21, but is easily overlooked or understood to only apply to page 21). (Actually, the slightly weaker assumption $a \geq b - 1$ is enough for your proof to work.)
- **Page 23, proof of Lemma 6.1:** Replace "We look at the first point $p_o = \begin{bmatrix} n \\ m \end{bmatrix}$" by "We look at the first point $p_o = \begin{bmatrix} m \\ n \end{bmatrix}$".
- **Page 23, proof of Lemma 6.1:** Replace "and n is the smallest" by "and m is the smallest".

- **Page 23, proof of Lemma 6.1:** Replace "so that if p_o is not the origin, then $h(p_o) > 0$; that is, p_o lies strictly above the diagonal" by "but $h(p_o) > 0$ (since $\mathcal{P}$ rises above the main diagonal); thus, p_o lies strictly above the main diagonal. In particular, p_o is not the origin.".

- **Page 23, proof of Lemma 6.1:** Replace "$\begin{bmatrix} m-1 \\ n \end{bmatrix}$" by "$\begin{bmatrix} m \\ n-1 \end{bmatrix}$" (three times in the proof).

- **Page 23, proof of Lemma 6.1:** Remove the words "the next move of $\mathcal{P}$ must be to the right, that is, the next point after p_o on $\mathcal{P}$ must be $\begin{bmatrix} m+1 \\ n \end{bmatrix}$". You never use this observation.

- **Page 23, proof of Lemma 6.1:** Replace "by shifting by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$" by "by shifting by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$".

- **Page 24, proof of Lemma 6.1:** Replace "$\begin{bmatrix} m' \\ n' \end{bmatrix}$on" by "$\begin{bmatrix} m' \\ n' \end{bmatrix}$ on".

- **Page 24, proof of Lemma 6.1:** There is a subtlety here that should (in my opinion) be made explicit. You speak of "the path $\mathcal{P}$ constructed in the previous paragraph from $\mathcal{P}'$". This construction of $\mathcal{P}$ from $\mathcal{P}'$ rests on one assumption: the assumption that the last point on the highest diagonal reached by $\mathcal{P}'$ is not the endpoint of the path $\mathcal{P}'$. [1] This assumption, of course, is obviously satisfied when your path $\mathcal{P}'$ results from an increasing path $\mathcal{P}$ by the algorithm you explained on page 23, but it is not completely obvious why it holds when the path $\mathcal{P}'$ is just some arbitrary increasing path from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. So let me prove it in the latter case. Let $\mathcal{P}'$ be an increasing path from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. Then, $a+1 > a \geq b-1$, so that the point $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$ lies (strictly) below the main diagonal. Therefore, the origin lies on a higher diagonal than the point $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. Hence, the path $\mathcal{P}'$ reaches a point on a higher diagonal than $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$ (namely, the origin). Therefore, the last point on the highest diagonal reached by $\mathcal{P}'$ is not $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$. In other words, the last point on the highest diagonal reached

[1] Indeed, you use this assumption (because you speak of "The move from this point", and this move only exists if this point is not the endpoint).

by $\mathcal{P}'$ is not the endpoint of the path $\mathcal{P}'$ (since the endpoint of the path $\mathcal{P}'$ is $\begin{bmatrix} a+1 \\ b-1 \end{bmatrix}$). This finishes the proof.

- **Page 24, proof of Lemma 6.1:** Replace "$P \to P'$" by "$\mathcal{P} \mapsto \mathcal{P}'$".
- **Page 24:** Replace "$\rho_n^D \times S^2 \otimes S^2$" by "$\rho_n^D \otimes S^2 \otimes S^2$".
- **Page 30:** Replace "we will show that how" by either "we will show that" or "we will show how".
- **Page 35:** Replace "that a tableaux" by "that a tableau".
- **Page 39:** "in a the cone" $\to$ "in the cone".
- **Page 39:** "to the Hibi ring $\mathbb{C}\left(\mathbb{Z}_{\geq}^{+}\right)\left(\mathrm{GT}_{(n,k,\ell)}\right)$" $\to$ "to the Hibi ring $\mathbb{C}\left(\mathbb{Z}_{\geq}^{+}\left(\mathrm{GT}_{(n,k,\ell)}\right)\right)$".
- **Page 40:** Replace "Knutson-Tau" by "Knutson-Tao".
- **Page 48, reference [Hum]:** Replace "Humphrey" by "Humphreys".
- **Page 48, reference [Rei]:** The title of this reference should be "Signed poset". The "Victor" is just the first name of the author.