A Simple Proof of an Identity Concerning Pfaffians of Skew Symmetric Matrices

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Abstract

In the present paper we prove an identity concerning Pfaffians similar to the well-known Grassmann–Plücker relations for determinants, using tools from multilinear algebra. More precisely, we shall derive the identity as a corollary to an equation concerning skew-symmetric bilinear forms and operators, acting on the exterior algebra $\Lambda(M)$ of an R-module M over some fixed commutative ring R with $1 \in R$.

1 Introduction

The Pfaffian $\mathrm{Pf}(A) = \mathrm{Pf}_n(A) = \mathrm{Pf}_n^R(A)$ of a skew-symmetric[1] $(n \times n)$-matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & a_{n-1,n} \\ -a_{1n} & -a_{2n} & \cdots & -a_{n-1,n} & 0 \end{pmatrix} \tag{1.1}$$

with coefficients a_{ij} $(1 \leq i < j \leq n)$ in some commutative ring R has played an important rôle in many branches of mathematics, in particular in determinant theory, in symplectic geometry and in differential geometry (see for example [A, Chapter III]; [K, Kapitel 9]; [St]; [DFN, Sect. 34]; and [Sp, Chap. 13, Sect. 3]).

There are many ways to define the Pfaffian. A rather conceptual way is the following:

Let $\mathrm{Skew}_n : \mathbf{rings} \to \mathbf{sets}$ denote the covariant functor from the category **rings** of commutative rings into the category **sets** of all sets which associates to each commutative ring R the set $\mathrm{Skew}_n(R)$ of all skew-symmetric $(n \times n)$-matrices over R and let $\mathrm{Forget} : \mathbf{rings} \to \mathbf{sets}$ denote the forgetful functor from rings into sets which associates to a commutative ring R the set R itself. Then

$$\mathrm{Pf} = \mathrm{Pf}_n = (\mathrm{Pf}_n^R)_{R \in \mathbf{rings}}$$

can be defined as the unique natural transformation from Skew_n back into Forget such that

$$\left(\mathrm{Pf}_n^R(A)\right)^2 = \det(A) \tag{1.2}$$

holds for every commutative ring R and every $A \in \mathrm{Skew}_n(R)$, normalized in case $n \equiv 0 \mod 2$ such that for

$$A := A_0 := \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$$

(I_m denoting the identity $(m \times m)$-matrix) one has[2]

$$\mathrm{Pf}(A_0) := (-1)^{(n/2)(n/2-1)/2}. \tag{1.3}$$

This means that the natural transformation $\mathrm{Pf}^2 : \mathrm{Skew}_n \to \mathrm{Forget}$ which maps for every R a matrix $A \in \mathrm{Skew}_n(R)$ onto $\left(\mathrm{Pf}_n^R(A)\right)^2$ coincides with the natural transformation "det" from Skew_n into Forget, given by forming the determinant.

In case $n \equiv 1 \mod 2$ we have of course $\mathrm{Pf}(A) = 0$ for all $A \in \mathrm{Skew}_n(R)$.

[1] *Comment by DG:* A *skew-symmetric* matrix here means a square matrix A that satisfies $A^T = -A$ and whose diagonal entries are 0. Several sources refer to such matrices as *alternating* instead.

[2] *Comment by DG:* The formula (1.3) has been corrected (the original had 1 on the right hand side).

In case $n \equiv 0 \mod 2$ one can describe the Pfaffian equivalently as a polynomial

$$\mathrm{Pf}(X_{12}, X_{13}, \ldots, X_{1n}, X_{23}, \ldots, X_{2n}, \ldots, X_{n-1,n}) \in \mathbb{Z}[X_{12}, X_{13}, \ldots, X_{n-1,n}]$$

such that for each commutative ring R and each $A \in \mathrm{Skew}_n(R)$ of the form (1.1) one has

$$\mathrm{Pf}(a_{12}, a_{13}, \ldots, a_{n-1,n})^2 = \det(A), \tag{1.2$'$}$$

normalized such that[3]

$$\mathrm{Pf}(a_{12}, a_{13}, \ldots, a_{n-1,n}) := (-1)^{(n/2)(n/2-1)/2} \quad \text{for} \quad a_{ij} := \begin{cases} 1, & \text{for } j - i = \dfrac{n}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3$'$}$$

Much more constructively, on the other hand, one can define, for given R, even n, and $A \in \mathrm{Skew}_n(R)$ of the form (1.1), the value of $\mathrm{Pf}_n(A)$ in a recursive way by

$$\mathrm{Pf}_2\left(\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}\right) := a_{12} \qquad (a_{12} \in R) \tag{1.4}$$

for $n = 2$ and by

$$\mathrm{Pf}_n(A) := \sum_{j=2}^{n} (-1)^j \cdot a_{1j} \cdot \mathrm{Pf}_{n-2}\big(A^{(j)}\big) \qquad \text{for } n \geq 4, \tag{1.5}$$

where $A^{(j)}$ denotes the $(n-2) \times (n-2)$-matrix one gets from A by eliminating the first and the jth row and column[4].

Of course, whatever definition one starts with, the point which needs proof is that both definitions are indeed equivalent, that is, that a natural transformation as described in the first definition exists[5] and can be constructed according to the second one. In this note we shall provide such a proof by starting with still another definition,

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[3] *Comment by DG:* The formula (1.3′) has been corrected (the original had 1 on the right hand side).

[4] *Comment by DG:* I would include the base case

$$\mathrm{Pf}_0\,(0 \times 0\text{-matrix}) := 1 \tag{1.4$'$}$$

for the sake of completeness. This way, (1.4) becomes a particular case of (1.5).

[5] *Comment by DG:* The uniqueness of this natural transformation follows from what was said above about polynomials: Any natural transformation from Skew_n to Forget is uniquely determined by its value on the "generic skew-symmetric matrix"

$$X = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & X_{1n} \\ -X_{12} & 0 & X_{23} & \cdots & X_{2n} \\ -X_{13} & -X_{23} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & X_{n-1,n} \\ -X_{1n} & -X_{2n} & \cdots & -X_{n-1,n} & 0 \end{pmatrix}$$

based on exterior algebra concepts. The reason for doing this, though, is not so much to provide a rather nice and conceptual proof of the above-mentioned facts but rather to establish—in the course of doing this—a certain identity concerning the Pfaffian which can be viewed as a straightforward generalization of the famous Grassmann–Plücker identities.

To be more explicit, consider as above a skew-symmetric $(n \times n)$-matrix A of the form (1.1) with coefficients in some commutative ring R—with n even or odd—and for each subset $I \subseteq \{1, \ldots, n\}$ of cardinality, say, m let $A(I)$ denote the skew-symmetric $(m \times m)$-matrix one gets from A by eliminating all rows and columns not indexed by indices from I. Moreover, put

$$p(I) := \operatorname{Pf}(A(I)) \qquad \text{for every } I \subseteq \{1, \ldots, n\}, \tag{1.6}$$

where—as usual—$\operatorname{Pf}(A(\varnothing)) := 1$. Then the following result holds:

Theorem 1. *For any two subsets $I_1, I_2 \subseteq \{1, \ldots, n\}$ of odd cardinality and elements $i_1, \ldots, i_t \in \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_t$ and*

$$\{i_1, i_2, \ldots, i_t\} = I_1 \bigtriangleup I_2 := (I_1 \setminus I_2) \cup (I_2 \setminus I_1)$$

one has

$$\sum_{\tau=1}^{t} (-1)^{\tau} \cdot p(I_1 \bigtriangleup \{i_\tau\}) \cdot p(I_2 \bigtriangleup \{i_\tau\}) = 0. \tag{1.7}$$

Note that in the case in which A is a skew-symmetric $(3m) \times (3m)$-matrix of the form

$$A = \begin{pmatrix} & & & a_{1,m+1} & \cdots & a_{1,3m} \\ & 0 & & \vdots & & \vdots \\ & & & a_{m,m+1} & \cdots & a_{m,3m} \\ -a_{1,m+1} & \cdots & -a_{m,m+1} & & & \\ \vdots & & \vdots & & 0 & \\ -a_{1,3m} & \cdots & -a_{m,3m} & & & \end{pmatrix},$$

over the polynomial ring $\mathbb{Z}[X_{12}, X_{13}, \ldots, X_{n-1,n}]$. For the natural transformation Pf_n, this value must be a polynomial $\operatorname{Pf}_n(X)$ that squares to $\det X$. This uniquely determines this polynomial up to sign (if such a polynomial exists). For $n \equiv 0 \mod 2$, the sign is then uniquely determined by the additional requirement (1.3′). For $n \equiv 1 \mod 2$, the sign does not matter, since a classical argument shows that $\det X = 0$ (in fact, $X^T = -X$ and thus $\det\left(X^T\right) = \det(-X) = -\det X$ because n is odd, and therefore $-\det X = \det\left(X^T\right) = \det X$, so that $\det X = 0$), and the zero polynomial 0 has only one square root, which of course is 0. Thus, in both cases, the polynomial $\operatorname{Pf}_n(X)$ is uniquely determined (if it exists); and this, in turn, determines the whole natural transformation Pf_n (since any skew-symmetric matrix A is the image of X under an appropriate ring homomorphism from $\mathbb{Z}[X_{12}, X_{13}, \ldots, X_{n-1,n}]$ to R).

I_1 equals $\{1, \ldots, 2m+1\}$, and I_2 equals $\{1, \ldots, m, 2m+2, \ldots, 3m\}$, the above identity indeed yields the Grassmann–Plücker identities[6] (for more details see [W4, Sect. 7]).

The above identity was discovered by the second named author in [W1, Proposition 2.3] while trying to understand the geometric algebra associated with Δ-matroids (cf. [B1, B2, BDH, DH, W2, W3, W4]). Though the identity looks very "classical," we have sofar been unable to find any reference to it in the existing literature. While the original proof given in [W1] and based on the recursive definition of the Pfaffian outlined above is rather computational, the new proof presented here is more conceptual and, we hope, transparent—at least, once one is willing to accept[7] exterior algebra concepts as something of that sort.

The outline of the paper is as follows:

In Section 2 we present the basic exterior algebra machinery we shall need later on. In Section 3 we associate with any skew-symmetric bilinear form $\beta : M \times M \to R$, defined on some R-module M over a commutative ring R, a canonical R-linear map, also called β, from the exterior algebra $\Lambda(M)$ of M back into R such that $\beta(m \wedge m') = \beta(m, m')$ for all $m, m' \in M$, and we establish

Theorem 2. *For all $m_1, \ldots, m_n \in M$ one has*

$$\beta(m_1 \wedge \cdots \wedge m_n) = \operatorname{Pf}\big((\beta(m_i, m_j))_{i,j=1,\ldots,n}\big).$$

Here we assume the Pfaffian to be defined recursively according to (1.4) and (1.5).

In Section 4 we prove a simple identity between the values of β, namely

Theorem 3. *For all $m_1, \ldots, m_k, n_1, \ldots, n_l \in M$ one has*

$$\sum_{i=1}^{k} (-1)^i \cdot \beta(m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_k) \cdot \beta(m_i \wedge n_1 \wedge \cdots \wedge n_l)$$
$$= -\sum_{j=1}^{l} (-1)^j \cdot \beta(n_1 \wedge \cdots \wedge n_{j-1} \wedge n_{j+1} \wedge \cdots \wedge n_l) \cdot \beta(n_j \wedge m_1 \wedge \cdots \wedge m_k).$$

Moreover, we will show that Theorem 1 is a simple consequence of Theorem 3, thus establishing the primary goal of this note. Even so, we go on in Section 5 to show as promised how the exterior algebra machinery we have set up before can also be used to give a rather conceptual proof of the identity (1.2), where again we assume that the Pfaffian is defined recursively according to (1.4) and (1.5).

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[6] *Comment by DG:* This conclusion rests on the known fact that every $m \times m$-matrix B satisfies

$$\operatorname{Pf}_{2m}\left(\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}\right) = (-1)^{m(m-1)/2} \det B.$$

[7] *Comment by DG:* In the original, "except".

2 Some Exterior Algebra

As above, let R denote a commutative ring and let M be a left R-module. Adopting standard notations we denote by $\Lambda(M)$ the *exterior algebra* of M, which we assume to contain R as a subring and M as an R-submodule so that—with "$\wedge$" denoting the multiplication in the exterior algebra—$\Lambda(M)$ can be considered as a *graded R-algebra*

$$\Lambda(M) = \bigoplus_{k=0}^{\infty} \Lambda_k(M) = \Lambda_0(M) \oplus \Lambda_1(M) \oplus \cdots \oplus \Lambda_k(M) \oplus \cdots \tag{2.1}$$

with $\Lambda_0(M) = R$, $\Lambda_1(M) = M$, and $\Lambda_k(M)$ the R-submodule of $\Lambda(M)$ generated[8] by all products of the form $m_1 \wedge \cdots \wedge m_k$ with $m_1, \ldots, m_k \in M$. As usual, we also put

$$\Lambda_+(M) := \bigoplus_{k \equiv 0 \bmod 2} \Lambda_k(M) \tag{2.2}$$

and

$$\Lambda_-(M) := \bigoplus_{k \equiv 1 \bmod 2} \Lambda_k(M), \tag{2.3}$$

so that

$$\Lambda(M) = \Lambda_+(M) \oplus \Lambda_-(M) \tag{2.4}$$

can also be viewed as a bigraded algebra[9].

For $x \in \Lambda(M)$ and $k \in \mathbb{N}_0$ we denote by $x^{(k)} \in \Lambda_k(M)$ its component in $\Lambda_k(M)$, that is, the unique element in $\Lambda_k(M)$ such that $x - x^{(k)} \in \bigoplus_{i \neq k} \Lambda_i(M)$. Hence, $x^{(k)} = 0$ for almost all $k \in \mathbb{N}_0$ and $x = \bigoplus_{k=0}^{\infty} x^{(k)}$. Similarly, we put

$$x^+ := \bigoplus_{k \equiv 0 \bmod 2} x^{(k)} \in \Lambda_+(M),$$
$$x^- := \bigoplus_{k \equiv 1 \bmod 2} x^{(k)} \in \Lambda_-(M),$$

and $\bar{x} := x^+ - x^-$, so that $x = x^+ \oplus x^-$ is the decomposition of x relative to the bigraded structure (2.4) of $\Lambda(M)$, and one has $\bar{\bar{x}} = x$ as well as

$$\overline{x \wedge y} = \bar{x} \wedge \bar{y} \qquad \text{for all } x, y \in \Lambda(M).$$

Recall that the R-multilinear map

$$\lambda_k : M^k \to \Lambda_k(M) : (m_1, \ldots, m_k) \mapsto m_1 \wedge \cdots \wedge m_k$$

satisfies $\lambda_k(m_1, \ldots, m_k) = 0$ whenever $m_i = m_{i+1}$ for some $i \in \{1, \ldots, k-1\}$, and that for every other R-multilinear map $\lambda : M^k \to N$ from M^k into some R-module N

[8] *Comment by DG:* Removed the comma before "generated" to make this less ambiguous.
[9] *Comment by DG:* The word "bigraded" means "$\mathbb{Z}_2$-graded" in this paper.

with $\lambda(m_1,\dots,m_k)=0$ whenever $m_i=m_{i+1}$ for some $i\in\{1,\dots,k-1\}$ there exists a unique *induced* R-linear map $\widehat{\lambda}:\Lambda_k(M)\to N$ such that the diagram

$$\begin{tikzcd} & M^k \arrow[dl, "\lambda_k"'] \arrow[dr, "\lambda"] & \\ \Lambda_k(M) \arrow[rr, dashed, "\widehat{\lambda}"'] & & N \end{tikzcd}$$

commutes. Equivalently, given an R-bilinear map $\lambda: M\times\Lambda_{k-1}(M)\to N$, there exists a unique induced R-linear map $\widehat{\lambda}:\Lambda_k(M)\to N$ with $\lambda(m,x)=\widehat{\lambda}(m\wedge x)$ for all $m\in M$ and $x\in\Lambda_{k-1}(M)$ if and only if $\lambda(m,m\wedge x)=0$ for all $m\in M$ and $x\in\Lambda_{k-2}(M)$. [10] This observation is the key for many recursive constructions of maps defined on $\Lambda(M)$: For instance, if $\varphi: M\to\Lambda(M)$ is an R-linear map, then there exists a unique R-linear extension $\widehat{\varphi}:\Lambda(M)\to\Lambda(M)$ satisfying $\widehat{\varphi}(R)=\{0\}$ and

$$\widehat{\varphi}(m\wedge x)=\varphi(m)\wedge x-m\wedge\widehat{\varphi}(x)\quad\text{for all } m\in M \text{ and } x\in\Lambda(M) \tag{2.5}$$

if and only if

$$\varphi(m)\wedge m=m\wedge\varphi(m)\quad\text{for all } m\in M. \tag{2.6}$$

Indeed, putting $x:=m$ in (2.5) yields that (2.6) must hold if φ can be extended to some $\widehat{\varphi}:\Lambda(M)\to\Lambda(M)$ such that (2.5) is satisfied. Vice versa, if (2.6) holds then $\widehat{\varphi}$ can be constructed recursively on each $\Lambda_k(M)$, $k\geq 2$, so that (2.5) holds, because, independent of the actual value of $\widehat{\varphi}(x)$, (2.6) implies

$$\varphi(m)\wedge(m\wedge x)-m\wedge\big(\varphi(m)\wedge x-m\wedge\widehat{\varphi}(x)\big)=0$$

for all $m\in M$ and $x\in\Lambda(M)$.

Note that (2.5) implies

$$\begin{aligned}\widehat{\varphi}(m_1\wedge\cdots\wedge m_k)&=\sum_{i=1}^{k}(-1)^{i-1}m_1\wedge\cdots\wedge m_{i-1}\wedge\varphi(m_i)\wedge m_{i+1}\wedge\cdots\wedge m_k\\&=\sum_{i=1}^{k}\overline{m_1\wedge\cdots\wedge m_{i-1}}\wedge\varphi(m_i)\wedge m_{i+1}\wedge\cdots\wedge m_k\end{aligned} \tag{2.7}$$

for all $m_1,\dots,m_k\in M$ and therefore also

$$\widehat{\varphi}(x\wedge y)=\widehat{\varphi}(x)\wedge y+\bar{x}\wedge\widehat{\varphi}(y)\quad\text{for all } x,y\in\Lambda(M). \tag{2.8}$$

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[10] *Comment by DG:* Indeed, the "only if" part is clear (since $m\wedge(m\wedge x)=0$), whereas the "if" part is an easy consequence of the previous sentence: If $\lambda(m,m\wedge x)=0$ for all $m\in M$ and $x\in\Lambda_{k-2}(M)$, then the R-multilinear map $\lambda': M^k\to N$ sending each $(m_1,\dots,m_k)\in M^k$ to $\lambda(m_1,m_2\wedge\cdots\wedge m_k)\in N$ satisfies the property that $\lambda'(m_1,\dots,m_k)=0$ whenever $m_i=m_{i+1}$ for some $i\in\{1,\dots,k-1\}$ (this is clear for $i>1$, and follows from the $\lambda(m,m\wedge x)=0$ assumption when $i=1$), and thus induces a unique R-linear map $\widehat{\lambda'}:\Lambda_k(M)\to N$; this latter map $\widehat{\lambda'}$ is precisely the $\widehat{\lambda}$ that we are looking for.

[11]

Note also that for any two elements $r_1, r_2 \in R$ and R-linear maps

$$\varphi_1, \varphi_2 : M \to \Lambda(M)$$

satisfying (2.6), R-linearity of the formulas (2.5) and (2.6) ensures also that $\varphi := r_1 \cdot \varphi_1 + r_2 \cdot \varphi_2 : M \to \Lambda(M)$ satisfies (2.6) and that

$$\widehat{\varphi} = r_1 \cdot \widehat{\varphi_1} + r_2 \cdot \widehat{\varphi_2}. \tag{2.9}$$

Finally, note that (2.6) holds in particular for all R-linear maps $\varphi : M \to R \subseteq \Lambda(M)$, so definitely all these maps can be extended as described above. For such maps φ we evidently have

$$\widehat{\varphi}(\Lambda_k(M)) \subseteq \Lambda_{k-1}(M) \quad \text{for all } k \in \mathbb{N}. \tag{2.10}$$

They also satisfy

$$\widehat{\varphi}(\varphi(m)) = 0 \qquad \text{and} \qquad \overline{\varphi(m)} = \varphi(m) \quad \text{for each } m \in M. \tag{2.11}$$

These conditions in turn imply[12]

$$\widehat{\varphi}(\widehat{\varphi}(x)) = 0 \quad \text{for all } x \in \Lambda(M), \tag{2.12}$$

[13] as follows directly by induction on

$$\min\left\{k \in \mathbb{N}_0 \mid x^{(\ell)} = 0 \text{ for all } \ell > k\right\}$$

[14] in view of the formula

$$\begin{aligned} \widehat{\varphi}(\widehat{\varphi}(m \wedge x)) &= \widehat{\varphi}\big(\varphi(m) \wedge x - m \wedge \widehat{\varphi}(x)\big) \\ &= \widehat{\varphi}(\varphi(m) \wedge x) - \widehat{\varphi}(m \wedge \widehat{\varphi}(x)) \\ &= \widehat{\varphi}(\varphi(m)) \wedge x + \overline{\varphi(m)} \wedge \widehat{\varphi}(x) \\ &\qquad - \varphi(m) \wedge \widehat{\varphi}(x) + m \wedge \widehat{\varphi}(\widehat{\varphi}(x)) \end{aligned} \tag{2.13}$$

[15] for all $m \in M$ and $x \in \Lambda(M)$.

[11] *Comment by DG:* A linear map $f : \Lambda(M) \to \Lambda(M)$ satisfying $f(x \wedge y) = f(x) \wedge y + \bar{x} \wedge f(y)$ for all $x, y \in \Lambda(M)$ is sometimes called a *superderivation* of $\Lambda(M)$. Thus, (2.8) shows that $\widehat{\varphi}$ is a superderivation of $\Lambda(M)$.

[12] *Comment by DG:* This is all under the condition that $\varphi : M \to R$.

[13] *Comment by DG:* Thus, $\widehat{\varphi}$ can be used to define a chain complex

$$\cdots \xrightarrow{\widehat{\varphi}} \Lambda_2(M) \xrightarrow{\widehat{\varphi}} \Lambda_1(M) \xrightarrow{\widehat{\varphi}} \Lambda_0(M) \longrightarrow 0.$$

[14] *Comment by DG:* Typo corrected in the above display, and min(...) replaced by min{...}.

[15] *Comment by DG:* Corrected the sign before the $m \wedge \widehat{\varphi}(\widehat{\varphi}(x))$ term (it was a $-$ in the original).

Hence, for R-linear maps[16] $\varphi_1, \varphi_2 : M \to R$ and $\varphi := \varphi_1 + \varphi_2$ we have

$$\begin{aligned} 0 &= \widehat{\varphi} \circ \widehat{\varphi} = (\widehat{\varphi_1} + \widehat{\varphi_2}) \circ (\widehat{\varphi_1} + \widehat{\varphi_2}) \\ &= \widehat{\varphi_1} \circ \widehat{\varphi_1} + \widehat{\varphi_1} \circ \widehat{\varphi_2} + \widehat{\varphi_2} \circ \widehat{\varphi_1} + \widehat{\varphi_2} \circ \widehat{\varphi_2} \\ &= \widehat{\varphi_1} \circ \widehat{\varphi_2} + \widehat{\varphi_2} \circ \widehat{\varphi_1} \end{aligned} \tag{2.14}$$

and therefore

$$\widehat{\varphi_1} \circ \widehat{\varphi_2} = -\widehat{\varphi_2} \circ \widehat{\varphi_1}. \tag{2.15}$$

Similarly, (2.7) implies by induction relative to k that for $m_1, \ldots, m_k \in M$ and for R-linear maps[17] $\varphi_1, \ldots, \varphi_k : M \to R$ one has

$$\widehat{\varphi_k} \circ \cdots \circ \widehat{\varphi_1}(m_1 \wedge \cdots \wedge m_k) = \det\big((\varphi_i(m_j))_{i,j=1,\ldots,k}\big) \tag{2.16}$$

and therefore, using (2.15),

$$\widehat{\varphi_1} \circ \cdots \circ \widehat{\varphi_k}(m_1 \wedge \cdots \wedge m_k) = (-1)^{k(k-1)/2} \det\big((\varphi_i(m_j))_{i,j=1,\ldots,k}\big). \tag{2.17}$$

3 Exterior Algebra and Skew-Symmetric Bilinear Forms

Continuing with our notations let us now assume that in addition to R and M we are given a skew-symmetric R-bilinear form[18] $\beta : M \times M \to R$ or—equivalently—an R-linear map $\beta : \Lambda_2(M) \to R$. As $\beta(m, \cdot) : M \to R \subseteq \Lambda(M) : m' \mapsto \beta(m, m')$ is R-linear, we can apply Section 2 to extend this map to an R-linear map $\widehat{\beta}(m, \cdot) : \Lambda(M) \to \Lambda(M) : x \mapsto \widehat{\beta}(m, x)$ satisfying[19]

$$\widehat{\beta}(m, m' \wedge x) = \beta(m, m') \cdot x - m' \wedge \widehat{\beta}(m, x) \tag{3.1}$$

for all $m, m' \in M$, $x \in \Lambda(M)$, as well as[20]

$$\widehat{\beta}(m, \widehat{\varphi}(x)) = -\widehat{\varphi}(\widehat{\beta}(m, x)) \tag{3.2}$$

[16] *Comment by DG:* Added "R-linear maps".

[17] *Comment by DG:* Added "for R-linear maps".

[18] *Comment by DG:* A *skew-symmetric R-bilinear form* (also known as an *alternating R-bilinear form*) means an R-bilinear form $\beta : M \times M \to R$ that satisfies $\beta(m, m) = 0$ for each $m \in M$. This automatically implies $\beta(m, n) = -\beta(n, m)$ for all $m, n \in M$. Each skew-symmetric R-bilinear form $\beta : M \times M \to R$ can be written as $\beta' \circ \lambda_2$ for a unique R-linear map $\beta' : \Lambda_2(M) \to R$. By abuse of notation, we denote the latter β' by β as well.

[19] *Comment by DG:* In the notations of Section 2, this map $\widehat{\beta}(m, \cdot)$ is $\widehat{\varphi}$, where φ is the R-linear map $\beta(m, \cdot)$.

[20] *Comment by DG:* Specifically: the equalities (3.2), (3.3), (3.4) and (3.5) follow from (2.15), (2.12), (2.10), (2.9), respectively.

[21] for all $m \in M$, R-linear maps $\varphi : M \to R$, and $x \in \Lambda(M)$,

$$\widehat{\beta}(m, \widehat{\beta}(m, x)) = 0, \tag{3.3}$$

$$\widehat{\beta}(m, \Lambda_k(M)) \subseteq \Lambda_{k-1}(M), \tag{3.4}$$

and

$$\widehat{\beta}(r_1 \cdot m_1 + r_2 \cdot m_2, x) = r_1 \cdot \widehat{\beta}(m_1, x) + r_2 \cdot \widehat{\beta}(m_2, x) \quad \text{for all } m, m_1, m_2 \in M,\ x \in \Lambda(M),\ r_1, r_2 \in R. \tag{3.5}$$

In consequence, we can extend $\beta : \Lambda_2(M) \to R$, considered as an R-linear form defined on $\Lambda_2(M)$, recursively to a uniquely determined R-linear form $\beta : \Lambda(M) \to R$ satisfying $\beta(r) = r$ for all $r \in R$ as well as

$$\beta(m \wedge x) = \beta(\widehat{\beta}(m, x)) \quad \text{for all } m \in M \text{ and } x \in \Lambda(M) \tag{3.6}$$

in view of the formula[22]

$$\widehat{\beta}(m, m \wedge x) = \beta(m, m) \wedge x - m \wedge \widehat{\beta}(m, x) = -m \wedge \widehat{\beta}(m, x) \tag{3.7}$$

[21] *Comment by DG:* Typo corrected ($\beta(m, x)$ replaced by $\widehat{\beta}(m, x)$).

[22] *Comment by DG:* Let me explain the proof of the well-definedness of this form $\beta : \Lambda(M) \to R$ in a bit more detail:

First, we define an R-linear form $\beta_k : \Lambda_k(M) \to R$ for each $k \geq 0$ by recursion on k, starting with $\beta_0 = \mathrm{id}$ and $\beta_1 = 0$. For each $k > 1$, if we assume that $\beta_{k-2} : \Lambda_{k-2}(M) \to R$ has already been defined, then we define $\beta_k : \Lambda_k(M) \to R$ by

$$\beta_k(m \wedge x) = \beta_{k-2}(\widehat{\beta}(m, x)) \quad \text{for all } m \in M \text{ and } x \in \Lambda_{k-1}(M); \tag{3.6$'$}$$

this is well-defined for the following reason: Recall from Section 2 that given an R-bilinear map $\lambda : M \times \Lambda_{k-1}(M) \to N$, there exists a unique induced R-linear map $\widehat{\lambda} : \Lambda_k(M) \to N$ with $\lambda(m, x) = \widehat{\lambda}(m \wedge x)$ for all $m \in M$ and $x \in \Lambda_{k-1}(M)$ if and only if $\lambda(m, m \wedge x) = 0$ for all $m \in M$ and $x \in \Lambda_{k-2}(M)$. We apply this construction to $N = R$ and to the R-bilinear map $\lambda : M \times \Lambda_{k-1}(M) \to R$ given by $\lambda(m, x) = \beta_{k-2}(\widehat{\beta}(m, x))$ for all $m \in M$ and $x \in \Lambda_{k-1}(M)$, which is well-defined by (3.4), and which satisfies $\lambda(m, m \wedge x) = 0$ for all $m \in M$ and $x \in \Lambda_{k-2}(M)$ because

$$\begin{aligned}
\lambda(m, m \wedge x) &= \beta_{k-2}(\widehat{\beta}(m, m \wedge x)) = \beta_{k-2}(-m \wedge \widehat{\beta}(m, x)) \\
&\qquad \Big(\text{since (3.1) yields } \widehat{\beta}(m, m \wedge x) = \beta(m, m) \cdot x - m \wedge \widehat{\beta}(m, x) = -m \wedge \widehat{\beta}(m, x)\Big) \\
&= -\beta_{k-2}(m \wedge \widehat{\beta}(m, x)) = -\beta_{k-4}(\widehat{\beta}(m, \widehat{\beta}(m, x))) \\
&\qquad \text{(by (3.6$'$) for } k-2 \text{ instead of } k \text{ (which holds by the inductive hypothesis))} \\
&= -\beta_{k-4}(0) \qquad \text{(by (3.3))} \\
&= 0
\end{aligned}$$

(here we have assumed $k > 3$ in the third-to-last equality sign; the case $k \leq 3$ can be done by hand). Thus, we obtain a unique induced R-linear map $\widehat{\lambda} : \Lambda_k(M) \to R$ with $\widehat{\lambda}(m \wedge x) = \lambda(m, x) = \beta_{k-2}(\widehat{\beta}(m, x))$ for all $m \in M$ and $x \in \Lambda_{k-1}(M)$. We denote this map $\widehat{\lambda}$ by β_k; then the equality just stated is (3.6$'$), and so the recursive definition of β_k is complete.

Now, the R-linear form $\beta : \Lambda(M) \to R$ is the sum of the forms $\beta_k : \Lambda_k(M) \to R$ over all $k \geq 0$ (that is, it sends each $x \in \Lambda_k(M)$ to $\beta_k(x)$ for each $k \geq 0$). By linearity, its property (3.6) follows from (3.6$'$).

which together with (3.3) and (3.4) implies that in the case β has been defined on $\bigoplus_{i\le k}\Lambda_i(M)$ such that[23] (3.6) holds, the resulting map

$$M\times\Lambda_k(M)\to R:(m,x)\mapsto\beta(\widehat{\beta}(m,x))$$

satisfies

$$\begin{aligned}\beta(\widehat{\beta}(m,m\wedge x))&=\beta(-m\wedge\widehat{\beta}(m,x))=-\beta(m\wedge\widehat{\beta}(m,x))\\&=-\beta(\widehat{\beta}(m,\widehat{\beta}(m,x)))=0\end{aligned}$$

for all $m\in M$ and $x\in\Lambda_{k-1}(M)$ [24].

Note that (3.6) implies that $\beta(m)=0$ for all $m\in M$, and thus $\beta(\Lambda_-(M))=0$, because, using (3.4), it implies easily by induction on k that $\beta(x)=0$ for all x in $\Lambda_k(M)$, $k\equiv 1\mod 2$. Thus β "lives" essentially on $\Lambda_+(M)$.

Obviously, for $m_1,m_2,\ldots,m_n\in M$ one has

$$\beta(m_1\wedge m_2)=\beta(m_1,m_2)\tag{3.8}$$

and

$$\begin{aligned}\beta(m_1\wedge\cdots\wedge m_n)&=\beta(\widehat{\beta}(m_1,m_2\wedge\cdots\wedge m_n))\\&=\beta\left(\sum_{j=2}^{n}(-1)^j\cdot\beta(m_1,m_j)\cdot m_2\wedge\cdots\wedge m_{j-1}\wedge m_{j+1}\wedge\cdots\wedge m_n\right)\\&=\sum_{j=2}^{n}(-1)^j\cdot\beta(m_1,m_j)\cdot\beta(m_2\wedge\cdots\wedge m_{j-1}\wedge m_{j+1}\wedge\cdots\wedge m_n)\end{aligned}\tag{3.9}$$

for $n\ge 2$, [25] so if the Pfaffian is defined recursively according to (1.4) and (1.5), we indeed have

Theorem 2. *For all $m_1,\ldots,m_n\in M$ one has*

$$\beta(m_1\wedge\cdots\wedge m_n)=\mathrm{Pf}_n^R\big((\beta(m_i,m_j))_{i,j=1,\ldots,n}\big).\tag{3.10}$$

4 The Pfaffian Identity

Continuing with our notations we now prove

[23] *Comment by DG:* "$\bigoplus_{i\le k}\Lambda_k(M)$" corrected to "$\bigoplus_{i\le k}\Lambda_i(M)$".

[24] *Comment by DG:* In the original, this was "$x\in\Lambda_k(M)$", which would not fit with the domain of the map.

[25] *Comment by DG:* Added "for $n\ge 2$".

Theorem 3. *With R, M, and $\beta : \Lambda_2(M) \to R$ as above we have*

$$\sum_{i=1}^{k}(-1)^i \cdot \beta(m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_k) \cdot \beta(m_i \wedge n_1 \wedge \cdots \wedge n_l)$$
$$= -\sum_{j=1}^{l}(-1)^j \cdot \beta(n_1 \wedge \cdots \wedge n_{j-1} \wedge n_{j+1} \wedge \cdots \wedge n_l) \cdot \beta(n_j \wedge m_1 \wedge \cdots \wedge m_k)$$

for all $m_1, \ldots, m_k, n_1, \ldots, n_l \in M$.

Proof. This formula follows immediately from the equations

$$\begin{aligned}\beta(m_i \wedge n_1 \wedge \cdots \wedge n_l) &= \beta(\widehat{\beta}(m_i, n_1 \wedge \cdots \wedge n_l)) \\ &= \beta\left(\sum_{j=1}^{l}(-1)^{j-1} \cdot \beta(m_i, n_j) \cdot n_1 \wedge \cdots \wedge n_{j-1} \wedge n_{j+1} \wedge \cdots \wedge n_l\right) \\ &= \sum_{j=1}^{l}(-1)^{j-1} \cdot \beta(m_i, n_j) \cdot \beta(n_1 \wedge \cdots \wedge n_{j-1} \wedge n_{j+1} \wedge \cdots \wedge n_l)\end{aligned}$$

for $1 \leq i \leq k$ and

$$\beta(n_j \wedge m_1 \wedge \cdots \wedge m_k) = \sum_{i=1}^{k}(-1)^{i-1} \cdot \beta(n_j, m_i) \cdot \beta(m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_k)$$

for $1 \leq j \leq l$. $\square$

Now assume that $x_1, \ldots, x_n \in M$ and put

$$p(I) := \beta(x_{a_1} \wedge \cdots \wedge x_{a_k})$$

for any subset $I = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\}$ with $a_1 < a_2 < \cdots < a_k$ [26].

Next assume that I_1, I_2 are two arbitrary subsets of the index set $I_0 := \{1, \ldots, n\}$. Write $I_1 = \{a_1, \ldots, a_k\}$, $I_2 = \{b_1, \ldots, b_l\}$, where $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_l$. Put $m_i := x_{a_i}$ for $i = 1, \ldots, k$ and $n_j := x_{b_j}$ for $j = 1, \ldots, l$. Then Theorem 3 implies

$$\sum_{i=1}^{k}(-1)^i \cdot \beta(x_{a_1} \wedge \cdots \wedge x_{a_{i-1}} \wedge x_{a_{i+1}} \wedge \cdots \wedge x_{a_k}) \cdot \beta(x_{a_i} \wedge x_{b_1} \wedge \cdots \wedge x_{b_l})$$
$$+ \sum_{j=1}^{l}(-1)^j \cdot \beta(x_{b_1} \wedge \cdots \wedge x_{b_{j-1}} \wedge x_{b_{j+1}} \wedge \cdots \wedge x_{b_l}) \cdot \beta(x_{b_j} \wedge x_{a_1} \wedge \cdots \wedge x_{a_k}) = 0.$$

[26] *Comment by DG:* In the original: "for $I = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\}$ and $a_1 < a_2 < \cdots < a_k$"; edited for additional clarity.

Note that for all $i = 1, \ldots, k$ one has

$$\beta(x_{a_i} \wedge x_{b_1} \wedge \cdots \wedge x_{b_l}) = \begin{cases} 0, & \text{if } a_i \in I_2, \\ (-1)^{\#\{j|b_j<a_i\}} \cdot p(I_2 \bigtriangleup \{a_i\}), & \text{otherwise,} \end{cases}$$

and for all $j = 1, \ldots, l$ one has similarly

$$\beta(x_{b_j} \wedge x_{a_1} \wedge \cdots \wedge x_{a_k}) = \begin{cases} 0, & \text{if } b_j \in I_1, \\ (-1)^{\#\{i|a_i<b_j\}} \cdot p(I_1 \bigtriangleup \{b_j\}), & \text{otherwise.} \end{cases}$$

Finally, note that with $I_1 \bigtriangleup I_2 = \{c_1, \ldots, c_t\}$, $c_1 < c_2 < \cdots < c_t$, and $a_i = c_\tau \in I_1 \setminus I_2$ for some $i \in \{1, \ldots, k\}$ and $\tau \in \{1, \ldots, t\}$ one has

$$(-1)^i \cdot (-1)^{\#\{j|b_j<a_i\}} = (-1)^\tau,$$

because

$$\begin{aligned}
\tau &= \#\{c \in I_1 \bigtriangleup I_2 \mid c \leq a_i\} \\
&= \#\{c \in I_1 \setminus I_2 \mid c \leq a_i\} + \#\{c \in I_2 \setminus I_1 \mid c \leq a_i\} \\
&\equiv \#\{c \in I_1 \setminus I_2 \mid c \leq a_i\} + \#\{c \in I_1 \cap I_2 \mid c \leq a_i\} \\
&\qquad + \#\{c \in I_2 \setminus I_1 \mid c \leq a_i\} + \#\{c \in I_2 \cap I_1 \mid c \leq a_i\} \\
&= \#\{c \in I_1 \mid c \leq a_i\} + \#\{c \in I_2 \mid c \leq a_i\} \\
&= i + \#\{j \mid b_j < a_i\} \mod 2.
\end{aligned}$$

Similarly, for $b_j = c_\tau \in I_2 \setminus I_1$ for some $j \in \{1, \ldots, l\}$ and $\tau \in \{1, \ldots, t\}$ one has

$$(-1)^j \cdot (-1)^{\#\{i|a_i<b_j\}} = (-1)^\tau.$$

Hence, Theorem 3 indeed yields that

$$\sum_{\tau=1}^{t} (-1)^\tau \cdot p(I_1 \bigtriangleup \{c_\tau\}) \cdot p(I_2 \bigtriangleup \{c_\tau\}) = 0.$$

[27] As this is the identity (1.7) for $A := (\beta(x_i, x_j))_{i,j=1,\ldots,n}$ and as any skew-symmetric $(n \times n)$-matrix $A \in \mathrm{Skew}_n(R)$ is of this form for some M, β, and $x_1, \ldots, x_n \in M$, Theorem 1 follows.

5 The Pfaffian and the Determinant

In this section we use our exterior algebra formalism to prove

[27] *Comment by DG:* The "c_τ"s in the preceding equality were called "i_τ"s in the original; fixed for consistency with the equality $I_1 \bigtriangleup I_2 = \{c_1, \ldots, c_t\}$.

Theorem 4 (J. F. Pfaff). *For R, M, β as above and $m_1, \dots, m_n \in M$ one has*

$$\beta(m_1 \wedge \cdots \wedge m_n)^2 = \det\big((\beta(m_i, m_j))_{i,j=1,\dots,n}\big). \tag{5.1}$$

Note that Theorem 2 and Theorem 4 yield the classical relation (1.2).

To prove Theorem 4 it is convenient to define an R-linear map $I_\beta : \Lambda(M) \to \Lambda(M)$ recursively by

$$I_\beta(r) := r \qquad \text{for } r \in R, \tag{5.2a}$$

$$I_\beta(m \wedge x) := m \wedge I_\beta(x) + \widehat{\beta}(m, I_\beta(x)) \qquad \text{for } m \in M, \quad x \in \Lambda(M). \tag{5.2b}$$

Since (3.1) and (3.3) imply

$$\begin{aligned} m \wedge \big(m \wedge I_\beta(x) + \widehat{\beta}(m, I_\beta(x))\big) + \widehat{\beta}\big(m, m \wedge I_\beta(x) + \widehat{\beta}(m, I_\beta(x))\big) \\ = m \wedge \widehat{\beta}(m, I_\beta(x)) + \beta(m,m) \cdot I_\beta(x) - m \wedge \widehat{\beta}(m, I_\beta(x)) + \widehat{\beta}(m, \widehat{\beta}(m, I_\beta(x))) = 0 \end{aligned}$$

—independent of the actual value of $I_\beta(x)$—the map I_β is indeed well defined.

In order to study I_β we first observe that for every R-linear[28] $\varphi : M \to R$ and its canonical extension $\widehat{\varphi} : \Lambda(M) \to \Lambda(M)$ one has

$$I_\beta(\widehat{\varphi}(x)) = \widehat{\varphi}(I_\beta(x)) \qquad \text{for all } x \in \Lambda(M). \tag{5.3}$$

[29] If $x \in R$, then both sides in (5.3) vanish; thus by induction it suffices to observe that by (2.5), (5.2b), and (3.2) the assumption $I_\beta(\widehat{\varphi}(x)) = \widehat{\varphi}(I_\beta(x))$ implies

$$\begin{aligned} I_\beta(\widehat{\varphi}(m \wedge x)) &= I_\beta(\varphi(m) \cdot x - m \wedge \widehat{\varphi}(x)) \\ &= \varphi(m) \cdot I_\beta(x) - m \wedge I_\beta(\widehat{\varphi}(x)) - \widehat{\beta}(m, I_\beta(\widehat{\varphi}(x))) \\ &= \varphi(m) \cdot I_\beta(x) - m \wedge \widehat{\varphi}(I_\beta(x)) - \widehat{\beta}(m, \widehat{\varphi}(I_\beta(x))) \\ &= \widehat{\varphi}(m \wedge I_\beta(x)) + \widehat{\varphi}(\widehat{\beta}(m, I_\beta(x))) \\ &= \widehat{\varphi}(I_\beta(m \wedge x)). \end{aligned}$$

Next we prove that with $P_0 : \Lambda(M) \twoheadrightarrow \Lambda_0(M) : x \mapsto x^{(0)}$ denoting the canonical projection from $\Lambda(M)$ onto $\Lambda_0(M) = R$ one has

$$P_0(I_\beta(x)) = \beta(x) \qquad \text{for all } x \in \Lambda(M). \tag{5.4}$$

For $x \in R$ we have indeed $P_0(I_\beta(x)) = x = \beta(x)$. By linearity we may now assume that $x = m_1 \wedge \cdots \wedge m_k$ for some $m_1, \dots, m_k \in M$ and that (5.4) is true for all $x' \in \bigoplus_{i=0}^{k-1} \Lambda_i(M)$. Then we get, by (5.2b), (5.3)[30], (3.4), and (3.6),

$$\begin{aligned} P_0(I_\beta(m_1 \wedge \cdots \wedge m_k)) &= P_0(\widehat{\beta}(m_1, I_\beta(m_2 \wedge \cdots \wedge m_k))) \\ &= P_0(I_\beta(\widehat{\beta}(m_1, m_2 \wedge \cdots \wedge m_k))) \\ &= \beta(\widehat{\beta}(m_1, m_2 \wedge \cdots \wedge m_k)) \\ &= \beta(m_1 \wedge \cdots \wedge m_k), \end{aligned}$$

[28] *Comment by DG:* Added "R-linear".

[29] *Comment by DG:* Removed "$m \in M$ and".

[30] *Comment by DG:* Specifically, (5.3) is being applied to $\varphi = \beta(m_1, \cdot)$ to obtain $I_\beta(\widehat{\beta}(m_1, x)) = \widehat{\beta}(m_1, I_\beta(x))$ for all $x \in \Lambda(M)$.

so (5.4) follows by induction.

Now let ${}^{\mathrm{t}}\beta : M \times M \to R$ denote the skew-symmetric R-bilinear form given by

$$ {}^{\mathrm{t}}\beta(m_1, m_2) := \beta(m_2, m_1) = -\beta(m_1, m_2) \qquad \text{for } m_1, m_2 \in M. $$

The induced R-linear form ${}^{\mathrm{t}}\beta : \Lambda(M) \to R$ satisfies ${}^{\mathrm{t}}\beta(x) = 0$ for all $x \in \Lambda_-(M)$, and (3.9) implies by induction for $k \equiv 0 \mod 2$ and all $m_1, \dots, m_k \in M$:

$$ \begin{aligned} {}^{\mathrm{t}}\beta(m_1 \wedge \cdots \wedge m_k) &= (-1)^{k\cdot(k-1)/2} \cdot \beta(m_1 \wedge \cdots \wedge m_k) \\ &= \begin{cases} \beta(m_1 \wedge \cdots \wedge m_k), & \text{for } k \equiv 0 \mod 4, \\ -\beta(m_1 \wedge \cdots \wedge m_k), & \text{for } k \equiv 2 \mod 4. \end{cases} \end{aligned} \tag{5.5} $$

Theorem 4 now follows directly from (5.4) and (5.5)[31] once we have proved that for all $k \in \mathbb{N}$ and $m_1, \dots, m_k \in M$ the following much stronger identity holds:

Theorem 4'. *For R, M, β as above and $m_1, \dots, m_k \in M$ one has*

$$ I_\beta(m_1 \wedge \cdots \wedge m_k) \wedge I_{{}^{\mathrm{t}}\beta}(m_1 \wedge \cdots \wedge m_k) = (-1)^{k\cdot(k-1)/2} \cdot \det\bigl((\beta(m_i, m_j))_{i,j=1,\dots,k}\bigr). $$

Proof. Note first that (2.8) and (2.9) imply

$$ \widehat{\beta}(m, x \wedge y) = \widehat{\beta}(m, x) \wedge y + \bar{x} \wedge \widehat{\beta}(m, y) \tag{5.6} $$

and

$$ \widehat{{}^{\mathrm{t}}\beta}(m, x) = -\widehat{\beta}(m, x) \tag{5.7} $$

for all $m \in M$ and $x, y \in \Lambda(M)$. Now assume that $m \in M$ and $y \in \Lambda(M)$ satisfy $m \wedge y = 0$. Then $I_{{}^{\mathrm{t}}\beta}(m \wedge y) = 0$, so that (5.2b) and (5.7) yield[32]

$$ m \wedge I_{{}^{\mathrm{t}}\beta}(y) = -\widehat{{}^{\mathrm{t}}\beta}(m, I_{{}^{\mathrm{t}}\beta}(y)) = \widehat{\beta}(m, I_{{}^{\mathrm{t}}\beta}(y)). $$

Hence, for any $x \in \Lambda(M)$ we have[33]

$$ \begin{aligned} I_\beta(m \wedge x) \wedge I_{{}^{\mathrm{t}}\beta}(y) &= \bigl(m \wedge I_\beta(x) + \widehat{\beta}(m, I_\beta(x))\bigr) \wedge I_{{}^{\mathrm{t}}\beta}(y) \\ &= \widehat{\beta}(m, I_\beta(x)) \wedge I_{{}^{\mathrm{t}}\beta}(y) + \overline{I_\beta(x)} \wedge m \wedge I_{{}^{\mathrm{t}}\beta}(y) \\ &= \widehat{\beta}(m, I_\beta(x)) \wedge I_{{}^{\mathrm{t}}\beta}(y) + \overline{I_\beta(x)} \wedge \widehat{\beta}(m, I_{{}^{\mathrm{t}}\beta}(y)) \end{aligned} $$

and thus by (5.6)[34]

$$ I_\beta(m \wedge x) \wedge I_{{}^{\mathrm{t}}\beta}(y) = \widehat{\beta}\bigl(m, I_\beta(x) \wedge I_{{}^{\mathrm{t}}\beta}(y)\bigr). \tag{5.8} $$

[31] *Comment by DG:* ... and from the fact that the projection P_0 is a ring homomorphism

[32] *Comment by DG:* Added some more details to this sentence.

[33] *Comment by DG:* The second equality sign in the following computation relies on the (easy) fact that every $m \in M$ and $z \in \Lambda(M)$ satisfy $m \wedge z = \overline{z} \wedge m$.

[34] *Comment by DG:* We are still assuming $m \wedge y = 0$ here.

It follows by induction that for $m_1, \dots, m_k \in M$, $y \in \Lambda(M)$, and $\varphi_i := \beta(m_i, \cdot)$ $(i = 1, \dots, k)$ [35] one has[36]

$$\begin{aligned}
&I_\beta(m_1 \wedge \cdots \wedge m_k) \wedge I_{{}^t\beta}(m_1 \wedge \cdots \wedge m_k \wedge y) \\
&\quad = \widehat{\beta}\big(m_1, I_\beta(m_2 \wedge \cdots \wedge m_k) \wedge I_{{}^t\beta}(m_1 \wedge \cdots \wedge m_k \wedge y)\big) \\
&\quad = \widehat{\varphi_1}\big(I_\beta(m_2 \wedge \cdots \wedge m_k) \wedge I_{{}^t\beta}(m_1 \wedge \cdots \wedge m_k \wedge y)\big) \\
&\quad = \cdots = \widehat{\varphi_1} \circ \cdots \circ \widehat{\varphi_k}\big(I_{{}^t\beta}(m_1 \wedge \cdots \wedge m_k \wedge y)\big) \\
&\quad = I_{{}^t\beta}\big(\widehat{\varphi_1} \circ \cdots \circ \widehat{\varphi_k}(m_1 \wedge \cdots \wedge m_k \wedge y)\big)
\end{aligned}$$

[37] and therefore, putting $y := 1$ and using (2.17),

$$\begin{aligned}
&I_\beta(m_1 \wedge \cdots \wedge m_k) \wedge I_{{}^t\beta}(m_1 \wedge \cdots \wedge m_k) \\
&\quad = I_{{}^t\beta}\big(\widehat{\varphi_1} \circ \cdots \circ \widehat{\varphi_k}(m_1 \wedge \cdots \wedge m_k)\big) \\
&\quad = I_{{}^t\beta}\left((-1)^{k(k-1)/2} \det\big((\varphi_i(m_j))_{i,j=1,\dots,k}\big)\right) \\
&\quad = (-1)^{k(k-1)/2} \det\big((\beta(m_i, m_j))_{i,j=1,\dots,k}\big). \qquad \square
\end{aligned}$$

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[35] *Comment by DG:* In the original, "$\varphi_i := \beta(m_i, \cdot)$" was "$\varphi_i := \beta(m_i, -)$". The change was made to ensure consistency with the "$\widehat{\beta}(m, \cdot)$" earlier on.

[36] *Comment by DG:* In the following computation, (5.8) is applied first to m_1, $m_2 \wedge \cdots \wedge m_k$ and $m_1 \wedge \cdots \wedge m_k \wedge y$ instead of m, x and y; then to m_2, $m_3 \wedge \cdots \wedge m_k$ and $m_1 \wedge \cdots \wedge m_k \wedge y$ instead of m, x and y; then to m_3, $m_4 \wedge \cdots \wedge m_k$ and $m_1 \wedge \cdots \wedge m_k \wedge y$ instead of m, x and y; and so on. Each time, we are using $m_i \wedge (m_1 \wedge \cdots \wedge m_k \wedge y) = 0$.

[37] *Comment by DG:* The last equality sign here used (5.3).

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