Higher Lie idempotents

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http://lacim.uqam.ca/~christo/idempotents.pdf (version November 18, 1998)

Errata and questions - I (version 2)

- **Page 1:** Typo: "caracteristic" should be "characteristic".
- **Pages 1 and 2:** Typo: "envelopping" should be "enveloping" (this typo appears several times).
- **Page 2 and further:** Typo: "familly" should be "family" (this typo appears several times).
- **Page 2:** Maybe "Given a familly of Lie idempotents" should be "Given an arbitrary Lie idempotent"? I think the constructions of the higher Lie idempotents depend only on one Lie idempotent ι and (in the case of higher Lie idempotents of the third kind) on a family of coefficients a_μ^ι.
- **Page 3:** Typo: "reodering" should be "reordering".
- **Page 4:** Between Definition 2.2 and the Example, you write that "the ι-descent algebra decomposes as a direct sum

  $$\mathcal{D}_\iota = \bigoplus_{n=0}^{\infty} \mathcal{D}_{\iota n}.$$

  ". It might be useful to notice here that this is a direct sum of vector spaces, not of algebras (under the convolution $*$).
- **Page 5:** In the proof of Lemma 3.1, you write: "More generally, for any $l \geq 2$ and $k \geq 3$, let Δ_{l2} be [...]". I don't see any reason to require $l \geq 2$ and $k \geq 3$ here; everything is just as correct for any $l \geq 0$ and $m \geq 0$.
- **Page 6:** In the proof of Lemma 3.1, the $\sum_{\sigma \in S_n}$ should be $\sum_{\sigma \in S_k}$.
- **Page 6:** In the proof of Lemma 3.1, you write: "If we apply Π_k to the whole sum" (in the fourth line of page 6). I think you are applying $\Pi_k^{\otimes k}$ here, not Π_k.
- **Page 6:** In the proof of Lemma 3.1, you have a typo: "Aplying" should be "Applying".
- **Page 6:** In the proof of Lemma 3.1, you write: "Now,this sum is equal to $\sum \left(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}\right) \circ \sigma \left(x_1 \otimes ... \otimes x_k\right)$, where σ denotes here the natural action of the symmetric group on $A^{\otimes n}$". First, there should be a whitespace after "Now,". Second, the $\sigma \left(x_1 \otimes ... \otimes x_k\right)$ should be a $\sigma^{-1} \left(x_1 \otimes ... \otimes x_k\right)$, because $\sigma \left(x_1 \otimes ... \otimes x_k\right)$ is $x_{\sigma^{-1}(1)} \otimes ... \otimes x_{\sigma^{-1}(k)}$ rather than $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$. Third, I think you mean $A^{\otimes k}$ instead of $A^{\otimes n}$ (unless you want to talk about general n).

- **Page 6:** In the proof of Lemma 3.1, you write: "Since the coproduct is cocommutative, we deduce that

$$(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ (\Pi_k^{\otimes k}) \circ \Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k}) = \sum (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \sigma \circ \Delta_k = \sum (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \Delta_k,$$

which implies (ii)." The σ here should be a σ^{-1}. (Also, what somewhat confused me is that cocommutativity is used in the passage from $\sum (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \sigma \circ \Delta_k$ to $\sum (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \Delta_k$, not in the passage from $(\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ (\Pi_k^{\otimes k}) \circ \Delta_{kk} \circ (\iota_{\lambda_1} \otimes ... \otimes \iota_{\lambda_k})$ to $\sum (\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}) \circ \sigma \circ \Delta_k$. It thus would probably better to mention cocommutativity after the long equation rather than before it.)

- **Page 7:** In the proof of Theorem 3.4, you write: "Thus f is idempotent if and only if [...]". But in general, only the "if" part of this is true (and fortunately, only the "if" part is needed), since nobody has told us that the ι_α are linearly independent.

- **Page 7:** In the proof of Theorem 3.4, it would be clearer if you replace $(n_1 + ... + n_k)!/n_1!...n_k!$ by $(n_1 + ... + n_k)!/(n_1!...n_k!)$. (I consider the notation $a/b_1b_2...b_k$ for $a/(b_1b_2...b_k)$ outdated and ambiguous, although it seems to be still in use.)

- **Page 7:** In Definition 4.1, I feel it would be good to point out three things explicitly:

  – The "1" in "$F_\lambda^\iota := \left(1 - \sum\limits_{l(\mu)<l(\lambda)} F_\mu^\iota\right) \circ E_\lambda^\iota$" means the identity map $\mathrm{id}_{A_n} \in \mathrm{End}(A_n)$, not the unity of the algebra $\mathcal{L}(A)$.

  – For $n = 0$, the element $F_{()}^\iota$ is defined as $E_{()}^\iota = \mathrm{id}_{A_0} = \eta \circ \epsilon$ (here we are using the identification of $\mathrm{End}(A_0)$ with the space of all graded endomorphisms of A whose image is $\subseteq A_0$). (While this can be seen as a consequence of the formula $F_\lambda^\iota := \left(1 - \sum\limits_{l(\mu)<l(\lambda)} F_\mu^\iota\right) \circ E_\lambda^\iota$ applied to $\lambda = ()$, it would be helpful to point this out explicitly).

  – The maps F_λ^ι are called the "higher Lie idempotents of the second kind".

- **Page 7:** In Definition 4.1, it wouldn't harm to say that the "induction base" $F_{(n)}^\iota := E_{(n)}^\iota = \iota_n$ is, itself, a particular case of the "induction step" $F_\lambda^\iota := \left(1 - \sum\limits_{l(\mu)<l(\lambda)} F_\mu^\iota\right) \circ E_\lambda^\iota$. In fact, if we substitute $\lambda = (n)$ in $F_\lambda^\iota := \left(1 - \sum\limits_{l(\mu)<l(\lambda)} F_\mu^\iota\right) \circ E_\lambda^\iota$, then we get $F_{(n)}^\iota = \left(1 - \sum\limits_{l(\mu)<l((n))} F_\mu^\iota\right) \circ E_{(n)}^\iota$, but the sum $\sum\limits_{l(\mu)<l((n))} F_\mu^\iota$ is empty since $l((n)) = 1$, and thus this becomes $F_{(n)}^\iota = E_{(n)}^\iota$.

  This fact allows us to use $F_\lambda^\iota = \left(1 - \sum\limits_{l(\mu)<l(\lambda)} F_\mu^\iota\right) \circ E_\lambda^\iota$ not only for $\lambda \neq (n)$ but also for all λ. This is used in several proofs in your paper.

- **Page 7:** In the Remark 1) at the end of page 7, you made a typo: "othogonal" should be "orthogonal".

- **Page 9:** On the first line of this page, you write: "$F^\iota_\mu \circ F^\iota_\beta = \delta_{\mu\beta}$". This should be $F^\iota_\mu \circ F^\iota_\beta = \delta_{\mu\beta} F^\iota_\mu$. (The only thing you actually use, though, is that $F^\iota_\mu \circ F^\iota_\beta = 0$ for $\mu \neq \beta$ when $l(\mu)$ and $l(\beta)$ are both $< k$.)

- **Page 9:** In the proof of Theorem 4.3, you write: "we have by Def.4.1 that $E^\iota_\lambda(x) = F^\iota_\lambda(x)$ plus a sum of $E^\iota_{\lambda_1} \circ ... \circ E^\iota_{\lambda_k}$". First, either you should replace the $E^\iota_\lambda(x)$ and $F^\iota_\lambda(x)$ here by E^ι_λ and F^ι_λ, or you should replace the $E^\iota_{\lambda_1} \circ ... \circ E^\iota_{\lambda_k}$ by an $\left(E^\iota_{\lambda_1} \circ ... \circ E^\iota_{\lambda_k}\right)(x)$. Second, "sum" is slightly imprecise; you mean a linear combination rather than a sum (the coefficients in this combination can be both $+1$ and -1).

- **Page 9:** In the proof of Theorem 4.3, you write: "the elements $(a_1, ..., a_k) = (1/k!) \sum_{k\in S_k} a_{\sigma(1)} ... a_{\sigma(k)}$". Replace $\sum_{k\in S_k}$ by $\sum_{\sigma\in S_k}$ here.

- **Page 9:** In the proof of Theorem 4.3, you write:
  "Since A is a graded cocommutative connected bialgebra of characteristic zero, it is by the Cartier-Milnor-Moore theorem isomorphic to the envelopping algebra of $Prim(A)$. Hence, by the Poincaré-Birkhoff-Witt theorem it is the direct sum of its subspaces A^λ, where for any partition λ, the latter subspace is spanned by the elements $(a_1, ..., a_k) = (1/k!) \sum_{k\in S_k} a_{\sigma(1)} ... a_{\sigma(k)}$, for any choice of homogeneous primitive elements a_i, with $\deg(a_i) = \lambda_i$ and $\lambda = (\lambda_1, ..., \lambda_k)$."
  This is a correct argument (up to the typos I mentioned above), but somewhat an overkill. In fact, you only need the easy part of the Cartier-Milnor-Moore theorem[1] and only the easy part of the Poincaré-Birkhoff-Witt theorem[2] to show that A is the sum of its subspaces A^λ (we don't yet know that it is the *direct* sum), and this is already enough for your proof of Theorem 4.3. (I can detail this argument better if you wish, but I have a feeling that you already know this). Maybe you need something stronger (like the direct sum assertion) to prove Corollary 4.4 though (I don't understand your proof at the moment), but I would always try to do without - maybe this will net us an explicit constructive proof of Poincaré-Birkhoff-Witt or Cartier-Milnor-Moore at the end...

- **Page 9:** In the proof of Theorem 4.3, you write: "It is equal to $\sum_\mu \Pi_k \circ \iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k} \circ \Delta_k(a_1 ... a_k)$". I would put the $\iota_{\mu_1} \otimes ... \otimes \iota_{\mu_k}$ term in brackets here.

- **Page 9:** In the last absatz of page 9, you write: "the *cofree cocommutative coalgebra* on a vector space V". But I think it is more common to say "over a vector space V" rather than "on a vector space V". (You yourself say "over" in Corollary 4.4.)

- **Page 10:** In Corollary 4.4, replace "$\bigoplus\limits_{n\in\mathbf{N}} \iota^{\otimes n} \circ \Delta_n$" by "$\bigoplus\limits_{n\in\mathbf{N}} \frac{1}{n!} \iota^{\otimes n} \circ \Delta_n$" (otherwise, this map would not be a coalgebra homomorphism).

- **Page 10:** In Corollary 4.4, replace the $\mapsto$ arrow by a $\rightarrow$ arrow.

[1] By the "easy part", I mean the statement that a graded cocommutative connected bialgebra over a field of characteristic 0 is always generated as an algebra by its primitive elements.

[2] Here, the "easy part" is the statement that the symmetrization map $S(\mathfrak{g}) \to U(\mathfrak{g})$ is surjective. (This only makes sense in characteristic 0.)

- **Page 10:** In Corollary 4.4, replace "$\frac{1}{l(\lambda)!}\left(1-\sum\limits_{l(\mu)<l(\lambda)}F_{\mu}^{\iota}\right)\circ\Pi_k$" by "$\left(1-\sum\limits_{l(\mu)<l(\lambda)}F_{\mu}^{\iota}\right)\circ\Pi_k$" (this change is needed to "balance out" the $\frac{1}{n!}$ factor I added to "$\bigoplus\limits_{n\in\mathbf{N}}\iota^{\otimes n}\circ\Delta_n$").

- **Page 10:** You write that "The corollary follows, once it is noted that $Sym^{\lambda}(Prim(A))$ is canonically isomorphic to A^{λ}, through the map Π_k". I do understand why $\operatorname{Sym}^{\lambda}(\operatorname{Prim}A)$ is canonically isomorphic to A^{λ} through the map Π_k [3]. But I don't understand how Corollary 4.4 follows from this! In particular, I don't see how the $\frac{1}{l(\lambda)!}\left(1-\sum\limits_{l(\mu)<l(\lambda)}F_{\mu}^{\iota}\right)$ term appears.

- **Page 11:** In the proof of Theorem 5.1, you write: "We multiply this by e_n on the right in $\mathcal{L}(A)$". I think this is confusing: Multiplying something in $\mathcal{L}(A)$

[3] In fact, let $\lambda=(\lambda_1,\lambda_2,...,\lambda_k)$. For every F-vector space V and every subset S of V, let $\langle S\rangle$ denote the F-linear span of the set S.

By the definition of $\operatorname{Sym}^{\lambda}(\operatorname{Prim}A)$, we know that $\operatorname{Sym}^{\lambda}(\operatorname{Prim}A)$ is the F-linear span of the elements $\frac{1}{k!}\sum\limits_{\sigma\in S_k}x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes...\otimes x_{\sigma(k)}$ where $(x_1,x_2,...,x_k)$ ranges over all k-tuples of homogeneous elements of $\operatorname{Prim}A$ satisfying ($\deg(x_i)=\lambda_i$ for all $i\in\{1,2,...,k\}$). In other words,

$$\begin{aligned}\operatorname{Sym}^{\lambda}(\operatorname{Prim}A) &= \left\langle\left\{\frac{1}{k!}\sum_{\sigma\in S_k}x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes...\otimes x_{\sigma(k)}\ \mid\ \text{all } x_i \text{ are homogeneous}\right.\right.\\ &\qquad\left.\left.\text{elements of } \operatorname{Prim}A \text{ and satisfy } \deg(x_i)=\lambda_i \text{ for all } i\in\{1,2,...,k\}\right\}\right\rangle\\ &= \left\langle\left\{\frac{1}{k!}\sum_{\sigma\in S_k}x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes...\otimes x_{\sigma(k)}\ \mid\ \text{all } x_i \text{ are primitive and homogeneous}\right.\right.\\ &\qquad\left.\left.\text{elements of } A \text{ and satisfy } \deg(x_i)=\lambda_i \text{ for all } i\in\{1,2,...,k\}\right\}\right\rangle\\ &= \left\langle\left\{\frac{1}{k!}\sum_{\sigma\in S_k}a_{\sigma(1)}\otimes a_{\sigma(2)}\otimes...\otimes a_{\sigma(k)}\ \mid\ \text{all } a_i \text{ are primitive and homogeneous}\right.\right.\\ &\qquad\left.\left.\text{elements of } A \text{ and satisfy } \deg(a_i)=\lambda_i \text{ for all } i\in\{1,2,...,k\}\right\}\right\rangle\end{aligned}$$

(here, we renamed x_i as a_i). In other words,

$$\begin{aligned}&\left\langle\left\{\frac{1}{k!}\sum_{\sigma\in S_k}a_{\sigma(1)}\otimes a_{\sigma(2)}\otimes...\otimes a_{\sigma(k)}\ \mid\ \text{all } a_i \text{ are primitive and homogeneous}\right.\right.\\ &\qquad\left.\left.\text{elements of } A \text{ and satisfy } \deg(a_i)=\lambda_i \text{ for all } i\in\{1,2,...,k\}\right\}\right\rangle\\ &= \operatorname{Sym}^{\lambda}(\operatorname{Prim}A). \qquad\text{(A1)}\end{aligned}$$

By the definition of A^{λ}, we know that A^{λ} is the F-linear span of the elements $\frac{1}{k!}\sum\limits_{\sigma\in S_k}a_{\sigma(1)}a_{\sigma(2)}...a_{\sigma(k)}$ where $(a_1,a_2,...,a_k)$ ranges over all k-tuples of primitive homogeneous elmeans convolution, but you want composition. Maybe you could just say "We compose this with e_n on the right"?

- **Page 11:** In the proof of Theorem 5.1, you write:
"Thus we obtain $e_n = \alpha e_n$, since $e_\mu \circ e_n = 0$ by Lemma 3.2. Thus, in case $e_n \neq 0$, $\alpha = 1$; and in case $e_n = 0$, we must have also $\iota_n = 0$, and we may take $\alpha = 1$ in (*)."
This argument is correct, but I think it can be simplified as follows:
"Thus we obtain $e_n = \alpha e_n$, since $e_\mu \circ e_n = 0$ by Lemma 3.2. Thus, we can replace αe_n by e_n in (*), and get $\iota_n = e_n + \sum_\mu * e_\mu$."

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ements of A satisfying ($\deg(a_i) = \lambda_i$ for all $i \in \{1, 2, ..., k\}$). In other words,

$$A^\lambda = \left\langle \left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous elements of } A \text{ and satisfy } \deg(a_i) = \lambda_i \text{ for all } i \in \{1, 2, ..., k\} \right\} \right\rangle$$

$$= \left\langle \left\{ \Pi_k \left(\frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes ... \otimes a_{\sigma(k)} \right) \mid \text{all } a_i \text{ are primitive and homogeneous elements of } A \text{ and satisfy } \deg(a_i) = \lambda_i \text{ for all } i \in \{1, 2, ..., k\} \right\} \right\rangle$$

$$\left(\begin{array}{c} \text{since } \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(k)} = \Pi_k \left(\frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes ... \otimes a_{\sigma(k)} \right) \\ \text{for any } (a_1, a_2, ..., a_k) \in A^k \end{array} \right)$$

$$= \left\langle \Pi_k \left(\left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes ... \otimes a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous elements of } A \text{ and satisfy } \deg(a_i) = \lambda_i \text{ for all } i \in \{1, 2, ..., k\} \right\} \right) \right\rangle$$

$$= \Pi_k \left(\left\langle \left\{ \frac{1}{k!} \sum_{\sigma \in S_k} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes ... \otimes a_{\sigma(k)} \mid \text{all } a_i \text{ are primitive and homogeneous elements of } A \text{ and satisfy } \deg(a_i) = \lambda_i \text{ for all } i \in \{1, 2, ..., k\} \right\} \right\rangle \right)$$

$$\text{(since } \Pi_k \text{ is } F\text{-linear)}$$

$$= \Pi_k \left(\operatorname{Sym}^\lambda (\operatorname{Prim} A) \right) \qquad \text{(by (A1))}.$$

Hence, Π_k restricts to a surjective homomorphism $\operatorname{Sym}^\lambda (\operatorname{Prim} A) \to A^\lambda$.

Moreover, let $\widetilde{\Pi}$ be the homomorphism $\bigoplus_{n \in \mathbb{N}} \Pi_n \mid_{((\operatorname{Prim} A)^{\otimes n})^{S_n}} : \bigoplus_{n \in \mathbb{N}} \left((\operatorname{Prim} A)^{\otimes n} \right)^{S_n} \to A$ (composed of the homomorphisms $\Pi_n \mid_{((\operatorname{Prim} A)^{\otimes n})^{S_n}} : \left((\operatorname{Prim} A)^{\otimes n} \right)^{S_n} \to A$ for all $n \in \mathbb{N}$). This homomorphism $\widetilde{\Pi}$ sends $\frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes ... \otimes a_{\sigma(n)}$ to $\frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} a_{\sigma(2)} ... a_{\sigma(n)}$ for every $n \in \mathbb{N}$ and $(a_1, a_2, ..., a_n) \in (\operatorname{Prim} A)^n$. According to the Poincaré-Birkhoff-Witt theorem, this homomorphism $\widetilde{\Pi}$ is an isomorphism (since the Cartier-Milnor-Moore theorem yields $A \cong U(\operatorname{Prim} A)$, and under the identification of A with $U(\operatorname{Prim} A)$ the homomorphism $\widetilde{\Pi}$ becomes the symmetrization map $S(\operatorname{Prim} A) \to U(\operatorname{Prim} A)$). Hence, $\widetilde{\Pi}$ is injective.

This simplified argument has the additional advantage of being valid when k is not necessarily a field.

- **Page 11:** In the proof of Theorem 5.1, you made a typo: "matrix fom" should be "matrix form".

- **Page 11:** In the proof of Theorem 5.1, you write: "It is clear that (i) implies (iv)". But is this really clear on its own, or is it clear using the fact that $\mathcal{D}(A)$ is closed under convolution (a consequence of Theorem 9.2 in [R2], but [R2] only considers the case when A is the tensor algebra of an alphabet)?

- **Page 12:** In the proof of Lemma 5.3, replace $\iota_{\mu 2}$ by ι_{μ_2} (you forgot to make the 2 an index).

- **Pages 12 and 13:** In the proof of Theorem 5.4, you write: "Moreover:

$$\left(\sum_{\mu<[n]} \mathcal{E}_\mu^\iota\right)^2 = \left(1 - \mathcal{E}_{[n]}^\iota\right)^2 = 1 - \mathcal{E}_{[n]}^\iota = \sum_{\mu<[n]} \mathcal{E}_\mu^\iota,$$

and:

$$\left(\sum_{\mu<[n]} \mathcal{E}_\mu^\iota\right) \circ \mathcal{E}_{[n]}^\iota = \left(1 - \mathcal{E}_{[n]}^\iota\right) \circ \mathcal{E}_{[n]}^\iota = 0.$$

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These formulas are not literally true, because $\sum\limits_{\mu<[n]} \mathcal{E}_\mu^\iota$ is $p_n - \mathcal{E}_{[n]}^\iota$ rather than

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Now, since $\operatorname{Sym}^\lambda(\operatorname{Prim} A) \subseteq \left((\operatorname{Prim} A)^{\otimes k}\right)^{S_k}$, we have

$$\begin{aligned}\widetilde{\Pi} \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)} &= \underbrace{\left(\widetilde{\Pi} \mid_{\left((\operatorname{Prim} A)^{\otimes k}\right)^{S_k}}\right)}_{\substack{=\Pi_k\mid_{\left((\operatorname{Prim} A)^{\otimes k}\right)^{S_k}} \\ \left(\text{since } \widetilde{\Pi} = \bigoplus\limits_{n\in\mathbb{N}} \Pi_n\mid_{\left((\operatorname{Prim} A)^{\otimes n}\right)^{S_n}}\right)}} \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)} \\ &= \left(\Pi_k \mid_{\left((\operatorname{Prim} A)^{\otimes k}\right)^{S_k}}\right) \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)} = \Pi_k \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)} .\end{aligned}$$

Since $\widetilde{\Pi} \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)}$ is injective (because $\widetilde{\Pi}$ is injective), this yields that $\Pi_k \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)}$ is injective.

Now, consider the surjective homomorphism $\operatorname{Sym}^\lambda(\operatorname{Prim} A) \to A^\lambda$ to which Π_k restricts. This homomorphism is also injective (since $\Pi_k \mid_{\operatorname{Sym}^\lambda(\operatorname{Prim} A)}$ is injective), and thus it is an isomorphism. Thus, Π_k restricts to an isomorphism $\operatorname{Sym}^\lambda(\operatorname{Prim} A) \to A^\lambda$. Hence, $\operatorname{Sym}^\lambda(\operatorname{Prim} A)$ is isomorphic to A^λ through the map Π_k, qed.

$1 - \mathcal{E}^{\iota}_{[n]}$ (since

$$\sum_{\mu<[n]} \mathcal{E}^{\iota}_{\mu} + \mathcal{E}^{\iota}_{[n]} = \sum_{\mu\leq[n]} \mathcal{E}^{\iota}_{\mu} = \sum_{\substack{\mu \text{ is a partition} \\ \text{of } n}} \mathcal{E}^{\iota}_{\mu} = \sum_{\substack{\lambda \text{ is a partition} \\ \text{of } n}} \underbrace{\mathcal{E}^{\iota}_{\lambda}}_{= \sum_{\substack{\mu \text{ is a composition of } n; \\ p(\mu)=\lambda}} a^{\iota}_{\mu}\cdot\iota_{\mu}}$$

$$= \underbrace{\sum_{\substack{\lambda \text{ is a partition} \\ \text{of } n}} \sum_{\substack{\mu \text{ is a composition of } n; \\ p(\mu)=\lambda}}}_{= \sum_{\mu \text{ is a composition of } n}} a^{\iota}_{\mu} \cdot \iota_{\mu} = \sum_{\mu \text{ is a composition of } n} a^{\iota}_{\mu} \cdot \iota_{\mu} = p_n$$

). Only if you restrict all maps to the n-th graded component of A, these equations become true. Alternatively, you could replace these equations by

$$\left(\sum_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right)^2 = \left(p_n - \mathcal{E}^{\iota}_{[n]}\right)^2 = \underbrace{p_n^2}_{=p_n} - \underbrace{\mathcal{E}^{\iota}_{[n]} \circ p_n}_{=\mathcal{E}^{\iota}_{[n]}} - \underbrace{p_n \circ \mathcal{E}^{\iota}_{[n]}}_{=\mathcal{E}^{\iota}_{[n]}} + \underbrace{\left(\mathcal{E}^{\iota}_{[n]}\right)^2}_{=\mathcal{E}^{\iota}_{[n]}} = p_n - \mathcal{E}^{\iota}_{[n]} = \sum_{\mu<[n]} \mathcal{E}^{\iota}_{\mu},$$

and:

$$\left(\sum_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right) \circ \mathcal{E}^{\iota}_{[n]} = \left(p_n - \mathcal{E}^{\iota}_{[n]}\right) \circ \mathcal{E}^{\iota}_{[n]} = \underbrace{p_n \circ \mathcal{E}^{\iota}_{[n]}}_{=\mathcal{E}^{\iota}_{[n]}} - \underbrace{\left(\mathcal{E}^{\iota}_{[n]}\right)^2}_{=\mathcal{E}^{\iota}_{[n]}} = 0.$$

A similar inaccuracy appears at the end of page 13: There you write

$$h \circ 1 = h \circ (h + g + k) = bh + h \circ g.$$

This is not wrong, but not exactly clear: Probably you want to say

$$h = h \circ p_n = h \circ (h + g + k) = bh + h \circ g.$$

- **Page 13:** You write: "In other words, $\mathcal{E}^{\iota}_{[n]}$ and $\sum\limits_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}$ are two orthogonal idempotents." But in order to show this, you must not only prove that $\left(\mathcal{E}^{\iota}_{[n]}\right)^2 = \mathcal{E}^{\iota}_{[n]}$, $\left(\sum\limits_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right)^2 = \sum\limits_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}$ and $\left(\sum\limits_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right) \circ \mathcal{E}^{\iota}_{[n]} = 0$ (this you have proven), but also prove that $\mathcal{E}^{\iota}_{[n]} \circ \left(\sum\limits_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right) = 0$. This is easy, of course:

$$\mathcal{E}^{\iota}_{[n]} \circ \left(\sum_{\mu<[n]} \mathcal{E}^{\iota}_{\mu}\right) = \mathcal{E}^{\iota}_{[n]} \circ \left(p_n - \mathcal{E}^{\iota}_{[n]}\right) = \underbrace{\mathcal{E}^{\iota}_{[n]} \circ p_n}_{=\mathcal{E}^{\iota}_{[n]}} - \underbrace{\left(\mathcal{E}^{\iota}_{[n]}\right)^2}_{=\mathcal{E}^{\iota}_{[n]}} = 0.$$

But it should be mentioned, I think.

- **Page 14:** You write: "It follows that the coefficients $a_\mu^{\iota^\epsilon}$ of the higher Lie idempotents of the third kind depend polynomially of ϵ."

  First, I don't understand how this follows from $p_n = \sum\limits_{|\mu|=n} F_\mu^{\iota^\epsilon}$. While all $F_\mu^{\iota^\epsilon}$ are (by definition) linear combinations (with constant coefficients) of *compositions* of various ι_ν^ϵ, it is not clear (to me) why they are linear combinations (with coefficients polynomial in ϵ) of *convolutions* of various ι_ν^ϵ. I do know that $\mathcal{D}_{\iota^\epsilon}$ is closed under convolution (by Theorem 5.1, since $\iota^\epsilon \in \langle \iota, e\rangle \subseteq \mathcal{D}(A)$), and this yields that they are linear combinations of convolutions of various ι_ν^ϵ, but why with coefficients polynomial in ϵ ?

  Second, even if we can show that we can write p_n as a linear combination of ι_μ^ϵ with coefficients polynomial in ϵ, then it is not clear to me why these coefficients, when specializing at $\epsilon = 1$, become our a_μ^ι - in fact, the a_μ^ι are not always uniquely determined by $p_n = \sum\limits_{|\mu|=n} a_\mu^\iota \iota_\mu$ (since the ι_μ are not always linearly independent), so the a_μ^ι you have started with might not be the same as the a_μ^ι you get by writing p_n as a linear combination of ι_μ^ϵ and specializing at $\epsilon = 1$ (although both families of a_μ^ι satisfy $p_n = \sum\limits_{|\mu|=n} a_\mu^\iota \iota_\mu$).

  I am interested in how you actually show that the $a_\mu^{\iota^\epsilon}$ depend polynomially of ϵ in such a way that specialization at $\epsilon = 1$ yields our initial a_μ^ι. I think I can show this (with some handwaving) under the additional condition that $a_{[n]}^\iota = 1$ for every n. Here is how my proof (roughly) goes:

  Start with the equations $p_n = \sum\limits_{|\mu|=n} a_\mu^\iota \iota_\mu$. By repeated convolution, these equations yield equations of the form $p_\nu = \sum\limits_{\substack{|\mu|=|\nu|;\\ \mu\geq\nu}} a_{\mu,\nu}^\iota \iota_\mu$ (with $a_{\mu,\nu}^\iota$ being scalars, and $a_{\mu,[n]}^\iota = a_\mu^\iota$) for all partitions ν, where $\mu \geq \nu$ means that the composition μ can be obtained by splitting some parts of ν into smaller parts (this defines a partial order $\geq$ on compositions). Since $a_{[n]}^\iota = 1$ for every n, we find that $a_{\nu,\nu}^\iota = 1$ for every composition ν. Now, the equations $p_\nu = \sum\limits_{\substack{|\mu|=|\nu|;\\ \mu\geq\nu}} a_{\mu,\nu}^\iota \iota_\mu$ show us that $\left(a_{\mu,\nu}^\iota\right)_{|\mu|=|\nu|=n}$ is an upper triangular matrix, and the equations $a_{\nu,\nu}^\iota = 1$ show that its diagonal entries are $= 1$. Hence, it has an inverse matrix $\left(b_{\mu,\nu}^\iota\right)_{|\mu|=|\nu|=n}$ which satisfies $\iota_\nu = \sum\limits_{\substack{|\mu|=|\nu|;\\ \mu\geq\nu}} b_{\mu,\nu}^\iota p_\mu$ for all compositions ν, and again is upper triangular and has diagonal entries $= 1$. The same argument, done for e instead of ι, shows that there exists a matrix $\left(b_{\mu,\nu}^e\right)_{|\mu|=|\nu|=n}$ which satisfies $e_\nu = \sum\limits_{\substack{|\mu|=|\nu|;\\ \mu\geq\nu}} b_{\mu,\nu}^e p_\mu$ for all compositions ν, and again is upper triangular and has its diagonal entries $= 1$. Now, the matrix $\left(\epsilon \cdot b^{\iota}_{\mu,\nu} + (1-\epsilon) \cdot b^{e}_{\mu,\nu}\right)_{|\mu|=|\nu|=n}$ satisfies

$$\iota^{\epsilon}_{\nu} = \epsilon \cdot \iota_{\nu} + (1-\epsilon) \cdot e_{\nu} = \epsilon \cdot \sum_{\substack{|\mu|=|\nu|;\\ \mu \geq \nu}} b^{\iota}_{\mu,\nu} p_{\mu} + (1-\epsilon) \cdot \sum_{\substack{|\mu|=|\nu|;\\ \mu \geq \nu}} b^{e}_{\mu,\nu} p_{\mu}$$
$$= \sum_{\substack{|\mu|=|\nu|;\\ \mu \geq \nu}} \left(\epsilon \cdot b^{\iota}_{\mu,\nu} + (1-\epsilon) \cdot b^{e}_{\mu,\nu}\right) p_{\mu}$$

for all compositions ν, and again is upper triangular and has its diagonal entries $= 1$. Hence, its inverse matrix $\left(a^{\iota^{\epsilon}}_{\mu,\nu}\right)_{|\mu|=|\nu|=n}$ satisfies $p_n = \sum\limits_{|\mu|=n} a^{\iota^{\epsilon}}_{\mu,[n]} \iota^{\epsilon}_{\mu}$, but its entries $a^{\iota^{\epsilon}}_{\mu,\nu}$ are polynomials in the entries of $\left(\epsilon \cdot b^{\iota}_{\mu,\nu} + (1-\epsilon) \cdot b^{e}_{\mu,\nu}\right)_{|\mu|=|\nu|=n}$ (because if C is an upper triangular matrix with diagonal entries $= 1$, then the entries of C^{-1} are polynomials in the entries of C), and thus polynomials in ϵ. This gives us what we want.

But I cannot get rid of the condition that $a^{\iota}_{[n]} = 1$ for every n (not only for the one we are working with, but also for the smaller n, because we need all $a^{\iota}_{\nu,\nu}$ to be 1).

HOWEVER, I think that I can modify your proof of Theorem 5.4 in a different way to make it valid:

First of all, let us generalize the results of Section 3 from one Lie idempotent to two Lie idempotents:[4]

Lemma 5.6. Let ι and ρ be two Lie idempotents. Then, any two compositions λ and μ such that $|\lambda| \neq |\mu|$ satisfy $\iota_{\lambda} \circ \rho_{\mu} = 0$.

This is a very obvious fact (it is obvious because the image of ρ_{μ} lies in the $|\mu|$-th graded component of H, whereas ι_{λ} sends every graded component of H except of the $|\lambda|$-th one to 0), and it generalizes the property $\iota_{\lambda} \circ \iota_{\mu} = 0$ for $|\lambda| \neq |\mu|$.

Less trivially, we have:

Lemma 5.7. Let ι and ρ be two Lie idempotents. Let μ and λ be two compositions of the same weight and the same length k.

(i) If $p(\lambda) \neq p(\mu)$, then $\iota_{\mu} \circ \rho_{\lambda} = 0$.

(ii) If $p(\lambda) = p(\mu)$, then $\iota_{\mu} \circ \rho_{\lambda} = N \rho_{\mu}$, where N is the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $p(\mu) = p(\lambda)$. (This number N only depends on $p(\lambda) = p(\mu)$, and will often be denoted by $N(p(\lambda))$ or by $N(\lambda)$.)

For the proof of Lemma 5.7, proceed in the same way as in the proof of Lemma 3.1. You will need the identity $\iota \circ \rho = \rho$, which follows from $\iota\mid_{\operatorname{Prim} A} = \operatorname{id}_{\operatorname{Prim} A}$ (because both ι and ρ are Lie idempotents, i. e., projections on $\operatorname{Prim} A$).

Similarly:

Lemma 5.8. Let ι and ρ be two Lie idempotents. Let μ and λ be two compositions of the same weight such that $l(\mu) > l(\lambda)$. Then $\iota_{\mu} \circ \rho_{\lambda} = 0$.

This is proven in the same way as Lemma 3.2.

Next, we need a kind of generalization of Lemma 5.3:

Lemma 5.9. Let ι and ρ be two Lie idempotents. Let λ be a partition. For

[4] In the following Lemmas 5.6, 5.7, 5.8 and 5.9, we don't assume that $\mathcal{D}(A) = \mathcal{D}_{\iota}$.

every composition μ with $p(\mu) = \lambda$, let b_μ^ι and b_μ^ρ be two scalars. Then,

$$\left(\sum_{p(\mu)=\lambda} b_\mu^\iota \iota_\mu \right) \circ \left(\sum_{p(\mu)=\lambda} b_\mu^\rho \rho_\mu \right) = \left(\sum_{p(\mu)=\lambda} b_\mu^\iota \right) N \left(\sum_{p(\mu)=\lambda} b_\mu^\rho \rho_\mu \right),$$

where N is the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence λ.

The proof of this lemma proceeds in the same way as the identity $\left(\sum\limits_{p(\mu)=\lambda} b_\mu \iota_\mu \right)^2 = \left(\sum\limits_{p(\mu)=\lambda} b_\mu \right) N \left(\sum\limits_{p(\mu)=\lambda} b_\mu \iota_\mu \right)$ was proven in the proof of Lemma 5.3. Here are the details of the proof:

Proof of Lemma 5.9. For every composition μ satisfying $p(\mu) = \lambda$, we know that N is the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence $p(\mu)$ (because N is defined as the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence λ, but we have $\lambda = p(\mu)$). Hence, for every composition μ satisfying $p(\mu) = \lambda$, we have $i_\mu \circ \rho_\mu = N \rho_\mu$ (by Lemma 5.7 **(ii)**, applied to μ instead of λ). Since composition of linear maps is bilinear, we have

$$\begin{aligned}
&\left(\sum_{p(\mu)=\lambda} b_\mu^\iota \iota_\mu \right) \circ \left(\sum_{p(\mu)=\lambda} b_\mu^\rho \rho_\mu \right) \\
&= \sum_{p(\mu)=\lambda} \sum_{p(\mu)=\lambda} b_\mu^\iota b_\mu^\rho \underbrace{\iota_\mu \circ \rho_\mu}_{=N\rho_\mu} = N \sum_{p(\mu)=\lambda} \sum_{p(\mu)=\lambda} b_\mu^\iota b_\mu^\rho \rho_\mu \\
&= \left(\sum_{p(\mu)=\lambda} b_\mu^\iota \right) N \left(\sum_{p(\mu)=\lambda} b_\mu^\rho \rho_\mu \right) \qquad \text{(since composition of linear maps is bilinear)}.
\end{aligned}$$

This proves Lemma 5.9.

Now to the *proof of Theorem 5.4.* We proceed in the same way as you do (with one exception: we don't have to assume $h \neq 0$) until your Claim 5.5 (which we cannot make anymore, since we haven't assumed that $h \neq 0$). Then, just as you, we prove $h \circ g = (1 - b) h$ and $k \circ g = (b - 1) h$. Now I am going to show that $h^2 = h$.

First of all, we have $p_n = \sum\limits_{|\mu|=n} \frac{1}{n!} e_\mu$ [5]. Let us define a scalar a_μ^e by $a_\mu^e = \frac{1}{n!}$ for every partition μ. Then, $p_n = \sum\limits_{|\mu|=n} \underbrace{\frac{1}{n!}}_{=a_\mu^e} e_\mu = \sum\limits_{|\mu|=n} a_\mu^e e_\mu$. Hence, in the same way as we defined an element $\mathcal{E}_\lambda^\iota$ for every partition λ in Definition 5.2, we can define

[5] This is a known fact (I knew it in the form $p_n = \sum\limits_{\ell=0}^{n} \frac{1}{\ell!} \sum\limits_{\substack{(a_1, a_2, ..., a_\ell) \in \{1,2,...,n\}^\ell; \\ n = a_1 + a_2 + ... + a_\ell}} (e_{a_1} * e_{a_2} * ... * e_{a_\ell})$). It can be easily derived from the fact that $e = \log_* (\mathrm{id})$, so that $\mathrm{id} = \exp_* e = \exp_* (e_1 + e_2 + e_3 + ...)$.

an element $\mathcal{E}_\lambda^e$ for every partition λ by the formula

$$\mathcal{E}_\lambda^e := \sum_{p(\mu)=\lambda} \underbrace{a_\mu^e}_{=\frac{1}{n!}} \cdot e_\mu = \sum_{p(\mu)=\lambda} \frac{1}{n!} e_\mu .$$

From Lemmas 5.7 and 5.8 (applied to e and ι instead of ι and ρ), we conclude that $\mathcal{E}_\lambda^e \circ \mathcal{E}_\mu^\iota = 0$ for every partition $\mu < \lambda$. Hence,

$$\mathcal{E}_\lambda^e \circ \underbrace{k}_{=\sum_{\mu<\lambda} \mathcal{E}_\mu^\iota} = \mathcal{E}_\lambda^e \circ \left(\sum_{\mu<\lambda} \mathcal{E}_\mu^\iota \right) = \sum_{\mu<\lambda} \underbrace{\mathcal{E}_\lambda^e \circ \mathcal{E}_\mu^\iota}_{\substack{=0 \\ (\text{since } \mu<\lambda)}} = 0.$$

On the other hand, for every partition λ, let $N(\lambda)$ denote the number of permutations of $\{1, 2, ..., k\}$ which act trivially on the sequence λ. We have $\mathcal{E}_\lambda^e = \sum_{p(\mu)=\lambda} \frac{1}{n!} e_\mu$ and $h = \mathcal{E}_\lambda^\iota = \sum_{p(\mu)=\lambda} a_\mu^\iota e_\mu$, so that

$$\mathcal{E}_\lambda^e \circ h = \left(\sum_{p(\mu)=\lambda} \frac{1}{n!} e_\mu \right) \circ \left(\sum_{p(\mu)=\lambda} a_\mu^\iota \iota_\mu \right) = \left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) \cdot \underbrace{\left(\sum_{p(\mu)=\lambda} a_\mu^\iota \iota_\mu \right)}_{=h}$$

$$\left(\text{by Lemma 5.9, applied to } N(\lambda), \frac{1}{n!}, a_\mu^\iota, e \text{ and } \iota \text{ instead of } N, b_\mu^\iota, b_\mu^\rho, \iota \text{ and } \rho \right)$$

$$= \left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) h.$$

Now, compare

$$\underbrace{\mathcal{E}_\lambda^e \circ k}_{=0} \circ g = 0 \circ g = 0$$

with

$$\mathcal{E}_\lambda^e \circ \underbrace{k \circ g}_{=(b-1)h} = (b-1) \underbrace{\mathcal{E}_\lambda^e \circ h}_{=\left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) h} = (b-1) \left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) h.$$

This yields

$$(b-1) \left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) h = 0.$$

Since $\left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda)$ is invertible in k (in fact, $\left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) \neq 0$ obviously; we can even prove that $\left(\sum_{p(\mu)=\lambda} \frac{1}{n!} \right) N(\lambda) = 1$, but we don't need this),

this becomes $(b-1)\,h = 0$, so that $h = bh$. Compared with $h \circ h = bh$ (which follows from the proof of Lemma 5.3), this yields $h \circ h = h$, so that h is an idempotent.
Since $g^2 = g$ (because $g = \sum\limits_{\mu > \lambda} \mathcal{E}_\mu^\iota$, and by the induction assumption the $\mathcal{E}_\mu^\iota$ are orthogonal idempotents), $h \circ g = (1-b)\,h = -\underbrace{(b-1)\,h}_{=0} = 0$ and $k \circ g = (b-1)\,h = 0$, we can continue the proof as you do after you prove Claim 5.5. This proves Theorem 5.4.

- **Page 14:** There is a typo: $b_\lambda^{i^\epsilon}$ should be $b_\lambda^{\iota^\epsilon}$.
- **Page 15:** You write: "and the proof of theorem 5.3 is complete". The theorem is Theorem 5.4, not 5.3.
- **Page 16:** In reference [R1], typo: "represntations".