An Introduction to Symmetric Functions and Their Combinatorics
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Errata and comments

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The following are all comments I have noted down when reading your text "*An Introduction to Symmetric Functions and Their Combinatorics*" (Student Mathematical Library #91, AMS 2019). My apologies for mixing real corrections with subjective suggestions, pedantic misreadings and the occasional alternative proof I couldn't help writing up. I also have never proofread the list below, so it might be in need of its own error hunter.

"Line $-n$" (for negative n) means "line n from the bottom". Lines in displayed formulas are counted as lines.

1. **page 1:** You write: "The algebra of symmetric polynomials in n variables is isomorphic to the algebra generated by the characters of the irreducible representations of the symmetric group S_n". I am pretty sure this is wrong in any possible interpretation. The former algebra is a polynomial ring; the latter is finite-dimensional. The closest thing that is true is that the algebra of symmetric functions(!) is isomorphic to the direct sum of the character rings of $S_0, S_1, S_2, \ldots$, equipped with a special multiplication obtained from induction (not just from multiplying characters pointwise).

2. **page 2:** I doubt that "the set of all symmetric functions with coefficients in $\mathbb{Q}$ is a finite-dimensional vector space over $\mathbb{Q}$". I assume you are talking about a graded component of this space.

3. **page 3:** When defining X_n, you should require $n \geq 0$ instead of $n \geq 1$. Indeed, you use $X_0 = \{\}$ (and the respective symmetric functions $e_k(X_0)$ and $h_k(X_0)$) in Proposition 3.1, for example (since Proposition 3.1 for $n = 1$ mentions X_{1-1}).

4. **page 9, proof of Proposition 1.11:** The expression "$\mu_1, \ldots, \mu_{j+1}, \mu_j, \ldots, \mu_n$" can be interpreted in two ways. The most obvious interpretation (namely, as the concatenation of the $(j+1)$-tuple $(\mu_1, \ldots, \mu_{j+1})$ with the $(n-j+1)$-tuple $(\mu_j, \ldots, \mu_n)$) is the wrong one; the right interpretation is "the n-tuple $(\mu_1, \ldots, \mu_n)$ with its j-th and $(j+1)$-st entries swapped". Worth pointing out.

5. **page 14:** When defining the total degree, replace "$a_1 + a_2 \cdots$" by "$a_1 + a_2 + \cdots$".

6. **page 17, between Definition 1.19 and Definition 1.20:** "we constructed the monomial symmetric functions" $\to$ "we constructed the monomial symmetric polynomials".

7. **page 17, Definition 1.19:** This definition of Λ has the consequence that $\prod_{i=1}^{\infty}(1+x_i)$ is a symmetric function. This conflicts with the standard definition of symmetric functions in the literature (which requires a symmetric function to be of bounded degree, so that $\prod_{i=1}^{\infty}(1+x_i)$ does not fit the bill). It also conflicts with your own Definition 7.5, because if you allow $f := \prod_{i=1}^{\infty}(1+x_i)$ to be an element of Λ, then the Hall inner product $\langle f, f\rangle$ is undefined. It makes sense to study both "versions" of symmetric functions (i.e., the symmetric power series of bounded degree, and the merely symmetric power series), but they need different names and different notations.

8. **page 17, solution to Exercise 1.21:** Replace "sum of the images of $x_1^2x_2$ under all permutations" by "sum of all monomials that can be obtained from $x_1^2x_2$ through permutations". (There is a subtle difference: The former wording suggests a sum over permutations, whereas the latter suggests a sum over monomials. The latter is right, because one and the same monomial can be obtained using different permutations and thus would be counted multiple times according to the former wording.)

9. **page 20, Exercise 1.21:** This exercise relies on a concept of infinite products of formal power series, which is not completely obvious. Since it is not easily found in algebra textbooks (usually, textbooks only define convergence of formal power series in finitely many variables, and not all of these definitions are easily adapted to the case of infinitely many variables; for example, $\prod_{i=1}^{\infty}(1+x_i)$ does **not** converge with respect to total degree, but rather converges with respect to the product topology), I suggest saying some words about how it is defined.

10. **page 24, between Example 2.1 and Definition 2.2:** "is the sum of all products" $\to$ "is (up to sign) the sum of all products".

11. **page 27, second paragraph:** "in any linear combination of monomial symmetric functions" $\to$ "in any linear combination of monomial symmetric polynomials (in a given set of variables X_n)". After this sentence, add "The same applies to monomial symmetric functions.".

12. **page 27, proof of Proposition 2.6:** The "If $n \geq |\lambda|$" here is misplaced: The sentence in which it appears does not require $n \geq |\lambda|$ (but the next sentence does).

13. **page 28, solution to Example 2.7:** I would replace "with distinct positive integers" by "with k distinct positive integers".

14. **page 30, solution to Example 2.7:** "case A4" $\to$ "case A3".

15. **page 31, proof of Proposition 2.10:** When you say "By (2.2) we have", you mean not (2.2) but rather the next displayed equality after (2.2).

16. **page 32, proof of Proposition 2.10:** Before "each such e_{λ_j} must contribute exactly one factor x_2", add "$\mu_1 = \lambda'_1$ and $\mu_2 = \lambda'_2$ imply that" (it is easy to lose track of the context here).

17. **page 34, Proposition 2.12 (c):** After "$l(\lambda)$ urns", I'd add "(which are distinguishable and labelled $1, 2, \ldots, l(\lambda)$)".

18. **page 37, proof of Theorem 2.17:** I'd mention that different monomials of f lead to different e_λ's in $f(e_1, e_2, \ldots, e_n)$; otherwise the same argument would show the obviously wrong claim that $e_1, e_1, e_1, \ldots$ are algebraically independent.

19. **page 38, lines 1–4:** What you are saying here is not a restatement of Theorem 2.17 alone, but also incorporates the fact that the $e_1, e_2, e_3, \ldots$ generate the $\mathbb{Q}$-algebra Λ. (This follows from Corollary 2.11, too.)

20. **page 39, Definition 2.19:** "*complete homogeneous polynomial*" $\to$ "*complete homogeneous symmetric polynomial*".

21. **page 39, Definition 2.19:** Usually your text isn't exactly suffering from a shortage of examples, but here is one place where an example would be really helpful: after this definition. The undefined concepts around multisets (e.g., what is the size of a multisubset? is a sum over $J \subseteq [[n]]$ itself a sum over a set or over a multisubset?) might confuse some readers; it would be helpful to dispel such confusion by showing (say) h_1, h_2, h_3 and $h_2(X_3)$.

22. **page 41:** "In Example 2.20 we found" $\to$ "In Example 2.22 we found".

23. **page 43, Combinatorial Proof of Proposition 2.24:** "then we just need to give an involution" might be a bit abrupt for those readers who aren't used to proofs by sign-reversing involutions. What about including a simple example for this method before (say, a proof of $\sum_{T \subseteq S} (-1)^{|T|} = 0$ for any nonempty finite set S) in an appendix?

    (Strangely enough, you do handhold the reader about sign-reversing involutions – but not until later, when you prove Proposition 3.2.)

24. **page 44, line 1:** "new way" $\to$ "way" (I don't think you showed this fact in any other way before).

25. **page 44, proof of Proposition 2.25:** It might be worth pointing out that the induction step tacitly uses the fact that $h_\lambda h_\mu = h_\nu$ for some partition ν (namely, for $\nu = \lambda \cup \mu$ as defined on page 13). Actually, it may help making this fact into a separate lemma; you use it on page 180 as well.

26. **page 45, Proposition 2.28:** The word "nonempty" isn't needed. (Likewise on the few lines before Proposition 2.28.)

27. **page 46, proof of Proposition 2.30 (i):** The fact that $\mu >_{\text{lex}} \lambda$ if μ is obtained from λ by merging parts and rearranging the resulting numbers into weakly decreasing order is not as obvious as it might appear; I would suggest at least an example to illustrate the reasons for it.

28. **page 48:** When you write "logarithms of formal power series behave in analogy with logarithms of polynomials", do you really mean "polynomials"? I think logarithms of polynomials aren't much simpler than logarithms of arbitrary formal power series.

29. **page 48, proof of Proposition 2.33:** I'd remove the "$P(t) =$" at the beginning of the displayed computation, since it's not clear which side of (2.16) it is referring to. (Do you even need the notation $P(t)$ anywhere? If not, I'd remove it from the statement of the proposition as well. It's not standard; Macdonald's book uses $P(t)$ for what you would call $\frac{d}{dt}P(t)$.)

30. **page 49, Problem 2.9:** The theorem you are asking the reader to prove here is the Gale–Ryser theorem, which is much harder than most other problems in this section. As far as symmetric functions are concerned, only the "only if" part is useful, while the "if" part is a nice curiosity. (Mark Wildon, in §5.2 of his *An Involutive Introduction to Symmetric Functions*, gives a proof of the "if" part using symmetric functions, but it uses more tools than at the reader's disposal in §2 of your book.)

31. **page 49, Problem 2.14:** Replace "$n \geq 0$" by "$n \geq 1$" twice in this problem, since there is no f_0.

32. **page 50, Problem 2.24:** Replace "$n - 1 - k$" by "$n - 1 + k$". Also worth pointing out that k can be any integer $\geq 1 - n$ here (with the standard convention that $h_i(X_n) = 0$ for all $i < 0$).

33. **page 51, Problem 2.27:** Why are you repeating the equality (2.15) here?

34. **page 51, Problem 2.28 (a):** Replace "$n \times n$" by "$k \times k$".

35. **page 51, Problem 2.28 (a):** "exactly one entry of column j is λ_j" is literally false if $\lambda_j = 0$. There are two ways to fix this: either you make the matrix $k \times l(\lambda)$ (as opposed to $k \times k$), or you replace "exactly one entry of column

j is λ_j and all other entries in column j are 0" by "column j has an entry equal to λ_j while all other entries of this column are 0".

36. **page 51, Problem 2.28 (b):** "for all" $\to$ "for each". (Otherwise, the "all of which have the same type" part sounds like all balls in all columns must have the same type.)

37. **page 51:** Additional problem suggestions: It is worth adding an exercise that asks to show that if $\lambda, \mu \vdash k$ satisfy $M_{\lambda,\mu}(p, m) \neq 0$, then $\lambda \trianglelefteq \mu$ in dominance order. (This would strengthen Proposition 2.30 (i).) Another exercise could ask to show that the relation $M_{\lambda,\mu}(p, m) \neq 0$ between λ and μ is in itself a partial order on the set of all partitions of k.

38. **page 54, proof of Proposition 3.2:** After "followed by a choice of one of the variables x_s in that term", I would add "(there are j possible choices, to explain the factor j in (3.4))".

39. **page 55, proof of Proposition 3.2:** "For example, when $k = 7$ and $j = 3$" $\to$ "For example, when $k = 7$ and $j = 4$".

40. **page 57, proof of Proposition 3.3:** "after the rightmost r" $\to$ "after the rightmost number $\leq r$" (there might not be an r in the first title).

41. **page 57, proof of Proposition 3.3:** I would suggest at least mentioning the existence of a simple and slick proof of (3.6) that uses neither bijections nor power series: Expand the p_j in $\sum_{j=1}^{k} h_{k-j} p_j$ using (3.5), then interchange the summation signs, then get rid of the inner sum using (2.12) to obtain kh_k. Probably (3.7) can be proved in the same way.

42. **page 58, proof of Proposition 3.5:** Here you are tacitly using the fact that evaluating the symmetric **polynomial** $e_j(X_n)$ at $\underbrace{1, 1, \ldots, 1}_{n \text{ times}}$ gives the same result as evaluating the symmetric **function** e_j at $\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, 0, \ldots$. (And the same fact about h_j.) This is close to obvious, but worth stating explicitly.

43. **page 58, between Proposition 3.5 and Proposition 3.6:** The restatement of (3.10) using multinomial coefficients is broken: The addend for $j = 0$ is undefined (there is a division by zero and a multinomial coefficient with a negative integer in it).

44. **page 59, after proof of Proposition 3.6:** The restatement of (3.11) using multinomial coefficients may involve addends in which the $n - k + j$ is negative. To make sense of them, $\binom{a+b+c}{a, b, c}$ should be defined to be 0 when one of a, b, c is negative.

45. **page 60, Definition 3.10:** "For all $n \geq 1$ and all $k \geq 1$" $\to$ "For all $n \geq 0$ and $k \in \mathbb{Z}$". (You use the 0-cases later on, e.g., in Proposition 3.13. The case of negative k is more of a luxury, but it has the advantage of making (3.17) work for all $k \geq 0$, which to me is (pardon) a plus. It is worth saying that $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $k < 0$.)

46. **page 60, Definition 3.12:** "For all $n \geq 1$ and all $k \geq 1$" $\to$ "For all $n \geq 0$ and $k \in \mathbb{Z}$". (I'm not sure if you use the 0-cases later on, but there is no reason to exclude them. Once again, the case of negative k is more of a luxury. It is worth saying that $\begin{Bmatrix} n \\ k \end{Bmatrix} = 0$ whenever $k < 0$.)

47. **page 61, Proposition 3.13:** I'd replace "for all $n \geq 2$ and k with $1 \leq k \leq n-1$" by "for all $n \geq 1$ and $k \in \mathbb{Z}$". There is no need to rule out the trivial cases; the proof works equally well in them.

48. **page 62, proof of Proposition 3.13:** When you say "remove the n from one of these permutations", you mean removing it from the cycle notation rather than (say) from the one-line notation. This should be said – it's the natural choice but not the only one.

49. **page 62, proof of Proposition 3.13:** "Therefore, we can construct each permutation of" $\to$ "Therefore, we can construct each second-type permutation of".

50. **page 62, proof of Proposition 3.13:** "wth" $\to$ "with".

51. **page 62, Proposition 3.14:** Again, I'd replace "for all $n \geq 2$ and all k with $2 \leq k \leq n-1$" by "for all $n \geq 1$ and $k \in \mathbb{Z}$". There is no difference to the proof, but the generality makes it easier to use the proposition.

52. **page 62, Proposition 3.15:** You can replace "$n \geq 1$" by "$n \geq 0$".

    You also don't want to require $k \leq n$ – certainly not for (3.18), but even for (3.17) it is not very helpful. (As it stands, it forces you to use $\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} = n!$ in the proof, which you haven't shown.)

    Assuming that you have changed Definitions 3.10 and 3.12 as I suggested above, you can simply replace "and all k with $0 \leq k \leq n$" here by "and all $k \geq 0$" here, and the proposition remains valid. The proof also remains valid, provided that you use induction on n and (in the induction step) treat the $k = 0$ case separately.

53. **page 63, (3.20):** "$(-1)^{j-1}$" should be "$(-1)^{j-1} j$" on the right hand side.

54. **page 64, (3.21):** Replace "j" by "k" on the left hand side.

55. **page 65, Example 3.21:** It is worth saying somewhere that "word" means "finite tuple of positive integers" in this book, and that every permutation $\pi \in S_n$ is identified with the word $(\pi(1), \pi(2), \ldots, \pi(n))$. (This becomes particularly important in Chapter 10.)

56. **page 69, proof of Proposition 3.26:** Your induction step uses $n \geq 2$ and $k \geq 1$, but you only have $n \geq 2$ **or** $k \geq 1$. It thus remains to deal with the case $n = 1$ and the case $k = 0$ separately. (Of course, both of these cases can be left to the reader, but the structure of the proof should still be adapted to mention them.)

57. **page 69, between Proposition 3.26 and Proposition 3.27:** "which can appear is $1 \cdot q \cdot q^2 \cdots q^k$" $\to$ "which can appear is $1 \cdot q \cdot q^2 \cdots q^{k-1}$".

58. **page 70, (3.28):** Replace "k" by "j" on the right hand side.

59. **page 71, Problem 3.3:** The "n" atop the summation sign should be a "k".

60. **page 75, §4.1:** Not sure what you mean by "In Chapter 2.16 and 2.17".

61. **page 81:** "seminstandard" $\to$ "semistandard".

62. **page 82, Definition 4.11:** Something tells me that you will use the notion of "content" not just for semistandard tableaux but for any fillings of a partition shape (and even more generally, as in the proof of Proposition 5.20), so it may be worth making this definition a bit more general.

63. **page 83:** "Therefore, in T we have" $\to$ "Therefore, in R we have".

64. **page 87:** "our solution to Example 4.12 raises" $\to$ "our solution to Example 4.9 raises".

65. **page 87, (4.6):** I'd say a couple of words about why this equality holds. (Namely, it follows from Proposition 4.15 using the same trick that was used in the proof of Proposition 2.6 and in the solution to Example 2.7.)

66. **page 90, proof of Proposition 4.23:** This is mis-organized: The displayed equality relies on $\mu_j = \mu_l$, so you should say "if $\mu_j = \mu_l$" in the first (not in the second) sentence of this proof. On the other hand, the second sentence needs an "if $f(X_n)$ is alternating".

67. **page 91, proof of Proposition 4.24:** "Then by Proposition 1.2(i),(ii)" $\to$ "Then by Proposition 1.2(i),(ii),(iii)".

68. **page 91, proof of Proposition 4.24:** In the second displayed equation of this proof, replace "$s_{\tau(n)}^{\mu_n}$" by "$x_{\tau(n)}^{\mu_n}$".

69. **page 92, line 1:** "is the sequence $n-1, n-2, \ldots, 2, 1$" $\to$ "is the sequence $n-1, n-2, \ldots, 2, 1, 0$".

70. **page 92, proof of Proposition 4.25:** It is worth pointing out that this proof can be skipped, as the result of Proposition 4.25 will not be used anywhere (or will it? I don't see any direct references).

71. **page 97, after the proof of Proposition 4.31:** "the left side of (4.12)" $\rightarrow$ "the left side of (4.13)".

72. **page 97, Example 4.33:** "π-tableaux" $\rightarrow$ "semistandard π-tableaux". The same mistake (lack of "semistandard" even though it is meant) is made a few times later on, so perhaps it is worth defining "π-tableau" to mean "semistandard π-tableau".

73. **page 100, Definition 4.37:** Replace "which are equal to $\pi\left(l_j+1\right)$" by "which are equal to $\pi\left(l_j-1\right)$", and replace "such that $t_j > r_j$" by "such that $t_j < r_j$". (Otherwise, you are only comparing the numbers of $\pi\left(l_j\right)$'s and $\pi\left(l_j+1\right)$'s in the tail segments that begin with a $\pi\left(l_j\right)$; but such segments are biased towards having more $\pi\left(l_j\right)$'s, whereas you are interested in them having as few $\pi\left(l_j\right)$'s as possible. In particular, if the word has no $\pi\left(l_j\right)$'s altogether but some $\pi\left(l_j+1\right)$'s, then it should not count as a Littlewood–Richardson π-word, but your current definition lets it pass as one.)

    With my suggested changes, you need to require that $l_j > 1$ (in "for each entry").

74. **page 101, line 2:** "each filling is a π-tableau" $\rightarrow$ "each filling is a semistandard π-tableau".

75. **page 102, line 4:** "will not start with $\pi(1)$" $\rightarrow$ "will not end with $\pi(1)$".

76. **page 103, Proposition 4.40:** Replace "$l(\lambda) \geq n$" by "$l(\lambda) \leq n$".

77. **page 103, Proposition 4.40 (ii):** Replace "leftmost" by "rightmost" in "it is the leftmost entry in word (T) which is in row j but is not equal to $\pi(j)$". For example, in the non-Littlewood–Richardson tableau

2	3	4	
1	1	1	1

    $(\text{for } \pi = \text{id})$,

    the π-climber is the 4 in row 2, which is not the leftmost entry in row 2 that doesn't equal $\pi(2) = 2$ (there is a 3 to its left).

78. **page 104, proof of Lemma 4.41:** Replace "the results follow" by "the result follows".

79. **page 105:** "isn't even a $\kappa(\pi)$-tableau" $\rightarrow$ "isn't even a semistandard $\kappa(\pi)$-tableau".

80. **pages 106–109:** Figures 4.16, 4.19 and 4.20 use the symbol "ℓ" for what is denoted by "l" in the main text. These look different enough to suggest different meanings.

81. **page 107:** "middle tableau" $\to$ "second tableau from the left".

82. **page 107:** "This means $k = 1$, $j = 2$, and $l = 3$" $\to$ "This means $k = 1$, $j = 3$, and $l = 3$".

83. **page 108, Lemma 4.43 (ii):** "not a semistandard π-tableau" $\to$ "not a Littlewood–Richardson π-tableau".

84. **page 110, proof of Lemma 4.43 (ii):** It is not true that "in each row other than the row of the π-climber, T and $\kappa(T)$ have the same number of k's and the same number of l's". Indeed, this analysis correctly accounts for the free k's and l's, but ignores the existence of paired k's and l's. The latter have to be handled separately: In each column other than the column of the π-climber, T and $\kappa(T)$ have the same number of paired k's and the same number of paired l's. Now it remains to add up the numbers for free and for paired entries.

85. **page 111, proof of Proposition 4.28:** In "over all Littlewood–Richardson π-tableau", replace "π-tableau" by "π-tableaux".

86. **page 111, proof of Proposition 4.28:** In the last sentence of this proof, replace "we have" by "this rewrites as".

87. **page 113, Problem 4.20:** Replace "$n \geq k$" by "$n > k$". Indeed, this doesn't hold for $n = k$ unless you interpret $s_{0,1^n}$ to mean 0.

88. **page 115, Problem 4.30:** What does "sign-reversing" mean? Do you mean that $\text{weight}(T') = -\text{weight}(T)$?

89. **page 115, Problem 4.31:** Why do you require $n \geq k$ here?

90. **page 116, Problem 4.36:** After "Let $\widetilde{\lambda}$ be the partition with", add "at most n parts and with".

91. **page 116, Problem 4.36:** Replace "λ_{r-j+1}" by "λ_{n-j+1}".

92. **page 117:** "done this" is an overstatement: I believe Billey, Rhoades and Tewari have only showed that there exist expansions with nonnegative coefficients, but have not found any formula for these coefficients.

93. **page 121:** After "By our definition of the skew Schur functions", add "(and by Proposition 5.4)".

94. **page 122, Proposition 5.7:** Here and in the following, you omit the word "skew" in "semistandard skew tableau". Worth mentioning, I think.

95. **page 126, proof of Proposition 5.11:** Remove "with $\alpha_1 + \cdots + \alpha_n = n$" (this requirement is unnecessary and distracting).

96. **page 126, proof of Proposition 5.11:** "a semistandard tableaux" → "a semistandard tableau".

97. **page 127, proof of Proposition 5.12:** "in the its" → "in its".

98. **page 128:** "We might hope" → "We might expect" (I don't see a reason to hope this – don't we want more identities?).

99. **page 130, Definition 5.16:** "over all set-valued tableaux" → "over all set-valued semistandard tableaux".

100. **page 132:** When you say "we adapt the Bender–Knuth involutions", it is worth explaining one little subtlety: If a cell a set-valued semistandard tableau contains both j and $j+1$, then these two entries are considered to be free, not paired (even though the definition of "paired" you give on page 83 would count them as paired).

101. **page 134:** After "Every semistandard tableau is a set-valued semistandard tableau", I would add "(by regarding each of its entries i as a singleton set $\{i\}$)".

102. **page 134:** When you say "the symmetric function G_λ", you are using a nonstandard concept of "symmetric function" that does not require bounded degree. Likewise, on page 135, the words "linear combination" seem to refer to an infinite linear combination, which is also nonstandard usage. At the risk of pedantry, I think this is worth a disclaimer.

103. **page 136, Definition 5.22:** "of elegant tableau" → "of elegant tableaux".

104. **page 136, Example 5.24:** I would require $n > 0$ here, since the answer is wrong for $k = n = 0$ at least.

105. **page 136, solution to Example 5.24:** On the last line of page 136, replace "$n + j$" by "$n + j - 1$".

106. **page 137, solution to Example 5.24:** "can be at most $n + j$" → "can be at most $n + j - 1$".

107. **page 138:** Add "of shape λ" before "with entries in $[n]$". Likewise, add "of shape λ" before "with entries in $\mathbb{P}$".

108. **page 141, first bullet point:** In "we replace each j with $j+1$ in the left column and we replace each $j+1$ with j in the right column", the words "left" and "right" should be interchanged.

109. **page 142:** Replace "be $\sum_k jv_k$" by "be $\sum_k kv_k$".

110. **page 143, solution to Example 5.31:** "Since a semistandard tableau T of shape (n) cannot have repeated entries in a column, these tableaux are exactly the reverse plane partitions of shape (n)" $\rightarrow$ "Since a reverse plane partition T of shape (n) cannot have repeated entries in a column, these reverse plane partitions are exactly the semistandard tableaux of shape (n)".

111. **page 143, Example 5.32:** You need to require $n \geq 1$ for the solution to work.

112. **page 143, solution to Example 5.32:** "with at least one j for $1 \leq j \leq k$" $\rightarrow$ "with at least one j for each $1 \leq j \leq k$".

113. **page 145, line 1:** I would replace "$n \geq 1$" by "$n \geq 0$". There is no reason to exclude the trivial case $n = 0$; it gives extra information about the chromatic polynomial (namely, that its constant term is 0 unless the graph has no vertices).

114. **page 146, shortly after the solution to Example 5.35:** The formulation "we can remove all of the vertices of a graph G by removing one vertex of degree 1 at a time" is, strictly speaking, incorrect: What you mean is that by successively removing one vertex of degree 1 at a time, you can end up with a 1-vertex graph. You cannot, of course, get rid of the final vertex, as it has degree 0 rather than 1.

     Also, you are using the notion of the degree of a vertex in a graph; this should be defined. (Namely: If v is a vertex of a graph G, then the *degree* of v means the number of edges of G that contain v.)

115. **page 146, Proposition 5.36:** Once again, you can replace "$n \geq 1$" by "$n \geq 0$". The proof works fine for $n = 0$: In this case, it is still true that any proper coloring of U can be extended to a proper coloring of T in $n - 1$ many ways (even though $n - 1$ is negative); indeed, this is vacuously true, since there is no proper coloring of U.

116. **page 146, proof of Proposition 5.36:** "for the endpoint" $\rightarrow$ "for the vertex".

117. **page 146, proof of Proposition 5.36:** "for the color of the endpoint" $\rightarrow$ "for the color of the remaining vertex".

118. **page 146, Definition 5.37:** Replace "proper coloring" by "coloring" here. Later (on page 147) you refer to the weight of any coloring (not necessarily proper).

119. **page 148, proof of Proposition 5.41:** On line 1 of the proof, replace "$x_{\lambda_{l(\lambda)}}^{\lambda_{l(\lambda)}}$" by "$x_{l(\lambda)}^{\lambda_{l(\lambda)}}$".

120. **page 148, proof of Proposition 5.41:** Replace "of m_λ in X_G)" by "of m_λ) in X_G".

121. **page 148, proof of Proposition 5.41:** After "by choosing a stable partition $V_1, V_2, \ldots$ of G", add "satisfying $|V_j| = \lambda_j$ for all j,".

122. **page 153, Problem 5.5:** Do you really mean to ask this as stated? Because the first row of λ is longer than that of μ, so the answer is 0.

123. **page 154, Problem 5.8:** I'd add a "for $n \geq 1$" before the comma here.

124. **page 155, Problem 5.22:** The left hand side of the equality needs a t^n factor (I believe).

125. **page 156:** "of graphs" → "of connected graphs".

126. **page 160:** After "is an infinite sequence of north $(0, 1)$ and east $(1, 0)$ steps which contains exactly n east steps", I suggest adding "and has its first step start at $(a, 1)$ whereas each further step starts at the endpoint of the previous step". (This is probably clear to everyone who has ever counted lattice paths, but I'm not sure whether you want to make this assumption on the reader's experience.)

127. **page 161:** After the first displayed equation, add "(where a is an arbitrary integer, which may even be chosen differently for each factor in the product)".

128. **pages 161, 164, 168, 171, 173, 174, 175, 176, 178:** I would replace every "$(-1)^{\mathrm{inv}(\pi)}$" by a "$\mathrm{sgn}\,(\pi)$" on these pages. Of course, these are synonyms, but you use the sgn notation in Lemma 6.5 and Proposition C.7, and you speak of a "sign-reversing" involution (as opposed to an "inversion-parity-reversing" one), so I think it makes sense to stick to one choice of notation.

129. **page 162:** Replace "moves $\lambda_j + \pi_j - j$" by "moves $\lambda_j + \pi(j) - j$".

130. **page 162:** "ending j units from the right" → "whose ending point is the j-th one from the right". (The ending points are usually not evenly spaced, so I wouldn't speak of "units".)

131. **page 167, proof of Lemma 6.5 (ii):** "same set of east steps" → "same multiset of east steps".

132. **page 167, Figure 6.8:** I suggest switching the colors of the red and blue paths here. After all, the tail-swapping procedure results in each path keeping its upper end (but possibly changing its lower end), so unless the colors of the paths do not correspond to the paths, it makes sense for the upper ends to preserve their colors.

133. **page 168, proof of Lemma 6.5 (iii):** I suggest replacing "decompose as α_m^-, α_w^+ and α_w^-, α_m^+" by "decompose as α_w^-, α_m^+ and α_m^-, α_w^+" in order not to swap the two paths unnecessarily.

134. **page 168, proof of Lemma 6.6:** The proof could be made clearer by some reorganization. You aren't cancelling the addends corresponding to the β's with $\pi \neq 12 \cdots k$; you are cancelling the addends corresponding to the β's that have intersection. The former are a subset of the latter, but tswp does not generally send the former to the former, so it is misleading to claim that it's the former that are getting cancelled.

    I think the best way to organize the proof is by first invoking tswp to cancel the intersecting β's, thus obtaining

    $$\sum_{\beta \in H_{\lambda,k}} \operatorname{sgn}(\pi) \operatorname{wt}_h(\beta) = \sum_{\beta \in II_{\lambda,k}} \operatorname{sgn}(\pi) \operatorname{wt}_h(\beta);$$

    and then arguing that each $\beta \in II_{\lambda,k}$ satisfies $\pi = 12 \cdots k$ and therefore $\operatorname{sgn}(\pi) = 1$, so the equality above simplifies to (6.5).

135. **page 168, proof of Lemma 6.7:** "Suppose $\beta \in H_{\lambda,k}$" $\to$ "Suppose $\beta \in II_{\lambda,k}$".

136. **page 168, proof of Lemma 6.7:** When you write "where $\alpha_m \in \Gamma_{-m,\lambda_m}$", you are implicitly using the fact that the permutation π corresponding to any $\beta \in II_{\lambda,k}$ is the identity $12 \cdots k$. Since you only stated this during the proof of Lemma 6.6, it is worth repeating it here.

137. **page 173:** "product of the weights" $\to$ "product of the e-weights".

138. **page 174:** At the end of the second paragraph of this page, I would add: "Thus each path α_m ends at $(\lambda'_m - m, \infty)$, which shows that α_m is the path with the m-th rightmost ending point among $\alpha_1, \alpha_2, \ldots, \alpha_k$ (and the ending points are distinct).".

139. **page 176, proof of Lemma 6.12:** "have the same sets of east steps" $\to$ "have the same multisets of east steps".

140. **page 176, proof of Lemma 6.12:** "has leftmost point (l, m)" $\to$ "has rightmost point (l, m)" (otherwise, you would need to increment the subscripts by 1).

141. **page 176, Lemma 6.14:** Replace "$\operatorname{SST}(\lambda')$" by "$\operatorname{SST}(\lambda)$".

142. **page 176, proof of Lemma 6.14:** "Ferrers diagram of λ'" $\to$ "Ferrers diagram of λ".

143. **page 176, proof of Lemma 6.14:** "the weights" $\to$ "the e-weights".

144. **page 176, proof of Lemma 6.14:** "column strict" $\to$ "column-strict".

145. **page 176, proof of Lemma 6.14:** "if and only if" $\to$ "because" (you assumed that $\beta \in \Xi_{\lambda,k}$, so you cannot have any intersecting paths in β).

146. **page 178, proof of Theorem 6.10:** "$T \in \operatorname{SST}(\lambda')$" $\to$ "$T \in \operatorname{SST}(\lambda)$".

147. **page 178, proof of Theorem 6.10:** "$s_{\lambda'}$" $\to$ "s_λ".

148. **page 179, proof of Proposition 6.16:** The equality $\omega_h\left(\det\left(h_{\lambda_i+j-i}\right)_{1\le i,j\le k}\right) = \det\left(e_{\lambda_i+j-i}\right)_{1\le i,j\le k}$ is not immediately obvious at this stage. It relies on three things: (1) Proposition C.7 (which shows that the determinant of a matrix is a very specific linear combination of products of its entries), (2) the fact that ω_h is linear, and (3) the fact that $\omega_h\left(h_{\nu_1}h_{\nu_2}\cdots h_{\nu_p}\right) = e_{\nu_1}e_{\nu_2}\cdots e_{\nu_p}$ for any finite list $(\nu_1,\nu_2,\ldots,\nu_p)$ of integers. Among these three things, (3) warrants some justification (in my opinion). Namely: If all entries of the list $(\nu_1,\nu_2,\ldots,\nu_p)$ are positive, then the equality in question ($\omega_h\left(h_{\nu_1}h_{\nu_2}\cdots h_{\nu_p}\right) = e_{\nu_1}e_{\nu_2}\cdots e_{\nu_p}$) can be rewritten as $\omega_h(h_\mu) = e_\mu$, where μ is the partition obtained by sorting the list $(\nu_1,\nu_2,\ldots,\nu_p)$ into weakly decreasing order; but the latter equality follows from the definition of ω_h. Thus, it remains to handle the case when not all entries of the list $(\nu_1,\nu_2,\ldots,\nu_p)$ are positive. If at least one entry of the list $(\nu_1,\nu_2,\ldots,\nu_p)$ is negative, then the equality in question ($\omega_h\left(h_{\nu_1}h_{\nu_2}\cdots h_{\nu_p}\right) = e_{\nu_1}e_{\nu_2}\cdots e_{\nu_p}$) boils down to the obvious equality $\omega_h(0) = 0$ (since $h_m = 0$ for every $m < 0$), and thus is true. Otherwise, at least one entry of the list $(\nu_1,\nu_2,\ldots,\nu_p)$ is 0; but then we can remove this entry from the list without changing the equality (since $h_0 = 1$), which reduces our problem to a simpler case (thus leading to an inductive proof over p).

149. **page 189, Problem 6.27:** I'd add "Let n be a positive integer." at the beginning of this problem.

150. **page 191, §7.1:** Worth saying that you will sometimes abbreviate $\delta_{\lambda,\mu}$ as $\delta_{\lambda\mu}$ (for example, in Proposition 7.4).

151. **page 193, Definition 7.2:** I don't think you need to define $\Lambda_k(X,Y)$; you can just as well work in the whole space of formal power series in $x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots$.

152. **page 194, proof of Proposition 7.3:** The second sentence of this proof makes no sense as stated. I suggest replacing it by the following argument:

You know that

$$\sum_{j=1}^{l} \sum_{m=1}^{n} A_{jm} u_j (X) v_m (Y) = 0. \tag{1}$$

You want to prove that each m with $1 \leq m \leq n$ satisfies

$$\sum_{j=1}^{l} A_{jm} u_j (X) = 0. \tag{2}$$

Fix any monomial $\mathfrak{m}$ in the variables $x_1, x_2, x_3, \ldots$. Then, if we compare coefficients before $\mathfrak{m}$ on both sides of (1) (while considering $y_1, y_2, y_3, \ldots$ as scalars), we obtain

$$\sum_{j=1}^{l} \sum_{m=1}^{n} A_{jm} \cdot \left(\text{the coefficient of } \mathfrak{m} \text{ in } u_j (X)\right) \cdot v_m (Y) = 0.$$

Renaming the variables $y_1, y_2, y_3, \ldots$ as $x_1, x_2, x_3, \ldots$ in this equality, we obtain

$$\sum_{j=1}^{l} \sum_{m=1}^{n} A_{jm} \cdot \left(\text{the coefficient of } \mathfrak{m} \text{ in } u_j (X)\right) \cdot v_m = 0.$$

In other words,

$$\sum_{m=1}^{n} \left(\sum_{j=1}^{l} A_{jm} \cdot \left(\text{the coefficient of } \mathfrak{m} \text{ in } u_j (X)\right) \right) \cdot v_m = 0.$$

Since $\{v_m \mid 1 \leq m \leq n\}$ is linearly independent, we thus conclude that each m with $1 \leq m \leq n$ satisfies

$$\sum_{j=1}^{l} A_{jm} \cdot \left(\text{the coefficient of } \mathfrak{m} \text{ in } u_j (X)\right) = 0.$$

Multiplying both sides of this equality by $\mathfrak{m}$, we obtain

$$\sum_{j=1}^{l} A_{jm} \cdot \left(\text{the coefficient of } \mathfrak{m} \text{ in } u_j (X)\right) \cdot \mathfrak{m} = 0. \tag{3}$$

Now, forget that we fixed $\mathfrak{m}$. We thus have proved the equality (3) for each monomial $\mathfrak{m}$ in the variables $x_1, x_2, x_3, \ldots$ and each m with $1 \leq m \leq n$. Summing these equalities over all monomials $\mathfrak{m}$ (while m is fixed), we obtain precisely (2). Thus, (2) is proven.

Note that this argument is completely bulletproof; there are no "technical issues" that you are leaving aside here, so the whole first paragraph of page 194 (= last paragraph of page 193) is unnecessary.

153. **page 194, the paragraph before Proposition 7.4:** "inner product" $\to$ "bilinear map on $\Lambda_k \times \Lambda_k$".

154. **page 194, Proposition 7.4:** "is the function" $\to$ "is the bilinear map" (twice).

155. **page 194, Proposition 7.4:** A reminder about the meaning of $p(k)$ would be useful (you haven't used that notation for a while).

156. **page 195, proof of Proposition 7.4:** "(by Definition A.15(i)–(iii))" $\to$ "(by Definition A.15(1)–(3))".

157. **page 195, proof of Proposition 7.4:** "function" $\to$ "bilinear function".

158. **page 195, proof of Proposition 7.4:** In the first sentence of the proof of (i) $\Longleftrightarrow$ (iii), replace "$B_{\mu\beta}$" by "$B_{\lambda\beta}$".

159. **page 195, proof of Proposition 7.4:** After "then we can simplify the innermost sum to", replace "get to" by "$\delta_{\alpha,\beta}$, whence the whole right hand side simplifies to".

160. **page 197, proof of Proposition 7.6:** The second computation relies on the fact that $\delta_{\lambda',\mu'} = \delta_{\lambda,\mu}$, which is fairly obvious but (I think) still worth mentioning.

161. **page 198:** An example of a generalized permutation and its two weights would be useful.

162. **pages 199–200, proof of Proposition 7.10:** In the first paragraph of the proof, both "$\mathbb{P} \times \mathbb{P}$ matrix" and "domino" should probably be explained. (Also, I'd speak of a "domino of π" rather than just of a "domino".)

163. **page 200, proof of Proposition 7.10:** When you apply Problem 2.17(a), you are assuming λ and μ to be partitions, which is not generally satisfied when λ and μ come from an arbitrary generalized permutation.

    The easiest way to fix this, I believe, is to recognize that both sides of (7.6) are symmetric in $x_1, x_2, x_3, \ldots$ and also symmetric in $y_1, y_2, y_3, \ldots$, and thus it suffices to verify that they have identical coefficients of $x^\lambda y^\mu$ whenever λ and μ are partitions. Thus, you can restrict yourself to the case when λ and μ are partitions. In this case, you can furthermore WLOG assume that the partitions λ and μ have the same size (because if $|\lambda| \neq |\mu|$, then the coefficients on both sides of (7.6) are easily seen to be 0). Thus, you can apply Problem 2.17(a).

    Restricting yourself to specific λ and μ also allows you to work with $n \times n$-matrices (for $n = |\lambda| = |\mu|$) instead of $\mathbb{P} \times \mathbb{P}$-matrices, which makes the application of Problem 2.17(a) a lot more direct.

164. **page 200, proof of Corollary 7.11:** In order to apply Proposition 7.4, you cannot immediately use the equality

$$\sum_{\lambda} m_\lambda(X) h_\lambda(Y) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y);$$

instead, you need its "finite" variant

$$\sum_{\lambda \vdash n} m_\lambda(X) h_\lambda(Y) = \sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(Y) \qquad \text{for } n \in \mathbb{N}.$$

So I would spend a sentence or so explaining how the latter can be derived from the former.

Likewise, your argument only gives you $\langle m_\lambda, h_\mu \rangle = 0$ in the case when $|\lambda| = |\mu|$ (since the bilinear forms in Proposition 7.4 are defined on $\Lambda_k \times \Lambda_k$, not on the whole $\Lambda \times \Lambda$). A few words should be said about why $\langle m_\lambda, h_\mu \rangle = 0$ also holds for even simpler reasons when $|\lambda| \neq |\mu|$ (maybe a good problem?).

165. **page 202, solution to Example 7.16:** I'd replace "$h_0 = p_0$" by "$h_0 = p_\varnothing$" in order to clarify that the "0" in "p_0" stands not for the number 0 (as p_0 would be undefined for the number 0) but for the empty partition $\varnothing$.

166. **pages 203–204, proof of Proposition 7.17:** It's probably a correct proof, but my head is spinning from trying to verify the mutual inverseness of the two maps, so let me suggest a simpler proof. I will use a tiny bit of algebra – namely, the orbit-stabilizer formula (in one of the most down-to-earth settings: the symmetric group S_n acting on n-tuples). Feel free to include the following proof (or split it into exercises).

Alternative proof of Proposition 7.17 (sketched). Fix $n \geq 0$.

Step 1: If $j = (j_1, j_2, \ldots, j_n) \in \mathbb{P}^n$ is an n-tuple, then we let x_j be the monomial $x_{j_1} x_{j_2} \cdots x_{j_n}$. An n-tuple $(j_1, j_2, \ldots, j_n) \in \mathbb{P}^n$ is said to be *weakly increasing* if $j_1 \leq j_2 \leq \cdots \leq j_n$. The definition of h_n yields

$$h_n = \sum_{\substack{j \in \mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} x_j. \tag{4}$$

Step 2: We let the symmetric group S_n act on the set $\mathbb{P}^n$ from the right by permuting the entries: i.e., we set

$$(k_1, k_2, \ldots, k_n) \cdot \sigma = \left(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(n)}\right) \tag{5}$$

for any $\sigma \in S_n$ and $(k_1, k_2, \ldots, k_n) \in \mathbb{P}^n$.

It is clear that if $k \in \mathbb{P}^n$ and $\sigma \in S_n$, then

$$x_{k \cdot \sigma} = x_k. \tag{6}$$

In other words, if $j \in \mathbb{P}^n$ and $k \in \mathbb{P}^n$ satisfy $j \in k \cdot S_n$ (where $k \cdot S_n$ denotes the orbit $\{k \cdot \sigma \mid \sigma \in S_n\}$ of k), then

$$x_j = x_k. \tag{7}$$

For each n-tuple $k \in \mathbb{P}^n$, the orbit $k \cdot S_n = \{k \cdot \sigma \mid \sigma \in S_n\}$ contains exactly one weakly increasing n-tuple (namely, the result of sorting the entries of k into increasing order). In other words, for each n-tuple $k \in \mathbb{P}^n$, there is exactly one weakly increasing n-tuple $j \in \mathbb{P}^n$ satisfying $j \in k \cdot S_n$.

The orbit-stabilizer theorem (applied to the action of S_n on $\mathbb{P}^n$) yields that each $k \in \mathbb{P}^n$ satisfies

$$|S_n| = |k \cdot S_n| \cdot |\{\sigma \in S_n \mid k \cdot \sigma = k\}|.$$

In view of $|S_n| = n!$, this rewrites as

$$n! = |k \cdot S_n| \cdot |\{\sigma \in S_n \mid k \cdot \sigma = k\}|. \tag{8}$$

Step 3: Let $\operatorname{type} \sigma$ denote the cycle type of any permutation $\sigma \in S_n$. (This is a partition of n.)

For any permutation $\sigma \in S_n$, we have

$$p_{\operatorname{type} \sigma} = \sum_{\substack{k \in \mathbb{P}^n; \\ k \cdot \sigma = k}} x_k. \tag{9}$$

[*Proof of (9):* Let $\sigma \in S_n$ be a permutation. Let $z_1, z_2, \ldots, z_m$ be all cycles of σ (including the 1-cycles), listed in the order of decreasing length (with ties broken in some arbitrary way, without repetitions). Then, $\operatorname{type} \sigma = (|z_1|, |z_2|, \ldots, |z_m|)$ (by the definition of $\operatorname{type} \sigma$).

Fix some $k = (k_1, k_2, \ldots, k_n) \in \mathbb{P}^n$ satisfying $k \cdot \sigma = k$. Then,

$$\begin{aligned} (k_1, k_2, \ldots, k_n) = k &= \underbrace{k}_{=(k_1, k_2, \ldots, k_n)} \cdot \sigma = (k_1, k_2, \ldots, k_n) \cdot \sigma \\ &= \left(k_{\sigma(1)}, k_{\sigma(2)}, \ldots, k_{\sigma(n)}\right) \qquad \text{(by (5))}. \end{aligned}$$

In other words, $k_i = k_{\sigma(i)}$ for each $i \in [n]$. Thus, if two elements i and j of $[n]$ can be obtained from one another by applying σ some number of times, then $k_i = k_j$. Hence, the values of k_j for all $j \in z_1$ are equal (since all $j \in z_1$ can be obtained from one another by applying σ some number of times[1]). In other words, there exists some $r_1 \in \mathbb{P}$ such that all $j \in z_1$ satisfy $k_j = r_1$. This r_1 is unique (since z_1 is nonempty). Consider this r_1. The same logic can be applied to the other cycles $z_2, z_3, \ldots, z_m$ of σ; thus, for

[1] because z_1 is a cycle of σ

each $i \in \{1, 2, \ldots, m\}$, we obtain a value $r_i \in \mathbb{P}$ such that all $j \in z_i$ satisfy $k_j = r_i$. It is now easy to see that the m-tuple $(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m$ satisfies $x_{r_1}^{|z_1|} x_{r_2}^{|z_2|} \cdots x_{r_m}^{|z_m|} = x_k$.

Forget that we fixed k. Thus, for each n-tuple $k = (k_1, k_2, \ldots, k_n) \in \mathbb{P}^n$ satisfying $k \cdot \sigma = k$, we have constructed an m-tuple $(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m$ satisfying

$$x_{r_1}^{|z_1|} x_{r_2}^{|z_2|} \cdots x_{r_m}^{|z_m|} = x_k. \tag{10}$$

(Namely, r_i is the unique positive integer such that all $j \in z_i$ satisfy $k_j = r_i$.) This defines a map

$$R : \{k \in \mathbb{P}^n \mid k \cdot \sigma = k\} \to \mathbb{P}^m,$$
$$k \mapsto (r_1, r_2, \ldots, r_m).$$

It is easy to see that this map R is injective[2] and surjective[3]; thus, R is bijective. Hence, we can substitute $R(k)$ for $(r_1, r_2, \ldots, r_m)$ in the sum $\sum_{(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m} x_{r_1}^{|z_1|} x_{r_2}^{|z_2|} \cdots x_{r_m}^{|z_m|}$, and thus obtain

$$\sum_{(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m} x_{r_1}^{|z_1|} x_{r_2}^{|z_2|} \cdots x_{r_m}^{|z_m|} = \sum_{\substack{k \in \mathbb{P}^n; \\ k \cdot \sigma = k}} x_k \qquad \text{(by (10))}.$$

But $\operatorname{type} \sigma = (|z_1|, |z_2|, \ldots, |z_m|)$; thus,

$$\begin{aligned} p_{\operatorname{type} \sigma} &= p_{(|z_1|, |z_2|, \ldots, |z_m|)} = p_{|z_1|} p_{|z_2|} \cdots p_{|z_m|} = \prod_{i=1}^{m} \underbrace{p_{|z_i|}}_{= \sum_{r \in \mathbb{P}} x_r^{|z_i|}} = \prod_{i=1}^{m} \sum_{r \in \mathbb{P}} x_r^{|z_i|} \\ &= \sum_{(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m} x_{r_1}^{|z_1|} x_{r_2}^{|z_2|} \cdots x_{r_m}^{|z_m|} \qquad \text{(by the product rule)} \\ &= \sum_{\substack{k \in \mathbb{P}^n; \\ k \cdot \sigma = k}} x_k. \end{aligned}$$

This proves (9).]

Step 4: Problem C.4 shows that each $\lambda \vdash n$ satisfies

$$(\text{the number of all } \sigma \in S_n \text{ satisfying } \operatorname{type} \sigma = \lambda) \cdot z_\lambda = n!. \tag{11}$$

[2]Indeed, any $k \in \mathbb{P}^n$ satisfying $k \cdot \sigma = k$ can be uniquely reconstructed from its image $R(k) = (r_1, r_2, \ldots, r_m)$: Indeed, the value r_i determines the entries k_j for all $j \in z_i$ (where $k = (k_1, k_2, \ldots, k_n)$), and thus all entries k_j of k are determined (since each $j \in [n]$ belongs to some z_i).

[3]Indeed, if $(r_1, r_2, \ldots, r_m) \in \mathbb{P}^m$ is given, then we can easily construct an n-tuple $k \in \mathbb{P}^n$ (satisfying $k \cdot \sigma = k$) that gets mapped to $(r_1, r_2, \ldots, r_m)$ under R. Namely, the j-th entry of this n-tuple k will be r_i where i is the unique element of $\{1, 2, \ldots, m\}$ that satisfies $j \in z_i$.

(Indeed, this is precisely the claim $c_\lambda z_\lambda = n!$ from Problem C.4, because c_λ stands for the number of all $\sigma \in S_n$ satisfying $\operatorname{type}\sigma = \lambda$ in Problem C.4.) Now, $\operatorname{type}\sigma \vdash n$ for each $\sigma \in S_n$; hence, we can split the sum $\sum\limits_{\sigma \in S_n} p_{\operatorname{type}\sigma}$ according to the value of $\operatorname{type}\sigma$ as follows:

$$\begin{aligned}
\sum_{\sigma\in S_n} p_{\operatorname{type}\sigma} &= \sum_{\lambda\vdash n} \sum_{\substack{\sigma\in S_n;\\ \operatorname{type}\sigma=\lambda}} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=p_\lambda \\ \text{(since } \operatorname{type}\sigma=\lambda)}} \\
&= \sum_{\lambda\vdash n} \underbrace{\sum_{\substack{\sigma\in S_n;\\ \operatorname{type}\sigma=\lambda}} p_\lambda}_{=(\text{the number of all } \sigma\in S_n \text{ satisfying } \operatorname{type}\sigma=\lambda)\cdot p_\lambda} \\
&= \sum_{\lambda\vdash n} \underbrace{(\text{the number of all } \sigma\in S_n \text{ satisfying } \operatorname{type}\sigma=\lambda)}_{\substack{=\frac{n!}{z_\lambda} \\ \text{(by (11))}}} \cdot p_\lambda \\
&= \sum_{\lambda\vdash n} \frac{n!}{z_\lambda} p_\lambda .
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{\lambda\vdash n} \frac{n!}{z_\lambda} p_\lambda &= \sum_{\sigma\in S_n} \underbrace{p_{\operatorname{type}\sigma}}_{\substack{=\sum\limits_{\substack{k\in\mathbb{P}^n;\\ k\cdot\sigma=k}} x_k \\ \text{(by (9))}}} = \underbrace{\sum_{\sigma\in S_n} \sum_{\substack{k\in\mathbb{P}^n;\\ k\cdot\sigma=k}}}_{=\sum\limits_{k\in\mathbb{P}^n} \sum\limits_{\substack{\sigma\in S_n;\\ k\cdot\sigma=k}}} x_k \\
&= \sum_{k\in\mathbb{P}^n} \underbrace{\sum_{\substack{\sigma\in S_n;\\ k\cdot\sigma=k}} x_k}_{=(\text{the number of all } \sigma\in S_n \text{ satisfying } k\cdot\sigma=k) x_k} \\
&= \sum_{k\in\mathbb{P}^n} \underbrace{(\text{the number of all } \sigma\in S_n \text{ satisfying } k\cdot\sigma=k)}_{\substack{=|\{\sigma\in S_n \ \mid\ k\cdot\sigma=k\}| \\ =\frac{n!}{|k\cdot S_n|} \\ \text{(by (8))}}} x_k \\
&= \sum_{k\in\mathbb{P}^n} \frac{n!}{|k\cdot S_n|} x_k .
\end{aligned} \tag{12}$$

But recall that (as we saw in Step 2) for each n-tuple $k \in \mathbb{P}^n$, there is exactly one weakly increasing n-tuple $j \in \mathbb{P}^n$ satisfying $j \in k \cdot S_n$. Thus, we can split the sum $\sum_{k\in\mathbb{P}^n} \dfrac{n!}{|k\cdot S_n|} x_k$ according to the value of this j. We thus find

$$\begin{aligned}
&\sum_{k\in\mathbb{P}^n} \frac{n!}{|k\cdot S_n|} x_k \\
&= \sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} \sum_{\substack{k\in\mathbb{P}^n; \\ j\in k\cdot S_n}} \underbrace{\frac{n!}{|k\cdot S_n|}}_{\substack{=\frac{n!}{|j\cdot S_n|} \\ \text{(since } k\cdot S_n = j\cdot S_n \\ \text{(because } j\in k\cdot S_n\text{))}}} \underbrace{x_k}_{\substack{=x_j \\ \text{(by (7))}}} \\
&= \sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} \underbrace{\sum_{\substack{k\in\mathbb{P}^n; \\ j\in k\cdot S_n}} \frac{n!}{|j\cdot S_n|} x_j}_{=(\text{the number of all } k\in\mathbb{P}^n \text{ satisfying } j\in k\cdot S_n)\cdot \frac{n!}{|j\cdot S_n|} x_j} \\
&= \sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} \underbrace{(\text{the number of all } k\in\mathbb{P}^n \text{ satisfying } j\in k\cdot S_n)}_{\substack{=(\text{the number of all } k\in\mathbb{P}^n \text{ satisfying } k\in j\cdot S_n) \\ \text{(since "} j\in k\cdot S_n \text{" is equivalent to "} k\in j\cdot S_n \text{")}}} \cdot \frac{n!}{|j\cdot S_n|} x_j \\
&= \sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} \underbrace{(\text{the number of all } k\in\mathbb{P}^n \text{ satisfying } k\in j\cdot S_n)}_{=|j\cdot S_n|} \cdot \frac{n!}{|j\cdot S_n|} x_j \\
&= \sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} \underbrace{|j\cdot S_n| \cdot \frac{n!}{|j\cdot S_n|}}_{=n!} x_j = n! \underbrace{\sum_{\substack{j\in\mathbb{P}^n \text{ is weakly} \\ \text{increasing}}} x_j}_{\substack{=h_n \\ \text{(by (4))}}} \\
&= n! h_n.
\end{aligned}$$

Hence, (12) becomes

$$\sum_{\lambda\vdash n} \frac{n!}{z_\lambda} p_\lambda = \sum_{k\in\mathbb{P}^n} \frac{n!}{|k\cdot S_n|} x_k = n! h_n.$$

Dividing both sides of this equality by $n!$, we obtain

$$\sum_{\lambda\vdash n} \frac{1}{z_\lambda} p_\lambda = h_n.$$

This proves Proposition 7.17 (again). ■

167. **page 206, proof of Proposition 7.15:** In "for any $n \geq 0$, we have $p_n(XY)$", replace "$n \geq 0$" by "$n \geq 1$".

168. **page 210, second paragraph:** I'd add a footnote reminding the reader that $\operatorname{sh}(T)$ means the shape of a tableau T. (You defined this notation long ago and never used it before.)

169. **page 213, first paragraph:** I suggest replacing "and we replace b with a" by "and we replace this b with a" to make it clearer that only one copy of b is getting replaced.

170. **page 218, Lemma 8.8:** I'd split the first sentence into two, in order for "$c \leq d$" and "$s_1, \ldots, s_n$" to be separated by more than a comma.

171. **page 218, proof of Lemma 8.8:** The sentence "Therefore, d_1 was to the right of c_1 in T, so $c_1 \leq d_1$" only makes sense if c_1 and d_1 are both defined. You probably also want to say that if d_1 exists (i.e., the letter d doesn't just end up as a new last entry in the first row), then so does c_1.

172. **pages 218–220, proof of Proposition 8.9:** This proof is not as clear as it could be.

    First of all, I believe it makes more sense to split it into two: First, show a lemma that $r_c(T)$ is a semistandard tableau whenever T is a semistandard tableau; then use this lemma to quickly obtain Proposition 8.9 by induction. (This is worth already because you are already using this lemma! On page 251, when you say "By Proposition 8.9", you really mean "By the lemma", because you aren't starting with an empty tableau there. Moreover, you are already stating the analogue of this lemma for reverse row insertion explicitly (as Proposition 8.11); thus, for symmetry reasons alone, it is worth giving it the same treatment.)

    Second, the last two paragraphs of your proof (which are where the real work is happening) are somewhat confusing. They refer to Figures 8.14 and 8.15, but the configurations shown there do not seem representative of all cases: what if there is no b? what if there is no c_{j-1}? what if there is no c_{j-2}? Finally, it is not completely obvious that the shape of the tableau remains a partition (i.e., columns don't grow holes) throughout the process, and this is somewhat implicit in your argument. If interpreted correctly, the proof works, but I think you can make it easier for the reader to interpret it correctly if you reorganize it as follows:

    After the sentence "In fact, we only change one entry of that column", add the following: " – namely, we insert c_{j-1} either into the position previously occupied by c_j (somewhere in the existing j-th row) or into a hitherto empty position (thus extending the j-th row). Thus, we need to prove that:

    (A) no "hole" is gaping below this position (i.e., there is an entry immediately below it), unless it is in the first row;

    (B) the number c_{j-1} (in its new position) is larger than the entry immediately below it, unless it is in the first row; and

(C) the number c_{j-1} (in its new position) is smaller than the entry immediately above it (if there is such an entry).

Among these three properties, (C) is trivial: If there is an entry b immediately above the new position, then the position into which we inserted c_{j-1} was previously occupied by c_j, and we must have $c_j < b$ (since the tableau was semistandard before the insertion), and thus by Lemma 8.6 we have $c_{j-1} < c_j < b$, which means precisely that c_{j-1} is smaller than the entry immediately above the position where it was inserted."

Now, of course, it remains to prove (A) and (B). Now you can forget about b, and the existence of c_{j-1} and c_{j-2} is not in question (we are assuming that we aren't in the first row anymore, so we have $j \geq 2$).

Obviously, my writing is neither economical nor very readable, but I think the organization of my argument is better.

173. **pages 220–221, first paragraph of page 221 (= last paragraph of page 220) and second paragraph of page 221:** I find this rather confusing. No clarity is gained by handling the $n = 2$ case separately from the $n \geq 3$ case; your description of the latter is actually clearer (at least to me), and the former is obscured by the fact that "the intermediate pair (S_{int}, U_{int}) we obtain by applying our bijection to the reverse plane partition consisting of just T_2" is simply the pair $(T_2, \varnothing)$, which makes the reader wonder whether she is misreading the text (or why otherwise you are introducing new names for old things).

    Also, "By Lemma 8.8, no two boxes of U are in the same column" is not quite correct: This is true not by Lemma 8.8, but rather by the very fact that we have only added one new box per column.

174. **page 221, last paragraph:** "add one" $\to$ "add 1". (The word "one" can easily be mistaken for a pronoun, which would even make some sense here but definitely not reflect your intent.)

175. **page 222, first paragraph:** You write: "This will fill some of the empty boxes, but by Lemma 8.8 we will only need the empty boxes we have already added". Be careful: It is true that the newly filled boxes form a horizontal strip (because of Lemma 8.8); but this only shows that the newly filled boxes in rows $2, 3, 4, \ldots$ do not stick out of λ. You need to argue separately for why the newly filled boxes in row 1 do not stick out of λ (after all, a horizontal strip could have an arbitrarily long first row). This follows from the fact that the bottom row of S is T_1; but you need to state this fact here (and prove it – frankly, I don't find it obvious enough to just leave as an exercise at this stage in the treatment of RSK).

    In general, I suggest first dealing with S completely (including showing that S is semistandard and does not stick out of λ), and only then defining U.

I would also refer to Figures 8.18 and 8.19 earlier on (not just at the end of the construction), and I would also say in the caption of Figure 8.19 what step of the construction each filling corresponds to, and what $\widehat{T}_1$ is (namely, $\widehat{T}_1 = \left(5 \;\; 5 \;\; 6 \right)$).

176. **page 223, first paragraph:** "subtract one" → "subtract 1". (Again, the word "one" looks too much like a pronoun.)

177. **page 226, between Proposition 8.11 and Definition 8.12:** "to ensure $x^{Q(\pi)} = x^{\text{topwt}(\pi)}$" → "to ensure $x^{Q(\pi)} = \text{topwt}(\pi)$".

178. **page 227, proof of Proposition 8.14:** After "Now by Lemma 8.8", add "and Lemma 8.7" (and, ideally, elaborate on the details). Indeed, you want to know that the bumping path that results in a_j ends further right than the bumping path that results in a_{j-r} does. Lemma 8.8 only shows that the former bumping path stays to the right of the latter in each row; but since the two paths can end in different rows, this does not mean that the former path ends further right than the latter. You only get to that conclusion if you know that bumping paths trend left (Lemma 8.7).

    Better yet, this consequence of Lemma 8.8 and Lemma 8.7 could be made an extra lemma, since you use it several times. (Note that you already state its analogue for reverse row insertion as a lemma – Lemma 8.17.)

179. **page 229, first paragraph:** Again, when you say "by Lemma 8.8", you mean not Lemma 8.8 but the consequence of Lemma 8.8 and Lemma 8.7 mentioned above (page 227).

180. **page 229, last paragraph:** "and building $E(P,Q)$" → "and building $E(P,Q) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{bmatrix}$" (since you are referring to a_j and b_j a few sentences further on).

181. **page 230, proof of Lemma 8.17:** In the first paragraph of this proof, it is not precise to say that c_0 and d_0 are moved to a lower row; they are rather being expelled from the tableau altogether. (Perhaps worth making precise in a footnote.)

182. **page 231, proof of Lemma 8.17:** The last paragraph of this proof is confusing: When you refer to "P", it is not always clear which version of P you mean (as P changes during the procedure that computes $E(P,Q)$). Moreover, you have a_j and a_{j+1} switched a few times. I suggest denoting the versions of P by $P_n, P_{n-1}, \ldots, P_0$, where P_i is the version of P obtained after removing $n-i$ boxes (so that i boxes remain). Also, some pseudo-formulas like

$$P_{j+2} \underset{\text{obtain } c_0}{\overset{\text{remove } c_k}{\longrightarrow}} P_{j+1} \underset{\text{obtain } d_0}{\overset{\text{remove } d_m}{\longrightarrow}} P_j$$

(if properly explained beforehand) might be easier to parse than long sentences.

183. **page 231, Theorem 8.19:** In the last sentence, replace "$x^{\operatorname{topwt}(\pi)} y^{\operatorname{bottomwt}(\pi)} = x^{Q(\pi)} y^{P(\pi)}$" by "$\operatorname{topwt}(\pi) = x^{Q(\pi)}$ and $\operatorname{bottomwt}(\pi) = y^{P(\pi)}$" (to get the notations right, but also to match the formulation with requirements on $\mathfrak{R}$ stated on page 210).

184. **page 232:** You write: "if π is a permutation in the usual sense". This assumes that the reader understands how a permutation can be viewed as a generalized permutation (viz., by writing it in two-line notation, with the top row being sorted to be $1, 2, \ldots, n$). Have you ever explained that? (I'd say it's worth doing so already in Chapter 7, when you define the concept of generalized permutations; it explains the name.)

185. **Page 233:** Your construction of α does not guarantee that α_j will be positive integers (or even nonnegative integers), nor does it properly explain when the sequence should end and what the j-th entry of a finite sequence should be understood as when the sequence has fewer than j entries. I would suggest rephrasing it in terms of infinite sequences of integers. Then you can show that what if you start with $\varnothing$'s on the left and bottom borders, then all the sequences you actually get are partitions (padded with infinitely many zeroes as usually). This can be proved by induction using a lemma which says that if, in the situation of Figure 8.31, all of λ, μ and ν are partitions such that μ/λ and ν/λ are horizontal strips, then (a) all of m, M and α are partitions as well (where m, M and α are as defined before – I suggest labeling the defining formulas so that you can refer to them cleanly), and (b) both α/μ and α/ν are horizontal strips.

186. **page 235, Lemma 8.22:** You probably want to restate this lemma so that it says that $\lambda_j \leq \mu_j$ whenever μ is the right or upper neighbor of λ in the growth diagram. Indeed, the way you are currently stating it, it does not claim that $\lambda_j \leq \mu_j$ when λ and μ are on the eastern or northern border of the growth diagram; but you do use this very case later (in the third paragraph of page 237, when you say "By Lemma 8.22").

187. **page 237, proof of Proposition 8.25:** You define $\operatorname{Ins}(a, b)$ thus: "For each vertex (a, b), let $\operatorname{Ins}(a, b)$ be the semistandard tableau we obtain by using the RSK correspondence to insert the entries of the bottom row of π which correspond to the squares below and to the left of the vertex (a, b) in the growth diagram."

     I find this confusing. First, the RSK correspondence yields two tableaux, so "the semistandard tableau we obtain by using the RSK correspondence" isn't as unambiguous as it might appear. Second, "below and to the left" may mean either a whole quadrant or just two half-axes; you mean the quadrant, but this isn't immediately clear. Finally, there is too much going on in a single sentence.

I suggest fixing this by defining the notation $\pi(a,b)$ as in the proof of Proposition 8.28 further below (except that the meaning of "below and to the left" should be clarified, and ideally illustrated on an example), and then defining $\operatorname{Ins}(a,b)$ to be $P(\pi(a,b))$. (Note that $\pi(a,b)$ can also be defined more directly as the generalized permutation obtained from π by removing all columns $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ with $a_i > a$ and removing all columns $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ with $b_i > b$.)

Such a definition of $\operatorname{Ins}(a,b)$ would have the additional advantage of making the similarity between the P-filling and the Q-filling more explicit.

188. **page 239, second paragraph:** After "Since $\operatorname{Ins}(W, H-1)$ is the tableau made by the numbers less than H", add "in the tableau $\operatorname{Ins}(W,H)$".

189. **page 239, second paragraph:** After "a copy of H in $\operatorname{Ins}(W-1,H)$ is bumped up to the next row if and only if it is in a box in μ that also appears in ν", I would add "(because it is bumped up to the next row if and only if its box is filled with a number less than H afterwards)".

190. **page 241, proof of Proposition 8.28:** In the last sentence of this proof, before "we must have", add "and since $\operatorname{Fill}(W,H)$ and $Q(W,H)$ have the same shape (because $\operatorname{Fill}(W,H)$ has the same shape as the tableau $\operatorname{Ins}(W,H)$ from the proof of Proposition 8.25, which in turn is identical to the tableau $P(W,H)$ from the same proof, which in turn has the same shape as $Q(W,H)$), ".

191. **page 242, proof of Theorem 8.29:** "over a line from the lower left corner to the upper right corner" $\to$ "over the line of slope 1 passing through the lower left corner". (This line will not pass through the upper right corner unless $W = H$.)

192. **page 242, proof of Theorem 8.29:** "the P-labeling and the Q-labeling" $\to$ "the P-filling and the Q-filling".

193. **page 243, Problem 8.6:** The words "with finite support" should be explained.

194. **page 243, Problem 8.9:** The word "hook" has never been defined.

195. **page 243, Problem 8.10:** "whose last entry is a_j" is ambiguous – you mean that the last entry is picked from the j-th position, not just that it equals a_j (which would be the literal interpretation).

196. **page 245, Problem 8.21:** The "for which boxes" formulation is somewhat ambiguous (the "boxes" looks like it refers to the "which"). I would replace it by "whose order relation is defined as follows: two boxes".

197. **page 246:** Replace "the hook lengths" by "the hook-length" (in order to match the hyphenation you use in Problem B.5).

198. **page 248, §9.1:** One line above Example 9.1, replace "$s_{11} = h_2$, and $s_2 = e_2$" by "$s_{11} = e_2$, and $s_2 = h_2$".

199. **page 251, proof of Theorem 9.3:** At the end of the first paragraph of this proof, replace "for which $\mu \subseteq \operatorname{sh}(P')$, $\operatorname{sh}(P') / \mu$ is a horizontal strip of length n, and $x^P \prod\limits_{j \in J} x_j = x^{P'}$" by "for which $\mu \subseteq \operatorname{sh}(P')$ and $\operatorname{sh}(P') / \mu$ is a horizontal strip of length n, which bijection has the property that $x^P \prod\limits_{j \in J} x_j = x^{P'}$". (Your current wording belies the fact that "$\mu \subseteq \operatorname{sh}(P')$" and "$\operatorname{sh}(P') / \mu$ is a horizontal strip of length n" are properties of the tableaux P' in the set, whereas "$x^P \prod\limits_{j \in J} x_j = x^{P'}$" is a property of the bijection.)

200. **page 251, proof of Theorem 9.3:** After "to obtain a filling P' of shape λ.", I'd add "Thus, $P' = r_{j_n}\left(r_{j_{n-1}}\left(\cdots\left(r_{j_1}(P)\right)\cdots\right)\right)$.".

201. **page 251, proof of Theorem 9.3:** When you write "combined with Lemma 8.8", you mean not Lemma 8.8 itself but rather the corollary of Lemma 8.8 and Lemma 8.7 that says that (in the notation of Lemma 8.8) the new cell in $r_d(r_c(T))$ is in a column further right than the new cell in $r_c(T)$. (As I already said above, I think it makes sense to state this corollary as a lemma in its own right; this is not the only place where you say you are using Lemma 8.8, but in truth are using that corollary.)

202. **page 252, proof of Proposition 9.4:** After "But a horizontal strip of length λ_1 has exactly one filling of content λ_1", add ", whereas any other skew shape has none". (You want to know that the sum in (9.2) has no other addends than the ones coming from horizontal strips of length λ_1.)

203. **page 252, proof of Proposition 9.4:** "we can group fillings of ν/μ of content λ" $\to$ "we can group the semistandard tableaux of shape ν/μ and content λ". (For arbitrary fillings, the boxes which contain l will not generally form a horizontal strip.)

204. **page 254, first paragraph:** In the first display on page 254, replace "$\sum\limits_{\zeta} K_{\nu,\zeta} m_\nu$" by "$\sum\limits_{\zeta} K_{\nu,\zeta} m_\zeta$". Besides, you are using the equality (4.6) here, which may be worth mentioning.

205. **page 254, last paragraph:** Replace "for which $\mu \subseteq \operatorname{sh}(P')$, $\operatorname{sh}(P') / \mu$ is a vertical strip of length n, and $x^P \prod_{j \in J} x_j = x^{P'}$" by "for which $\mu \subseteq \operatorname{sh}(P')$ and $\operatorname{sh}(P') / \mu$ is a vertical strip of length n, which bijection has the property that $x^P \prod_{j \in J} x_j = x^{P'}$". (This is the same issue as on page 251 above.)

206. **page 260, (9.5):** Replace the outer summation sign "$\sum_{\mu_{vh} \supseteq \mu}$" by "$\sum_{\substack{\mu_{vh} \supseteq \mu; \\ |\mu_{vh}/\mu| = n}}$" (since otherwise, the condition $|\mu_{vh}/\mu| = n$ is lost).

207. **page 260:** Before "We suspect many terms", I'd add "Each such choice of μ_v partitions the skew shape μ_{vh}/μ into two subshapes μ_v/μ and μ_{vh}/μ_v, which shall be called the *inner vertical strip* and the *outer horizontal strip*, respectively.". (This introduces the terminology that is used later on.)

208. **page 264, first paragraph:** Here it should also be explained why the traversal of the component starting at the head actually is a traversal (i.e., it passes through each box).

209. **page 264, Lemma 9.16:** In the first sentence of the lemma: "Suppose $\mu \subseteq \mu_{vh}$ are partitions and μ_{vh}/μ can be separated into an inner vertical strip and an outer horizontal strip" $\to$ "Suppose $\mu \subseteq \mu_{vh}$ are partitions such that μ_{vh}/μ is a border strip".

    In the last sentence of the lemma, after "there is a separation of μ_{vh}/μ", add "into an inner vertical strip and an outer horizontal strip".

210. **page 264, proof of Lemma 9.16:** In the first paragraph of this proof, you say that "we can assume without loss of generality that μ_{vh}/μ is connected". This hinges on some tacit arguments that is not completely trivial. As you correctly notice, one of these arguments is saying that two boxes in different connected components cannot lie in the same row or column. But here is another: If we separate each connected component of a skew shape into an inner and an outer part, then we get a separation of the whole skew shape into the union of all inner parts and the union of all outer parts. (That is, the union of all inner parts is a skew shape, and the union of all outer parts is a skew shape.) The easiest way to prove this is by arguing that if we do not get a separation this way, then we must have some inner-part box that is a right or top neighbor of some outer-part box; but due to their adjacency, these two boxes have to belong to the same connected component, and thus cause a contradiction.

211. **page 264, proof of Lemma 9.16:** In the third paragraph of this proof, replace "is to the left (resp., below) the box" by "is to the left of (resp., below) the box".

212. **page 265, first line:** On the first line of page 265, replace "Lemma 9.16 tells" by "Lemma 9.15 and Lemma 9.16 tell" (you need Lemma 9.15 to know that all μ_{vh}/μ's appearing in (9.5) are border strips).

213. **page 265, (9.6):** Replace the outer summation sign "$\sum\limits_{\mu_{vh}\supseteq\mu}$" by "$\sum\limits_{\substack{\mu_{vh}\supseteq\mu; \\ |\mu_{vh}/\mu|=n; \\ \mu_{vh}/\mu \text{ is a border strip}}}$".

    The same change should be done in (9.7).

214. **page 265, Theorem 9.17:** Replace "$n \geq 0$" by "$n \geq 1$" here. Even if you did define p_0 to be 1, the theorem would still fail for $n = 0$ due to an incorrect sign.

215. **page 266, Definition 9.18:** Before "$\lambda(j+1)/\lambda(j)$ is a connected border strip", I would add "the set $\lambda(j)$ is (the Ferrers diagram of) a partition and". (This might be redundant – I am not 100% sure, but I'm pretty sure it makes everything clearer; you haven't even properly defined what α/β is when α and β are not partition shapes. Equivalently, you can require that the entries of the tableau increase weakly left-to-right and top-to-bottom.)

216. **page 266, Definition 9.18:** The upper bound of the product in the displayed formula should be $l(\mu) - 1$ rather than $l(\mu)$.

    I would also replace "for all j with $0 \leq j \leq l(\mu)$" by "for all j with $0 \leq j \leq l(\mu) - 1$". (Strictly speaking, this is unnecessary, at least if you consider an empty skew shape to be a connected border strip; but I think it is natural to talk of $0 \leq j \leq l(\mu) - 1$.)

    Finally, I would replace "we set" by "we define the *sign* $\operatorname{sgn}(T)$ of T by" (so that the word "sign" is explained).

217. **page 267, proof of Theorem 9.20:** A better induction base for this proof would be the case $l(\lambda) = 0$ (which has to be handled either way). There is nothing in your induction step that really requires $l(\lambda) \geq 2$ (as opposed to just $l(\lambda) \geq 1$).

218. **page 268, proof of Theorem 9.20:** After "When we combine T_α with μ/ν", I would add "(filling the boxes in μ/ν with the number l and adding them to T_α)". I would also replace "combine T_α" by "combine a $T_\alpha \in BST(\nu, \alpha)$".

219. **page 269, second paragraph:** Wouldn't it be helpful to actually cite a proof of this claim about irreducible characters? One place where this is proved is §5.5 of Mark Wildon's nice (albeit much less detailed than your book) notes `http://www.ma.rhul.ac.uk/~uvah099/Maths/Sym/SymFuncs2017.pdf` . Another is Proposition 5.21.1 in Pavel Etingof et al., *Introduction to representation theory*, AMS 2011 (updated version 2018).

220. **page 272, solution to Example 10.2:** It would be helpful to clarify whether the rows are T_1 and the columns are T_2, or vice versa.

221. **page 272, (10.3):** You are missing an $s_{411}(X_3)$ term on the right hand side.

222. **page 273:** After "will have the shapes $(4,2)$," add "$(4,1,1)$,".

223. **page 270, Problem 9.10:** Are you sure you don't want to require the border strips in the decompositions to be connected?

224. **page 274, second paragraph:** In the description of jeu de taquin, you write: "There is a unique way to slide a box into the blank space so that the resulting object is column strict and row nondecreasing". This is not quite obvious: The uniqueness is easy, but the existence (specifically, the proof that the sliding entry will not be smaller than its new left neighbor or smaller-or-equal to its new bottom neighbor) needs a proof. And this proof is not completely trivial; it relies on the fact that there is only one empty box in an otherwise filled skew diagram (otherwise it wouldn't work, which is why the tableau switching paper of Benkart/Sottile/Stroomer requires the rather technical "staircases" condition).

225. **page 274:** "column strict and row nondecreasing" $\to$ "column-strict and row-nondecreasing".

226. **page 278, §10.2:** In the second paragraph of §10.2, replace "suppose word (T_1) has just one entry" by "suppose word (T_2) has just one entry".

227. **page 279, third paragraph:** Replace "In one type of step we have consecutive entries x and z" by "In one type of step we have consecutive entries z and x". (It is a bit confusing to label consecutive steps starting with the second one.)

228. **page 279, third paragraph:** Replace "for some $i \geq j$" by "for some $i > j$". (You aren't switching c with a_j.)

229. **page 279, third paragraph:** Replace "as well as $x = a_i$ and $y = a_{i+1}$ for some $i < j$" by "as well as $x = a_i$ and $y = \begin{cases} a_{i+1}, & \text{if } i < j-1; \\ c, & \text{if } i = j-1 \end{cases}$ for some $i < j$". (When you first swap a_j with a_{j-1}, the catalyzing neighbor on the right is c, not a_j.)

230. **page 279, third paragraph:** I think an example of the traveler moving through a row would be helpful here (showing what x, y, z are in each step).

231. **page 280:** After Definition 10.10, I'd add a sentence along the lines of "This relation $\sim_K$ is an equivalence relation, since any elementary Knuth transformation can be undone by another elementary Knuth transformation.".

232. **page 281, first paragraph:** In the sentence "In general, our definition of Knuth equivalence guarantees that the Knuth equivalence classes for a set of words are exactly the connected components of the corresponding graph on those words", replace "a set of words" by "a set of words closed under elementary Knuth transformations"[4]. For example, if S is the set of all words that correspond to even permutations $\pi \in S_n$, then the graph corresponding to this set S is totally disconnected (i.e., each vertex is isolated), but the Knuth equivalence classes are nontrivial (for large n).

233. **page 281, second paragraph:** "as a product on words" $\to$ "as a product on tableaux".

234. **page 282:** One line above Corollary 10.13, replace "tableaux" by "tableau".

235. **page 283, proof of Theorem 10.14:** "using using".

236. **page 283, proof of Theorem 10.14:** When you refer to Figures 10.8 and 10.9, I suggest explaining that the part left of a and the part right of d_k are not shown (i.e., there may be more filled boxes left of a or right of d_k, but they do not matter). I would actually remove the a from the figures as well, and only re-include it when it is needed in the proof later on (so I would duplicate the figures, removing the a's from the first copies but leaving them in the second).

237. **pages 283–285, proof of Theorem 10.14:** I have given up trying to understand this proof. I think it is too long and yet too terse. I understand the idea, but I'm not sure I can fill in the details. I am also a bit skeptical: You start by moving the c_j's past the b_i's using x as a catalyst, and you finish by claiming that "we can reverse our initial steps" – but x is no longer available as a catalyst at that point. (Something tells me that we need more than just the single x entry – I would expect the upper neighbors of $c_1, c_2, \ldots, c_m$ to all come useful.)

    I believe there is a good case for such a proof, but I believe it should be organized differently. It lends itself ideally to being factored into a sequence of relatively simple lemmas such as "If $c_1 \leq c_2 \leq \cdots \leq c_m < x \leq b_1 \leq b_2 \leq \cdots \leq b_k$ is a sequence of letters, then $xb_1 \cdots b_k c_1 \cdots c_m \sim_K xc_1 \cdots c_m b_1 \cdots b_k$" (I'm not actually if this one is true, but I think you get the gist), and I see no reason not to factor it. Moreover, these proofs can preferably be done by induction rather than by showing the first step and handwaving the rest

[4]This is not the best condition, but the easiest one to state. In truth, it suffices to have a "Knuth-convex" set of words, i.e., a set of words such that any two Knuth-equivalent words in the set are connected by a sequence of elementary Knuth transformations with all intermediate words being in the set as well.

However, given the position in which Theorem 10.14 stands in your book, I would suggest a different way of proof:

Alternative proof of Theorem 10.14 (sketched). It suffices to prove the following lemma:

Lemma 10.14a. Let $a_1 \leq a_2 \leq \cdots \leq a_m \leq x \leq b_1 \leq b_2 \leq \cdots \leq b_k$ and $c_1 \leq c_2 \leq \cdots \leq c_m \leq x \leq d_1 \leq d_2 \leq \cdots \leq d_k$ be two weakly increasing numbers such that

- every i satisfies $c_i < a_i$;
- every j satisfies $d_j < b_j$.

Then,

$$a_1 \cdots a_m x b_1 \cdots b_k c_1 \cdots c_m d_1 \cdots d_k \sim_K a_1 \cdots a_m b_1 \cdots b_k c_1 \cdots c_m x d_1 \cdots d_k.$$

[*Proof of Lemma 10.14a.* Let T be the tableau that consists of just one row, with entries $a_1, a_2, \ldots, a_m, x, b_1, \ldots, b_k$ (from left to right). This is clearly a semistandard tableau, and satisfies

$$\operatorname{word}(T) = a_1 \cdots a_m x b_1 \cdots b_k. \tag{13}$$

Now, let us insert the numbers $c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_k$ into T (using RSK insertion, in this order). Here is what happens:

- The insertion of c_1 bumps a_1 into the 2nd row (since $c_1 < a_1$). The first row becomes $c_1 a_2 \cdots a_m x b_1 \cdots b_k$.
- The insertion of c_2 bumps a_2 into the 2nd row (since $c_1 \leq c_2 < a_2$). The first row becomes $c_1 c_2 a_3 \cdots a_m x b_1 \cdots b_k$.
- The insertion of c_3 bumps a_3 into the 2nd row (since $c_2 \leq c_3 < a_3$). The first row becomes $c_1 c_2 c_3 a_4 \cdots a_m x b_1 \cdots b_k$.
- And so on. After $c_1, c_2, \ldots, c_k$ have been inserted, we obtain the tableau

a_1	a_2	$\cdots$	a_m					
c_1	c_2	$\cdots$	c_m	x	b_1	b_2	$\cdots$	b_k

.

- The insertion of d_1 bumps b_1 into the 2nd row (since $x \leq d_1 < b_1$). The first row becomes $c_1 \cdots c_m x d_1 b_2 \cdots b_k$.
- The insertion of d_2 bumps b_2 into the 2nd row (since $d_1 \leq d_2 < b_2$). The first row becomes $c_1 \cdots c_m x d_1 d_2 b_3 \cdots b_k$.
- The insertion of d_3 bumps b_3 into the 2nd row (since $d_2 \leq d_3 < b_3$). The first row becomes $c_1 \cdots c_m x d_1 d_2 d_3 b_4 \cdots b_k$.
- And so on. After $d_1, d_2, \ldots, d_k$ have been inserted, we obtain the tableau

a_1	a_2	$\cdots$	a_m	b_1	b_2	$\cdots$	b_k	
c_1	c_2	$\cdots$	c_m	x	d_1	d_2	$\cdots$	d_k

.

Thus,

$$r_{d_k}\left(r_{d_{k-1}}\left(\cdots\left(r_{d_1}\left(r_{c_m}\left(r_{c_{m-1}}\left(\cdots\left(r_{c_1}\left(T\right)\right)\right)\right)\right)\right)\right)\right)$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|c}\hline a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_k & \\ \hline c_1 & c_2 & \cdots & c_m & x & d_1 & d_2 & \cdots & \multicolumn{1}{c|}{d_k} \\ \hline \end{array},$$

so that

$$\text{word}\left(r_{d_k}\left(r_{d_{k-1}}\left(\cdots\left(r_{d_1}\left(r_{c_m}\left(r_{c_{m-1}}\left(\cdots\left(r_{c_1}\left(T\right)\right)\right)\right)\right)\right)\right)\right)\right)$$
$$= a_1 \cdots a_m b_1 \cdots b_k c_1 \cdots c_m x d_1 \cdots d_k. \qquad (14)$$

But Theorem 10.11 (applied to $(c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_k)$ instead of $(c_1, c_2, \ldots, c_n)$) yields

$$\text{word}\left(T\right) c_1 \cdots c_m d_1 \cdots d_k$$
$$\sim_K \text{word}\left(r_{d_k}\left(r_{d_{k-1}}\left(\cdots\left(r_{d_1}\left(r_{c_m}\left(r_{c_{m-1}}\left(\cdots\left(r_{c_1}\left(T\right)\right)\right)\right)\right)\right)\right)\right)\right).$$

In view of (13) and (14), this rewrites as

$$a_1 \cdots a_m x b_1 \cdots b_k c_1 \cdots c_m d_1 \cdots d_k \sim_K a_1 \cdots a_m b_1 \cdots b_k c_1 \cdots c_m x d_1 \cdots d_k.$$

This proves Lemma 10.14a. ■]

238. **page 285, §10.3:** It is worth explaining that a *permutation* of a multiset A is defined to be a tuple of elements of A such that every element of A appears in this tuple as often as it appears in A.

239. **page 286:** "identical to the left of x and to the right of z" → "identical to the left and to the right of the respective yzx or yxz blocks".

240. **page 286:** I think this argument is somewhat incomplete. In order to obtain $r_x\left(r_z\left(r_y\left(T\right)\right)\right) = r_z\left(r_x\left(r_y\left(T\right)\right)\right)$, it isn't enough to show that the bumping paths of z and x in $r_y\left(T\right)$ have no common boxes; it also needs to be shown that

    - the bumping path of z in $r_y\left(T\right)$ is identical to the bumping path of z in $r_x\left(r_y\left(T\right)\right)$, and
    - the bumping path of x in $r_y\left(T\right)$ is identical to the bumping path of x in $r_z\left(r_y\left(T\right)\right)$.

    The second of these two statements is an automatic consequence of the disjointness of the bumping paths, but the first is not, since the bumping procedure is influenced not only by the numbers in the boxes on the bumping path itself but also by the numbers in their left neighbor boxes. Fortunately, bumping an entry can only replace it by a smaller entry, so whatever entries have changed from $r_y\left(T\right)$ to $r_x\left(r_y\left(T\right)\right)$ will not affect the bumping path of z. Thus, the first statement holds as well. But I believe this should be explained.

241. **pages 287–288:** I think the case distinction in the "$r_y(r_z(r_x(T))) = r_y(r_x(r_z(T)))$" part of the proof can be much improved. While each case is indeed reasonably straightforward, I would argue that identifying the right cases to distinguish between is not. Thus, I suggest listing all the cases with their outcomes.

Here is my version of this list:

We suppose that the entries of the first row of T are $a_1, \ldots, a_{j+k+l+n}$, where

$$\begin{aligned} a_1 \leq \cdots \leq a_j \leq x, \\ x < a_{j+1} \leq \cdots \leq a_{j+k} \leq y, \\ y < a_{j+k+1} \leq \cdots \leq a_{j+k+l} \leq z, \\ z < a_{j+k+l+1} \leq \cdots \leq a_{j+k+l+n}. \end{aligned}$$

Here, if $k = 0$, then the chain of inequalities $x < a_{j+1} \leq \cdots \leq a_{j+k} \leq y$ should be interpreted as being vacuous; it is thus not saying that $x < y$. (The same applies to the chain $y < a_{j+k+1} \leq \cdots \leq a_{j+k+l} \leq z$ when $l = 0$, but this does not matter, since we already are given that $y < z$.) Now, the cases I distinguish are the following:

- *Case 1:* We have $n = 0$.
  - *Subcase 1.1:* We have $k + l = 0$. In this case, the first row of T has the form $a_1 \cdots a_j$. Hence, when we construct $r_y(r_z(r_x(T)))$, we first have x insert itself at the end of the first row; then z inserts itself at the end of the first row; then y bumps out z. When we construct $r_y(r_x(r_z(T)))$, we first have z insert itself at the end of the first row; then x bumps out z; then y inserts itself at the end of the first row. Thus, the first rows of $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$ are identical, and both times the same letter is being bumped into the next row.
  - *Subcase 1.2:* We have $k + l > 0$. In this case, when we construct $r_z(r_x(T))$, we first have x bump out a_{j+1}; then z inserts itself at the end of the first row. When we construct $r_x(r_z(T))$, we first have z insert itself at the end of the first row; then x bumps out a_{j+1}. Thus, the first rows of $r_z(r_x(T))$ and $r_x(r_z(T))$ are identical, and both times the same letter is being bumped into the next row. Obviously, this identity does not change when we apply r_y again; thus, the same holds for $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$.
- *Case 2:* We have $n > 0$.
  - *Subcase 2.1:* We have $k + l = 0$.
    - *Subsubcase 2.1.1:* We have $n = 1$. In this case, $j + k + l = j$. Hence, the first row of T has the form $a_1 a_2 \cdots a_j a_{j+1}$, with

$x \leq y < z < a_{j+1}$. Hence, when we construct $r_y(r_z(r_x(T)))$, we first have x bump out a_{j+1}; then z inserts itself at the end of the first row; then y bumps out z. When we construct $r_y(r_x(r_z(T)))$, we first have z bump out a_{j+1}; then x bumps out z; then y inserts itself at the end of the first row. Thus, the first rows of $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$ are identical, and both times the same two letters are being bumped into the next row (in the same order).

* *Subsubcase 2.1.2:* We have $n > 1$. This is the second case you consider on page 288.

– *Subcase 2.2:* We have $k + l > 0$.

* *Subsubcase 2.2.1:* We have $l = 0$. In this case, $k + l = k$. Hence, the first row of T has the form $a_1 a_2 \cdots a_j a_{j+1} \cdots a_{j+k} a_{j+k+1} \cdots a_{j+k+n}$, with $y < z < a_{j+k+1} \leq \cdots \leq a_{j+k+n}$. Thus, when we construct $r_y(r_z(r_x(T)))$, we first have x bump out a_{j+1} (which is strictly left of a_{j+k+1}, because $k = k + l > 0$); then z bumps out a_{j+k+1}; then y bumps out z. When we construct $r_y(r_x(r_z(T)))$, we first have z bump out a_{j+k+1}; then x bumps out a_{j+1} (which is strictly left of a_{j+k+1}); then y bumps out z. Thus, the first rows of $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$ are identical, and the letters that are bumped out from these first row are (respectively) a_{j+1}, a_{j+k+1}, z (in this order) and a_{j+k+1}, a_{j+1}, z (in this order). Since we know that $a_{j+1} \leq x < z$, we can conclude from the induction hypothesis that inserting these bumped-out letters into the remaining rows of T will again have the same result.

* *Subsubcase 2.2.2:* We have $k = 0$. In this case, $k + l = l$. Hence, the first row of T has the form $a_1 a_2 \cdots a_j a_{j+1} \cdots a_{j+l} a_{j+l+1} \cdots a_{j+l+n}$, with $x \leq y < a_{j+1} \leq \cdots \leq a_{k+l} \leq z$. Thus, when we construct $r_y(r_z(r_x(T)))$, we first have x bump out a_{j+1} (which is strictly left of a_{j+l+1}, because $l = k + l > 0$); then z bumps out a_{j+l+1}; then y bumps out a'_{j+2}, where $a'_{j+2} = \begin{cases} a_{j+2}, & \text{if } l \geq 2; \\ z, & \text{if } l = 1 \end{cases}$ is the entry currently occupying position $j + 2$ (here, we are using the facts that $y < a_{j+1} \leq a_{j+2}$ and that $y < z$). When we construct $r_y(r_x(r_z(T)))$, we first have z bump out a_{j+l+1}; then x bumps out a_{j+1} (which is strictly left of a_{j+l+1}); then y bumps out a'_{j+2}, where a'_{j+2} is the same as in the previous sentence. Thus, the first rows of $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$ are identical, and the letters that are bumped out from these first row are (respectively) $a_{j+1}, a_{j+l+1}, a'_{j+2}$ (in this order) and $a_{j+l+1}, a_{j+1}, a'_{j+2}$ (in this order). Since we know that $a_{j+1} \leq a'_{j+2} < a_{j+l+1}$ (indeed, if $l \geq 2$, then this follows from $a_{j+1} \leq$

$a_{j+2} \leq z < a_{j+l+1}$; but if $l = 1$, then this follows from $a_{j+1} \leq z < a_{j+l+1}$), we can conclude from the induction hypothesis that inserting these bumped-out letters into the remaining rows of T will again have the same result.

∗ *Subsubcase 2.2.3:* We have $k > 0$ and $l > 0$. This is the "most generic" case, as all the relevant intervals $[j+1, j+k]$, $[j+k+1, j+k+l]$ and $[j+k+l, j+k+n]$ are nonempty. Thus, when we construct $r_y(r_z(r_x(T)))$, we first have x bump out a_{j+1}; then z bumps out $a_{j+k+l+1}$; then y bumps out a_{j+k+1}. When we construct $r_y(r_x(r_z(T)))$, we first have z bump out $a_{j+k+l+1}$; then x bumps out a_{j+1}; then y bumps out a_{j+k+1}. Thus, the first rows of $r_y(r_z(r_x(T)))$ and $r_y(r_x(r_z(T)))$ are identical, and the letters that are bumped out from these first row are (respectively) $a_{j+1}, a_{j+k+l+1}, a_{j+k+1}$ (in this order) and $a_{j+k+l+1}, a_{j+1}, a_{j+k+1}$ (in this order). Since we know that $a_{j+1} \leq a_{j+k+1} < a_{j+k+l+1}$ (indeed, if $a_{j+k+1} < a_{j+k+l+1}$ follows from $a_{j+k+1} \leq z < a_{j+k+l+1}$), we can conclude from the induction hypothesis that inserting these bumped-out letters into the remaining rows of T will again have the same result.

Note that I am **not** distinguishing between the cases $x = y$ and $x < y$, as I don't see a reason to do so. I believe the same arguments apply to both $x = y$ and $x < y$, as long as you make sure to interpret the chain of $x < a_{j+1} \leq \cdots \leq a_{j+k} \leq y$ as vacuous when $k = 0$.

242. **page 289, proof of Theorem 10.17:** "to its right" → "to its left".

243. **page 289, proof of Theorem 10.17:** Before "each entry of r_{l-1} bumps", add "when we insert r_{l-1} into $P(r_l)$ using RSK insertion,".

244. **page 290, proof of Corollary 10.22:** Since you haven't mentioned it before but are using it now, it feels reasonable to mention the monoidal property of Knuth equivalence here (i.e., the fact that if $\alpha_1 \sim_K \alpha_2$ and $\beta_1 \sim_K \beta_2$, then $\alpha_1\beta_1 \sim_K \alpha_2\beta_2$).

245. **page 292, line 7:** "tableaux" → "tableau".

246. **page 292:** Two lines above Definition 10.25, you write "into the new boxes of λ/μ". This presupposes the following two facts:

    - The boxes created by inserting $b_1, \ldots, b_{|\nu|}$ into U are actually the boxes of λ/μ.
    - Inserting $s_1, \ldots, s_{|\nu|}$ into these boxes (in the order in which they appear) yields a semistandard skew tableau.

These facts are not hard to check, but at least something should be said about the proofs. (The first fact comes from observing that

$$\begin{aligned}
& r_{b_{|\nu|}}\left(r_{b_{|\nu|-1}}\left(\cdots r_{b_1}\left(U\right)\right)\right) \\
& = \left(\text{result of inserting the word } b_1 \cdots b_{|\nu|} \text{ into } U\right) \\
& = \left(\text{result of inserting the word } \operatorname{word}\left(V\right) \text{ into } U\right) \\
& \qquad \left(\begin{array}{c} \text{by an analogue of Theorem 10.16,} \\ \text{since } b_1 \cdots b_{|\nu|} \sim_K \operatorname{word}\left(\underbrace{P\left(b_1 \cdots b_{|\nu|}\right)}_{=P(\sigma)=V}\right) = \operatorname{word}\left(V\right) \end{array}\right) \\
& = U \star_r V \qquad \left(\text{by the definition of } U \star_r V\right) \\
& = U \star V = T \qquad \left(\text{since } \left(U, V\right) \in \mathcal{T}\left(\mu, \nu, T\right)\right)
\end{aligned}$$

is a tableau of shape λ. Here, the "analogue of Theorem 10.16" means the fact that Knuth-equivalent words not only produce the same P-tableau, but also lead to the same result when inserted in an already existing tableau. While this fact follows easily from Theorem 10.16 (just write the already existing tableau as the P-tableau of some word), I find this fact sufficiently useful to state in its own right. The second fact presupposed above is proved in the same way as Proposition 8.14.)

247. **page 294:** You write: "Fill in the boxes of μ in S with entries which are smaller than all entries of S to create a semistandard tableau US of shape λ". I think this is only possible if you allow negative integers as entries. Example 10.28 illustrates why: The only reason why you are able to avoid negative (or zero) entries in Figure 10.17 is that you are not taking the "smaller than all entries of S" requirement seriously (the 2 you add in row 2 is not smaller than the 2 in S).

    You should probably say that you are going to allow tableaux (and generalized permutations) with negative entries when necessary, seeing that you use them in the proof of Lemma 10.30 as well.

248. **pages 294–295:** I don't understand your argument for why U is independent of the choice of entries to fill μ with. (I think you are using the letter U for two different tableaux in that argument.) I also am missing a proof for why V is independent of this choice. Fortunately, I don't believe these arguments are necessary: You can instead fix a choice of entries to fill μ with, and define the map Ω using this choice; then Theorem 10.33 entails that Ω is actually independent of this choice because it is the inverse of the map Ψ (and inverses are unique).

249. **page 295, Example 10.28:** As I said above, your US in Figure 10.17 does not quite fit the "entries smaller than all entries of S" bill.

250. **page 296, proof of Lemma 10.29:** After "let v_j' be v_j with b removed", I'd add "(where b refers to the leftmost appearance of b in each word)".

251. **page 297, proof of Lemma 10.30:** You are tacitly using that the array τ is actually a generalized permutation. This follows from the fact that $c_j < a_k$ for all j and k (which, in turn, follows from the fact that $c_1, \ldots, c_m$ are the entries of $Q(\sigma) = T^-$, which are nonpositive).

252. **page 298, proof of Proposition 10.31:** I'd replace "$\operatorname{rect}(\Psi(U, V)) = W$" by "$\operatorname{rect}(\Psi(U, V)) = Q(\sigma) = W$" just in order to make the reasoning clearer.

253. **page 298, proof of Proposition 10.32:** After "the entries in μ must be $u_1, \ldots, u_{|\mu|}$", add "(since π is a generalized permutation, so that $u_1 \leq \cdots \leq u_{|\mu|} \leq t_1 \leq \cdots \leq t_{|\lambda|-|\mu|}$, and this entails that $u_1, \ldots, u_{|\mu|}$ are the smallest $|\mu|$ many entries of US)".

254. **page 298, proof of Proposition 10.32:** After "In particular, the shape of U is μ", I'd add "(since $U = P\left(a_1, \ldots, a_{|\mu|}\right)$ is the intermediate result in the construction of $P(\pi)$ achieved after the first $|\mu|$ many insertions, and the Q-tableau achieved at that point is precisely the part of $Q(\pi) = US$ that contains the smallest $|\mu|$ many entries of US; but clearly the P-tableau and the Q-tableau always have the same shape)".

255. **page 298, proof of Proposition 10.32:** Replace "By Lemma 10.30" by "Note that the tableau $P(\pi)$ can be obtained from U by inserting $c_1, \ldots, c_{|\lambda|-|\mu|}$ in this order; and the S part of US is obtained by filling the boxes of λ/μ with $t_1, \ldots, t_{|\lambda|-|\mu|}$ in the order in which they are created during this insertion process. Hence, by Lemma 10.30 (applied to τ, $|\lambda| - |\mu|$, t_i, c_i, U and $P(\pi)$ instead of π, n, a_i, b_i, T and U)".

256. **page 298, proof of Theorem 10.33:** On the second line of this proof, replace ", and let $Q(\sigma) = W$" by "and $Q(\sigma) = W$". (This is part of the definition of σ, not of W.)

257. **page 299, proof of Theorem 10.33:** You are tacitly using the fact that the array π' (which you define as the concatenation $\sigma'\sigma$) is actually a generalized permutation. This follows from the fact that $v_1, \ldots, v_{|\mu|}$ (which are the entries of U_c, since $Q(\sigma') = U_c$) are smaller than $s_1, \ldots, s_{|\nu|}$ (which are the entries of S).

258. **page 299, proof of Theorem 10.33:** After "so we must have $\pi = \pi'$.", I would add "Thus, $\left(d_1, d_2, \ldots, d_{|\mu|}\right) = \left(a_1, a_2, \ldots, a_{|\mu|}\right)$ and $\left(b_1, b_2, \ldots, b_{|\nu|}\right) = \left(c_1, c_2, \ldots, c_{|\lambda|-|\mu|}\right)$."

259. **page 299, proof of Theorem 10.33:** Instead of saying "by Lemma 10.30 and our construction", I'd just say "as we have seen in the proof of Proposition 10.32 above". After all, your notations here are exactly the same as in that proof, so there is no need to redo anything.

260. **page 299, proof of Theorem 10.33:** In the last displayed equation on page 299, replace "$W = \operatorname{rect}(S_1) = Q(\sigma)$" by "$W = Q(\sigma)$". Indeed, the $\operatorname{rect}(S_1)$ term is useless and misleading (the equality $W = Q(\sigma)$ comes immediately from the definition of σ).

261. **page 299, proof of Theorem 10.33:** After "Therefore, $\tau = \sigma$", I would add ", so that $\pi = \begin{bmatrix} u_1 & \cdots & u_{|\mu|} & s_1 & \cdots & s_{|\nu|} \\ a_1 & \cdots & a_{|\mu|} & b_1 & \cdots & b_{|\nu|} \end{bmatrix}$". (This is closer to what you actually use in the next paragraph.)

262. **page 301, proof of Theorem 10.39:** I'd replace "$\operatorname{word}(S) \sim_K n \cdots n \cdots 1 \cdots 1$" by "$\operatorname{word}(S) \sim_K \operatorname{word}(W) = n \cdots n \cdots 1 \cdots 1$" (since you are talking about $\operatorname{word}(W)$ in the next paragraph).

263. **page 302, second paragraph:** "is if xyw_2 is not a Littlewood–Richardson word" $\rightarrow$ "is if the word xyw_2 contains more copies of x than of $x-1$ despite $x > 1$".

264. **page 302, third paragraph:** "would be if zyw_2 were not a Littlewood–Richardson word" $\rightarrow$ "would be if the word zyw_2 contained more copies of z than of $z-1$".

265. **page 302, third paragraph:** I'm skeptical about the claim that "xyw_2 has exactly the same number of x's as $x+1$'s". Fortunately, a weaker claim suffices: Since w_2 is Littlewood–Richardson, and since $y = x$, the word xyw_2 must have at least two more x's than it has $x+1$'s. In view of $z = x+1$, this rewrites as follows: The word xyw_2 must have at least two more $z-1$'s than it has z's. Therefore, the word zyw_2 (which has one more z and one fewer $x = z-1$ than xyw_2) must have at least as many $z-1$'s than it has z's. But this contradicts the fact that the word zyw_2 contains more copies of z than of $z-1$.

266. **page 304, Problem 10.9:** Remove the spurious "2" at the end of this exercise.

267. **page 307, Problem 10.12:** The last sentence should not be part of the last bullet point.

268. **page 307, Figure 10.20:** Are you sure the labelling of the middle rhombus is correct? Here b and c are two adjacent vertices, and so are a and d; the inequality $b + c \geq a + d$ then doesn't seem to correspond to the hive inequality in the works of Knutson and Tao. (But I'm no expert on those works.)

269. **page 315, proof of Proposition A.14:** On line -3 of the page, replace "F" by "F^n".

270. **page 317, line -4:** "we say $u_1, \ldots, u_n$" $\to$ "we say $\overrightarrow{u}_1, \ldots, \overrightarrow{u}_n$".

271. **page 318, Proposition A.18:** It is worth saying that the field is supposed to be $\mathbb{Q}$ or $\mathbb{R}$ here (but not $\mathbb{C}$, which would render axiom (4) meaningless and axiom (5) false).

272. **page 319, proof of Proposition A.18:** On the first line of the displayed equation, replace "u_k" by "$\overrightarrow{u}_k$".

273. **page 319, proof of Proposition A.18:** On line 10 of the proof (and of the page), replace "and $w =$" by "and $\overrightarrow{w} =$".

274. **page 320, line 2:** "function" $\to$ "bilinear map" (which should be defined). If it was just an arbitrary function, then the values of $\langle \overrightarrow{u}_j, \overrightarrow{v}_k \rangle$ would not uniquely determine it.

275. **page 321, Problem A.12:** On line 2 of the problem, replace "$\overrightarrow{w}, \ldots, \overrightarrow{w}_n$" by "$\overrightarrow{w}_1, \ldots, \overrightarrow{w}_n$".

276. **page 321, Problem A.13 (a):** "$\overrightarrow{v} = \langle a, b \rangle$" $\to$ "$\overrightarrow{v} = (a, b)$".

277. **page 324 (ca.):** It is worth pointing out that any partition λ of k has $l(\lambda) \leq k$ and thus can be written as $(\lambda_1, \lambda_2, \ldots, \lambda_k)$. This little notational trick gets tacitly used in the text many times, but may be somewhat confusing to readers who aren't used to it (they may wonder whether λ actually has k nonzero entries).

278. **page 333, proof of Proposition C.7:** Replace "of the form $1, j$" by "of the form $(1, j)$".

279. **page 335:** I'd add an exercise asking to prove that $\operatorname{inv}(\pi\sigma) \leq \operatorname{inv}\pi + \operatorname{inv}\sigma$ for any $\pi, \sigma \in S_n$.

280. **back cover:** "the involution Ω" $\to$ "the involution ω".