On a Polynomial Identity for $n \times n$ Matrices*

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Abstract

We prove that the polynomial

$$h_k(x_1, \dots, x_k, y_1, \dots, y_k) = \sum_{\sigma,\tau \in S_k} (\operatorname{sgn} \sigma\tau) x_{\sigma(1)} y_{\tau(1)} \cdots x_{\sigma(k)} y_{\tau(k)}$$

vanishes on $n \times n$ matrices over a commutative ring for $k = 2n$ and for no smaller value of k.

Let C be a commutative ring with 1 and $M_n(C)$ the ring of $n \times n$ matrices over C. If $\{x_1, \dots, x_k, \dots\}$ and $\{y_1, \dots, y_k, \dots\}$ are two distinct sets of non-commuting variables, for each $k \geq 1$ we define the polynomial

$$h_k(x_1, \dots, x_k, y_1, \dots, y_k) = \sum_{\sigma,\tau \in S_k} (\operatorname{sgn} \sigma\tau) x_{\sigma(1)} y_{\tau(1)} \cdots x_{\sigma(k)} y_{\tau(k)},$$

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where S_k is the symmetric group of degree k.

It is clear that, for some k, h_k is a polynomial identity for[1] $M_n(C)$; in fact, since h_k is alternating in the x_i (and also in the y_i), h_{n^2+1} is an identity[2] for $M_n(C)$.

The purpose of this note is to prove that $2n$ is the smallest value of k for which h_k is a polynomial identity for $M_n(C)$. This answers a question of Formanek.

We have the following:

Theorem. *h_{2n} is a polynomial identity for $M_n(C)$. Moreover if h_k is a polynomial identity for $M_n(C)$, then $k \geq 2n$.*

Our approach will be based on a proof of the Amitsur–Levitzki theorem given by Rosset in [1].

Before proceeding to the proof of this theorem we need some preliminaries.

Let E be the exterior algebra on a $4n$-dimensional vector space V over the field of rational numbers $\mathbb{Q}$ and let $\{v_1, \dots, v_{4n}\}$ be a basis of V over $\mathbb{Q}$. E may be viewed as the free algebra on V modulo the relations $v_i v_j = -v_j v_i$. We write $E = E_0 + E_1$, where E_0 is the subalgebra generated by $\mathbb{Q}$ and the monomials in the v_i of even degree and E_1 is the space generated by the monomials of odd degree.

In the following proposition we study some properties of the algebra

$$M_k(E) \simeq M_k(\mathbb{Q}) \otimes_{\mathbb{Q}} E.$$

Proposition. (i) *If $U \in M_k(E_0)$ is such that $\operatorname{tr}(U) = \operatorname{tr}(U^2) = \cdots = \operatorname{tr}(U^k) = 0$, then $U^k = 0$.*

(ii) *If $U, T \in M_k(E_1)$, then $\operatorname{tr}(UT) = -\operatorname{tr}(TU)$.*

Proof. Since E_0 is a commutative algebra over $\mathbb{Q}$, (i) follows from Newton's formulas for symmetric functions (see [1][3]). To prove (ii), write $U = \sum A_i w_i$, $T = \sum B_i w_i$, where $A_i, B_i \in M_k(\mathbb{Q})$ and the w_i are monomials in E_1. Recalling that tr is a symmetric bilinear form on $M_k(\mathbb{Q})$ and $w_i w_j = -w_j w_i$, we have[4]

$$\operatorname{tr}(UT) = \sum \operatorname{tr}(A_i B_j) w_i w_j = -\sum \operatorname{tr}(B_j A_i) w_j w_i = -\operatorname{tr}(TU).$$

□

Proof of the Theorem. Since, for $k > 1$,

$$h_k(x_1, \dots, x_k, y_1, \dots, y_k) = \sum_{i,j=1}^{k} (-1)^{i+j} x_i y_j \, h_{k-1}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_k),$$

[1] *Comment by DG:* "$M_n(C)$" was "$M_k(C)$" in the original, but the intended meaning was likely "$M_n(C)$".

[2] *Comment by DG:* Corrected "h_{n^2}" to "h_{n^2+1}".

[3] *Comment by DG:* See Corollary 4.1 **(b)** in Darij Grinberg, *The trace Cayley-Hamilton theorem*, arXiv:2510.20689v2 for the precise fact being used.

[4] *Comment by DG:* Corrected "$\operatorname{tr}(AB)$" and "$\operatorname{tr}(BA)$" to "$\operatorname{tr}(UT)$" and "$\operatorname{tr}(TU)$".

to prove the second part of the theorem it is enough to check that h_{2n-1} is not an identity for $M_n(C)$. To this end, consider the substitution (double staircase)

$$x_1 = e_{11}, \quad y_1 = e_{12}, \quad x_2 = e_{22}, \quad y_2 = e_{23}, \quad \dots, \quad x_n = e_{nn},$$

$$y_n = e_{nn}, \quad x_{n+1} = e_{n,n-1}, \quad y_{n+1} = e_{n-1,n-1}, \quad \dots, \quad x_{2n-1} = e_{21}, \quad y_{2n-1} = e_{11}.$$

Then

$$h_{2n-1}(e_{11}, \dots, e_{21}, e_{12}, \dots, e_{11}) = e_{11} + 2\sum_{i>1} e_{ii} \neq 0$$

[5] and h_{2n-1} is not an identity for $M_n(C)$.

For the first part of the proof notice that, since h_{2n} is multilinear and each monomial has coefficient ± 1, it is enough to prove that h_{2n} vanishes on $M_n(\mathbb{Q})$ (see [2]).

Let $A_1, \dots, A_{2n}, B_1, \dots, B_{2n} \in M_n(\mathbb{Q})$ and let

$$A = \sum_{i=1}^{2n} A_i v_i, \qquad B = \sum_{i=1}^{2n} B_i v_{2n+i}.$$

Then $A, B \in M_n(E_1)$ and

$$(AB)^{2n} = h_{2n}(A_1, \dots, A_{2n}, B_1, \dots, B_{2n}) v_1 v_{2n+1} v_2 v_{2n+2} \cdots v_{2n} v_{4n}.$$

This last equality can be verified by noticing that $v_k^2 = 0$ ($k = 1, \dots, 4n$) and for $\sigma, \tau \in S_{2n}$,

$$v_{\sigma(1)} v_{2n+\tau(1)} v_{\sigma(2)} v_{2n+\tau(2)} \cdots v_{\sigma(2n)} v_{2n+\tau(2n)}$$
$$= (\operatorname{sgn} \sigma\tau) v_1 v_{2n+1} v_2 v_{2n+2} \cdots v_{2n} v_{4n}.$$

Take now the matrix

$$D = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \in M_{2n}(E_0).$$

Since for $i \geq 1$, we have[6] $(BA)^{i-1}B$, $A \in M_n(E_1)$, by Proposition (ii), $\operatorname{tr}((AB)^i) = -\operatorname{tr}((BA)^i)$ [7]; thus

$$\operatorname{tr}(D^i) = \operatorname{tr}((AB)^i) + \operatorname{tr}((BA)^i) = 0.$$

But then Proposition (i) forces $D^{2n} = 0$ and so $(AB)^{2n} = 0$. This last equality is equivalent to $h_{2n}(A_1, \dots, A_{2n}, B_1, \dots, B_{2n}) = 0$. The proof is now complete. □

It has come to our attention that a different proof of the above theorem has been announced by Qing Chang.[8]

[5] *Comment by DG:* Corrected "$\sum e_{ii}$" to "$e_{11} + 2\sum_{i>1} e_{ii}$".

[6] *Comment by DG:* Corrected "$(BA)^i B$" to "$(BA)^{i-1} B$", and added "we have".

[7] *Comment by DG:* This follows by applying Proposition (ii) to $U = A$ and $T = (BA)^{i-1}B$, and observing that $TU = (BA)^{i-1}BA = (BA)^i$ and $UT = A(BA)^{i-1}B = (AB)^i$.

[8] *Comment by DG:* This is likely a reference to: Qing Chang, *Some consequences of the standard polynomial*, Proc. Amer. Math. Soc. **104** (1988), pp. 707–710.

References

[1] S. Rosset, A new proof of the Amitsur–Levitzki identity, *Israel J. Math.* **23** (1976), 187–188.

[2] L. H. Rowen, "*Polynomial Identities in Ring Theory*", Academic Press, New York, 1980.