**From iterated integrals and chronological calculus to Hopf and Rota-Baxter algebras**

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**Errata and addenda by Darij Grinberg**

1. **page 5, very last display:** The $\int\limits_{0\leq t_1\leq t_2\leq t}$ sign should be a double integral sign: $\iint\limits_{0\leq t_1\leq t_2\leq t}$.

2. **page 6, middle of the page (two paragraphs above Definition 2):** What exactly do you mean by "(looking for example at the boundary of $\Delta_t^\nu$)"?

3. **page 7, first display:** Here I would also add the alternative definition

$$\mathbf{L}_\sigma(t) = \iiint\limits_{0\leq t_1\leq t_2\leq\cdots\leq t_n\leq t} dt_1\cdots dt_n L_1\left(t_{\sigma(1)}\right)\cdots L_n\left(t_{\sigma(n)}\right).$$

   This is the form you often use below.

4. **page 7, Lemma 3:** This looks wrong. Shouldn't it be $\mathbf{L}_\alpha(t)\,\mathbf{K}_\beta(t) = (\mathbf{L}\cdot\mathbf{K})_{\alpha*\beta}(t)$ ?

5. **page 7, Lemma 4:** Replace "$\gamma(n+1) = n+1$" by "$\gamma(n+1) = n+m+1$", or so I believe.

6. **page 8, Proposition 8:** "basis" $\to$ "bases".

7. **page 8, Remark 10:** "availble".

8. **page 8, Theorem 11:** Here, you are letting the group algebra of the symmetric group act "formally" on the operators, to make sense of expressions like $X_n(t)\cdot w$ for some $w$ in the group algebra. This is worth explaining.

9. **page 9, proof of Theorem 11:** Shouldn't the "$(-1)^{l-1}$" be a "$(-1)^l$"?

10. **page 9, proof of Theorem 11:** The last equality sign here behooves more explanation, I believe.

11. **page 9, Remark 12:** Shouldn't the "321" be "312"?

12. **page 10:** "branche" $\to$ "branch".

13. **page 10:** In "inequalities $t_i \leq t_j \iff j \leq_T i$", the "$\iff$" sign should be a "$\Longleftarrow$" sign, right?

14. **page 12, last paragraph:** The connection between Chen's rule and (6) is not very clear to me.

15. **page 13, Corollary 24:** "over all admissible" $\to$ "over all nonempty admissible".

16. **page 14, first displayed equation:** The right hand side has a closing parenthesis ) too much.

17. **page 16, Definition 25:** To me, the word "shuffle algebra" means the bialgebra $V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \cdots$ with the shuffle product as multiplication and with deconcatenation as comultiplication. What you call a "shuffle algebra" is what I know as a "dendriform algebra". Wouldn't it be good to point out the alternative language here rather than just in Remark 30?

18. **page 16, Remark 27:** Maybe better to use the $\prec$ and $\succ$ symbols for the operations here, seeing that this remark is about general shuffle algebras rather than just the integral ones.

19. **page 17, proof sketch to Lemma 29:** In the second displayed equation, the right hand side should be $((y \succ x \prec z) \cdot_A y') \succ x \prec z'$. (So no two $x$'es, and no extra ) parenthesis at the end.)

20. **page 17, after Definition 31:** "Caley" $\to$ "Cayley".

21. **page 19, Definition 35:** Add "$\{\} v = v$", since otherwise the brace map is not defined on the 0-th degree component of $T(V)$.

22. **page 19, Definition 35:** Does "$\{w_1, \ldots, w_n\}$" really mean "$\{w_1 \ldots w_n\}$"?

    I also don't see an immediate reason why this is well-defined. Symmetric tensors like $w_1 \ldots w_n$ come with no canonical ordering of their factors, so $w_n$ could as well be any other $w_i$, but the right hand side is not obviously symmetric. I see a long proof by induction on $n$ (using the pre-Lie axiom), but this should be cited to a place in the literature.

23. **page 19:** "Recall that the time-ordered product": Why "Recall"? Has the time-ordered product been defined before?

24. **page 20, Theorem 38:** Shouldn't "$V := \mathcal{PL}(L)$" be "$L := \mathcal{PL}(V)$"?

25. **page 21, second paragraph:** In the definition of $G\left(\widehat{H}\right)$, add the extra condition $\epsilon(x) = 1$ (otherwise, 0 would be an element).

26. **page 21, second paragraph:** "Notice that $\widehat{k[V]}$ is such a Hopf algebra" $\to$ "Notice that $\widehat{k[V]}$ is such a completed Hopf algebra".

27. **pages 20–21:** Do you really need $V$ to be a **free** pre-Lie algebra for these Magnus computations to work? I don't understand them in detail, but my impression is that any graded (in positive degrees) pre-Lie algebra would work.

28. **page 21, Definition 41:** You don't use the words "2-group" and "2-pre-Lie algebra" anywhere outside of this definition. (I also would suggest using something other than "2-group", as that word already has two different meanings – sadly, 2 is prime...)

29. **page 21, Definition 41:** Shouldn't the $\triangleleft$ sign be a $\triangleright$ sign?

30. **page 26, Lemma 47:** This requires the algebra to be commutative, or at least some weaker version thereof. For example, for $n = 1$ and $m = 1$, this lemma is saying that $R(f)R(g) = R(fR(g)) + R(gR(f))$, which differs from the Rota–Baxter identity (17) in that $R(R(f)g)$ is replaced by $R(gR(f))$.

31. **page 27, Example 50:** "In fact, any projector which satisfies relation (15) is of weight $\theta = 1$": This is not literally true, since (e.g.) a projector onto a square-zero ideal along a subalgebra will satisfy the relation (15) for any $\theta$.

32. **page 27, last paragraph:** I don't see why you write "$(m, n; r)$-quasi-shuffle of type $\max(m, n) \leq r \leq m + n$" when all the information is contained in the "$(m, n; r)$-quasi-shuffle" part. In this context, the "type" part is only confusing. Also, the first time you use the formulation, you misspell the second "$\leq$" sign as a "$<$" sign, which is wrong.

33. **page 28, Lemma 52:** The "$x_i$" at the end should be an "$x_j$".

34. **page 28, Definition 53:** "operator on $A$" $\rightarrow$ "operator $R$ on $A$".

35. **page 30, Lemma 64:** The comma after "weight $\theta$" should be a period.

36. **page 31:** "possesses more properties" $\rightarrow$ "possess more properties".

37. **page 31:** "seen in Theorem 61" $\rightarrow$ "seen in Proposition 61" (or turn the proposition into a theorem).

38. **page 32, (37):** What are these equations for? Are the two "exp" elements sought? Are they supposed to live in $A[[\lambda]]$ ?

39. **page 34, (39):** The sum on the left hand side should be over "$m > 0$", not over "$n > 0$".

40. **page 35, after Proposition 73:** Why is the "obvious identity" obvious?

41. **page 36, (45):** I assume the $\prod_{i=1}^{m_j-1}$ product is taken in the order of $i$ increasing from left to right?

42. **page 36, §4.6:** With just a bit more work, you can prove many of the claims stated here.

    To wit, consider the free commutative Rota–Baxter algebra $RB(x)$ on a single generator $x$. Let $\Psi : RB(x) \to A^{\mathbb{N}}$ be the Rota–Baxter algebra homomorphism sending $x$ to $x$ (of course, a different $x$). Let the *chain monomials* be the elements of the form $x^{a_0} R\left(x^{a_1} R\left(x^{a_2} R\left(\cdots R\left(x^{a_k}\right)\right)\right)\right)$ in $RB(x)$ for $(a_0, a_1, \ldots, a_k) \in \mathbb{N}^k$ (where $0 \in \mathbb{N}$). It is easy to see (by induction on $k$) that the image of such a standard monomial $x^{a_0} R\left(x^{a_1} R\left(x^{a_2} R\left(\cdots R\left(x^{a_k}\right)\right)\right)\right)$ under $\Psi$ is the sequence $(u_0, u_1, u_2, \ldots) \in A^{\mathbb{N}}$, where

    $$u_i = \sum_{i=j_0>j_1>j_2>\cdots>j_k} x_{j_0}^{a_0} x_{j_1}^{a_1} \cdots x_{j_k}^{a_k} \qquad \text{for each } i \in \mathbb{N}.$$

    These images are linearly independent (since you can reconstruct $(a_0, a_1, \ldots, a_k)$ from the monomials appearing in $u_{k+1}$). Thus, the chain monomials in $RB(x)$ are also linearly independent. Since they furthermore span $RB(x)$ (since Lemma 52 lets us expand the product of two such monomials as a sum of such monomials, and it is easy to apply $R$ to such a monomial), it follows that they form a basis of $RB(x)$. Since their images under $\Psi$ are linearly independent, this entails that the map $\Psi$ is injective, and therefore the Rota–Baxter subalgebra of $A^{\mathbb{N}}$ generated by $x$ (that is, the image of $\Psi$) is also free as a commutative Rota–Baxter algebra.

    Let $\Gamma : \mathcal{S} \to RB(x)$ be the algebra homomorphism that sends each $e_n \in \mathcal{S}$ to $R^{(n)}(x) \in RB(x)$. This is the embedding of $\mathcal{S}$ into $RB(x)$ that you are talking about, but we have yet to prove that it is an embedding.

    Now, let $\Phi : \mathcal{S} \to A^{\mathbb{N}}$ be the map that sends each $f \in \mathcal{S}$ to the sequence $(v_0, v_1, v_2, \ldots) \in A^{\mathbb{N}}$, where

    $$v_i = f(x_0, x_1, \ldots, x_{i-1}, 0, 0, 0, \ldots) \qquad \text{for each } i \in \mathbb{N}.$$

    This is also an algebra homomorphism, and is easily seen to be injective. Moreover, it is easy to see (by induction on $n$) that $\Phi(e_n) = R^{(n)}(x)$ for each $n \in \mathbb{N}$. Hence, it follows easily that $\Phi = \Psi \circ \Gamma$. This shows that $\Gamma$ is injective (since $\Phi$ is injective), i.e., an embedding.

    It is moreover trivial to see that $\Phi(h_n) = R(x^n)$ for each $n > 0$ for the power-sum symmetric functions $h_n$ (which everyone I know calls $p_n$, by the way). Therefore, by "un-applying" the injective map $\Psi$, we obtain $\Gamma(h_n) = R(x^n)$ as well (since $\Psi$ sends $\Gamma(h_n)$ to $\Phi(h_n)$ and sends $R(x^n) \in RB(x)$ to $R(x^n) \in A^{\mathbb{N}}$). This is one of the properties of the embedding/correspondence between $\mathcal{S}$ and $RB(x)$ that you have stated.

(I have learned the above ideas in some form from two papers by Glanffrwd Thomas: `https://doi.org/10.1007/bfb0090016` and `https://doi.org/10.1016/B978-0-12-428780-8.50012-X` .)

43. **page 37, Definition 74:** "by the $R^{(n)}(x)$" should be "by the $R(x^n)$" here.

44. **page 40, Definition 77:** "by the $R^{(n)}(x)$" should be "by the $R(x^n)$" here in the second paragraph.

    However, there is another problem: I don't see how to express $R\left(x^2\right)$ in terms of the generators $R^{(n)}(x)$, since the equality $\underbrace{R(x)\cdot R(x)}_{=\left(R^{(1)}(x)\right)^2} = \underbrace{R(R(x)\,x)}_{=R^{(2)}(x)} + R(xR(x)) + R\left(x^2\right)$ contains two "unknowns" (we can no longer equate $R(xR(x))$ with $R(R(x)\,x)$). What am I missing?

45. **References:** Why is the title of a paper sometimes formatted in *italics* and sometimes in *slanted*?