Frobenius's last proof
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Errata and comments

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- **page 6, §4:** The recurrence $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + x^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$ holds only for $n > 0$ (since you decided to set $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for all $n < 0$). This is worth saying.

- **page 7:** The "alternative recurrence" also requires $n > 0$.

  (I would also suggest giving the two recurrences labels, and referring to them in the later proofs that use them.)

- **page 7, proof of Proposition 7:** Before the computation, add "For any $n > 1$, we have" (since the computation is not true for $n \leq 1$).

- **page 8, proof of Proposition 7:** After "This establishes the recurrence" (the last sentence of the proof), I would add ", since combining consecutive addends in the definition of $P(n)$ yields

  $$P(n) = (-1)^n \sum_{\substack{\lambda \in \mathbb{Z}; \\ \lambda \equiv n \bmod 2}} \left(x^{a(\lambda)} \begin{bmatrix} n \\ \left\lfloor \dfrac{n+5\lambda}{2} \right\rfloor \end{bmatrix} - x^{a(\lambda+1)} \begin{bmatrix} n \\ \left\lfloor \dfrac{n+5(\lambda+1)}{2} \right\rfloor \end{bmatrix} \right)$$

  and similarly

  $$\begin{aligned} & P(n-1) \\ &= (-1)^n \sum_{\substack{\lambda \in \mathbb{Z}; \\ \lambda \equiv n \bmod 2}} \left(x^{a(\lambda)} \begin{bmatrix} n-1 \\ \left\lfloor \dfrac{(n-1)+5\lambda}{2} \right\rfloor \end{bmatrix} - x^{a(\lambda+1)} \begin{bmatrix} n-1 \\ \left\lfloor \dfrac{(n-1)+5(\lambda+1)}{2} \right\rfloor \end{bmatrix} \right) \\ &= (-1)^n \sum_{\substack{\lambda \in \mathbb{Z}; \\ \lambda \equiv n \bmod 2}} \left(x^{a(\lambda)} \begin{bmatrix} n-1 \\ \left\lfloor \dfrac{n+5\lambda}{2} \right\rfloor - 1 \end{bmatrix} - x^{a(\lambda+1)} \begin{bmatrix} n-1 \\ \left\lfloor \dfrac{n+5(\lambda+1)}{2} \right\rfloor \end{bmatrix} \right) \end{aligned}$$

and

$$P(n-2) = (-1)^n \sum_{\substack{\lambda\in\mathbb{Z};\\ \lambda\equiv n \bmod 2}} \left(x^{a(\lambda)} \left[\begin{matrix} n-2 \\ \left\lfloor \dfrac{(n-2)+5\lambda}{2} \right\rfloor \end{matrix} \right] - x^{a(\lambda+1)} \left[\begin{matrix} n-2 \\ \left\lfloor \dfrac{(n-2)+5(\lambda+1)}{2} \right\rfloor \end{matrix} \right] \right)$$

$$= (-1)^n \sum_{\substack{\lambda\in\mathbb{Z};\\ \lambda\equiv n \bmod 2}} \left(x^{a(\lambda)} \left[\begin{matrix} n-2 \\ \left\lfloor \dfrac{n+5\lambda}{2} \right\rfloor - 1 \end{matrix} \right] - x^{a(\lambda+1)} \left[\begin{matrix} n-2 \\ \left\lfloor \dfrac{n+5(\lambda+1)}{2} \right\rfloor - 1 \end{matrix} \right] \right).$$

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- **page 9, Proposition 5:** Replace "$+Q(n-2)$" by "$+x^{n-1}Q(n-2)$".
- **page 9, Proposition 6:** Under the summation sign, replace the "$-\left\lfloor \frac{n}{5} \right\rfloor \leq \lambda \leq \left\lfloor \frac{n+1}{5} \right\rfloor$" by "$-\left\lfloor \frac{n}{3} \right\rfloor \leq \lambda \leq \left\lfloor \frac{n+1}{3} \right\rfloor$".
- **page 10, proof of Proposition 6:** At the beginning of the proof, add "For each $n > 0$, we have". Also, add a period after the computation that follows.
- **page 10, proof of Proposition 6:** Replace "with $\alpha = c(\lambda)$, $b = c(\lambda+1)$" by "with $a = x^{c(\lambda)}$, $b = x^{c(\lambda+1)}$".
- **page 10, §5:** Add a "with" before the chain of inequalities "$a_1 \geq a_2 \geq \ldots \geq a_n \geq 1$", and add a comma after this chain of inequalities.
- **page 11:** I would replace "GS1" by "Proposition 4" on the off-chance someone won't get the abbreviation. Similarly for "GS2".
- **page 11:** In the last displayed equation on this page, replace "$\prod_{k\geq 0} (1-x^{5k+2})(1-x^{5k+3})$" by "$\prod_{k\geq 0} \frac{1}{(1-x^{5k+2})(1-x^{5k+3})}$".
- **page 12:** You seem to use "all-but-equal" by "differing by 1"; this might not be standard usage.
- I found it curious that the polynomials $R(n)$ in your Proposition 6 look very similar to the $C_{2n}(q)$ and $C_{2n+1}(q)$ in

  A. A. Kirillov, A. Melnikov, *On a Remarkable Sequence of Polynomials*, 1995.

  and in

Shalosh B. Ekhad, Doron Zeilberger, *The Number of Solutions of* $X^2 = 0$ *in Triangular Matrices over* $GF(q)$, The Electronic Journal of Combinatorics **3** (1996), #R2.

Do you see any closer connection?