Polynomial Representations of GL_n
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Errata and addenda by Darij Grinberg

14. Corrections

This is a list of corrections to "Polynomial Representations of GL_n" by J. A. Green (2nd edition 2007). Most of it concerns the Appendix on Schensted correspondence and Littelmann paths by K. Erdmann, J.A. Green and M. Schocker (freely available from SpringerLink). I have read almost everything in this Appendix up until the end of (D.7). After the list of corrections, I give an alternative proof of the Littlewood–Richardson rule that avoids any representation theory (see Section 15 below).

14.1. Corrections to §A

1. **page 74, (A.2b):** It is worth mentioning that what is called a "standard" tableau here is often called a "semistandard" tableau in more combinatorial-minded sources.

2. **page 75, §A.3:** Add "Let $i = i_1 i_2 \ldots i_r \in I(n,r)$ be a word." after "we need some preliminary definitions".

3. **page 75, (A.3a):** The equality $h_c^i(t) = \omega(i_1) + \cdots + \omega(i_t)$ is actually true for all $t \in \{0, 1, \ldots, r\}$, including $t = 0$.

4. **page 75, (A.3b):** Again, this is actually true for all $t \in \{0, 1, \ldots, r\}$, including $t = 0$.

5. **page 76, (A.3e):** It is worth saying that $M^i = 0$ is equivalent to $q = 0$ (since the least $t \in \{0, 1, \ldots, r\}$ satisfying $h_c^i(t) = 0$ is clearly 0).

6. **page 76, (A.3f):** It is worth saying that $M^i = h_c^i(r)$ is equivalent to $\overline{q} = r$ (since the greatest $t \in \{0, 1, \ldots, r\}$ satisfying $h_c^i(t) = h_c^i(r)$ is clearly r). Hence, if we don't have $M^i = h_c^i(r)$, then we don't have $\overline{q} = r$, and thus we have $\overline{q} < r$, so that speaking of $s_{\overline{q}+1}$ makes sense.

7. **page 76, (A.3):** There is an urgent need for examples here! Here is one:

   Let $n = 5$ and $c = 2$ and $r = 10$ and $i = 4232133253$. Then, the height function $h_c^i = h_2^i$ sends the numbers $0, 1, \ldots, 10$ to $0, 0, 1, 0, 1, 1, 0, -1, 0, 0, -1$, respectively. Hence, $M = M_c^i = 1$ and $q = q_c^i = 2$ and $\overline{q} = \overline{q}_c^i = 5$. Thus,

$$\widetilde{f}_c(i) = 4\underline{3}32133253 \qquad \text{and}$$
$$\widetilde{e}_c(i) = 42321\underline{3}3253,$$

where we have underlined the unique entry of i that got changed.

Now let $j = \widetilde{f}_c(i) = 4332133253$. Then, the height function $h_c^j = h_2^j$ sends the numbers $0, 1, \ldots, 10$ to $0, 0, -1, -2, -1, -1, -2, -3, -2, -2, -3$, respectively. Hence, $M = M_c^j = 0$ and $q = q_c^j = 0$ and $\overline{q} = \overline{q}_c^j = 1$. Thus,

$$\widetilde{f}_c(j) = \infty \text{ (since } M_c^j = 0\text{)} \qquad \text{and}$$
$$\widetilde{e}_c(j) = 4\underline{2}32133253 = i,$$

which confirms (A.3g)(5).

8. **page 76, (A.3g) (6):** It is worth saying that

$$h_c^{i|j}(t) = \begin{cases} h_c^i(t), & \text{if } t \leq r; \\ h_c^i(r) + h_c^i(t-r), & \text{if } t \geq r \end{cases} \qquad \text{for all } t \in \{0, 1, \ldots, r+s\}$$

(note that the case $t = r$ is covered by both cases here), and therefore

$$M_c^{i|j} = \max\left\{M_c^i,\ h_c^i(r) + M_c^j\right\} \qquad \text{and}$$
$$q_c^{i|j} = \begin{cases} q_c^i, & \text{if } M_c^i \geq h_c^i(r) + M_c^j; \\ h_c^i(r) + q_c^j, & \text{if } M_c^i < h_c^i(r) + M_c^j \end{cases} \qquad \text{and}$$
$$\overline{q}_c^{i|j} = \begin{cases} \overline{q}_c^i, & \text{if } M_c^i > h_c^i(r) + M_c^j; \\ h_c^i(r) + \overline{q}_c^j, & \text{if } M_c^i \leq h_c^i(r) + M_c^j. \end{cases}$$

These facts are easy to verify, and are the reason why the formulas for $\widetilde{f}_c(i \mid j)$ and $\widetilde{e}_c(i \mid j)$ hold.

9. **page 79, Definition:** Replace "$y_{m,\lambda_{m-1}}$" by "$y_{m-1,\lambda_{m-1}}$".

10. **page 79:** "proved in §C.4" $\to$ "proved in §C.6".

14.2. Corrections to §B

1. **page 80, §B.1:** What is called a "*standard*" tableau here is usually called a "*semistandard*" tableau in texts on combinatorics.

2. **page 83, Step 1:** Replace "$(< x_{1,\mu(1)+1} = \infty)$" by "$(< u_{1,\mu(1)+1} = \infty)$".

3. **page 83:** It should be said that the notation μ_a is used as synonym for $\mu(a)$.

4. **page 84, Definition of z:** It should be explained why $z \leq n$ (otherwise, the vector ε_z in (B.3d) is undefined; and earlier even, the $k(a)$ in the inductive step would be undefined if $a = z + 1$).

There is an easy explanation: We WLOG assume that $z > 1$ (since the claim is obvious for $z = 1$). Then, $x_z = u_{z-1,k(z-1)}$ is an entry of the tableau u, and hence belongs to $\underline{n}$. Thus, $x_z \leq n$.

We have $1 \leq x_1 < x_2 < \cdots < x_z$ by the definition of the x_a. Thus, $x_a \geq a$ for each a (by induction on a). In particular, $x_z \geq z$, so that $z \leq x_z \leq n$, qed.

5. **page 84, (B.3d):** "We shall show in (B.5b)" $\to$ "We shall show in (B.5a)".

6. **page 85:** Replace "we define $x_a := k_{a-1,k(a-1)}$" by "we define $x_a := u_{a-1,k(a-1)}$".

7. **page 85:** Replace "between the entries $u_{a,k(a-1)}$" by "between the entries $u_{a,k(a)-1}$".

8. **page 85:** It would be good to introduce another piece of standard terminology: The sequence of places

$$((1, k(1)), (2, k(2)), \ldots, (z, k(z)))$$

(these are the places in which P differs from U) is called the *bumping route* (or *bumping path*) of the insertion $U \leftarrow x_1$.

9. **page 85, (B.4a):** Replace "$P = P(i)$" by just "P". (There is no "i" involved yet.)

10. **page 85, (B.4a):** Remove the comma in "The row (z) of P, is".

11. **page 86, (B.4a):** Remove the comma in "3 in row (1) of U, is bumped".

12. **page 86, last paragraph:** Remove the comma in "$P(i_1 i_2, \cdots i_r)$".

13. **page 88, proof of (B.5b):** "then by (B.5a)" $\to$ "then by (B.3d)".

14. **page 89, proof of (B.5b):** It took me a while to understand how "$u_{a,k(a+1)} \leq x_a$" is obtained here. Namely: Combining $k(a+1) \leq k(a)$ with $k(a+1) = h \neq k(a)$, we obtain $k(a+1) < k(a)$, hence $k(a+1) \leq k(a) - 1$. By the row-standardness of U, we thus have $u_{a,k(a+1)} \leq u_{a,k(a)-1} \leq x_a$ by (B.3c).

15. **page 90, Step 1 of the construction of the extrusion sequence:** I would replace "$w_z := p_{z,\lambda_z}$" by "$w_z := p_{z,l(z)} = p_{z,\lambda_z}$", in order to make it clear where the "$w_{a+1} := p_{a+1,l(a+1)}$" in the Inductive step comes from.

16. **page 90, Step 1 of the construction of the extrusion sequence:** Remove the comma in "the entry in P, at the place".

17. **page 90, Step 2 of the construction of the extrusion sequence:** Worth reminding the reader that p_{k,λ_k+1} is understood to be ∞, and that $p_{k,0}$ is understood to be 0 for each k.

18. **page 91, proof of (B.6h):** The appeal to analogy in the last sentence of this proof is a bit of a stretch, so let me explain in some detail how the standardness of U is proved:

    First, we show that

    $$l(1) \geq l(2) \geq \cdots \geq l(z). \tag{1}$$

    This is an analogue of (B.3f), and is proved rather similarly:

    [*Proof of (1):* Let $a \in \{1,2,\ldots,z-1\}$. We must prove that $l(a) \geq l(a+1)$. Since P is column-standard, we have $p_{a,l(a+1)} < p_{a+1,l(a+1)} = w_{a+1}$ (by the definition of w_{a+1}). But $l(a)$ is the largest $l \in \{1,2,\ldots,\lambda_a\}$ such that $p_{a,l} < w_{a+1}$. Hence, from $p_{a,l(a+1)} < w_{a+1}$, we obtain $l(a) \geq l(a+1)$. Thus, (1) is proved.]

    Next, we observe that

    $$w_1 < w_2 < \cdots < w_z, \tag{2}$$

    because each $a \in \{1,2,\ldots,z-1\}$ satisfies $w_a = p_{a,l(a)} < w_{a+1}$ (by (B.6f)).

    Now, we can show that U is standard:

    First, we shall show that U is row-standard, i.e., that $u_{a,h-1} \leq u_{a,h}$ for all adjacent pairs $(a,h-1)$, (a,h) of places in any row (a) of $[\mu]$.

    [*Proof:* This is very similar to the proof of (i) in the proof of (B.5b). Let $(a,h-1)$ and (a,h) be two adjacent places in any row of $[\mu]$. Since P is row-standard, we have $p_{a,h-1} \leq p_{a,h}$. But the only entry in row a that differs between U and P is $u_{a,l(a)} \neq p_{a,l(a)}$. Thus, our claim $u_{a,h-1} \leq u_{a,h}$ follows from $p_{a,h-1} \leq p_{a,h}$, unless $l(a)$ is one of $h-1$ and h. So it remains to consider these two cases $l(a) = h-1$ and $l(a) = h$. If $l(a) = h-1$, then we must prove that $u_{a,l(a)} \leq u_{a,l(a)+1}$; but this follows from $u_{a,l(a)} = w_{a+1} \leq p_{a,l(a)+1}$ (by (B.6f)) and $p_{a,l(a)+1} = u_{a,l(a)+1}$. If $l(a) = h$, then we must prove that $u_{a,l(a)-1} \leq u_{a,l(a)}$; but this follows from $u_{a,l(a)-1} = p_{a,l(a)-1} \leq p_{a,l(a)} < w_{a+1}$ (by (B.6f)) and $w_{a+1} = u_{a,l(a)}$. In either case, $u_{a,h-1} \leq u_{a,h}$ is proved.]

    Thus, we have shown that U is row-standard. It remains to prove that U is column-standard, i.e., that if $(a-1,h)$ and (a,h) are two adjacent places in the same column of $[\mu]$, then

    $$u_{a-1,h} < u_{a,h}. \tag{3}$$

    [*Proof:* Let $(a-1,h)$ and (a,h) be two adjacent places in the same column of $[\mu]$. If $h \neq l(a)$ and $h \neq l(a-1)$, then $u_{a,h} = p_{a,h}$ and $u_{a-1,h} \neq p_{a-1,h}$, then (3) follows from $p_{a-1,h} < p_{a,h}$, which holds because P is column-standard.

    Now consider the case when $h = l(a)$ and $h \neq l(a-1)$. Then, $l(a) = h \neq l(a-1)$. Moreover, $l(a-1) \geq l(a)$ (by (1)), so that $l(a-1) > l(a)$ (since $l(a-1) \neq l(a)$). Now, from $h = l(a)$, we obtain $p_{a,h} = p_{a,l(a)} = w_a < w_{a+1}$ (by (2)). But the column-standardness of P yields $p_{a-1,h} < p_{a,h}$. Since

$h \neq l(a-1)$, we have $u_{a-1,h} = p_{a-1,h} < p_{a,h} < w_{a+1} = u_{a,l(a)} = u_{a,h}$ (since $l(a) = h$). Thus, (3) is proved in the case when $h = l(a)$ and $h \neq l(a+1)$.

Now consider the case when $h \neq l(a)$ and $h = l(a-1)$. Thus, $h = l(a-1) \geq l(a)$ (by (1)). Combining this with $h \neq l(a)$, we find $h > l(a)$. Therefore, $h \geq l(a)+1$, so that $p_{a,h} \geq p_{a,l(a)+1}$ (since P is row-standard). Moreover, (B.6f) yields $p_{a,l(a)+1} \geq w_{a+1} > w_a$ (by (2)). Now, $h \neq l(a)$, so that $u_{a,h} = p_{a,h} \geq p_{a,l(a)+1} > w_a = u_{a-1,l(a-1)} = u_{a-1,h}$ (since $l(a-1) = h$). That is, $u_{a-1,h} < u_{a,h}$. Thus, (3) is proved in the case when $h \neq l(a)$ and $h = l(a+1)$.

Finally, consider the case when $h = l(a)$ and $h = l(a-1)$. Thus, $u_{a-1,h} = u_{a-1,l(a-1)} = w_a < w_{a+1}$ (by (2)) and $u_{a,h} = u_{a,l(a)} = w_{a+1}$, so that $u_{a-1,h} < w_{a+1} = u_{a,h}$. Thus, (3) is proved in the case when $h = l(a)$ and $h = l(a+1)$.

Hence, we have proved (3) in all four cases.]

We thus have shown that U is column-standard. Since U is also row-standard, we thus know that U is standard.

19. **page 91, (B.6j):** Remove the comma at the end of the insertion sequence.

20. **page 92, proof of (i):** Instead of "$x_a = p_{a,k(a)} < w_{a+1} \leq u_{a,k(a)+1} = p_{a,k(a)+1}$", it would be logically clearer to write "$p_{a,k(a)} = x_a < x_{a+1} = w_{a+1} = x_{a+1} \leq u_{a,k(a)+1} = p_{a,k(a)+1}$".

21. **page 92, proof of (i):** Replace "$\mathsf{E}(\mathsf{J}((\mu, U, V), x_1))))$" by "$\mathsf{E}(\mathsf{J}((\mu, U, V), x_1))$".

22. **page 92, proof of (i):** To prove that $\mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), x_1)$, we can proceed as follows: Each $a \in \{1, 2, \ldots, z\}$ satisfies

$$\begin{aligned} u_{a,l(a)} &= u_{a,k(a)} && (\text{since } l(a) = k(a) \text{ by (B.6m)}) \\ &= x_{a+1} && (\text{by (B.3b)}) \\ &= w_{a+1} && (\text{by (B.6m) again}). \end{aligned}$$

Thus, when the entries of P in the places $(a, l(a))$ get replaced by w_{a+1} in the construction of $\mathsf{E}((\lambda, P, Q))$, they merely revert to their original values $u_{a,l(a)}$ that they used to have in U. All the other entries of P were already equal to the respective entries of U (since $l(a) = k(a)$ for all a), and remain so. Hence, the tableau P reverts back to U in the construction of $\mathsf{E}((\lambda, P, Q))$. Moreover, the dominant weight λ reverts back to μ (since λ was $\mu + \varepsilon_z$ and now becomes $(\mu + \varepsilon_z) - \varepsilon_z = \mu$), and the tableau Q reverts back to V (since it loses the entry r in cell (z, λ_z)). Hence, $\mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), x_1)$.

23. **page 92, proof of (ii):** This proof relies on the tacit assumption that the z at the beginning of the extrusion sequence (B.6k) is the same as the z at the

end of the insertion sequence (B.6o). This is indeed the case, but it is not obvious from the get-go, so the notation needs to be changed and some justification added. In more detail:

After "Let (B.6k) be the extrusion sequence which defines $\mathsf{E}((\lambda, P, Q)) = ((\mu, U, V), w_1)$ (see (B.6g)).", add "Thus, $\mu = \lambda - \varepsilon_z$.".

Replace both "z"s in (B.6o) by "x"s.

In (B.6p), add "and $a \leq x$" after "$w_a = x_a$".

After "holds for some a", add "$< z$".

In the proof of (B.6p), replace "unique element of $\{1, 2, , \ldots, z\}$" by "unique element of $\{1, 2, \ldots, \lambda_a\}$" (this is a separate typo).

After "However this proves that $l(a) = k(a)$, from (B.3c).", add: "Thus, $k(a) = l(a) \in \{1, 2, \ldots, \lambda_a\} = \{1, 2, \ldots, \mu_a\}$ (because $a \neq z$ and therefore $\lambda_a = \mu_a$). This entails $k(a) \neq \mu_a + 1$. Thus, by (B.3b), the insertion sequence (B.6o) does not stop at $k(a)$; in other words, $a < z'$. Hence, $a + 1 \leq z'$.

In the sentence "Consequently $w_{a+1} = p_{a,l(a)} = p_{a,k(a)} = x_{a+1}$ (see (B.3b); we are here using the insertion of x_1 into (μ, U, V))", replace both "p"s by "u"s.

Replace "Now we can prove, by induction on a" by "Thus, we have proved (B.6p) by induction on a. As a side-result, we have obtained".

After (B.6r), add "In particular, this shows that $l(z) = k(z)$, so that $k(z) = l(z) = \lambda_z = \mu_z + 1$. Therefore, by (B.3b), the insertion sequence (B.6o) stops at $k(z)$, meaning that we have $z' = z$.".

24. **page 94:** Remove the colon at the end of the second display of the page. (This is the display saying "$\mathsf{J}_{r-s} : T(n, r-s-1) \times I(n, s+1) \longrightarrow T(n, r-s) \times I(n, s)$:".)

14.3. Corrections to §C

1. **page 96, §C.1:** "in exactly one place; see (A.3g)(2)" should reference (A.3g)(3) rather than (A.3g)(2).

2. **page 97, after (C.2b):** It is worth mentioning that the Knuth unwinding KY is also known as the *reading word* or the *row word* of Y.

3. **page 97, proof of part (i) of Proposition (C.2c):** In "Now suppose that $m > 1$ and that Proposition (C.2c) holds", replace "(C.2c)" by "(C.2c)(i)". (Part (ii) won't be proved until a while later.)

4. **page 98, Diagram C.1:** All the "0"s here should be "∞"s. The same applies to Diagram C.2.

5. **page 99:** "given the element $x_a = y_{a,t}$" should just be "given the element x_a", since the equality $x_a = y_{a,t}$ is not obvious at this point.

   The right time to state $x_a = y_{a,t}$ is after proving (2). Indeed, once (2) has been proved, we can show that

   $$x_a = y_{a,t} \qquad \text{and} \qquad k(a) = t \qquad \text{and} \qquad z \geq a \tag{4}$$

   for each $a \in \{1, 2, \ldots, \beta_t\}$ (where z is as in (B.3c)). This is proved by induction on a: For $a = 1$, it is clear (at least $x_1 = y_{1,t}$ and $z \geq 1$ are clear; but $k(1) = t$ follows from $y_{1,t-1} \leq y_{1,t} = x_1 < y_{2,t}$). Now fix an $a \in \{2, 3, \ldots, \beta_t\}$ and assume as induction hypothesis that (1) holds for $a - 1$ instead of a; in other words, assume that

   $$x_{a-1} = y_{a-1,t} \qquad \text{and} \qquad k(a-1) = t \qquad \text{and} \qquad z \geq a - 1.$$

   We must prove that (1) holds for a; that is, we must prove that

   $$x_a = y_{a,t} \qquad \text{and} \qquad k(a) = t \qquad \text{and} \qquad z \geq a.$$

   For this purpose, we note that the definition of insertion yields $x_a = u_{a-1,k(a-1)} = u_{a-1,t}$ (since $k(a-1) = t$). But we have $U = X[t-1]$, so that $u_{a-1,t} = y_{a,t}$ and $u_{a,t-1} = y_{a,t-1}$ and $u_{a,t} = y_{a+1,t}$. Hence, $x_a = u_{a-1,t} = y_{a,t}$. Furthermore, the equation (2) on page 99 can be rewritten as $u_{a,t-1} \leq x_a < u_{a,t}$ (since $u_{a,t-1} = y_{a,t-1}$ and $u_{a,t} = y_{a+1,t}$ and $x_a = y_{a,t}$). But $k(a)$ is defined to be the unique element of $\{1, 2, \ldots, \mu_a + 1\}$ such that $u_{a,k(a)-1} \leq x_a < u_{a,k(a)}$. Comparing this with $u_{a,t-1} \leq x_a < u_{a,t}$, we see that t fits this description, so that we have $k(a) = t$. It remains to show that $z \geq a$. To do so, we observe that the t-th column of U has $\beta_t - 1$ entries. Thus, it has an $(a-1)$-st entry (since $a \leq \beta_t$ and thus $a - 1 \leq \beta_t - 1$). In other words, $\mu_{a-1} \geq t$. But the definition of z yields $k(z) = \mu_z + 1$. If we had $z = a - 1$, then we could rewrite this as $k(a-1) = \mu_{a-1} + 1 > \mu_{a-1} \geq t$, which would contradict $k(a-1) = t$. Hence, $z \neq a - 1$. Combined with $z \geq a - 1$, this leads us to $z > a - 1$ and thus to $z \geq a$. Thus, the induction step is complete, and we have proved (1) for all $a \in \{1, 2, \ldots, \beta_t\}$.

   This confirms all the values claimed in Table C.1. To ensure that this is a complete analysis of the Schensted insertion $X[t-1] \leftarrow x_1$, we need to show that $z = \beta_t$ (meaning that there is no further bumping after the β_t-th row). But this is easy: The t-th column of U has $\beta_t - 1$ entries, but the $(t-1)$-st column of U has $\beta_{t-1} \geq \beta_t$ entries. Thus, the β_t-th row of U spans the first $t - 1$ columns but no further. In other words, $\mu_{\beta_t} = t - 1$. But applying (1) to $a = \beta_t$, we obtain $k(\beta_t) = t = \mu_{\beta_t} + 1$ (since $\mu_{\beta_t} = t - 1$). This shows that $z = \beta_t$ (since z is characterized as the first positive integer for which $k(z) = \mu_z + 1$).

6. **page 100, (C2.h):** The tableau Y needs to be standard (i.e., semistandard, if we use the combinatorialists' lingo) in order for this to hold. While this normally goes without saying (semistandardness is assumed by default according to §B.1), this might be worth a reminder, since a non-standard tableau has just been constructed a few lines above.

7. **page 101, Table C.2:** The tableau $X \leftarrow y_{1,1}$ is not an $(\lambda^* + \varepsilon_1)$-tableau, as claimed here, but rather an $\left(\lambda^* + \varepsilon_{\beta_1}\right)$-tableau.

8. **page 101, Table C.2:** Again, it should be explained that the "0"s actually stand for ∞ (or should just be disregarded completely). The same applies to Diagram C.3.

9. **page 103, definition of basic moves:** This is all good, but the rules in (C.3c) and (C.3d) are somewhat hard to memorize. Here is a more memorable way to restate the definition:

   - Given three letters $x, y, z \in \underline{n}$, we say that the letter z is *numerically intermediate* between x and y if and only if $\min \{x, y\} \leq z \leq \max \{x, y\}$.
   - A *basic move of type K'* is a transformation that swaps two consecutive letters x and y in a word under the condition that a letter z numerically intermediate between x and y lies immediately to the right of y, and that we have neither $x \leq y \leq z$ nor $y \leq x \leq z$ (that is, the three letters don't form a weakly increasing subword before or after the move).
   - A *basic move of type K''* is a transformation that swaps two consecutive letters x and y in a word under the condition that a letter z numerically intermediate between x and y lies immediately to the left of x, and that we have neither $z \leq x \leq y$ nor $z \leq y \leq x$ (that is, the three letters don't form a weakly increasing subword before or after the move).

   So the strict inequalities $a < b$ in (C.3c) and $b < c$ in (C.3d) are preventing our three relevant letters from forming a weakly increasing subword before or after the move.

10. **page 104, the paragraph above (C3.h):** Remove the comma in "of the tableau U, is identical to the".

11. **pages 104–106, proof of (C3.f) (including all items (C3.g)–(C3.m)):** This proof tries too hard to micromanage the entries of the tableau during the transformation. As a result, the proof is hard to follow and suffers from some minor imprecisions (e.g., Proposition (C3.m) (ii) requires $z > 1$, and the proof of Proposition (C3.m) (i) presumes that $k(1) < \mu_1$, which is not guaranteed).

A cleaner proof (although following essentially the same idea) can be made by breaking up the Knuth unwindings of U and P row by row (rather than letter by letter). Here is this proof:[1]

We let $I(n)$ be the set of all (finite) words with entries in $\underline{n}$. Thus, $I(n) = \bigsqcup_{r \geq 0} I(n,r)$.

On the set $I(n)$, we define a binary relation $\equiv$ by as follows: Two words $i, j \in I(n)$ shall satisfy $i \equiv j$ if and only if there is a finite sequence of words $i(1), i(2), \ldots, i(s)$ such that $i(1) = i$ and $i(s) = j$ and each consecutive pair of words $i(\sigma - 1)$, $i(\sigma)$ is connected by a basic move of type K' or K''. In other words, $i \equiv j$ if and only if the word i can be transformed into j by a sequence of basic moves.

Clearly, the relation $\equiv$ is an equivalence relation (it is symmetric since the basic moves are symmetric). Moreover, it is *monoidal*, meaning that if two words $i, j \in I(n)$ satisfy $i \equiv j$, then any further word $k \in I(n)$ satisfies $i \mid k \equiv j \mid k$ and $k \mid i \equiv k \mid j$. (This is because any basic move that can be applied to a word $w \in I(n)$ can also be applied to any appearance of w as a segment in a longer word.) These properties of the relation $\equiv$ will be used without explicit mention.

We have the following two easy properties of the relation $\equiv$:

> **(C3.g′) Lemma.** If $b_1, b_2, \ldots, b_s, x \in \underline{n}$ are letters such that $s \geq 1$ and $x < b_1 \leq b_2 \leq \cdots \leq b_s$, then $b_1 b_2 \ldots b_s x \equiv b_1 x b_2 b_3 \ldots b_s$.

Proof. We induct on s. The base case ($s = 1$) is trivial, since it just claims $b_1 x \equiv b_1 x$. For the induction step, we fix $s > 1$ and assume that $b_1 b_2 \ldots b_{s-1} x \equiv b_1 x b_2 b_3 \ldots b_{s-1}$ has already been proved. We must now show that $b_1 b_2 \ldots b_s x \equiv b_1 x b_2 b_3 \ldots b_s$. But we have $x < b_1 \leq b_2 \leq \cdots \leq b_s$ and thus $x < b_{s-1} \leq b_s$. Hence, we can transform the word $b_1 b_2 \ldots b_s x$ into the word $b_1 b_2 \ldots b_{s-1} x b_s$ by a basic move of type K' (applied to the last three letters). Thus,

$$b_1 b_2 \ldots b_s x \equiv \underbrace{b_1 b_2 \ldots b_{s-1} x}_{\equiv b_1 x b_2 b_3 \ldots b_{s-1}} b_s \equiv b_1 x b_2 b_3 \ldots b_{s-1} b_s = b_1 x b_2 b_3 \ldots b_s.$$

This completes the induction step. Thus, (C3.g′) is proved.

> **(C3.h′) Lemma.** If $b_1, b_2, \ldots, b_s, x \in \underline{n}$ are letters such that $s \geq 1$ and $b_1 \leq b_2 \leq \cdots \leq b_s < x$, then $b_1 b_2 \ldots b_{s-1} x b_s \equiv x b_1 b_2 \ldots b_s$.

Proof. We induct on s. The base case ($s = 1$) is trivial, since it just claims $x b_1 \equiv x b_1$. For the induction step, we fix $s > 1$ and assume that $b_1 b_2 \ldots b_{s-2} x b_{s-1} \equiv x b_1 b_2 \ldots b_{s-1}$ has already been proved. We must now show that $b_1 b_2 \ldots b_{s-1} x b_s \equiv x b_1 b_2 \ldots b_s$. But we have $b_1 \leq b_2 \leq \cdots \leq$

[1] That said, the notation defined in (C3.g) is used elsewhere, so it is not worth skipping.

$b_s < x$ and thus $b_{s-1} \leq b_s < x$. Hence, we can transform the word $b_1 b_2 \ldots b_{s-1} x b_s$ into the word $b_1 b_2 \ldots b_{s-2} x b_{s-1} b_s$ by a basic move of type K'' (applied to the last three letters). Thus,

$$b_1 b_2 \ldots b_{s-1} x b_s \equiv \underbrace{b_1 b_2 \ldots b_{s-2} x b_{s-1}}_{\equiv x b_1 b_2 \ldots b_{s-1}} b_s \equiv x b_1 b_2 \ldots b_{s-1} b_s = x b_1 b_2 \ldots b_s.$$

This completes the induction step. Thus, (C3.h') is proved.

Now, we are ready for the *proof of (C3.f):* Let $P = (p_{a,b})_{(a,b) \in [\lambda]}$ be the tableau $U \leftarrow x$. Write the tableau U as $U = (u_{a,b})_{(a,b) \in [\mu]}$. For any $a \in \underline{n}$, we let P_a be the word $p_{a,1} p_{a,2} \ldots p_{a,\lambda_a}$ (obtained by reading the row a of P from left to right), and we let U_a be the word $u_{a,1} u_{a,2} \ldots u_{a,\mu_a}$ (obtained by reading the row a of U from left to right). Note that the word P_a is empty if $\lambda_a = 0$, and likewise for U_a. By the definition of Knuth unwinding, we have $KU = U_n \mid U_{n-1} \mid \cdots \mid U_1$ and $KP = P_n \mid P_{n-1} \mid \cdots \mid P_1$.

Now, set $x_1 = x$ and consider the "parameters" of the insertion $U \leftarrow x_1$ as defined in (B.3b): the number z (which is the number for which $\lambda = \mu + \varepsilon_z$), the numbers $k(1), k(2), \ldots, k(z)$ (which describe in which positions U differs from P) and the letters $x_1, x_2, \ldots, x_z$ (which are the bumped entries). We note that the tableau $P = U \leftarrow x_1$ differs from U only in its first z rows (since the bumping ends in the z-th row). Thus, $P_a = U_a$ for each $a > z$. Hence,

$$P_n \mid P_{n-1} \mid \cdots \mid P_{z+1} = U_n \mid U_{n-1} \mid \cdots \mid U_{z+1}. \tag{5}$$

Furthermore, row z of $P = U \leftarrow x_1$ is just row z of U with a new entry x_z adjoined at its end (since (B.3d) says $p_{z,\mu_z+1} = x_z$). Hence,

$$P_z = U_z \mid x_z. \tag{6}$$

Now we claim the following:

(C3.i') For any $a \in \underline{z-1}$, we have

$$x_{a+1} \mid P_a \equiv U_a \mid x_a. \tag{7}$$

Proof. Let $a \in \underline{z-1}$. We have $P_a = p_{a,1} p_{a,2} \ldots p_{a,\lambda_a}$ and $U_a = u_{a,1} u_{a,2} \ldots u_{a,\mu_a}$. From $a \in \underline{z-1}$, we obtain $a \neq z$ and thus $\lambda_a = \mu_a$ (since $\lambda = \mu + \varepsilon_z$ differs from μ only in its z-th entry). From (B.3d), we know that row a of P differs from row a of U only in its $k(a)$-th entry, which is $p_{a,k(a)} = x_a$ instead of $u_{a,k(a)} = x_{a+1}$. Hence, $p_{a,j} = u_{a,j}$ for all $j \neq k(a)$.

From (B.3), we know that $x_1 < x_2 < x_3 < \cdots$, so that $x_a < x_{a+1} = u_{a,k(a)} \leq u_{a,k(a)+1} \leq u_{a,k(a)+2} \leq \cdots \leq u_{a,\mu_a}$ (since U is row-standard). Hence, (C3.g') (applied to $x = x_a$ and $(b_1, b_2, \ldots, b_s) = \left(u_{a,k(a)}, u_{a,k(a)+1}, \ldots, u_{a,\mu_a}\right)$) yields

$$\begin{aligned} & u_{a,k(a)} u_{a,k(a)+1} \cdots u_{a,\mu_a} x_a \\ & \equiv \underbrace{u_{a,k(a)}}_{= x_{a+1}} x_a u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a} \\ & = x_{a+1} x_a u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a}. \end{aligned} \tag{8}$$

From (B.3c), we know that $u_{a,k(a)-1} \leq x_a$. Since U is row-standard, we have $u_{a,1} \leq u_{a,2} \leq \cdots \leq u_{a,k(a)-1} \leq x_a < x_{a+1}$. Hence, (C3.h′) (applied to $x = x_{a+1}$ and $(b_1, b_2, \ldots, b_s) = \left(u_{a,1}, u_{a,2}, \ldots, u_{a,k(a)-1}, x_a\right)$) yields

$$\begin{aligned} & u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_{a+1} x_a \\ & \equiv x_{a+1} u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_a. \end{aligned} \tag{9}$$

Now, we recall that

$$\begin{aligned} P_a &= p_{a,1} p_{a,2} \cdots p_{a,\lambda_a} = p_{a,1} p_{a,2} \cdots p_{a,\mu_a} \qquad (\text{since } \lambda_a = \mu_a) \\ &= \underbrace{p_{a,1} p_{a,2} \cdots p_{a,k(a)-1}}_{\substack{= u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} \\ (\text{since } p_{a,j} = u_{a,j} \text{ for all } j \neq k(a))}} \underbrace{p_{a,k(a)}}_{= x_a} \underbrace{p_{a,k(a)+1} p_{a,k(a)+2} \cdots p_{a,\mu_a}}_{\substack{= u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a} \\ (\text{since } p_{a,j} = u_{a,j} \text{ for all } j \neq k(a))}} \\ &= u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_a u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a}. \end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned} U_a &= u_{a,1} u_{a,2} \cdots u_{a,\mu_a} \\ &= u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} u_{a,k(a)} u_{a,k(a)+1} \cdots u_{a,\mu_a}, \end{aligned}$$

so that

$$\begin{aligned} U_a \mid x_a &= u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} \underbrace{u_{a,k(a)} u_{a,k(a)+1} \cdots u_{a,\mu_a} x_a}_{\substack{\equiv x_{a+1} x_a u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a} \\ (\text{by } (8))}} \\ &\equiv \underbrace{u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_{a+1} x_a}_{\substack{\equiv x_{a+1} u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_a \\ (\text{by } (9))}} u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a} \\ &\equiv x_{a+1} \underbrace{u_{a,1} u_{a,2} \cdots u_{a,k(a)-1} x_a u_{a,k(a)+1} u_{a,k(a)+2} \cdots u_{a,\mu_a}}_{\substack{= P_a \\ (\text{by } (10))}} = x_{a+1} \mid P_a. \end{aligned}$$

This proves (7).

Now, we have

$$\begin{aligned}
KP &= P_n \mid P_{n-1} \mid \cdots \mid P_1 \\
&= \underbrace{P_n \mid P_{n-1} \mid \cdots \mid P_{z+1}}_{\substack{=U_n|U_{n-1}|\cdots|U_{z+1} \\ \text{(by (5))}}} \mid \underbrace{P_z}_{\substack{=U_z|x_z \\ \text{(by (6))}}} \mid P_{z-1} \mid P_{z-2} \mid P_{z-3} \mid \cdots \mid P_1 \\
&= U_n \mid U_{n-1} \mid \cdots \mid U_{z+1} \mid U_z \mid \underbrace{x_z \mid P_{z-1}}_{\substack{\equiv U_{z-1}|x_{z-1} \\ \text{(by (7)} \\ \text{for } a=z-1)}} \mid P_{z-2} \mid P_{z-3} \mid \cdots \mid P_1 \\
&\equiv U_n \mid U_{n-1} \mid \cdots \mid U_{z+1} \mid U_z \mid U_{z-1} \mid \underbrace{x_{z-1} \mid P_{z-2}}_{\substack{\equiv U_{z-2}|x_{z-2} \\ \text{(by (7)} \\ \text{for } a=z-2)}} \mid P_{z-3} \mid \cdots \mid P_1 \\
&\equiv U_n \mid U_{n-1} \mid \cdots \mid U_{z+1} \mid U_z \mid U_{z-1} \mid U_{z-2} \mid \underbrace{x_{z-2} \mid P_{z-3}}_{\substack{\equiv U_{z-3}|x_{z-3} \\ \text{(by (7)} \\ \text{for } a=z-3)}} \mid \cdots \mid P_1 \\
&\equiv \cdots \\
&\equiv \underbrace{U_n \mid U_{n-1} \mid \cdots \mid U_{z+1} \mid U_z \mid U_{z-1} \mid U_{z-2} \mid U_{z-3} \mid \cdots \mid U_1}_{=U_n|U_{n-1}|\cdots|U_1=KU} \mid \underbrace{x_1}_{=x} \\
&= KU \mid x = w.
\end{aligned}$$

In view of $P = U \leftarrow x$, we can rewrite this as $K(U \leftarrow x) \equiv w$. In other words, $K(U \leftarrow x)$ can be transformed into w by a sequence of basic moves. This proves (C3.f).

12. **page 106, proof of the "only if" part of Knuth's theorem:** The shorthands P_{s-1} and P_s for $P_{s-1}(i)$ and $P_s(i)$ should be explained. Maybe it is better to just drop the "(i)" argument altogether (since i is fixed here), and simply write P_s for $P(i_1 \ldots i_s)$.

    Furthermore it would be nice to remind the reader that $P_s = P_{s-1} \leftarrow i_s$ by the construction of the P-symbol.

13. **page 110, item (ii):** Remove the comma in "the first row of W, is $\widetilde{W} = \widetilde{U} \leftarrow y$".

14. **page 110:** When you say "(The case $p = \mu_1 + 2$ only occurs when $k = \mu_1 + 1$.)", I think you mean "$l = \mu_1 + 1$" rather than "$k = \mu_1 + 1$".

15. **page 111:** In the second paragraph of this page (line 8), replace "such that $s \geq l$" by "such that $s > l$". Also remove the comma that follows these words.

16. **page 111:** In the second paragraph of this page (line 12), replace "to $W'' = W \leftarrow b$" by "to $W'' = W \leftarrow c$".

17. **page 112, Case 1:** Remove the comma after "of V with $s > k$".

18. **page 112, Case 1:** Replace "bumps the same letter $y = v_{1,l} = w_{1,l}$" by "bumps the same letter y from place $(1, l)$" (this letter is the $(1, l)$-th entry of $V \leftarrow c$ and $W \leftarrow a$, and it does equal $w_{1,l}$, but it is not always the same as $v_{1,l}$).

19. **page 112, Table C.8:** Not pictured on this table is the case $l = p$. In this case, there is no letter c in the first row any more (since it has been bumped into the lower part of the tableau under the guise of z), but otherwise the picture looks the same (in particular, the entry b occupies the $l = p$-th place in row (1)).

20. **page 113, Case 1:** The proof of $x \leq y < z$ in the first paragraph of this page looks questionable to me. I would rather argue as follows:

    By definition, y is the l-th letter in the 1-st row of $V \leftarrow c$. Meanwhile, c is the p-th letter in this row. Since $l \leq p$, we thus obtain $y \leq c$ (since $V \leftarrow c$ is row-standard). Furthermore, $c < u_{1,p} = z$. Hence, $y \leq c < z$.

    By definition, y is the l-th letter in the 1-st row of $W \leftarrow a$. Since the 1-st rows of $W \leftarrow a$ and W differ only in their k-th letter (because a bumps out the k-th letter of the 1-st row of W), this entails that y is the l-th letter in the 1-st row of W as well (since $k < l$ and thus $l \neq k$). In other words, $y = w_{1,l}$. But $x = w_{1,k}$. Since W is row-standard, we have $w_{1,k} \leq w_{1,l}$ (since $k < l$), and thus $x = w_{1,k} \leq w_{1,l} = y$. Hence, altogether, $x \leq y < z$.

21. **page 113, Case 2:** Better to replace the first "=" sign in "bumps the letter $w = v_{1,k+1} = u_{1,k+1}$" by a ":=" sign, since the letter "w" (albeit with subscripts) has already been used for a different purpose. Or replace this letter "w" by an unused letter (such as "t").

22. **page 113, Case 2:** Here you write "note that $w_{1,p-1} = u_{1,p-1} \leq a < c = w_{1,p}$". It is perhaps clearer if this is replaced by "note that $p = k$ and thus $w_{1,p-1} = w_{1,k-1} = u_{1,k-1} \leq a < c = w_{1,p}$".

23. **page 113, Case 2:** Replace "From $v_{1,k} = a \leq b < c = v_{1,k+1}$" by "Let $(v \leftarrow c)_{r,s}$ denote the (r, s)-th letter of $V \leftarrow c$. Then, from $(v \leftarrow c)_{1,k} = a \leq b < c = (v \leftarrow c)_{1,k+1}$".

24. **page 113, Case 2:** Replace "From $w_{1,k} = a \leq b < c < u_{1,k} \leq u_{1,k+1} = w_{1,k+1}$" by "Let $(w \leftarrow a)_{r,s}$ denote the (r, s)-th letter of $W \leftarrow a$. Then, from $(w \leftarrow a)_{1,k} = a \leq b < c < u_{1,p} = u_{1,k} \leq u_{1,k+1} = (w \leftarrow a)_{1,k+1}$".

25. **page 114, the paragraph below (C.5a):** In "the set $\{0\} \cup \left\{h_c^{KP} : t \in [\lambda]\right\}$", replace the "$h_c^{KP}$" by "$h_c^{KP}(t)$". Moreover, it is worth pointing out that this set can also be written as $\left\{h_c^{KP}(t) : t \in [\lambda] \cup \{0\}\right\}$.

26. **page 114, footnote [3]:** After "view theorem (C.5b)", add "and Proposition (C.2c)".

27. **page 115, (C.5b):** Replace "such that $h((a,b)) = M$" by "such that $h^{KP}((a,b)) = M$".

28. **page 115, Table C.11:** I would replace the "$> c+1$" in row $a+1$ and column $b+1$ by a "$\geq c+1$" (see below for why).

29. **page 115, proof of (C.5b):** It is not clear to me why "Since $a \geq c+1$, entries in row $(a+1)$ to the right of column (b) are all $> c+1$". But I do understand why these entries are all $\geq c+1$ (since $p_{a+1,b} = c+1$ and since P is row-standard). Fortunately, this is sufficient for the proof (with a few little modifications, listed below).

30. **page 115, proof of (C.5b):** In the equality "$Y^* = \sum\limits_{1 \leq x \leq b'} \omega(p_{a,x})$", the "$b'$" under the summation sign should be "$b'-1$". Moreover, add a comma after this equality.

31. **page 115, proof of (C.5b):** Replace "But for $b+1 \leq x \leq \lambda_{a+1}$ all the entries $p_{a+1,x}$ are $> c+1$, hence all the summands $\omega(p_{a+1,x}) = 0$, therefore $Y = 0$" by "However, all the entries $p_{a+1,x}$ with $b+1 \leq x \leq \lambda_{a+1}$ are $\geq c+1$ and thus satisfy $\omega(p_{a+1,x}) \in \{0, -1\}$; hence, $Y \leq 0$".

32. **page 115, proof of (C.5b):** Replace "$h^{KP}(a,b) = h^{KP}(a+1, b'-1)$" by "$h^{KP}(a,b) \leq h^{KP}(a+1, b'-1)$. Since $h^{KP}(a,b) = M$ is the largest of all $h^{KP}(t)$, we thus conclude that $h^{KP}(a,b) = h^{KP}(a+1, b'-1)$".

33. **page 116, after the proof of (C.5b):** Add the sentence "In the situation of Theorem (C.5b), we denote the tableau $\widetilde{P}$ by $\widetilde{f}_c P$. Thus, $K\left(\widetilde{f}_c P\right) = \widetilde{f}_c(KP)$.".

34. **page 116, (C.5d):** "two set" → "two sets".

35. **page 116, Proposition B:** One of the claims made here is that if $\widetilde{f}_c(i) \neq \infty$, then $\widetilde{f}_c(KP(i)) \neq \infty$. But the converse claim should also be made: If $\widetilde{f}_c(KP(i)) \neq \infty$, then $\widetilde{f}_c(i) \neq \infty$. (This is used later on, in the proof of (D.1g).)

    Both claims follow from Lemma (C.6c). (Indeed, by repeatedly applying Lemma (C.6c), we see that if two words i and j in $I(n,r)$ are connected by a sequence of basic moves, then the statements $\widetilde{f}_c(i) \neq \infty$ and $\widetilde{f}_c(j) \neq \infty$ are equivalent. Now it just remains to apply this to $j = KP(i)$.)

36. **page 116, Lemma (C.6b):** Add "Let $c \in \{1, 2, \ldots, n-1\}$." at the beginning of the lemma.

37. **page 117, proof of Lemma (C.6b):** After the first sentence, I would add the following: "Thus, $j_k = z$ and $j_{k+1} = x$. The definition of a height function yields

$$\begin{aligned} h_c^i(k+1) &= h_c^i(k) + \omega_c(i_{k+1}) = h_c^i(k) + \omega_c(z) && \text{and} \\ h_c^i(k-1) &= h_c^i(k) - \omega_c(i_k) = h_c^i(k) - \omega_c(x) && \text{and} \\ h_c^j(k+1) &= h_c^j(k) + \omega_c(j_{k+1}) = h_c^j(k) + \omega_c(x) && \text{and} \\ h_c^j(k-1) &= h_c^j(k) - \omega_c(j_k) = h_c^j(k) - \omega_c(z) \end{aligned}$$

(where ω_c is short for $\omega_{c,c+1}$)."

38. **page 117, proof of (ii) in the proof of Lemma (C.6b):** This proof can be redone in a nicer way (at least to my taste):

Proof of (ii): By definition,

$$M^i = \max\left\{h_c^i(\nu) \mid \nu \in \{0,1,\ldots,r\}\right\}. \tag{11}$$

We shall now show that

$$M^i = \max\left\{h_c^i(\nu) \mid \nu \in \{0,1,\ldots,r\} \setminus \{k\}\right\} \tag{12}$$

holds as well. Indeed, assume the contrary. Thus, M^i is the maximum of the set $\left\{h_c^i(\nu) \mid \nu \in \{0,1,\ldots,r\}\right\}$, but not the maximum of the (only slightly smaller) set $\left\{h_c^i(\nu) \mid \nu \in \{0,1,\ldots,r\} \setminus \{k\}\right\}$. In other words, M^i is the largest of all the numbers $h_c^i(0), h_c^i(1), \ldots, h_c^i(r)$, but ceases to be so if we remove the number $h_c^i(k)$ from this list. Obviously, the only way this can happen is if M^i equals $h_c^i(k)$ but is larger than all the other numbers in the list $h_c^i(0), h_c^i(1), \ldots, h_c^i(r)$. So we must be in this exact situation. Thus, $M^i = h_c^i(k)$ but

$$M^i > h_c^i(\nu) \qquad \text{for all } \nu \neq k. \tag{13}$$

In particular, from (13), we obtain $M^i > h_c^i(k+1)$ and $M^i > h_c^i(k-1)$ (here we use the fact that $k \geq 1$). Thus, $h_c^i(k) = M^i > h_c^i(k+1) = h_c^i(k) + \omega_c(i_{k+1})$, so that $\omega_c(i_{k+1}) < 0$ and thus $i_{k+1} = c+1$ (by the definition of ω_c). Likewise, we can show (using $M^i > h_c^i(k-1)$) that $i_k = c$. Thus, the word j is obtained from i by swapping the consecutive letters $i_k = c$ and $i_{k+1} = c+1$. Moreover, this swap must be a basic move. If this basic move is of type K', then we must have $k \geq 2$ and $i_k < i_{k-1} \leq i_{k+1}$ (since $i_k = c < c+1 = i_{k+1}$), so that $i_{k-1} \in \left(\underbrace{i_k}_{=c}, \underbrace{i_{k+1}}_{=c+1}\right] = (c, c+1]$ and thus

$i_{k-1} = c+1$, whence

$$\begin{aligned} M = h_c^i(k) &= h_c^i(k-2) + \omega_c\left(\underbrace{i_{k-1}}_{=c+1}\right) + \omega_c\left(\underbrace{i_k}_{=c}\right) \\ &= h_c^i(k-2) + \underbrace{\omega_c(c+1)}_{=-1} + \underbrace{\omega_c(c)}_{=1} \\ &= h_c^i(k-2) + (-1) + 1 = h_c^i(k-2), \end{aligned}$$

and this contradicts (13). Thus, this basic move cannot be of type K'. Hence, it must be of type K''. Thus, we must have $k \leq r-2$ and $i_k \leq i_{k+2} < i_{k+1}$ (since $i_k = c < c+1 = i_{k+1}$), so that $i_{k+2} \in \left[\underbrace{i_k}_{=c}, \underbrace{i_{k+1}}_{=c+1}\right) = [c, c+1)$ and thus $i_{k+2} = c$, whence

$$\begin{aligned} M &> h_c^i(k+2) \qquad \text{(by (13))} \\ &= h_c^i(k) + \omega_c\left(\underbrace{i_{k+1}}_{=c+1}\right) + \omega_c\left(\underbrace{i_{k+2}}_{=c}\right) \\ &= h_c^i(k) + \underbrace{\omega_c(c+1)}_{=-1} + \underbrace{\omega_c(c)}_{=1} \\ &= h_c^i(k) + (-1) + 1 = h_c^i(k), \end{aligned}$$

and this contradicts $M = h_c^i(k)$. This contradiction shows that our assumption was false. Thus, (12) is proved.

The same argument, applied to the word j instead of i, shows that

$$M^j = \max\left\{h_c^j(\nu) \mid \nu \in \{0, 1, \ldots, r\} \setminus \{k\}\right\} \tag{14}$$

(because we have not used the assumption $i_k < i_{k+1}$ in our above argument).

But the right hand sides of the equalities (12) and (14) are equal (since (i) shows that $h_c^i(\nu) = h_c^j(\nu)$ for all $\nu \neq k$). Hence, so are the left hand sides. In other words, $M^i = M^j$.

39. **page 117, proof of Lemma (C.6b):** After the proof of (ii), I would add another claim: "Next we claim that

    (iii) we have $q^i \geq k$ if and only if $q^j \geq k$.

    Indeed, if $q^i < k$, then (i) yields $h_c^j(q^i) = h_c^i(q^i) = M^i = M^j$ by (ii), and thus $q^j \leq q^i$ by the minimality of q^j; but this entails $q^j \leq q^i < k$. Hence, $q^i < k$ implies $q^j < k$. Similarly, the converse is true. Hence, we have $q^i < k$ if and only if $q^j < k$. By taking the contrapositive, we obtain (iii)."

40. **page 117, proof of Lemma (C.6b):** I struggle to understand how you prove part (a) of the lemma. So here is my proof of part (a):

    "Now, for the proof of (a), suppose that $q^i \notin \{k, k+1\}$. We must show that $q^j = q^i$.

    We have $q^i \neq k$ (since $q^i \notin \{k, k+1\}$). Thus, (i) shows that $h^j_c(q^i) = h^i_c(q^i) = M^i$ (by the definition of q^i). Thus, $h^j_c(q^i) = M^i = M^j$ by (ii). Therefore, $q^i \geq q^j$, by the minimality of q^j.

    If we also have $q^j \neq k$, then the same argument (with the roles of i and j interchanged) shows that $q^j \geq q^i$, and thus $q^j = q^i$ (since $q^i \geq q^j$). Hence, we are done in this case. Thus, from now on, we WLOG assume that $q^j = k$.

    Hence, $k = q^j$ is the first place where the height function h^j_c reaches its maximum (by the definition of q^j). Hence, $h^j_c(k-1) < h^j_c(k)$. In other words, $\omega_c(z) > 0$ (since $h^j_c(k-1) = h^j_c(k) - \omega_c(z)$). Hence, $z = c$. Thus, $x < z = c$, so that $\omega_c(x) = 0$. Now, (i) yields $h^j_c(k+1) = h^i_c(k+1)$, whence

    $$\begin{aligned} h^i_c(k+1) = h^j_c(k+1) = h^j_c\left(\underbrace{k}_{=q^j}\right) + \underbrace{\omega_c(x)}_{=0} = h^j_c\left(q^j\right) \\ &= M^j \qquad \left(\text{by the definition of } M^j\right) \\ &= M^i. \end{aligned}$$

    Hence, $q^i \leq k+1$ (by the minimality of q^i). Since $q^i \notin \{k, k+1\}$, we can conclude that $q^i < k$, which contradicts $q^i \geq q^j = k$. This contradiction shows that the case we are considering (that is, when $q^j = k$) is impossible, and the proof of (i) is complete."

41. **page 117, proof of Lemma (C.6b):** In the proof of (b), you write "In fact, by (i), $q^j = k$". To me, this is unclear, so I'd rather say "But (iii) shows that $q^j \geq k$, so that $q^j = k$".

42. **page 117, proof of Lemma (C.6b):** In the proof of (c), you write "Then $x = c$, and $q^j \geq k$, by (ii)". Again, it is not clear to me how you obtain $q^j \geq k$ here, but you can obtain it immediately from (iii).

43. **page 117, proof of Lemma (C.6b):** In the proof of (c), after "Then $z > c+1$", I would add "(since $z > x = c$)".

44. **page 117, Lemma (C.6c):** Add "Let $c \in \{1, 2, \ldots, n-1\}$." at the beginning of the lemma.

45. **page 117, Lemma (C.6c):** Replace "If $\widetilde{f_c}(i) \neq \infty$, then $\widetilde{f_c}(j) \neq \infty$, and $\widetilde{f_c}(j)$ is obtained from $\widetilde{f_c}(i)$ by a basic move" by "Then, the inequality $\widetilde{f_c}(i) \neq \infty$ holds if and only if $\widetilde{f_c}(j) \neq \infty$. Moreover, if it holds, then $\widetilde{f_c}(j)$ is obtained from $\widetilde{f_c}(i)$ by a basic move".

    In fact, in this form, the statement is symmetric in i and j, which is used in the proof of the lemma.

46. **page 117, Lemma (C.6c):** The last claim of this lemma ("There is a corresponding statement (and proof), with $\widetilde{e_c}$ replacing $\widetilde{f_c}$") is quite an imposition on the reader, as it requires constructing analogues of both Lemma (C.6b) and Lemma (C.6c) and their already rather patience-demanding proofs.

    An easier way to prove this claim would be to use the results of §D.3 (which are completely independent), specifically Lemma (D.3e). This can be done as follows. First, we show a simple property of the operator C:

    > **(D.3i') Lemma.** Let $i, j \in I(n, r)$ be two words such that j is obtained from i by a basic move. Then, the word $C(j)$ is obtained from $C(i)$ by a basic move as well.

    Proof. If $i\ K'\ j$, then $i = (\ldots, b, c, a, \ldots)$ and $j = (\ldots, b, a, c, \ldots)$ for some letters $a < b \leq c$, and thus

    $$C(i) = (\ldots,\ n+1-a,\ n+1-c,\ n+1-b,\ \ldots) \qquad \text{and}$$
    $$C(j) = (\ldots,\ n+1-c,\ n+1-a,\ n+1-b,\ \ldots),$$

    which shows that $C(i)\ \ K''\ C(j)$ (since $a < b \leq c$ entails $n+1-c \leq n+1-b < n+1-a$). Similarly, if $i\ K''\ j$, then $C(i)\ \ K'\ C(j)$. In both cases, Lemma (D.3i') is proved.

    > **(D.3j') Lemma.** The analogue of Lemma (C.6c) for $\widetilde{e_c}$ instead of $\widetilde{f_c}$ holds. In other words:
    >
    > Let $c \in \{1, 2, \ldots, n-1\}$. Let $i, j \in I(n, r)$, and suppose j is obtained from i by a basic move. Then, the inequality $\widetilde{e_c}(i) \neq \infty$ holds if and only if $\widetilde{e_c}(j) \neq \infty$. Moreover, if it holds, then $\widetilde{e_c}(j)$ is obtained from $\widetilde{e_c}(i)$ by a basic move.

    Proof. Lemma (D.3i') shows that the word Cj is obtained from Ci by a basic move. Hence, Lemma (C.6c) (applied to $n-c$, Ci and Cj instead of c, i and j) shows that $\widetilde{f_{n-c}}(Ci) \neq \infty$ holds if and only if $\widetilde{f_{n-c}}(Cj) \neq \infty$, and furthermore that the word $\widetilde{f_{n-c}}(Cj)$ is obtained from $\widetilde{f_{n-c}}(Ci)$ by a basic move (if these words are not ∞).

    For convenience, set $C(\infty) := \infty$. Lemma (D.3e) yields $C(\widetilde{e_c}(i)) = \widetilde{f_{n-c}}(Ci)$ and similarly $C(\widetilde{e_c}(j)) = \widetilde{f_{n-c}}(Cj)$. Thus, we have the following chain of equivalences:

$$\begin{aligned}
\left(\widetilde{e}_c\left(i\right) \neq \infty\right) &\iff \left(C\left(\widetilde{e}_c\left(i\right)\right) \neq \infty\right) && \left(\begin{array}{c}\text{since the map } C \text{ is bijective} \\ \text{and sends } \infty \text{ to } \infty\end{array}\right) \\
&\iff \left(\widetilde{f}_{n-c}\left(Ci\right) \neq \infty\right) && \left(\text{since } C\left(\widetilde{e}_c\left(i\right)\right) = \widetilde{f}_{n-c}\left(Ci\right)\right) \\
&\iff \left(\widetilde{f}_{n-c}\left(Cj\right) \neq \infty\right) && \left(\begin{array}{c}\text{since } \widetilde{f}_{n-c}\left(Ci\right) \neq \infty \text{ holds} \\ \text{if and only if } \widetilde{f}_{n-c}\left(Cj\right) \neq \infty\end{array}\right) \\
&\iff \left(C\left(\widetilde{e}_c\left(j\right)\right) \neq \infty\right) && \left(\text{since } C\left(\widetilde{e}_c\left(j\right)\right) = \widetilde{f}_{n-c}\left(Cj\right)\right) \\
&\iff \left(\widetilde{e}_c\left(j\right) \neq \infty\right) && \left(\begin{array}{c}\text{since the map } C \text{ is bijective} \\ \text{and sends } \infty \text{ to } \infty\end{array}\right).
\end{aligned}$$

In other words, $\widetilde{e}_c\left(i\right) \neq \infty$ holds if and only if $\widetilde{e}_c\left(j\right) \neq \infty$. Moreover, if it holds, then $\widetilde{f}_{n-c}\left(Ci\right) \neq \infty$ and $\widetilde{f}_{n-c}\left(Cj\right) \neq \infty$ hold as well (by the above equivalence), and thus the word $\widetilde{f}_{n-c}\left(Cj\right)$ is obtained from $\widetilde{f}_{n-c}\left(Ci\right)$ by a basic move (as we have shown above). In other words, the word $C\left(\widetilde{e}_c\left(j\right)\right)$ is obtained from $C\left(\widetilde{e}_c\left(i\right)\right)$ by a basic move (since $C\left(\widetilde{e}_c\left(i\right)\right) = \widetilde{f}_{n-c}\left(Ci\right)$ and $C\left(\widetilde{e}_c\left(j\right)\right) = \widetilde{f}_{n-c}\left(Cj\right)$). Hence, Lemma (C.3i') shows that the word $C\left(C\left(\widetilde{e}_c\left(j\right)\right)\right)$ is obtained from $C\left(C\left(\widetilde{e}_c\left(i\right)\right)\right)$ by a basic move as well. Since $C\left(C\left(\widetilde{e}_c\left(j\right)\right)\right) = \widetilde{e}_c\left(j\right)$ (because $C^2 = \operatorname{id}$) and $C\left(C\left(\widetilde{e}_c\left(i\right)\right)\right) = \widetilde{e}_c\left(i\right)$ (similarly), we can rewrite this as follows: The word $\widetilde{e}_c\left(j\right)$ is obtained from $\widetilde{e}_c\left(i\right)$ by a basic move. Thus, Lemma (D.3j') is proved.

47. **page 118, proof of Lemma (C.6c):** Before going into the cases, it is worth reminding the reader that $\widetilde{f}_c\left(i\right)$ is defined to be the word i with the letter c at position q^i replaced by $c+1$. Hence, $\widetilde{f}_c\left(i\right)$ differs from i only in position q^i.

48. **page 118, Case (a):** After "The claim follows directly if y is not changed, either.", add "Thus we restrict ourselves to the case when it is changed. Then, $y = c$, and q^i is either $k-1$ or $k+2$ (depending on the type of the basic move).".

49. **page 118, Case (a):** After "we get $h_c^j\left(k\right) = h_c^j\left(k-1\right)+1$", add "$= h_c^j\left(q^j\right)+1$ (since $k-1 = q^i = q^j$)".

50. **page 118, Case (a):** After "we get $h_c^i\left(k\right) = h_c^i\left(k+2\right)$", add "$= h_c^i\left(q^i\right)$ (since $k+2 = q^i$)".

51. **page 118, Case (c):** Replace "since otherwise $h_c^i\left(k+2\right) = h_c^i\left(k\right)+1$" by "since otherwise $y = c$ (since $y \geq x = c$) and therefore $h_c^i\left(k+2\right) = h_c^i\left(k\right) + 1$ (because $z > c+1$) in contradiction to the fact that $h_c^i\left(k\right) = h_c^i\left(q^i\right) = M^i$ is the largest value of h_c^i".

52. **page 118, Case (c):** After "since otherwise $h_c^i(k-2) = h_c^i(k)$", add "in contradiction to the fact that $k = q^i$ is the first time the function h_c^i attains its maximum value".

14.4. Corrections to §D

1. **page 121:** Here is a *proof of Lemma (D.1a):*

   (i) Fix $i \in I(n,r)$ and $c \in \{1,2,\ldots,n-1\}$. We have the following chain of logical equivalences:

$$\begin{aligned}
&\left(\widetilde{f}_c(i) = \infty\right) \\
\iff &\left(M_c^i = 0\right) \qquad \text{(by (A.3e))} \\
\iff &\left(h_c^i(t) \leq h_c^i(0) \text{ for all } t \in \{0,1,\ldots,r\}\right) \\
&\qquad \left(\text{since } M_c^i \text{ is defined as the maximum value of } h_c^i\right) \\
\iff &\left(h_c^i(t) \leq h_c^i(0) \text{ for all } t \in \{1,2,\ldots,r\}\right) \\
&\qquad \left(\begin{array}{c}\text{since the inequality } h_c^i(t) \leq h_c^i(0) \\ \text{automatically holds for } t = 0\end{array}\right) \\
\iff &\left(h_c^i(t) \leq 0 \text{ for all } t \in \{1,2,\ldots,r\}\right) \qquad \left(\text{since } h_c^i(0) = 0\right) \\
\iff &\left(\#\{\nu \leq t : i_\nu = c\} - \#\{\nu \leq t : i_\nu = c+1\} \leq 0 \text{ for all } t \in \{1,2,\ldots,r\}\right) \\
&\qquad \left(\begin{array}{l}\text{since } h_c^i(t) = \#\{\nu \leq t : i_\nu = c\} - \#\{\nu \leq t : i_\nu = c+1\} \\ \text{for all } t \in \{0,1,\ldots,r\} \text{ (this is a restatement of (A.3b))}\end{array}\right) \\
\iff &\left(\#\{\nu \leq t : i_\nu = c\} \leq \#\{\nu \leq t : i_\nu = c+1\} \text{ for all } t \in \{1,2,\ldots,r\}\right).
\end{aligned}$$

   This proves Lemma (D.1a) (i).

   (ii) Fix $i \in I(n,r)$ and $c \in \{1,2,\ldots,n-1\}$. For each $s \in \{1,2,\ldots,r\}$, we have

$$\begin{aligned}
&\underbrace{h_c^i(r)}_{\substack{=\omega(i_1)+\omega(i_2)+\cdots+\omega(i_r) \\ \text{(by the definition of height function)}}} - \underbrace{h_c^i(s-1)}_{\substack{=\omega(i_1)+\omega(i_2)+\cdots+\omega(i_{s-1}) \\ \text{(by the definition of height function)}}} \\
&= (\omega(i_1) + \omega(i_2) + \cdots + \omega(i_r)) - (\omega(i_1) + \omega(i_2) + \cdots + \omega(i_{s-1})) \\
&= \omega(i_s) + \omega(i_{s+1}) + \cdots + \omega(i_r) \\
&= \#\{\nu \geq s : i_\nu = c\} - \#\{\nu \geq s : i_\nu = c+1\} \qquad (15)
\end{aligned}$$

   (by the definition of $\omega = \omega_{c,c+1}$). Now, we have the following chain of logical equivalences:

$$
\begin{aligned}
& \left(\widetilde{e}_c\left(i\right) = \infty\right) \\
\Longleftrightarrow\ & \left(M_c^i = h_c^i\left(r\right)\right) \qquad \text{(by (A.3f))} \\
\Longleftrightarrow\ & \left(h_c^i\left(t\right) \leq h_c^i\left(r\right) \text{ for all } t \in \left\{0,1,\ldots,r\right\}\right) \\
& \qquad \left(\text{since } M_c^i \text{ is defined as the maximum value of } h_c^i\right) \\
\Longleftrightarrow\ & \left(h_c^i\left(t\right) \leq h_c^i\left(r\right) \text{ for all } t \in \left\{0,1,\ldots,r-1\right\}\right) \\
& \qquad \left(\text{since the inequality } h_c^i\left(t\right) \leq h_c^i\left(r\right) \text{ holds automatically for } t = r\right) \\
\Longleftrightarrow\ & \left(h_c^i\left(s-1\right) \leq h_c^i\left(r\right) \text{ for all } s \in \left\{1,2,\ldots,r\right\}\right) \\
& \qquad \left(\text{here, we have substituted } s-1 \text{ for } t\right) \\
\Longleftrightarrow\ & \left(h_c^i\left(r\right) - h_c^i\left(s-1\right) \geq 0 \text{ for all } s \in \left\{1,2,\ldots,r\right\}\right) \\
\Longleftrightarrow\ & \left(\#\left\{\nu \geq s : i_\nu = c\right\} - \#\left\{\nu \geq s : i_\nu = c+1\right\} \geq 0 \text{ for all } s \in \left\{1,2,\ldots,r\right\}\right) \\
& \qquad \text{(by (15))} \\
\Longleftrightarrow\ & \left(\#\left\{\nu \geq s : i_\nu = c\right\} \geq \#\left\{\nu \geq s : i_\nu = c+1\right\} \text{ for all } s \in \left\{1,2,\ldots,r\right\}\right).
\end{aligned}
$$

This proves Lemma (D.1a) (ii).

2. **page 122:** It is worth saying that the operators W, B and C are known as *complementation*, *reversal* and *reverse-complementation*.

3. **page 122, proof of (D.1c):** "Prove similarly that $i \in \mathsf{T}$ implies $C\left(i\right) \in \mathsf{Y}$" should be "Prove similarly that $i \in \mathsf{Y}$ implies $C\left(i\right) \in \mathsf{T}$. Thus, C restricts to a map $\mathsf{T} \to \mathsf{Y}$ and also to a map $\mathsf{Y} \to \mathsf{T}$. These two maps are mutually inverse, since $C^2 = \operatorname{id}$.".

4. **page 122, (D.1e):** I would replace "column t of Z_λ" by "column t of $\left[\lambda\right]$", so as to avoid an impression of a circular definition.

5. **page 123, proof of Theorem (D.1g):** "Let $c \in \left\{1,2,\ldots,n\right\}$" should be "Let $c \in \left\{1,2,\ldots,n-1\right\}$".

6. **page 123, proof of Theorem (D.1g):** "From §A.3" could better be "From (A.3f)".

7. **page 123, proof of Theorem (D.1g):** Remove the "$= M_c^i$" part from "Hence $h_c^{KP(i)}\left(r\right) = h_c^i\left(r\right) = M_c^i$".

8. **page 123, proof of Theorem (D.1g):** Instead of "We can calculate $M_c^{KP(i)} = M_c^{KT_\lambda}$ easily; it is $\lambda_c - \lambda_{c+1}$, and it is attained at the last place $\left(1,\lambda_1\right)$ of $KP\left(i\right)$. Therefore the maximum M_c^i of h_c^i is also attained at the last place

of i", it would be clearer to say: "The height function $h_c^{KP(i)} = h_c^{KT_\lambda}$ starts out at 0, then decreases by 1 for each place in the $(c+1)$-st row of λ, then increases by 1 for each place in the c-th row of λ; then it remains constant with the value $\lambda_c - \lambda_{c+1}$ ever after. Since $\lambda_c \geq \lambda_{c+1}$, we conclude that its maximum value is $\lambda_c - \lambda_{c+1} = h_c^{KP(i)}(r) = h_c^i(r)$. In other words, $M_c^{KP(i)} = h_c^i(r)$. Hence, $M_c^i = M_c^{KP(i)} = h_c^i(r)$ as well".

9. **page 123, proof of Theorem (D.1g):** You write: "We know $\widetilde{e}_c(KP(j)) = KP(\widetilde{e}_c(j))$, by Proposition B, hence $\widetilde{e}_c(KP(j)) = \infty$ for all c". This uses a variation of Proposition B instead of Proposition B proper (Proposition B is about $\widetilde{f}_c$, not about $\widetilde{e}_c$). But it is easier to replace the argument by a different one: "Let $c \in \{1, 2, \ldots, n-1\}$. From $j \in \mathsf{T}$, we obtain $\widetilde{e}_c(j) = \infty$ and thus $M_c^j = h_c^j(r)$ by (A.3f). But $M_c^{KP(j)} = M_c^j$ by Lemma (C.6b) and Proposition (C.3p), and furthermore $h_c^{KP(j)}(r) = h_c^j(r)$ since the letters of the word $KP(j)$ are a permutation of the letters of j (and both words have length r, so that the value of the height function at r is accounting for all their letters). Thus, $h_c^{KP(j)}(r) = h_c^j(r) = M_c^j = M_c^{KP(j)}$, and therefore $\widetilde{e}_c(KP(j)) = \infty$ by (A.3f).".

10. **page 124, proof of Theorem (D.1g):** After "Hence all entries of the first row of $P(j)$ are equal to 1", I would add "(since $P(j)$ is row-standard)".

11. **page 124, proof of Theorem (D.1g):** When you say "Next consider the last entry, t say, in the s^{th} row of $P(j)$", you should explain that you are arguing by strong induction on s (since you use the induction hypothesis a few sentences later). So I would instead write: "We shall now show that all entries of the s^{th} row of $P(j)$ are equal to s for all $s \in \{1, 2, \ldots, n\}$. We will prove this by strong induction on s. So we assume this is proved for all entries of rows $1, 2, \ldots, s-1$, and we consider the last entry, t say, in the s^{th} row of $P(j)$.".

12. **page 124, proof of Theorem (D.1g):** After "it is constant on the letters of rows 1 up to $s-1$", add "(by the induction hypothesis, since $s-1 < t-1$)".

13. **page 124, proof of Theorem (D.1g):** Remove ", say x," (you never use the notation x).

14. **page 124, proof of Theorem (D.1g):** After "This is a contradiction.", I would add " Hence, $t \leq s$. Since $P(j)$ is row-standard, this shows that all entries in the s-th row of $P(j)$ are $\leq s$. Since $P(j)$ is column-standard, they cannot be $< s$, and thus they are all $= s$.".

15. **page 124, proof of Theorem (D.1g):** You write: "Using Lemma (C.6b) and Proposition B, as in the proof of (i), it is quite easy to see that $i \in \mathsf{Y}$".

I think this needs more details (it is indeed similar to the corresponding part of (i), but not completely analogous):

We want to show that $i \in \mathrm{Y}$. Let $c \in \{1,2,\ldots,n-1\}$. We must prove that $\widetilde{f}_c(i) = \infty$. Equivalently (by (A.3e)), we must prove that $M_c^i = 0$. But Proposition (C.3p) shows that the words i and $KP(i)$ are connected by a sequence of basic moves; hence, Lemma (C.6b) shows that their height functions have the same maximum: $M_c^i = M_c^{KP(i)}$. Thus, it remains to show that $M_c^{KP(i)} = 0$. In other words, it remains to show that all values of the height function $h_c^{KP(i)}$ are ≤ 0. In other words, it remains to show that $h_c^{KP(i)}(t) \leq 0$ for each $t \in \{0,1,\ldots,r\}$.

So let us show this. Fix $t \in \{0,1,\ldots,r\}$. The construction of the tableau Z_λ ensures that each appearance of c in Z_λ is followed by an appearance of $c+1$ one box further south in the same column (since the entries of each column are $k+1, k+2, \ldots, n$ for some k). In the Knuth unwinding KZ_λ, the latter $c+1$ appears earlier than the former c (since it lies further south in Z_λ). Thus, among the first t letters of KZ_λ, there must be at least as many $(c+1)$'s as there are c's (since each column that contributes a c to these first t letters must also contribute a $c+1$ to them). In other words,

$$\begin{aligned}&(\text{number of } (c+1)\text{'s among the first } t \text{ letters of } KZ_\lambda)\\ &\geq (\text{number of } c\text{'s among the first } t \text{ letters of } KZ_\lambda).\end{aligned}$$

But (A.3b) yields

$$\begin{aligned}h_c^{KZ_\lambda}(t) &= (\text{number of } c\text{'s among the first } t \text{ letters of } KZ_\lambda)\\ &\qquad - (\text{number of } (c+1)\text{'s among the first } t \text{ letters of } KZ_\lambda)\\ &\leq 0\end{aligned}$$

(by the preceding sentence). Since $P(i) = Z_\lambda$, we can rewrite this as $h_c^{KP(i)}(t) \leq 0$. This is precisely what we needed to show. Thus, $i \in \mathrm{Y}$ is proved.

16. **page 124, (D.1i):** The "$\mathsf{Q}(\lambda)$" here means the tableau $\mathsf{Q}^{(\lambda)}$ defined in (C.2h). This should be said.

17. **page 124, proof of Proposition (D.1i):** Here is this proof in a bit more detail:

From (D.1d), we have $i^\lambda = KT_\lambda$, and thus $P(i^\lambda) = P(KT_\lambda) = T_\lambda$ (by (C.2c) (i)) and $Q(i^\lambda) = Q(KT_\lambda) = \mathsf{Q}^{(\lambda)}$ (by (C.2h)). On the other hand, $P\left(i^{\mathsf{Q}(\lambda)}\right) = T_\lambda$ (by (D.1g) (i) and (D.1h)) and $Q\left(i^{\mathsf{Q}(\lambda)}\right) = \mathsf{Q}(\lambda)$ (by (D.1h)). Hence, $P(i^\lambda) = T_\lambda = P\left(i^{\mathsf{Q}(\lambda)}\right)$ and $Q(i^\lambda) = \mathsf{Q}^{(\lambda)} = \mathsf{Q}(\lambda) = Q\left(i^{\mathsf{Q}(\lambda)}\right)$.

Since any word i is uniquely determined by the pair $(P(i), Q(i))$ (by (B.6a)), we thus obtain $i^\lambda = i^{\mathsf{Q}(\lambda)}$. The same argument, using Z_λ and i_λ instead of T_λ and i^λ, shows that $i_\lambda = i_{\mathsf{Q}(\lambda)}$.

18. **page 124, §D.2:** "(see (A.4a))" should be "(see (A.4c))".

19. **page 125, proof of Proposition (D.2b):** At the very beginning of this proof, add the following: "By Remark (A.3g) (5), the cases $\widetilde{f}_c(i) = j$ and $\widetilde{e}_c(i) = j$ can be transformed into each other by swapping i with j. Thus, it suffices to handle one of them.".

20. **page 125, proof of Proposition (D.2b):** "By Proposition B (see (C.6b))" should be "By Proposition B (see (C.6a))".

21. **page 125, proof of Proposition (D.2b):** After "Now take $P = P(i)$ in Theorem (C.5b)", add "(this is applicable, since $\widetilde{f}_c(KP(i)) = KP(j) \neq \infty$ guarantees that $M_c^{KP(i)} \neq 0$)".

22. **page 125, proof of Proposition (D.2b):** After "Therefore $KP(j) = K\widetilde{P}$", add ", and thus $P(j) = \widetilde{P}$ since any tableau is uniquely determined by its Knuth unwinding (see (C.2c) (i))".

23. **page 125, proof of Proposition (D.2b):** After "If $q < r$, then $j' = \widetilde{f}_c(i')$", add "(since $q < r$ shows that $M^{i'} = M^i$ and thus $q^{i'} = q^i = q$, so that the construction of $\widetilde{f}_c(i')$ from i' changes the same c as the construction of $\widetilde{f}_c(i)$ from i)".

24. **page 125, proof of the "only if" part of Theorem 1:** Replace "$i(1), (2), \ldots, i(s)$" by "$i(1), i(2), \ldots, i(s)$".

25. **page 125, last paragraph:** It is worth reminding that ∞ does not count as a word (and thus is not included in $\mathcal{S}(w)$).

26. **page 125, last paragraph:** After "would be an element of $\mathcal{S}(w)$ of size $S - 1$." (the last words on page 125), add "Thus, $w' \in \mathsf{T}$.".

27. **page 126, first paragraph:** After "But then Theorem (D.1g)", add "(i)".

28. **page 126, proof of Proposition (D.2d):** I would say a few words about how the implication (1) $\Longrightarrow$ (3) in part (iii) is proven (all the other claims do indeed follow from the things above). Namely, we proceed similarly to the proof of the implication (1) $\Longrightarrow$ (2): Let $w \in I(Q, \approx)$. Define $\mathcal{S}(w)$ to be the set of all words of the form $\widetilde{f}_{c_1}\widetilde{f}_{c_2}\cdots\widetilde{f}_{c_t}(w)$, where $c_1, c_2, \ldots, c_t$ are arbitrary elements of $\{1, 2, \ldots, n-1\}$ (again, we allow t to be 0). This set $\mathcal{S}(w)$ is finite (being a subset of the finite set $I(n, r)$), and thus there exists an element w' of this set with largest size. This element w' must

then satisfy $\widetilde{f}_c(w') = \infty$ for all $c \in \{1, 2, \ldots, n-1\}$ (since otherwise, $\widetilde{f}_c(w')$ would have even larger size than w'), and thus lies in Y. By Proposition (D.2b), it satisfies $Q(w') = Q(w) = Q$ (since $w \in I(Q, \approx)$), so that $w' = i_Q$ (by Theorem (D.1g) (ii), since $w' \in$ Y). Hence, $i_Q = w' = \widetilde{f}_{c_1}\widetilde{f}_{c_2}\cdots\widetilde{f}_{c_t}(w)$ for some $c_1, c_2, \ldots, c_t$ (since $w' \in \mathcal{S}(w)$). Therefore, $w = \widetilde{e}_{c_t}\cdots\widetilde{e}_{c_2}\widetilde{e}_{c_1}(i_Q)$. Thus, the implication (1) $\Longrightarrow$ (3) is proved.

29. **page 126, Weights:** "Remember (see (A.3g)(3), or §3.1)" should be "Remember (see (A.3g)(4), or §3.1)".

30. **page 127, last line:** "We shall see in §D.4" $\to$ "We shall see in §D.8".

31. **page 127, proof of Proposition (D.2f) (ii):** "For each $t \in \underline{n}$" should be "For each positive integer t" (the tableau Z_λ can have more than n columns).

32. **page 127, proof of Proposition (D.2f) (iii) and (iv):** "By (D.2d) we know that" should perhaps be "By (D.2d) (iii) we know that".

33. **page 127, proof of Proposition (D.2f) (iii) and (iv):** "From (A.3g)(3)" should be "From (A.3g)(4)".

34. **page 127, proof of Proposition (D.2f) (iii) and (iv):** "Therefore $i \trianglelefteq i^Q$" should be "Therefore $\operatorname{wt}(i) \trianglelefteq \operatorname{wt}\left(i^Q\right)$".

35. **page 128, between (D.3b) and (D.3c):** I would replace "and $n - i_\nu = n - c$" by "and $n - i_\nu + 1 = n - c + 1$" (after all, the "$+1$"s are in the formula that you are rewriting).

36. **page 128, between (D.3b) and (D.3c):** After "So (D.3b) gives", add "(upon substutituting ν for $r - \rho + 1$ and rewriting the conditions)".

37. **page 128, after (D.3c):** "for every subset Π of $\{1, \ldots, s\}$" should be "for every subset Π of $\{1, \ldots, t\}$".

38. **page 128, Lemma (D.3e):** It should be pointed out that you set $C\infty := \infty$, so that the statement makes sense even if some of the operators yield ∞.

39. **page 128, proof of Lemma (D.3e):** "in the vertical line $x = r$" should be "in the vertical line $x = r/2$" (at least if reflection is understood in the sense of elementary geometry).

40. **pages 128–129, proof of Lemma (D.3e):** The shorthands "h" and "$\widetilde{h}$" for h_c^i and h_{n-c}^{Ci} are unnecessary and only confusing in this proof; they should both be replaced by "h_c^i" and "h_{n-c}^{Ci}" everywhere they are used.

41. **page 129, proof of Lemma (D.3e):** Remove the comma before "becomes the first maximum of".

42. **page 129, proof of Lemma (D.3e):** After "assumes its maximum at place $r - \overline{q}$ for the first time.", add "In other words, $q_{n-c}^{Ci} = r - \overline{q}$.".

43. **page 129, proof of Lemma (D.3g):** "the weight of $C\left(i^Q\right)$ is $\left(n^{\lambda_1}, (n-1)^{\lambda_2}, \ldots\right)$" should be "the weight of $C\left(i^Q\right)$ is $(\lambda_n, \ldots, \lambda_1)$. In other words, the weight of i_R is $(\lambda_n, \ldots, \lambda_1)$ (since $i_R = C(i) = C\left(i^Q\right)$)".

44. **page 129, proof of Lemma (D.3g):** This whole proof is confusingly phrased and awkward with its unnecessary separation of two cases that are really two parts of the same argument. Here is how I would write it:

    Let $i \in I(n, r)$ have shape λ. We must show that Ci has shape λ as well.

    Let $Q = Q(i)$, so that $Q \in \mathcal{Q}(\lambda)$. By the definition of i^Q, we have $Q\left(i^Q\right) = Q$ and $i^Q \in \mathsf{T}$. The latter yields $Ci^Q \in C(\mathsf{T}) = \mathrm{Y}$ by Lemma (D.1c). But each word $w \in \mathrm{Y}$ has the form $w = i_R$ for some standard tableau R with entries $1, 2, \ldots, r$ (indeed, if we set $R := Q(w)$, then i_R is defined as the unique word in Y whose Q-symbol is R; but this word must be precisely w because $w \in \mathrm{Y}$ and $Q(w) = R$). Hence, the word Ci^Q has this form (since $Ci^Q \in \mathrm{Y}$). In other words, $Ci^Q = i_R$ for some standard tableau R. Consider this R.

    By Proposition (D.2f) (i), we know that $\mathrm{wt}\left(i^Q\right) = \lambda$. Hence, $\mathrm{wt}\left(Ci^Q\right) = (\lambda_n, \ldots, \lambda_1)$. In other words, $\mathrm{wt}(i_R) = (\lambda_n, \ldots, \lambda_1)$, since $C\left(i^Q\right) = i_R$. But Proposition (D.2f) (ii) shows that $\mathrm{wt}(i_R) = (\mu_n, \ldots, \mu_1)$, where μ is the shape of R. Comparing these, we find $(\lambda_n, \ldots, \lambda_1) = (\mu_n, \ldots, \mu_1)$, hence $\lambda = \mu$. Thus, λ is the shape of R as well.

    From Proposition (D.2d) (iii), we see that there are $c_1, c_2, \ldots, c_t \in \{1, 2, \ldots, n-1\}$ such that $i = \widetilde{f}_{c_1} \cdots \widetilde{f}_{c_t}\left(i^Q\right)$. Hence, by Lemma (D.3e), we conclude that $Ci = \widetilde{e}_{n-c_1} \cdots \widetilde{e}_{n-c_t}\left(Ci^Q\right) = \widetilde{e}_{n-c_1} \cdots \widetilde{e}_{n-c_t}(i_R)$ (since $Ci^Q = i_R$). Since the operators $\widetilde{e}_c$ preserve the shape of a word (because Proposition (D.2b) shows that they even preserve its Q-symbol), we thus conclude that the shape of Ci is the shape of i_R. Hence, the shape of Ci is λ, since the shape of i_R is λ. This completes the proof.

45. **page 130, definition of the Littelmann algebra:** It is worth saying that algebras (and thus subalgebras) are not required to be unital here.

46. **page 130, (D.4b):** "spanned the set" should be "spanned by the set".

47. **page 130, proof of Proposition (D.4e):** "the basis $\{v_i : i \in I\}$" should be "the basis $\{v_i : i \in I(n, r)\}$".

48. **page 130, proof of Proposition (D.4e):** It is worth saying that fact (iv) follows by induction on w using fact (iii).

49. **page 130, proof of Proposition (D.4e):** After the final formula on this page, add "Thus, fact (iv) shows that $D_S \in L$, since fact (ii) shows that each $D_{Z(c)}$ and each $D_{Y(c)}$ lies in L.".

50. **page 131, proof of Proposition (D.4e):** Replace "$\mathsf{End}_F(S,S)$" by "$\mathsf{End}_F(S)$".

51. **page 131, §D.5:** "By Proposition B, this is an L-submodule of $V^{\otimes r}$" should be "By Proposition (D.2b), this is an L-submodule of $V^{\otimes r}$".

52. **page 131, §D.5:** After "For $z = \sum_i \xi_i v_i \in V^{\otimes r}$", add "(with $\xi_i \in F$)".

53. **page 132, Lemma (D.5b):** "lies in the set" should be "is a subset of ".

54. **page 132, Corollary (D.5c):** "lies in" should be "is a subset of ".

55. **page 132, the paragraph below Corollary (D.5c):** "lies in" should be "is a subset of ".

56. **page 132, definition of an L-module:** After "so that $x(ym) = (xy)m$", add "and $(x+y)m = xm + ym$". Also require that $(\lambda x)m = \lambda(xm)$ for all $\lambda \in F$ and $x \in L$ and $m \in M$.

57. **page 133, after (D.5f):** "stated in (A.3g)(4)" should be "stated in (A.3g)(5)".

58. **page 133, (D.5g):** "such that $\mathsf{sz}(i) \leq \mathsf{sz}(j)$ implies that $i \leq j$" should be "such that $\mathsf{sz}(i) < \mathsf{sz}(j)$ implies that $i \leq j$" (otherwise, the total order would not exist, since different words can have the same size).

59. **page 133, (D.5g):** Replace "upper triangular" and "lower triangular" by "strictly upper triangular" and "strictly lower triangular", respectively.

60. **page 133, Corollary (D.5h):** Remove the comma after "subalgebra of L".

61. **page 133, (D.5j):** This argumentation is a bit sloppy here. It has been shown that the L-module $V^{\otimes r}$ is a direct sum of finitely many simple L-modules (indeed, this follows from Lemma (D.5i) and from $V^{\otimes r} = \bigoplus_{\lambda \in \Lambda^+(n,r)} \bigoplus_{Q \in \mathcal{Q}(\lambda)} M_Q$).

    In other words, $V^{\otimes r}$ is completely reducible as an L-module. But this does not immediately yield that L itself is completely reducible as an L-module, which would be necessary to apply [11, Theorem (25.2), page 164]. Instead, the complete reducibility of L can be justified as follows:

    The L-module $V^{\otimes r}$ is faithful (since L is defined as an F-algebra of endomorphisms of $V^{\otimes r}$). Hence, the canonical F-algebra morphism $L \to \operatorname{End}_F(V^{\otimes r})$ is injective. Moreover, this morphism is a morphism of L-representations, if we equip L with the left regular L-module structure and equip $\operatorname{End}_F(V^{\otimes r})$ with the L-module structure given by post-action (i.e., the L-action given by $(\ell\alpha)(v) = \ell \cdot \alpha(v)$ for all $\ell \in L$ and $\alpha \in \operatorname{End}_F(V^{\otimes r})$

and $v \in V^{\otimes r}$). Hence, the left regular L-module L can be identified with an L-submodule of $\operatorname{End}_F(V^{\otimes r})$. But the L-module $\operatorname{End}_F(V^{\otimes r})$ is isomorphic to $\bigoplus_{i \in I(n,r)} V^{\otimes r}$ (by the isomorphism that sends each $\alpha \in \operatorname{End}_F(V^{\otimes r})$ to the family $(\alpha(v_i))_{i \in I(n,r)} \in \bigoplus_{i \in I(n,r)} V^{\otimes r}$), and thus is completely reducible (since $V^{\otimes r}$ is completely reducible as an L-module, and since complete reducibility is inherited by finite direct sums). Hence, the left regular L-module L is completely reducible as well (since any submodule of a completely reducible L-module is completely reducible – see [11, (15.2)] for a proof). Now, [11, Theorem (25.2)] really does show that L is semisimple.

62. **page 135, proof of (D.7c):** You write: "But $KP(i)$ equals $KP(KP(i))$, hence its Q-symbol is $\mathsf{Q}^{(\lambda)}$ (see (C.2i))". It would be easier to argue that Proposition (C.2h) yields $Q(KP(i)) = \mathsf{Q}^{(\lambda)}$.

63. **page 136, (D.7e):** In this diagram, both "$\widetilde{f}_{c(i)}$"s should be "$\widetilde{f}_c(i)$"s.

64. **page 136, after (D.7e):** You write: "In the same way, one has a diagram like (D.7e), with $\widetilde{e}_c$ replacing $\widetilde{f}_c$". This requires the $\widetilde{e}_c$-analogue of Proposition B, which has never been stated explicitly and which would need to be proved. Here is a quick way to do this:

    (D.7e′) Proposition B′. Let $i \in I(n,r)$ and $c \in \{1, 2, \ldots, n-1\}$. Then, $\widetilde{e}_c(i) \neq \infty$ holds if and only if $\widetilde{e}_c(KP(i)) \neq \infty$. Moreover, if these two inequalities hold, then $\widetilde{e}_c(KP(i)) = KP(\widetilde{e}_c(i))$.

    Proof. We distinguish the cases $\widetilde{e}_c(i) \neq \infty$ and $\widetilde{e}_c(i) = \infty$:

    - *Case 1:* We have $\widetilde{e}_c(i) \neq \infty$. Then, we must show that $\widetilde{e}_c(KP(i)) = KP(\widetilde{e}_c(i))$ and $\widetilde{e}_c(KP(i)) \neq \infty$.

      Indeed, $\widetilde{e}_c(i)$ is a word (since $\widetilde{e}_c(i) \neq \infty$). Let us denote this word by j. Thus, $j = \widetilde{e}_c(i)$, so that $\widetilde{f}_c(j) = i \neq \infty$. Therefore, (C.6a) (applied to j instead of i) yields $\widetilde{f}_c(KP(j)) \neq \infty$ and $\widetilde{f}_c(KP(j)) = KP\left(\widetilde{f}_c(j)\right)$. The latter equality entails

      $$\begin{aligned} KP(j) &= \widetilde{e}_c\left(KP\left(\underbrace{\widetilde{f}_c(j)}_{=i}\right)\right) \qquad \left(\text{since } \widetilde{e}_c \text{ undoes } \widetilde{f}_c\right) \\ &= \widetilde{e}_c(KP(i)). \end{aligned}$$

      Hence, $\widetilde{e}_c(KP(i)) = KP\left(\underbrace{j}_{=\widetilde{e}_c(i)}\right) = KP(\widetilde{e}_c(i)) \neq \infty$. This proves (D.7e′) in Case 1.

- *Case 2:* We have $\widetilde{e}_c(i) = \infty$. Then, we must show that $\widetilde{e}_c(KP(i)) = \infty$ as well.

  The word $KP(i)$ can be obtained from i by a sequence of basic moves (by Proposition (C.3p)). Each basic move might change the height function of the word, but leaves the maximum of this height function unchanged (by the first claim in Lemma (C.6b)). Hence, the height functions of the words $KP(i)$ and i have the same maximum. In other words, $M_c^{KP(i)} = M_c^i$. Moreover, $h_c^i(r)$ is the total weight of all letters of i (since i has r letters), whereas $h_c^{KP(i)}(r)$ is the total weight of all letters of $KP(i)$ (for similar reasons). Since the words i and $KP(i)$ have the same letters (just in a different order), this shows that $h_c^i(r) = h_c^{KP(i)}(r)$.

  But $\widetilde{e}_c(i) = \infty$, and thus $M_c^i = h_c^i(r)$ (by the definition of $\widetilde{e}_c$ in (A.3f)). Hence, $M_c^{KP(i)} = M_c^i = h_c^i(r) = h_c^{KP(i)}(r)$. Therefore, $\widetilde{e}_c(KP(i)) = \infty$ (again by the definition of $\widetilde{e}_c$). Thus, (D.7e') is proved in Case 2. ■

65. **page 136, proof of (D.7h):** Replace "Hom" by "Hom" (different font).

66. **page 141, (D.10b):** Replace "$t \geq 0$" by "$t \geq 1$".

67. **page 142:** On the first line, replace "tableaux" by "tableau".

68. **page 142:** The first displayed equation on this page is saying

$$\text{“ } t^{r_{tt}} (t-1)^{r_{t-1,t}} \dots 1^{r_{1t}} \text{ ”},$$

    but it should be the other way round:

$$\text{“ } 1^{r_{1t}} 2^{r_{2t}} \cdots t^{r_{tt}} \text{ ”}.$$

69. **page 142, item (1):** "the sum of the entries in row s" $\to$ "the number of the entries in row s".

70. **page 142, item (3):** Remove the period at the end of the displayed inequality.

71. **page 142, last paragraph:** This is a bit trickier than you make it sound: Why exactly is the matrix U upper-triangular?

    In truth, the triangularity of the matrix U is a red herring, and it would do no harm to ignore it completely; it would only simplify the proof.

72. **page 143, §D.11:** "Chapter 6 of the collective work" should be "Chapter 5 of the collective work". Likewise, all references to LLT in this section suffer from the same numbering shift: any number starting with "6." should be changed to start with "5.".

Also, it is worth warning the reader that LLT use the notation 0 for what is called ∞ in this appendix (that is, the undefined value of $\widetilde{f}_c$ and $\widetilde{e}_c$), and that the definition of σ_i in LLT's Section 5.5 suffers from a typo (it says "$\sigma_i\left(a_i^r a_{i+1}^s\right) = a_i^s a_{i+1}^s$" while meaning "$\sigma_i\left(a_i^r a_{i+1}^s\right) = a_i^s a_{i+1}^r$").

73. **page 143, (D.11e):** "*plactid*" $\to$ "*plactic*".

74. **page 144, (D.11e):** "implies $uu' \sim vv'$" should be "implies $uv \sim u'v'$".

75. **page 144, (D.11e):** Remove the comma after "with the $\sim$-class of u'".

76. **page 144, (D.11f):** It would be worth mentioning that $\mathbb{Z}[M]$ is known as the *monoid ring* of M.

77. **page 145, (D.11g):** Remove the comma after "Then a further generalization".

78. **page 145, (D.11g):** The last sentence here ("Notice that d_λ appears in the "λ-rectangle" (D.6d)") looks wrong to me: The number of plactic classes of **weight** λ is not the same as the number of plactic classes of **shape** λ (which is the d_λ from (D.6d)).

79. **page 146, (D.11i):** After "is the last place", add "of k".

80. **page 146, (D.11i):** After "is the place", add "of k".

14.5. Corrections to §E

1. **page 149:** "the directed graph Γ in (D.11i)" should be "the directed graph Γ in (D.11h)".

2. **page 149:** On the last line, "(C.2c)" should be "(C.2h)".

15. A shorter path to the Littlewood–Richardson rule

pages 129–140, sections D.4 till D.9: This is all nice and interesting, but the Littlewood–Richardson rule (D.10a) can be proved quite easily without any use of the Littelmann algebra. Here is how this proof goes:

(D.12a) Lemma. Let $i \in I(n,r)$ and $j \in I(n,s)$ be two words. Then, any word that is $\approx$-equivalent to the concatenation $i \mid j$ must itself be a concatenation $i' \mid j'$, where $i' \in I(n,r)$ and $j' \in I(n,s)$ are two words with $i' \approx i$ and $j' \approx j$.

Proof. Let w be a word that is $\approx$-equivalent to the concatenation $i \mid j$. We must prove that w is itself a concatenation $i' \mid j'$, where $i' \in I(n,r)$ and $j' \in I(n,s)$ are two words with $i' \approx i$ and $j' \approx j$.

By Theorem A (see (D.2a)), the word w can be obtained from $i \mid j$ by a sequence of operations of the form $\widetilde{e_c}$ and $\widetilde{f_c}$. Hence, it suffices to consider the case when $w = \widetilde{e_c}(i \mid j)$ or $w = \widetilde{f_c}(i \mid j)$ for some $c \in \{1, 2, \ldots, n-1\}$ (the general case will then follow by induction). Let us thus assume that we are in this case. Fix the $c \in \{1, 2, \ldots, n-1\}$ such that $w = \widetilde{e_c}(i \mid j)$ or $w = \widetilde{f_c}(i \mid j)$.

Now, (A.3g) (6) shows that $\widetilde{f_c}(i \mid j)$ is either $\widetilde{f_c}(i) \mid j$ or $i \mid \widetilde{f_c}(j)$ (or ∞). Thus, if $w = \widetilde{f_c}(i \mid j)$, then $w = \widetilde{f_c}(i) \mid j$ or $w = i \mid \widetilde{f_c}(j)$. In either case, w thus has the form $i' \mid j'$, where $i' \in I(n,r)$ and $j' \in I(n,s)$ are two words with $i' \approx i$ and $j' \approx j$ (because Theorem A shows that $\widetilde{f_c}(i) \approx i$ and $\widetilde{f_c}(j) \approx j$). Hence, we are done in the case when $w = \widetilde{f_c}(i \mid j)$.

The case when $w = \widetilde{e_c}(i \mid j)$ is completely analogous. Hence, we are done in all cases, and Lemma (D.12a) is proved. ■

Now, we let A^* denote the set of all words with letters in $\underline{n}$. Thus, $A^* = \bigcup\limits_{r \geq 0} I(n,r)$, and this is a disjoint union. The set A^* is a monoid with respect to concatenation ($ij = i \mid j$ for any two words $i, j \in A^*$), and thus has a monoid ring $\mathbb{Z}[A^*]$. The elements of the latter ring $\mathbb{Z}[A^*]$ are formal $\mathbb{Z}$-linear combinations of words.

An element $f \in \mathbb{Z}[A^*]$ is said to be *$\approx$-invariant* if it has the property that whenever $i, j \in A^*$ are two words satisfying $i \approx j$, the coefficients of i and j in f are equal. For instance, the element

$$2 \cdot [133] - 3 \cdot [331] - 3 \cdot [313] + 5 \cdot [1234]$$

(here we put words in square brackets, to avoid confusing them with the scalar factors in front of them) is $\approx$-invariant, but the element [331] itself is not (since the word 313 is $\approx$-equivalent to 331 but appears with a different coefficient in [331]).

Let $\mathbb{Z}[A^*]_{\approx}$ denote the set of all $\approx$-invariant elements of $\mathbb{Z}[A^*]$.

We use the notations $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ and $P : Q$ defined in (D.6a) and (D.6b) and (D.6c). For any $\lambda \in \Lambda^+(n,r)$ and any $Q \in \mathcal{Q}(\lambda)$, we let

$$(* : Q) := \sum_{P \in \mathcal{P}(\lambda)} (P : Q) = \sum_{\substack{i \in A^*; \\ Q(i) = Q}} i = \sum_{i \in I_\lambda(Q, \approx)} i \in \mathbb{Z}[A^*].$$

This element $(* : Q)$ is the sum of all words in the $\approx$-equivalence class $I_\lambda(Q, \approx)$; hence, all words in this class appear in it with coefficient 1, while all other words appear with coefficient 0. Therefore, this element $(* : Q)$ is $\approx$-invariant, and thus belongs to $\mathbb{Z}[A^*]_{\approx}$. Moreover:

> **(D.12b) Lemma.** The set $\mathbb{Z}[A^*]_{\approx}$ is the $\mathbb{Z}$-linear span of the elements $(* : Q)$ for all $\lambda \in \Lambda^+(n,r)$ and $Q \in \mathcal{Q}(\lambda)$.

Proof. These elements $(* : Q)$ are precisely the sums of the $\approx$-equivalence classes in A^*. But the span of these sums is precisely $\mathbb{Z}[A^*]_{\approx}$, by the definition of $\mathbb{Z}[A^*]_{\approx}$. ■

(D.12c) Lemma. The set $\mathbb{Z}[A^*]_{\approx}$ is a unital $\mathbb{Z}$-subalgebra of $\mathbb{Z}[A^*]$.

Proof. Clearly, $\mathbb{Z}[A^*]_{\approx}$ is a $\mathbb{Z}$-submodule, and contains the unity of $\mathbb{Z}[A^*]$ (which is the empty word). It remains to show that it is closed under multiplication. By Lemma (D.12b), it suffices to show that $(* : Q)(* : S) \in \mathbb{Z}[A^*]_{\approx}$ for all $\lambda \in \Lambda^+(n,r)$, $Q \in \mathcal{Q}(\lambda)$, $\mu \in \Lambda^+(n,s)$ and $S \in \mathcal{Q}(\mu)$. So let us consider such λ, Q, μ and S.

Recall that $(* : Q) = \sum_{i \in I_\lambda(Q,\approx)} i$ and $(* : S) = \sum_{j \in I_\mu(S,\approx)} j$ (similarly). Hence,

$$(* : Q)(* : S) = \left(\sum_{i \in I_\lambda(Q,\approx)} i \right) \left(\sum_{j \in I_\mu(S,\approx)} j \right) = \sum_{i \in I_\lambda(Q,\approx)} \sum_{j \in I_\mu(S,\approx)} i \mid j.$$

Hence, $(* : Q)(* : S)$ is the sum of all concatenations $i \mid j$ with $i \in I_\lambda(Q,\approx)$ and $j \in I_\mu(S,\approx)$. Note that any two such concatenations are distinct (because all words in $I_\lambda(Q,\approx)$ have length r and all words in $I_\mu(S,\approx)$ have length s), and thus this sum has no repeated addends. Therefore,

$$(* : Q)(* : S) = \sum_{w \in X} w, \tag{16}$$

where X is the set of all words $w \in A^*$ that can be written as concatenations $i \mid j$ with $i \in I_\lambda(Q,\approx)$ and $j \in I_\mu(S,\approx)$.

But Lemma (D.12a) shows that if $i \mid j$ is such a concatenation, then so is every word w that is $\approx$-equivalent to $i \mid j$ (because Lemma (D.12a) tells us that any such word w can be written as a concatenation $i' \mid j'$ with $i' \approx i$ and $j' \approx j$, and of course these $\approx$-relations entail that $i' \in I_\lambda(Q,\approx)$ and $j' \in I_\mu(S,\approx)$ because of $i \in I_\lambda(Q,\approx)$ and $j \in I_\mu(S,\approx)$). In other words, the set X is closed under $\approx$ (that is, every element that is $\approx$-equivalent to an element of X must itself lie in X). In other words, X is a union of $\approx$-equivalence classes. Hence, the sum $\sum_{w \in X} w$ is $\approx$-invariant, i.e., belongs to $\mathbb{Z}[A^*]_{\approx}$. In view of (16), we can rewrite this as follows: $(* : Q)(* : S)$ belongs to $\mathbb{Z}[A^*]_{\approx}$. As explained above, this completes the proof of Lemma (D.12c). ■

(D.12d) Definition. Let $\Lambda^+(n,*) := \bigsqcup_{r \in \mathbb{N}} \Lambda^+(n,r)$ and $\mathcal{Q}(*) := \bigsqcup_{\lambda \in \Lambda^+(n,*)} \mathcal{Q}(\lambda)$.

(D.12e) Proposition. The map

$$\begin{aligned} \mathsf{T} &\to \mathcal{Q}(*), \\ w &\mapsto Q(w) \end{aligned}$$

is a bijection.

Proof. Clearly, this map is well-defined. It remains to prove that each $Q \in \mathcal{Q}(*)$ has exactly one preimage under it. In other words, it remains to prove that for each $Q \in \mathcal{Q}(*)$, there is exactly one $w \in \mathsf{T}$ satisfying $Q(w) = Q$. In other words, it remains to prove that for each $Q \in \mathcal{Q}(*)$, there is exactly one $w \in \mathsf{T}$ lying in the set $I(Q, \approx)$. In other words, it remains to prove that for each $Q \in \mathcal{Q}(*)$, there is exactly one $w \in I(Q, \approx)$ lying in the set T. But this is part of Proposition (D.2d) (i). ■

(D.12f) Definition. For any $\lambda \in \Lambda^+(n,r)$, we define the *Schur polynomial* $s_\lambda \in \mathbb{Z}[X_1, X_2, \ldots, X_n]$ by

$$s_\lambda := \sum_{P \in \mathcal{P}(\lambda)} X_P, \qquad \text{where } X_P := \prod_{c \in [\lambda]} X_{P(c)}.$$

Splitting this sum by the weight of P, we can rewrite this formula as

$$\begin{aligned} s_\lambda &:= \sum_{\alpha \in \Lambda(n,r)} \ \sum_{P \in \mathcal{P}(\lambda) \text{ has weight } \alpha} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \\ &= \sum_{\alpha \in \Lambda(n,r)} \left| \{ P \in \mathcal{P}(\lambda) \mid P \text{ has weight } \alpha \} \right| \cdot X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}. \end{aligned}$$

This polynomial s_λ is precisely the formal character Φ_{M_λ} from §D.9 and also is precisely the formal character Φ_{V_λ} from §3.4 and §5. This is shown in the proof of (D.9a). As a consequence, the equality

$$\Phi_{V_\lambda} \cdot \Phi_{V_\mu} = \sum_\nu c^\nu_{\lambda,\mu} \Phi_{V_\nu}$$

(which defines the Littlewood–Richardson coefficients $c^\nu_{\lambda,\mu}$) can be rewritten as

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu. \tag{17}$$

We shall now obtain a different formula for $s_\lambda s_\mu$, which will then (by comparing coefficients) yield an expression for $c^\nu_{\lambda,\mu}$.

(D.12g) Theorem (Littlewood–Richardson rule, general form). Let $\lambda \in \Lambda^+(n,r)$ and $\mu \in \Lambda^+(n,s)$. Let $Q \in \mathcal{Q}(\lambda)$ and $S \in \mathcal{Q}(\mu)$. Let $\mathcal{W}$ be the set of all words $k \in I(n, r+s)$ of the form $k = i \mid j$, where

(a1) the word $i \in I(n,r)$ satisfies $Q(i) = Q$;

(b1) the word $j \in I(n,s)$ satisfies $Q(j) = S$;

(c1) the word k belongs to T.

Then,

$$s_\lambda s_\mu = \sum_{k \in \mathcal{W}} s_{\mathsf{wt}(k)}.$$

Here, $\mathsf{wt}(k)$ denotes the weight of a word k.

Proof. Consider the surjective unital $\mathbb{Z}$-algebra morphism $\pi : \mathbb{Z}[A^*] \to \mathbb{Z}[X_1, X_2, \ldots, X_n]$ that sends each word $i_1 i_2 \cdots i_r$ to the (commutative) monomial $X_{i_1} X_{i_2} \cdots X_{i_r}$. Then,

$$\begin{aligned}
\pi((* : Q)) &= \pi\left(\sum_{P \in \mathcal{P}(\lambda)} (P : Q)\right) && \left(\text{since } (* : Q) = \sum_{P \in \mathcal{P}(\lambda)} (P : Q)\right) \\
&= \sum_{P \in \mathcal{P}(\lambda)} \underbrace{\pi((P : Q))}_{\substack{= X_P \\ \text{(since the letters of } P:Q \\ \text{are the entries of } P)}} && (\text{since the map } \pi \text{ is } \mathbb{Z}\text{-linear}) \\
&= \sum_{P \in \mathcal{P}(\lambda)} X_P = s_\lambda && (\text{by the definition of } s_\lambda).
\end{aligned}$$

Similarly,

$$\pi((* : S)) = s_\mu.$$

Multiplying these two equalities, we find

$$\pi((* : Q)) \, \pi((* : S)) = s_\lambda s_\mu,$$

so that

$$\begin{aligned}
s_\lambda s_\mu &= \pi((* : Q)) \, \pi((* : S)) \\
&= \pi((* : Q)(* : S))
\end{aligned} \tag{18}$$

(since π is an algebra morphism).

But the product $(* : Q)(* : S)$ lies in the subalgebra $\mathbb{Z}[A^*]_\approx$ (by Lemma (D.12c)). Thus, it can be written as a linear combination

$$(* : Q)(* : S) = \sum_{W \in \mathcal{Q}(*)} c_W (* : W) \tag{19}$$

with coefficients $c_W \in \mathbb{Z}$ (since Lemma (D.12b) says that $\mathbb{Z}[A^*]_\approx$ is the $\mathbb{Z}$-linear span of the elements $(* : W)$ for all $W \in \mathcal{Q}(*)$).

Now, let $\pi_\mathsf{T} : \mathbb{Z}[A^*] \to \mathbb{Z}[A^*]$ be the $\mathbb{Z}$-linear map that

$$\text{sends each word } w \in A^* \text{ to } \begin{cases} w, & \text{if } w \in \mathsf{T}; \\ 0, & \text{else.} \end{cases}$$

Thus, π_T is the canonical projection from $\mathbb{Z}[A^*]$ to the span of the basis vectors w that belong to T. While π_T is not a $\mathbb{Z}$-algebra morphism, we can notice a few useful properties. Most importantly, applying π_T to a sum of words has the effect of filtering out all the words that don't belong to T from this sum, leaving only the words that do belong to T in place. Thus,

$$\pi_\mathsf{T}((* : Q)) = i^Q \tag{20}$$

(because among all the words $i \in I_\lambda(Q, \approx)$ that make up the sum $(* : Q) = \sum_{i \in I_\lambda(Q,\approx)} i$, there is only one that belongs to T, namely the word i^Q, as we know from Proposition (D.2d) (i)). The same argument (applied to W instead of Q) shows that for any $W \in \mathcal{Q}(*)$, we have

$$\pi_{\mathsf{T}}((* : W)) = i^W. \tag{21}$$

Hence, applying the map π_{T} to the equality (19), we find

$$\begin{aligned} \pi_{\mathsf{T}}((* : Q)(* : S)) &= \pi_{\mathsf{T}}\left(\sum_{W \in \mathcal{Q}(*)} c_W (* : W)\right) \\ &= \sum_{W \in \mathcal{Q}(*)} c_W \underbrace{\pi_{\mathsf{T}}((* : W))}_{\substack{= i^W \\ \text{(by (21))}}} \qquad (\text{since } \pi_{\mathsf{T}} \text{ is } \mathbb{Z}\text{-linear}) \\ &= \sum_{W \in \mathcal{Q}(*)} c_W i^W. \end{aligned} \tag{22}$$

On the other hand,

$$(* : Q) = \sum_{\substack{i \in A^*; \\ Q(i) = Q}} i = \sum_{\substack{i \in I(n,r); \\ Q(i) = Q}} i$$

(since any word $i \in A^*$ satisfying $Q(i) = Q$ must have $|Q| = |\lambda| = r$ letters and thus belong to $I(n,r)$) and similarly

$$(* : S) = \sum_{\substack{j \in I(n,s); \\ Q(j) = S}} j.$$

Multiplying these two equalities, we find

$$\begin{aligned} (* : Q)(* : S) &= \left(\sum_{\substack{i \in I(n,r); \\ Q(i) = Q}} i\right)\left(\sum_{\substack{j \in I(n,s); \\ Q(j) = S}} j\right) \\ &= \sum_{\substack{i \in I(n,r); \\ Q(i) = Q}} \sum_{\substack{j \in I(n,s); \\ Q(j) = S}} i \mid j. \end{aligned} \tag{23}$$

This sum has no repeated addends (since we can uniquely reconstruct the two words i and j from the concatenation $i \mid j$ when we know that i lies in $I(n,r)$), and thus can be rewritten as the sum of all words $k \in A^*$ that can be written as $i \mid j$ where the word $i \in I(n,r)$ satisfies $Q(i) = Q$ and where the word $j \in I(n,s)$ satisfies $Q(j) = S$. Hence, (23) rewrites as follows:

$$(* : Q)(* : S) = \sum_{\substack{k \in A^*; \\ k \text{ can be written as } i \mid j \\ \text{with } i \in I(n,r) \text{ and } j \in I(n,s) \\ \text{and } Q(i) = Q \text{ and } Q(j) = S}} k = \sum_{\substack{k \in A^*; \\ k \text{ can be written as } i \mid j \\ \text{with } Q(i) = Q \text{ and } Q(j) = S}} k$$

(here we have dropped the requirements "$i \in I(n,r)$" and "$j \in I(n,s)$", since they follow automatically from $Q(i) = Q$ and $Q(j) = S$). Applying the map π_{T} to this equality, we find

$$\pi_{\mathsf{T}}((* : Q)(* : S)) = \pi_{\mathsf{T}}\left(\sum_{\substack{k \in A^*; \\ k \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S}} k \right) = \sum_{\substack{k \in \mathsf{T}; \\ k \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S}} k$$

(since π_{T} filters the addends for belonging to T). Comparing this with (22), we obtain

$$\sum_{W \in \mathcal{Q}(*)} c_W i^W = \sum_{\substack{k \in \mathsf{T}; \\ k \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S}} k.$$

Comparing the coefficients of the word i^W on both sides of this equality (for a given $W \in \mathcal{Q}(*)$), we obtain

$$c_W = \begin{cases} 1, & \text{if } i^W \text{ can be written as } i \mid j \text{ with } Q(i) = Q \text{ and } Q(j) = S; \\ 0, & \text{else} \end{cases} \tag{24}$$

for any $W \in \mathcal{Q}(*)$ (since Proposition (D.2d) (i) shows that $i^W \in \mathsf{T}$).

For any tableau W, we let $\operatorname{shape}(W)$ denote the shape of W. Thus, if $W \in \mathcal{Q}(\nu)$ for some ν, then $\operatorname{shape}(W) = \nu$.

Now, let us apply the algebra morphism π to both sides of (19). We find

$$\begin{aligned} \pi((* : Q)(* : S)) &= \pi\left(\sum_{W \in \mathcal{Q}(*)} c_W (* : W) \right) \\ &= \sum_{W \in \mathcal{Q}(*)} c_W \underbrace{\pi((* : W))}_{\substack{= s_{\operatorname{shape}(W)} \\ \text{(this is proved just like} \\ \text{we showed } \pi((*:Q)) = s_\lambda)}} \qquad \text{(since } \pi \text{ is linear)} \\ &= \sum_{W \in \mathcal{Q}(*)} c_W s_{\operatorname{shape}(W)}. \end{aligned}$$

In view of (18), we can rewrite this as

$$\begin{aligned} s_\lambda s_\mu &= \sum_{W \in \mathcal{Q}(*)} c_W s_{\operatorname{shape}(W)} \\ &= \sum_{w \in \mathsf{T}} c_{Q(w)} s_{\operatorname{shape}(Q(w))} \end{aligned} \tag{25}$$

(here, we substituted $Q(w)$ for W in the sum, using the bijection from Proposition (D.12e)).

Now, let $w \in \mathsf{T}$. Then, $i^{Q(w)} = w$ (indeed, $i^{Q(w)}$ is defined as the unique word in $I(Q(w), \approx)$ lying in T; but the word w itself also lies in $I(Q(w), \approx)$ and in T, and therefore we conclude from the uniqueness that w must be $i^{Q(w)}$). But (24) yields

$$c_{Q(w)} = \begin{cases} 1, & \text{if } i^{Q(w)} \text{ can be written as } i \mid j \text{ with } Q(i) = Q \text{ and } Q(j) = S; \\ 0, & \text{else} \end{cases}$$
$$= \begin{cases} 1, & \text{if } w \text{ can be written as } i \mid j \text{ with } Q(i) = Q \text{ and } Q(j) = S; \\ 0, & \text{else} \end{cases} \tag{26}$$

(since $i^{Q(w)} = w$). Furthermore, Proposition (D.2f) (i) yields $\operatorname{wt}\left(i^{Q(w)}\right) = \operatorname{shape}(Q(w))$, and thus

$$\operatorname{shape}(Q(w)) = \operatorname{wt}\left(i^{Q(w)}\right) = \operatorname{wt}(w) \tag{27}$$

(since $i^{Q(w)} = w$).

Forget that we fixed w. We thus have shown that each $w \in \mathsf{T}$ satisfies (26) and (27). Hence, we can rewrite (25) as

$$s_\lambda s_\mu = \sum_{w \in \mathsf{T}} \begin{cases} 1, & \text{if } w \text{ can be written as } i \mid j \text{ with } Q(i) = Q \text{ and } Q(j) = S; \\ 0, & \text{else} \end{cases} s_{\operatorname{wt}(w)}$$
$$= \sum_{\substack{w \in \mathsf{T}; \\ w \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S}} s_{\operatorname{wt}(w)} = \sum_{\substack{k \in \mathsf{T}; \\ k \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S}} s_{\operatorname{wt}(k)}$$
$$= \sum_{\substack{k \in I(n,r+s); \\ k \text{ can be written as } i|j \\ \text{with } Q(i)=Q \text{ and } Q(j)=S; \\ k \in \mathsf{T}}} s_{\operatorname{wt}(k)} \tag{28}$$

(here, we have restricted the sum to $k \in I(n, r+s)$ only, because only a word of length $r+s$ has any chance of being written as $i \mid j$ with $Q(i) = Q$ and $Q(j) = S$). But the conditions under the summation sign in (28) are precisely the conditions (a1), (b1) and (c1) in the definition of the set $\mathcal{W}$ in Theorem (D.12g). Hence, the summation sign can be rewritten as "$\sum_{k \in \mathcal{W}}$".

Hence, (28) rewrites as

$$s_\lambda s_\mu = \sum_{k \in \mathcal{W}} s_{\operatorname{wt}(k)}.$$

This proves Theorem (D.12g). ■

(D.12h) Theorem (Littlewood–Richardson rule, restated form). Let $\lambda \in \Lambda^+(n, r)$ and $\mu \in \Lambda^+(n, s)$. Let $Q \in \mathcal{Q}(\lambda)$ and $S \in \mathcal{Q}(\mu)$. Let $\mathcal{W}$ be the set of all words $k \in I(n, r+s)$ of the form $k = i \mid j$, where

(a2) the word $i \in I(n,r)$ satisfies $Q(i) = Q$;

(b2) the word j is i^S;

(c2) the reverse $B(k)$ of the word k is a lattice permutation.

Then,

$$s_\lambda s_\mu = \sum_{k \in \mathcal{W}} s_{\mathrm{wt}(k)}.$$

Here, $\mathrm{wt}(k)$ denotes the weight of a word k.

Proof. This will follow from Theorem (D.12g), once we can show that the set $\mathcal{W}$ defined in Theorem (D.12g) is precisely the set $\mathcal{W}$ defined in Theorem (D.12h). For this purpose, we must show that the conditions (a1), (b1) and (c1) from Theorem (D.12g) (taken together) are equivalent to the conditions (a2), (b2) and (c2) from Theorem (D.12h).

Let us do this. Consider two words i and j and their concatenation $k = i \mid j \in I(n, r+s)$.

We observed in §D.1 that a word i belongs to T if and only if $B(i)$ is a lattice permutation. Hence, $k \in \mathsf{T}$ if and only if $B(k)$ is a lattice permutation. In other words, condition (c1) is equivalent to condition (c2). Thus, we WLOG assume that both conditions (c1) and (c2) hold (since otherwise, neither of them holds, and we are done). Hence, $k \in \mathsf{T}$, and the word $B(k)$ is a lattice permutation.

If a word $w = w_1 w_2 \cdots w_p$ is a lattice permutation, then any prefix of w (that is, any word of the form $w_1 w_2 \cdots w_q$ with $q \leq p$) is a lattice permutation again (by the definition of "lattice permutation"). Hence, any prefix of $B(k)$ is a lattice permutation (since $B(k)$ is a lattice permutation). But $B(j)$ is a prefix of $B(k)$ (since $k = i \mid j$ and thus $B(k) = B(i \mid j) = B(j) \mid B(i)$). Thus, we conclude that $B(j)$ is a lattice permutation. In other words, $j \in \mathsf{T}$ (since a word i belongs to T if and only if $B(i)$ is a lattice permutation).

Therefore, if $Q(j) = S$, then $j \in I(S, \approx)$ and thus $j = i^S$ (because Proposition (D.2d) (i) tells us that i^S is the only word in $I(S, \approx)$ that belongs to T). Hence, condition (b1) implies condition (b2).

Conversely, if the word j is i^S, then $Q(j) = Q(i^S) = S$ (by the definition of i^S) and thus also $j \in I(n,s)$ (since $Q(j) = S$ shows that j has as many letters as S has entries; but $S \in \mathcal{Q}(\mu)$ shows that S has $|\mu| = s$ many entries). Hence, condition (b2) implies (b1). Thus, we have shown that the two conditions (b1) and (b2) imply one another; in other words, they are equivalent.

Finally, conditions (a1) and (a2) are equivalent, since they say the same thing.

Altogether, we now have seen that the conditions (a1), (b1) and (c1) from Theorem (D.12g) (taken together) are equivalent to the conditions (a2), (b2) and (c2) from Theorem (D.12h). This completes our proof of (D.12h). ■

(D.12i) Theorem (Littlewood–Richardson rule, special form). Let $\lambda \in \Lambda^+(n,r)$ and $\mu \in \Lambda^+(n,s)$. Let $\mathcal{W}$ be the set of all words $k \in I(n, r+s)$ of the form $k = i \mid j$, where

(a3) the word i satisfies $KP(i) = i$, and the tableau $P(i)$ has shape λ;

(b3) the word j is i^{μ};

(c3) the reverse $B(k)$ of the word k is a lattice permutation.

Then,

$$s_\lambda s_\mu = \sum_{k \in \mathcal{W}} s_{\mathrm{wt}(k)}.$$

Here, $\mathrm{wt}(k)$ denotes the weight of a word k.

Proof. We set $Q := \mathsf{Q}^{(\lambda)}$ (see (C.2h) for the definition of this) and $S := \mathsf{Q}^{(\mu)}$. Then, we apply Theorem (D.12h). As a result, we obtain

$$s_\lambda s_\mu = \sum_{k \in \mathcal{W}} s_{\mathrm{wt}(k)},$$

where the set $\mathcal{W}$ is the one defined in Theorem (D.12h). It remains to show that the set $\mathcal{W}$ defined in Theorem (D.12h) is precisely the set $\mathcal{W}$ defined in Theorem (D.12i). For this purpose, we must show that the conditions (a2), (b2) and (c2) from Theorem (D.12h) (taken together) are equivalent to the conditions (a3), (b3) and (c3) from Theorem (D.12i).

Let us do this. Consider two words i and j and their concatenation $k = i \mid j \in I(n, r+s)$.

We have $i^{\mu} = i^{\mathsf{Q}^{(\mu)}}$ by Proposition (D.1i). In other words, $i^{\mu} = i^{S}$ (since $S = \mathsf{Q}^{(\mu)}$). Hence, condition (b2) is equivalent to (b3). Moreover, condition (c2) is equivalent to (c3), since they are literally the same.

We know from Exercise (C.2i) that a word i of shape λ satisfies $KP(i) = i$ if and only if $Q(i) = \mathsf{Q}^{(\lambda)}$. In other words, a word i of shape λ satisfies $KP(i) = i$ if and only if $Q(i) = Q$ (since $Q = \mathsf{Q}^{(\lambda)}$). Hence, if a word i of shape λ satisfies $KP(i) = i$, then $Q(i) = Q$ and therefore also $i \in I(n,r)$ (since $Q(i) = Q$ shows that i has as many letters as Q has entries; but $Q = \mathsf{Q}^{(\lambda)}$ shows that Q has $|\lambda| = r$ many entries). In other words, condition (a3) implies (a2). Conversely, condition (a2) implies (a3), since $Q(i) = Q = \mathsf{Q}^{(\lambda)} \in \mathcal{Q}(\lambda)$ entails that i has shape λ and thus satisfies $KP(i) = i$ (by Exercise (C.2i), since $Q(i) = \mathsf{Q}^{(\lambda)}$). Thus, the two conditions (a2) and (a3) mutually imply one another. In other words, they are equivalent.

Altogether, we now have seen that the conditions (a2), (b2) and (c2) from Theorem (D.12h) (taken together) are equivalent to the conditions (a3), (b3) and (c3) from Theorem (D.12i). This completes our proof of (D.12i). ∎

(D.12j) Theorem (Littlewood–Richardson rule, rewritten special form). Let $\lambda \in \Lambda^+(n,r)$ and $\mu \in \Lambda^+(n,s)$ and $\nu \in \Lambda^+(n, r+s)$. Let $\mathcal{W}_\nu$ be the set of all words $k \in I(n, r+s)$ of the form $k = i \mid j$, where

(a3) the word i satisfies $KP(i) = i$, and the tableau $P(i)$ has shape λ;

(b3) the word j is i^{μ};

(c3') the reverse $B(k)$ of the word k is a lattice permutation of weight ν.

Then,

$$c_{\lambda,\mu}^{\nu} = |\mathcal{W}_{\nu}|.$$

Proof. Take the formula $s_{\lambda} s_{\mu} = \sum\limits_{k \in \mathcal{W}} s_{\mathsf{wt}(k)}$ from Theorem (D.12i), and compare the coefficients of s_{ν} on both of its sides. The coefficient on the left hand side is $c_{\lambda,\mu}^{\nu}$ (by (17)), while the coefficient on the right hand side is $|\mathcal{W}_{\nu}|$. ■

Proof of (D.10a): The claim of (D.10a) is just Theorem (D.12j), with the words k, i, j and the set $\mathcal{W}_{\nu}$ renamed as i, j, k and $\mathcal{W}$. ■