Group Characters, Symmetric Functions, and the Hecke Algebras
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Chapter 1

- It would not harm to explain that whenever you say "module", you usually mean "right module". People are often used to left modules instead.

- **1.4 (Rieffel-Wedderburn):** The arrow in "natural map $B \mapsto \operatorname{End}_D(I)$" should be an $\rightarrow$ arrow rather than an $\mapsto$ arrow.

- **Between 1.4 and 1.5:** You write: "We remark that if I is a minimal right ideal above then D must be a division ring. For if $0 \neq \varphi \in D$, then $\ker(\varphi)$ and $\operatorname{im}(\varphi)$ are both right ideals of A, [...]". Actually they are right ideals of B, not of A.

- **Between 1.5 and 1.6:** You write: "In particular, $\dim_D(eB) = n$ so eB is a minimal right ideal." Could you explain the word "so" in this sentence? Because for me, the real reason why eB is a minimal right ideal is that $B \cong D^{n\times n}$ acts transitively on the set $eB \setminus 0 \cong D^n \setminus 0$ (since, just as in linear algebra, any nonzero row vector can be mapped to any other nonzero row vector of the same size by a suitably chosen square matrix).

Chapter 2

- **Page 5 and some times further in the book:** Your notation concerning matrix rings and general linear groups is not 100% consistent: the notations $M_n(\mathbb{C})$ and $M(n,\mathbb{C})$ are used for one and the same thing (the ring of $n\times n$ matrices over $\mathbb{C}$); the notations $\mathbf{GL}_n(\mathbb{C})$ and $\mathbf{GL}(n,\mathbb{C})$ are used for one and the same thing (the multiplicative group of units of $M_n(\mathbb{C})$).

- **The long formula on page 5:** In the formula

$$\left(\sum_{g\in G}\alpha(g)\,g\right)\left(\sum_{h\in G}\beta(h)\,h\right) = \sum_{g,h}\alpha(g)\,\beta(h)\,gh = \sum_x\sum_g\alpha(x)\,\beta\left(xg^{-1}\right)x,$$

  the term $\beta(xg^{-1})$ should be $\beta(x^{-1}g)$.

- **Proof of (2.1):** In the sentence "[...] where P is the matrix of the permutation $g \rightarrow g^{-1}$ ", the $\rightarrow$ arrow should be a $\mapsto$ arrow.

- **Between (2.5) and (2.6):** You write: "Indeed, the same argument shows more: the class sums are a basis for the integral group ring $\mathbb{Z}G \subseteq \mathbb{C}G$." I believe you meant this the right way, but it is just asking to be misunderstood: of course, the class sums are a basis for the *center of* the integral group ring, not for the group ring itself.

- **Proof of (2.8):** You write: "Let $\mathbf{X}$ be the $s \times s$ matrix whose (i, j) entry is $\chi_i(x_j)$." Here, $\chi_i(x_j)$ should be replaced by $\chi_j(x_i)$ (otherwise the rest of the proof doesn't work - unless I have miscalculated $\mathbf{X}^* D \mathbf{X}$.)

- **Page 10, between (2.8) and (2.9):** In the formula

$$\phi = \sum_{\chi \in \operatorname{Irr}(g)} (\chi, \phi) \chi,$$

the $\operatorname{Irr}(g)$ should be an $\operatorname{Irr}(G)$.

Chapter 3

- **(3.2):** In the statement of (3.2), you write: "The functions $\omega_i : \mathbf{Z}(\mathbb{C}G) \to \mathbb{C}$ are algebra homomorphisms whose values are algebraic integers." To be precise, "values" should be "values at elements of $\mathbf{Z}(\mathbb{Z}G)$" here.

Chapter 4

- **Proof of 4.4:** You write: "In particular, the set of products $\{x_i h_{ij} \mid 1 \le i \le t,\ 1 \le j \le T_i\}$ is a set of right coset representatives for K in G." The T_i in $\{x_i h_{ij} \mid 1 \le i \le t,\ 1 \le j \le T_i\}$ should be a t_i (with lowercase t).

- **4.5 (Frobenius):** In the statement of 4.5, it would be better to make the conditions more precise: $H \subseteq G$ is supposed to be a subgroup of G (not just some arbitrary subset), and "for $g \in G \setminus H$" should be "for all $g \in G \setminus H$" (and not just for *some* $g \in G \setminus H$).

- **Between (4.8) and (4.9):** You write: "This observation, originally due to Burnside, is useful in certain enumeration problems." Indeed, this observation is known to the whole world as Burnside's lemma, so the mention of Burnside is appropriate - but it should also be mentioned that Burnside was *not* the original author of this lemma.[1]

Chapter 5

- **Between (5.2) and (5.3):** You write: "We first argue that $A = \mathbf{C}_G(A)$, for if not then $\mathbf{C}_G(A) / A$ is a proper normal subgroup of the p-group G/A and therefore meets the center of G/A nontrivially." Here, the word "proper" should be "nontrivial", in my opinion.

- **Between (5.3) and (5.4):** You write: "it suffices by induction to show that every nonlinear character χ of G is induced from a proper subgroup of G." Here, "character" should be replaced by "irreducible character".

[1] cf. the historical remarks on `http://en.wikipedia.org/wiki/Burnside%27s_lemma`

- **Between (5.4) and (5.5):** You write: "and let $\mathcal{B}(G;\mathcal{H})$ be the set of permutation characters $\left\{1_H^G \mid H\in\mathcal{H}\right\}$". This is wrong - you don't want the *set* of these characters, but you want the *abelian group generated by this set*.

- **(5.5):** Maybe it wouldn't hurt to mention that "ring" means "not necessarily unital ring" here.

- **Proof of (5.6):** On the last line of page 20, you write $I_g = R$. The R should be a $\mathbb{Z}$ here.

- **Page 21, before (5.7):** You write: "It is clear that any subgroup of a quasi-elementary group is itself quasi-elementary". But why is this clear? The shortest proof I can think of is nontrivial[2].

- **Proof of (5.7):** You write: "Let P be a Sylow p-subgroup of $N = \mathbf{N}_G(C)$ containing g". But a p-group cannot contain g in general (the order of g is not always a power of p). I guess you want P to be a Sylow p-subgroup of $N = \mathbf{N}_G(C)$ containing g^n (this is possible and this leads to $g\in H$ afterwards).

- **Proof of (5.7):** You write: "Namely, choose coset representatives $\{x_1,...,x_t\}$ for H in G." I would say "*right* coset representatives" here to be more precise.

[2]The proof mainly consists of showing the following lemma:

Lemma. Let G be a finite group. Then, G is quasi-elementary if and only if the subset $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}$ of G is a cyclic subgroup for some prime p. Here, $\operatorname{ord} g$ denotes the order of the element g in G.

Proof of the Lemma. $\Longrightarrow$: Assume that G is quasi-elementary. Then, there is some prime p such that G is a semidirect product PC of some p-subgroup P of G with some cyclic normal subgroup C of G of order prime to p. Thus, $|G| = |PC| = |P|\cdot|C|$, so that $|G/C| = |G|/|C| = |P|\cdot|C|/|C| = |P|$ is a power of p, so that G/C is a p-group. Now, let $g\in G$ be some element such that $\operatorname{ord} g$ is prime to p. Then, the order of the element $\overline{g}$ of the quotient group G/C is prime to p as well (because the order of $\overline{g}$ in G/C divides the order of g in G). But the order of the element $\overline{g}$ of the quotient group G/C must be a power of p (since G/C is a p-group). Hence, the order of the element $\overline{g}$ of the quotient group G/C is 1 (because it is both prime to p and a power of p), and thus $\overline{g} = 1$, so that $g\in C$. We have thus shown that every element $g\in G$ such that $\operatorname{ord} g$ is prime to p must lie in C. Therefore, $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}\subseteq C$. Combining this with $C\subseteq\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}$ (which is clear because $\operatorname{ord} g$ is prime to p for every element $g\in C$, since the order of C is prime to p), we obtain $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\} = C$, so that $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}$ is a cyclic subgroup. This proves the $\Longrightarrow$ direction of the Lemma.

$\Longleftarrow$: Assume that the subset $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}$ of G is a cyclic subgroup for some prime p. Denote this subset $\{g\in G \mid \operatorname{ord} g \text{ is prime to } p\}$ by C. Clearly, C is a normal subgroup of G (it is in fact a characteristic subgroup). Now we are going to show that the quotient group G/C is a p-group. In fact, assume that it is not. Then, $|G/C|$ is not a power of p, so there must be a prime $q\neq p$ such that $q \mid |G/C|$. Thus, by Cauchy's theorem, there exists an element of G/C which has order q. Let $\overline{g}$ be this element (where $g\in G$). Then, $\overline{g}^q = 1$, so that $g^q\in C$. By the definition of C, this yields that $\operatorname{ord}(g^q)$ is prime to p, and thus $\operatorname{ord} g$ is prime to p as well (since $\operatorname{ord} g \mid q\cdot\operatorname{ord}(g^q)$ and since q is prime to p), so that $g\in C$ and thus $\overline{g} = 1$. This contradicts to our fact that the order of $\overline{g}$ is q. This contradiction shows that our assumption was wrong, and thus G/C is a p-group. Thus, $|G/C|$ is some power of p dividing $|G|$. Now let P be a Sylow p-subgroup of G. Then, $|P|\geq|G/C|$ (since $|G/C|$ is some power of p dividing $|G|$, while $|P|$ is the greatest power of p dividing $|G|$). Besides, $P\cap C = 1$ (since P is a p-group, while all elements of C have order prime to p) and obviously $|P|\geq|G/C|$ yields $|P|\cdot|C|\geq|G/C|\cdot|C| = |G|$, so that $|PC| = \dfrac{|P|\cdot|C|}{|P\cap C|}\geq\dfrac{|G|}{1} = |G|$ and thus $PC = G$. Hence, G is a semidirect product of P and C. This shows that G is quasi-elementary, and therefore the $\Longleftarrow$ direction of the Lemma is proven as well.

- **Proof of (5.8):** On the fourth line from the bottom of page 20, you write "[...] where λ is a linear character of some $H \in \mathcal{H}$." I think the $\mathcal{H}$ here is supposed to mean $\mathcal{E}$.

- **Between (5.10) and 5.11:** In the formula

$$1_G = 1_N^G - \sum_{i>0} a_i \lambda_i^G,$$

the minus sign should be a plus sign.

- **5.11:** In the statement of Brauer's theorem 5.11, maybe you should replace "power of p" by "nontrivial power of p" for better clarity.

- **Page 23:** You write: "Since cyclic groups are direct products of cyclic groups of prime power order, H is of the form $P \times Q$ where P is a p-group and $|Q| \not\equiv 0 \,(\operatorname{mod} p)$." But I think this follows directly from H being elementary - where are you using the fact that cyclic groups are direct products of cyclic groups of prime power order?

Chapter 6

- **Page 27:** You write: "In a slight departure from usual terminology, we will mean by a partition of Ω an ordered collection of pairwise disjoint nonempty subsets $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_r\}$ such that [...]". Actually, I would propose to write $(\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_r)$ instead of $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_r\}$ here, because $\{\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_r\}$ looks too much like "the set with elements $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_r$" which denies any ordering on the partitions, while you want the partitions to be ordered.

- **Proof of (6.2):** On the second line from the bottom of page 28, a full stop is missing after $\mathcal{C}(T) = \mathcal{Q}$ (but maybe it is just missing on the scanned version of the book).

- **Page 29:** On the third line before Gale-Ryser's theorem (6.3), you write: "It is clear that if $\lambda \leq \mu$, then $\lambda \ll \mu$." I think it's the other way round: if $\lambda \ll \mu$, then $\lambda \leq \mu$.

- **Proof of (6.4):** You write: "We say that σ is a *k-cycle* if $k_1 = k$ and $k_2 = 1$." To be completely rigorous here, I would replace this by "We say that σ is a *k-cycle* if $k_1 = k$ and $k_2 = 1$ or $r = 1$".

- **Proof of (6.4):** You write: "The usual notation for a k-cycle σ is $(m_0 m_1 ... m_{k-1})$ where $m_i^\sigma = m_{i+1}$ $(0 \leq i < k)$." I would add an explanation that m_k denotes m_0 here.

- **Between (6.4) and (6.5):** There is an empty space in the word "Then" in the sentence "T hen the σ_i are *disjoint* [...]".

- **Proof of (6.6):** On page 31, you list three facts (i), (ii) and (iii). Fact (iii) is wrong - instead, it should be

$$C_i \cap N_i = \mathbf{C}_{N_i}(\sigma_i) = \langle \sigma_{i,1}, \sigma_{i,2}, ..., \sigma_{i,m_i} \rangle ,$$

where $\sigma_{i,1}$, $\sigma_{i,2}$, ..., σ_{i,m_i} are the m_i disjoint i-cycles that σ_i consists of.

Chapter 7

- **Page 33:** There is an abuse of notation here: you denote by (-1) the signature character, while one could think that it means the additive inverse of the trivial character. Maybe it would not harm to introduce this notation explicitly.

- **Proof of (7.1):** On the fourth line from the bottom of page 33, you write $\mathcal{Q}^{\sigma h} = \mathcal{P}$. It seems that you mean $\mathcal{Q}^{\sigma h} = \mathcal{Q}$ here.

- **Between (7.1) and (7.2):** On page 34, you write: "Since the lexicographic order $\geq$ is a refinement of the partial order $\gg$, [...]". It would be better if you would define what you mean by "refinement", since this is not a standardized notion and can be understood in two opposite ways. I think the preferred word for what you mean by "refinement" is "extension".

- **Page 35:** You write: "As an example of how (7.3) is used to compute Y, [...]". The letter Y should be a boldfaced $\mathbf{Y}$ here.

Chapter 8

- **Proof of (8.1):** You write: "Since $\frac{1}{|S_{\mathcal{Q}}|}\tau_{\mathcal{Q}}$ is the primitive central idempotent of $\mathbb{C}S_{\mathcal{Q}}$ corresponding to $(-1)_H$, [...]". The $(-1)_H$ here should be a $(-1)_{S_{\mathcal{Q}}}$.

- **Proof of (8.1):** You write: "Similarly, $B_\chi J_{\pi'} = 0$ unless $(\chi, \phi_\pi) \neq 0$." I don't see how this follows "similarly" or how this is supposed to be true at all. Instead, I see why we "similarly" have $J_{\pi'} B_\chi = 0$ unless $(\chi, \phi_{\pi'}) \neq 0$ (note that I have switched the order of $J_{\pi'}$ and B_χ and replaced ϕ_π by $\phi_{\pi'}$).

- **Proof of (8.1):** You write: "Since $\mathbb{C}G$ is the sum of its minimal 2-sided ideals, we have first that

$$I_\pi \subseteq \sum_{(\chi,\psi_\pi)\neq 0} B_\chi,$$

[...]". This is true, but this doesn't help proving that $I_\pi J_{\pi'} \subseteq B_{\chi_\pi}$. What you actually seem to use is

$$J_{\pi'} \subseteq \sum_{(\chi,\phi_{\pi'})\neq 0} B_\chi$$

(which follows from $J_{\pi'} B_\chi = 0$ unless $(\chi, \phi_{\pi'}) \neq 0$, since $\mathbb{C}G$ is the sum of its minimal 2-sided ideals).

- The symmetric group S^n is sometimes denoted by G and sometimes by S in this chapter. It wouldn't harm to use consistent notation or to define these notations explicitly.

- **Between (8.2) and (8.3):** You write: "Since the $v(T)$ are permuted by S, they must span an S-submodule of X_π, but since X_π is irreducible, they span X_π." Here, you are silently using that the $v(T)$ are not all 0; this is not hard to see (in fact, every T satisfies

$$v(T) = f_{\mathcal{R}}\tau_{\mathcal{C}} = f_{\mathcal{R}} \sum_{g \in S_{\mathcal{C}}} (-1)^g g = \sum_{g \in S_{\mathcal{C}}} (-1)^g f_{\mathcal{R}} g = \sum_{g \in S_{\mathcal{C}}} (-1)^g f_{\mathcal{R}^g} \neq 0$$

because the $f_{\mathcal{R}^g}$ are pairwisely different vectors in the basis $\left\{ f_{\mathcal{P}} \mid \overline{\mathcal{P}} = \pi \right\}$ of M_π) but some readers probably won't notice the necessity unless it be pointed out.

- **Between (8.3) and (8.4):** You write: "We first observe that if $\sigma \in C(T)$ then $\tau_{\mathcal{C}^\sigma} = (-1)^\sigma \tau_{\mathcal{C}}$ [...]". It is a mystery to me where the $(-1)^\sigma$ factor comes from. Shouldn't it be $\tau_{\mathcal{C}^\sigma} = \tau_{\mathcal{C}}$ instead?

- **(8.4):** Here, again, I think the $(-1)^\sigma$ factor should be removed.

- **Between (8.6) and (8.7):** You write: "But every element of Ω_0 in column k is bigger than every element of Ω_0 in column j, so that the largest element of T which is moved by any $x \in X$ is moved to a lower-numbered column." The word "any" here is ambiguous; it would be better if you replace "by any $x \in X$" by "by x" and put a "Let $x \in X$ be arbitrary." before the whole sentence.

- **(8.8):** On the right hand side of the formula (8.8), there is a minor typo: The R in $(-1)^g f_{R_i^g}$ should be a calligraphic $\mathcal{R}$.

- **Proof of (8.7):** You write: "This implies that the coefficient of $f_{\mathcal{R}_1}$ in (8.8) is a_1 and that the coefficient of $f_{\mathcal{P}}$ is 0 for $\mathcal{P} > \mathcal{C}_1$." Here, $\mathcal{P} > \mathcal{C}_1$ should be $\mathcal{P} > \mathcal{R}_1$ instead.

- **Last line of page 39:** The $T_{\mathcal{Q}}$ here should be $\mathcal{Q}$ (or, more precisely, "standard Specht vectors which are smaller than $T_{\mathcal{Q}}$" should be "standard Specht vectors of standard tableaux whose row partitions are smaller than $\mathcal{Q}$").

- **Proof of (8.9):** Two lines above the statement of (8.10), you speak of "the map $T \to T'$ ". The $\to$ arrow here should be a $\mapsto$ arrow.

- **Proof of (8.9):** One line above the statement of (8.10), you write: "the disjoint union of the standard tableaus of type λ_j $(1 \leq j \leq s)$". The λ_j here should be $\lambda^{(j)}$ instead.

Chapter 9

- **(9.1):** When you write "the e_r are algebraically independent", you mean only the e_r for $r > 0$ (although you have defined e_0 as well). I think this is worth a mention.

- **(9.3):** Similarly, when you write "the h_r are algebraically independent", you mean only the h_r for $r > 0$ (although you have defined h_0 as well).

Chapter 10

- **Between (10.3) and (10.4):** "Note that if $\alpha_j < 0$ for some j, then the jth column of $\mathbf{H}_\alpha$ is zero, so we define $a_\alpha = 0$ if any $\alpha_j = 0$." Of course, you mean $\alpha_j < 0$ when you write $\alpha_j = 0$ here.

- **(10.6):** You write: "In particular, the Schur functions of degree n are a Z-basis for Λ^n." The Z here should be a boldface $\mathbb{Z}$.

- **(10.8):** In this formula, $\sum\limits_{\lambda} h_\lambda(z)$ should be $\sum\limits_{k} h_k(z)$.

- **(10.11):** It seems to me that the condition $\mu < \lambda$ should be $\mu > \lambda$ here.

Chapter 11

- **Proof of (11.2):** You refer to the "Frobenius reciprocity (3.1)". But the Frobenius reciprocity was (4.1), not (3.1).

- **Proof of (11.2):** In the formula

$$\mathbf{ch}(fg) = \left(f\sharp g^{S^{n+m}}, \rho\right) = (f\sharp g, \rho_{S^n \times S^m}) = \frac{1}{n!m!}\sum_{x,y} f(x)\, g(y)\, \rho(x,y),$$

  the term $f\sharp g^{S^{n+m}}$ should be $(f\sharp g)^{S^{n+m}}$.

- **Between (11.3) and (11.4):** When you say "and moreover $([\lambda], \psi_\mu) = 0$ for $\mu < \lambda$", it seems to me that you mean $\mu > \lambda$ instead of $\mu < \lambda$.

- **Between (11.3) and (11.4):** You write: "It now follows easily from (7.2) that $[\lambda] = \chi_\lambda$ [...]". But this doesn't seem that easy to me. The simplest argument I can come up with is the following: Since $[\lambda]$ is an irreducible character of S^n, there exists some partition $\rho(\lambda)$ of n such that $[\lambda] = \chi_{\rho(\lambda)}$. Thus we have defined a map ρ from the set of all partitions of n to itself. This map ρ is injective (since for any two partitions λ and μ such that $\rho(\lambda) = \rho(\mu)$, we have $[\lambda] = \chi_{\rho(\lambda)} = \chi_{\rho(\mu)} = [\mu]$ and thus $1 = ([\lambda], [\lambda]) = ([\lambda], [\mu]) = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$, so that $\lambda = \mu$), and thus a permutation of the set of partitions of n (since this set is finite). Since $\left(\psi_\lambda, \chi_{\rho(\lambda)}\right) = \left(\chi_{\rho(\lambda)}, \psi_\lambda\right) = ([\lambda], \psi_\lambda) = 1 \neq 0$, we have $\rho(\lambda) \gg \lambda$ for every λ by (7.2). Since $\gg$ is a partial order and ρ is a permutation, this yields $\rho(\lambda) = \lambda$ for every λ, and thus $[\lambda] = \chi_{\rho(\lambda)} = \chi_\lambda$, qed. Is there some simpler argument that I fail to see?

- **Between (11.4) and (11.5):** You write: "Moreover, since $\lambda_1 + r - 1 \leq n$ with equality iff $\lambda_2 = \lambda_3 = \cdots = \lambda_r = 1$, there can be at most one term equal to $[n]$ in the expansion of any determinant of the form (11.3), and it occurs in the expansion of exactly one such determinant [...]". I would add "(for fixed r)" here (because if we do not fix r, then it can occur in the expansion of *more than one* such determinant).

- **Between (11.4) and (11.5):** In the determinant

$$\det \begin{bmatrix} [n-r+1] & 1 & 0 & \cdots & 0 \\ [n-r+2] & [1] & 1 & \cdots & 0 \\ & & [1] & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ [n] & [r-2] & & \cdots & [1] \end{bmatrix},$$

the $[r-2]$ should be $[r-1]$.

- **Page 54:** You write: "In general, we need to evaluate a determinant of the form $\det(f_i(\mu_j))$ where f_i is a monic polynomial of degree i." Here I would rather say "of degree $r-i$", because otherwise you have to label the rows from 0 to $n-1$ rather than from 1 to n which is a bit unusual.

Chapter 12

- **Last line of page 55:** Here you write $h_{ij}(\lambda') = h_{i,j}(\lambda)$. The notations h_{ij} and $h_{i,j}$ denote one and the same thing; it would be best to decide for one of them throughout the text (I personally favor $h_{i,j}$ because it is less ambiguous).
- **Last line of page 56:** Here, "for any integer $\langle a\rangle$" should be "for any integer a".
- **Between (12.4) and (12.5):** On page 57, you write: "[...] we may as well assume that there is some index $i \geq k$ such that $\mu_i > \mu_k - m > \mu_{i+1}$, [...]". For the sake of completeness, it should be added here that μ_{r+1} is supposed to mean -1.
- **Page 57, one line above the picture of the Young diagram:** You write: "and $\lambda^{(k)} = 0$ for all other k". The equation $\lambda^{(k)} = 0$ is supposed to mean "$\lambda^{(k)}$ is not a partition". The same mistake is repeated three lines below the picture of the Young diagram.
- **After 12.6:** You notice correctly that the Murnaghan-Nakayama formula generalizes (8.9). I would add that it also generalizes (11.5).

Chapter 13

- **First sentence of Chapter 13:** "In this section we define the Hecke algebra (of type A_{n-1}) and prove that it is isomorphic to the group algebra $\mathbb{Q}[t]S^n$." But I don't think it is isomorphic to $\mathbb{Q}[t]S^n$. Maybe it becomes isomorphic when tensored with an appropriate field.
- **(13.1):** There are two mistakes here: First, $1 \leq i \leq n$ should be $1 \leq i < n$. Besides, $1 \leq i < n$ (in (13.1)) should be $1 \leq i < n-1$. The second of these mistakes is also repeated further below (between (13.1) and (13.2)).

- **Proof of (13.1):** You leave out the proof of (13.1). However, you are not winning much space by doing this, since it is very easy: Let Γ be the group with generators γ_1, γ_2, ..., γ_{n-1} subject to the relations

$$\begin{aligned} \gamma_i\gamma_{i+1}\gamma_i &= \gamma_{i+1}\gamma_i\gamma_{i+1} \text{ for each } 1 \leq i < n-1; \\ \gamma_i\gamma_j &= \gamma_j\gamma_i \text{ for all } i \text{ and } j \text{ satisfying } |i-j| \geq 2; \\ \gamma_i^2 &= 1 \text{ for all } 1 \leq i < n. \end{aligned}$$

Then, define a group homomorphism $P : \Gamma \to S_n$ by $(P(\gamma_i) = \sigma_i$ for every $1 \leq i < n)$ (this homomorphism is well-defined, because the transpositions σ_1, σ_2, ..., σ_{n-1} are easily seen to satisfy the relations (i), (ii) and (iii) of (13.1)). This homomorphism Γ is surjective (because $S_n = \langle \sigma_1, \sigma_2, ..., \sigma_{n-1} \rangle$). Now we can prove a discrete analogue of (13.2):
(13.2') Any element of Γ has the form $w\gamma_{1,i}$ for some $i \geq 0$, where $w \in \langle \gamma_1, \gamma_2, ..., \gamma_{n-2} \rangle$ and where $\gamma_{i,j}$ means $\begin{cases} \gamma_{n-i}\gamma_{n-i-1}\cdots\gamma_{n-j}, \text{ if } i \leq j, \\ 1, \text{ if } i > j \end{cases}$.
In order to obtain a proof of (13.2'), it is enough to read the proof of (13.2) with the following changes:
- replace every g_i by γ_i;
- replace $(t-1)\, wg_{n-1} + tw$ by w;
- replace H_n by Γ;
- read "word of the form" instead of "$\mathbb{Q}[t]$-linear combination of words of the form";
- read "consists of" instead of "is spanned by".
Now, (13.2') yields $|\Gamma| \leq n\, |\langle \gamma_1, \gamma_2, ..., \gamma_{n-2} \rangle|$, and thus by induction $|\Gamma| \leq n(n-1)\cdots 1 = n! = |S_n|$, so that the group homomorphism $P : \Gamma \to S_n$ must be bijective (since it is surjective), and thus $\Gamma \cong S_n$. This proves (13.1).

- **The definition of a standard word (directly above (13.3)):** You write: "Inductively, we define $w \in H_n$ to be a *standard word* if it is of the form $w_1 g_{1,i}$ for some $i \geq 0$, where w_1 is a standard word in $\langle g_1, ..., g_{n-2} \rangle$." This is slightly ambiguous - namely, if we would blindly follow this definition, we would believe that a standard word in $\langle g_1, ..., g_{n-2} \rangle$ means a word of the form $w_2 g_{1,i}$ for some $i \geq 0$, where w_2 is a standard word in $\langle g_1, ..., g_{n-3} \rangle$. But this makes no sense ($g_{1,i}$ is not in $\langle g_1, ..., g_{n-2} \rangle$ at all). Instead, a standard word in $\langle g_1, ..., g_{n-2} \rangle$ means a word of the form $w_2 g_{2,j}$ for some $j \geq 1$. It wouldn't hurt to warn the reader about this pitfall.

- **Proof of (13.3):** In order to prove that the standard words w_σ are linearly independent, you write: "Moreover, if there were a relation

$$\sum_\sigma p_\sigma(t)\, w_\sigma = 0$$

with $p_\sigma(t) \in \mathbb{Q}[t]$ and $\gcd\{p_\sigma(t)\} = 1$, [...]". But why can you assume that $\gcd\{p_\sigma(t)\} = 1$ here? If the $p_\sigma(t)$ have a common factor and you want to cancel it from them, you need the Hecke algebra to be torsionfree; is this trivial or have you silently proven this somewhere?

- **First line of page 64:** You write: "is really not necessary in the sequel". But don't you use (13.6) on page 69?

- **(13.7):** In the statement of (13.7), you write: "If $\sigma \in S^n$ fixes $\{1, ..., k\}$ and $\rho \in S^n$ fixes $\{k+1, ..., n\}$, [...]". However, as the statement (iii) shows, you want it exactly the other way round: you want σ to fix $\{k+1, ..., n\}$ (so that $\sigma \in S^k$) and ρ to fix $\{1, ..., k\}$ (so that $\rho \in S^{n-k}$).

Chapter 14

- **Proof of (14.1):** You write: "namely we define

$$\tau_n \left(w g_{1,i} \right) = s \tau_{n-1} \left(w g_{2,i} \right)$$

where w is a standard word in H_{n-1}". Here you should add "and $i > 0$", because for $i = 0$ this is wrong (and there is no need to define $\tau_n \left(w g_{1,i} \right)$ for $i = 0$, because $w g_{1,i} \in H_{n-1}$ for $i = 0$ and τ_n is supposed to extend τ_{n-1}).

- **Between (14.3) and (14.4):** You write: "So we may assume that $w = w_1 g_{1,j}$ in (14.3) for some standard word w_1 in H_{n-1} and some $j \geq 2$." Here, $j \geq 2$ should be $j \geq 1$.

- **The end of page 68:** When you write "Now if we define θ by the equation $\tau \left(\theta g_i \right) = \tau \left(\theta^{-1} g_i^{-1} \right)$, [...]", it would be nice to add that the τ that you are using here is actually a base extended version of the τ that you have defined before (namely, it is extended to a map $\tau : \mathbb{Q} \left(s, t, \theta \right) \to \mathbb{Q} \left(s, t, \theta \right)$).

- **First line of page 69:** I think $\pi_n : B_n \to H_n$ should be $\pi_n : B_n \to H_n \otimes \mathbb{Q} \left(s, t, \theta \right)$ here.

- **Example (about the trefoil) on page 69:** In this example, you obtain the formula

$$\widehat{\tau} \left(g_1^3 \right) = \left(\theta s \right) \theta^3 \tau \left(\left(t - 1 \right) g_1^2 + t g_1 \right) = \theta^4 s^3 \left[\left(t - 1 \right)^2 s + \left(t - 1 \right) t + t s \right].$$

Where does the s^3 come from?

- **Proof of (14.9):** The last equation on page 70,

$$e_n g_{n-1} = \left(e_{n-1} + g_{n-1} \rho_{n-1} \right) g_{n-1} = t e_n,$$

should be

$$e_n g_{n-1} = \left(e_{n-1} + e_{n-1} g_{n-1} \rho_{n-1} \right) g_{n-1} = t e_n.$$

- **Proof of (14.9):** In the equation

$$e_n g_{n-i} = e_{n-1} \rho_n g_{n-1} = t e_{n-1} \rho_n = t e_n$$

(this is the last equation in the proof), the g_{n-1} should be g_{n-i}.

- **Page 71:** Here you say: "From (14.9) we see that $e_n H_n$ is a one-dimensional right ideal [...]". But speaking about dimension makes sense only over a field.