On finite sequences satisfying linear recursions
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Errata and addenda by Darij Grinberg

Errata

The numbering of pages in the following errata matches the numbering of pages in the arXiv version (v2).

The errata cover everything until after the proof of the Corollary on page 8.

- **page 1, Abstract:** In the first sentence, the subsentence "subset of sequences satisfying a degree-m linear recursion, i.e. for which there exist $a_0, \ldots, {}_m \in k$" is repeated twice. (Only in the actual PDF; not in the arXiv metadata abstract.)

- **page 2:** You write: "linear subspaces of dimensions m and $m+1$ in W_n". I think the "$m+1$" should be an "$m-1$" here. The same applies to the abstract (and this one also appears in the arXiv metadata abstract).

- **page 3:** "Let W_n as the dual" $\to$ "Let W_n be the dual".

- **page 3, Lemma 1:** I would replace the word "adjoint" by "dual map", since the word "adjoint" has too many different meanings.

- **page 3, proof of Lemma 1:** In the first displayed equation of this proof, an "equals" sign is missing; i.e., the equation should be

$$P(Y, Z) = \sum_{j=0}^{n-m} b_j Y^j Z^{n-m-j}.$$

- **page 4, The ideal I_x:** It may be helpful to mention that I_x is the annihilator of x, where we let the graded k-algebra $k[Y, Z] = \bigoplus_{n \geq 0} V_n$ act on its graded dual $\bigoplus_{n \geq 0} V_n^* = \bigoplus_{n \geq 0} W_n$ in the obvious way (i.e., any $Q \in k[Y, Z]$ acts on $x \in \bigoplus_{n \geq 0} V_n^*$ by pre-composition). This explains why I_x is a homogeneous ideal, provided that one knows the basic properties of graded algebras.

- **page 4, Proposition 1:** The "V_{m-m_0}" on the right hand side of (11) needs to be understood as 0 if $m - m_0 < 0$. In view of this, it may be worth (at least in Proposition 1) defining $V_i = W_i = 0$ for all $i < 0$, not just for $i = -1$.

- **page 5, Lemma 4:** Here you are using without mention and proof the following facts about bivariate polynomials:

  1. Any two nonzero polynomials $Q_1, Q_2 \in k[Y,Z]$ have a gcd and an lcm. These are unique up to multiplication by a nonzero scalar, and can be chosen to satisfy $\gcd(Q_1, Q_2) \cdot \operatorname{lcm}(Q_1, Q_2) = Q_1 Q_2$.
  2. If Q_1 and Q_2 are nonzero homogeneous polynomials in $k[Y,Z]$, then $\gcd(Q_1, Q_2)$ and $\operatorname{lcm}(Q_1, Q_2)$ are homogeneous polynomials as well.

  It took me a while to figure out why these facts are true. In hindsight, it was pretty easy: Fact 1 holds in any unique factorization domain (and $k[Y,Z]$ is known to be a unique factorization domain); Fact 2 follows from the (easily provable) fact that any polynomial that divides a nonzero homogeneous polynomial in $k[Y,Z]$ is itself homogeneous. Alternatively, Facts 1 and 2 can be proved together rather nicely as follows: Define a map Φ from

  $$\{\text{nonzero homogeneous polynomials in } k[Y,Z]\}$$

  to

  $$\mathbb{N} \times \{\text{homogeneous polynomials in } k[X]\}$$

  which sends each nonzero homogeneous polynomial $P \in k[Y,Z]$ to the pair $(v_Y(P), P(1,X))$, where $v_Y(P)$ is the largest $i \in \mathbb{N}$ satisfying $Y^i \mid P$. This map Φ is easily seen to be a monoid isomorphism from

  $$(\{\text{nonzero homogeneous polynomials in } k[Y,Z]\}, \cdot)$$

  to

  $$(\mathbb{N}, +) \times (\{\text{homogeneous polynomials in } k[X]\}, \cdot);$$

  thus, it induces an isomorphism between the divisibility posets of these two monoids. Since the divisibility poset of the latter monoid is a distributive lattice (because it is a direct product of two distributive lattices), it thus follows that so is the divisibility poset of the former poset. But this means that nonzero homogeneous polynomials in $k[Y,Z]$ have gcds and lcms in the set of all nonzero homogeneous polynomials in $k[Y,Z]$. It is not hard to check that these gcds and lcms are also gcds and lcms in the entire polynomial ring $k[Y,Z]$. Thus, Facts 1 and 2 easily follow.

  This was a fun exercise, but I feel that some readers might prefer a reference.

- **page 6, proof of Proposition 1:** After "We can now easily prove Prop. 1.", I would add: "There exists a nonzero $Q \in I_x \cap V_{m_0}$, since I_x is a homogeneous ideal." (Otherwise, per se, we only know that there is a nonzero polynomial $Q \in I_x$ of degree m_0, not necessarily a homogeneous one.)

- **page 6, proof of Proposition 1:** Replace "and $Q \in I_x \cap V_{m_0} - \{0\}$ then" by "and $Q \in I_x \cap V_m - \{0\}$ then".

- **page 6, proof of Proposition 1:** Replace "thus $I_x \cap V_{m_0}$ consists" by "thus $I_x \cap V_m$ consists".
- **page 8:** After proving the Corollary, you write "has only m rows". I think this should be "has only m columns".
- **page 9, proof of Lemma 5:** Replace "$\widehat{\chi}_K(y)$" by "$\widehat{\chi_K}(a)$".
- **general notations:** You are using the $\geq$ (\geq) and $\geqslant$ (\geqslant) symbols interchangeably. It is probably best to replace each "$\geq$" by "$\geqslant$" for the sake of consistency with "$\leqslant$" (which you use consistently).