A Self-Dual Hopf Algebra on Double Partially Ordered Sets
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "A Self-Dual Hopf Algebra on Double Partially Ordered Sets" by the numbers under which they appear in this article (specifically, in its version of 21 May 2009, posted on the arXiv as arXiv preprint arXiv:0905.3508v1).

5. Errata

- **Various places (including Theorem 2.1):** You claim that $\mathbb{Z}\mathbf{D}$ is a self-dual Hopf algebra. The notion of self-duality that you are using, however, differs from the notions of self-duality commonly used in literature. In particular, a standard definition of a self-dual Hopf algebra H requires it to have a bilinear form which provides a Hopf algebra isomorphism $H \to H^*$. This is not satisfied for $\mathbb{Z}\mathbf{D}$: In fact, the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}\mathbf{D}$ gives rise to a Hopf algebra homomorphism $\mathbb{Z}\mathbf{D} \to (\mathbb{Z}\mathbf{D})^*$ which is not bijective. (It is in fact injective, but you do not prove this; instead, this is a consequence of [Foissy11, Theorem 36 1.].)

- **Page 2, §2:** Replace "if $x \in J$ and $x < y$" by "if $x \in S$ and $x < y$".

- **Page 3:** Replace "We define a pairing $\langle , \rangle : \mathbb{Z}\mathbf{D} \times \mathbb{Z}\mathbf{D} \to \mathbb{Z}$ for any double posets E, F by:

$$\langle E, F \rangle := |\{\alpha : E \to F,\ \alpha \text{ is a picture}\}|$$

" by "We define a pairing $\langle , \rangle : \mathbb{Z}\mathbf{D} \times \mathbb{Z}\mathbf{D} \to \mathbb{Z}$ by:

$$\langle E, F \rangle := |\{\alpha : E \to F,\ \alpha \text{ is a picture}\}| \qquad \text{for any double posets } E, F.$$

".

- **Page 3:** Replace "$\langle E \otimes G, \delta G \rangle$" by "$\langle E \otimes F, \delta G \rangle$".

- **Page 3, proof of Theorem 2.1:** Replace "$I \cap F$ is an inferior ideal of $(E, <_1)$" by "$I \cap F$ is an inferior ideal of $(F, <_1)$".

- **Page 4, proof of Theorem 2.1:** In "I is an inferior ideal of G", replace the "G" by a "G" (in mathmode).

- **Page 4, proof of Theorem 2.1:** In "take g, g' in G", replace the "G" by a "G" (in mathmode).

- **Page 4, proof of Theorem 2.1:** Replace "ordre" by "order".
- **Page 4, proof of Theorem 2.1:** Replace "a bijection $\alpha : E \to G$" by "a bijection $\alpha : EF \to G$".
- **Page 4, proof of Theorem 2.1:** Replace "doubles posets" by "double posets".
- **Page 4, §2.2:** Replace "over all π-partitions" by "over all π-partitions x into X".
- **Page 4, §2.2:** Replace "algebra of quasi-symmetric function" by "algebra of quasi-symmetric functions".
- **Page 4, §2.2:** Replace "commuting variables Y" by "commuting variables Y" (the "Y" should be in mathmode).
- **Page 5, §2.3:** Replace "Let $\phi : (E, <_1) \to (F, <_2)$ be a bijection" by "Let $\phi : E \to F$ be a bijection". (You really mean a bijection between sets here, and poset structures are not relevant to it.)
- **Page 5, §2.3:** The internal product is non-unital (i.e., there exists no element e of $\mathbb{Z}\mathbf{D}$ such that every $b \in \mathbb{Z}\mathbf{D}$ satisfies $e \circ b = b \circ e = e$). When I hear "product", I normally tend to expect that it has a unit, which is why I think this should be pointed out explicitly.

  On the other hand, what is true is that the internal product is associative. See Proposition 6.1 below for a proof.
- **Page 5, Lemma 2.1:** Replace "and and" by "and".
- **Page 5, Lemma 2.1:** Replace "and β a picture" by "and β is a picture".
- **Page 5, proof of Lemma 2.1:** Replace "the projection $F \times G \to F$" by "the projection $F \times_\psi G \to F$".
- **Page 5, proof of Lemma 2.1:** Replace "the projection $E \times F \to F$" by "the projection $E \times_\psi F \to F$".
- **Page 5, proof of Lemma 2.1:** In "Now β maps bijectively E into $F \times_\phi G$", replace "ϕ" by "ψ".
- **Page 5, proof of Lemma 2.1:** Replace "$\alpha^{-1}(\psi(f) = (\phi^{-1}(f), f)$" by "$\alpha^{-1}(\psi(f)) = (\phi^{-1}(f), f)$".
- **Page 5, proof of Lemma 2.1:** Replace "$\beta(e) = (\phi(e), \psi(\phi(e))$" by "$\beta(e) = (\phi(e), \psi(\phi(e)))$".
- **Page 6, proof of Lemma 2.1:** Replace "$\beta' = \beta'$" by "$\beta' = \beta$".

- **Pages 6-7, §3.1:** Here you define a bialgebra structure on $\mathbb{Z}S$ and claim that it is the one constructed in [MR]. It would be helpful to make this somewhat more precise: In [MR] you have defined two different (albeit isomorphic) bialgebra structures on $\mathbb{Z}S$, and the one that you are talking about is the one that you have called $(\mathbb{Z}S, *', \Delta')$ in [MR] (rather than the one you have called $(\mathbb{Z}S, *, \Delta)$ or simply $\mathbb{Z}S$ in [MR]).

- **Page 7:** Replace "each special double poset on the sum" to "each special double poset to the sum".

- **Page 7:** Replace "diagramm" by "diagram".

- **Page 7, proof of Theorem 3.1:** "a lower" should be "an inferior".

- **Page 8, proof of Theorem 3.1:** Replace "$st(v) = \sigma$" by "$st(v) = \beta$".

- **Page 8, proof of Lemma 3.1:** The proof should start with the sentence "Write the special double posets π and π' in the forms $\pi = (P, <_1, <_2)$ and $\pi' = (P', <_1, <_2)$, respectively", so that P and P' are defined.

- **Page 8, proof of Theorem 3.2:** Replace "let ϕ be increasing $(E, <_1) \to (F, <_2) = \{1, \ldots, n\}$" by "let ϕ be an increasing bijection $(E, <_1) \to (F, <_2) = \{1, \ldots, n\}$".

- **Page 8, proof of Theorem 3.2:** Replace "as in Section 2.2" by "as in Section 2.3".

- **Page 8, proof of Theorem 3.2:** Replace "$\sigma - 1 \circ \phi^{-1}$" by "$\sigma^{-1} \circ \phi^{-1}$".

- **Page 8, proof of Theorem 3.2:** Replace "$L(\pi) L(\pi')$" by "$L(\pi) \circ L(\pi')$".

- **Page 9, §3.1:** Replace "$c_1 ..., c_k$" by "$c_1, ..., c_k$".

- **Page 9, §3.1:** Replace "Section 3" in "Section 2.2" in "Recall that the bialgebra homomorphism $\Gamma : \mathbb{Z}\mathbf{D} \to \mathbf{QSym}$ has been defined in Section 3".

- **Page 9, §3.2:** The formulation "the word obtained from w by exchanging 1 and k in w , then 2 and $k-1$, and so on" is ambiguous (the "and so on" could be misunderstood to end at "k and 1", which would entail that the complement of w is w itself). I think a clearer definition of the complement would be: If $w = w_1 w_2 \ldots w_n$ is a word (with $w_1, w_2, \ldots, w_n$ being its letters) whose letters belong to $\{1, 2, 3, \ldots\}$, and k is the highest letter appearing in w, then the *complement* of w is defined as the word $(k+1-w_1)(k+1-w_2)\ldots(k+1-w_n)$.

- **Page 9, §3.2:** You define the weight of a word as follows: "The *weight* of a word w is the partition $\nu = 1^{n_1} 2^{n_2} ...$, where n_i is the number of i's in w. For the word above, it is the partition $1^4 2^3 3^1$." This notion of a weight (while clearly well-defined) is very unlikely to be the one that you want: indeed, it makes some of your statements below ("there is a well-known bijection between standard Young tableaux of shape ν and lattice permutations of weight ν", and Theorem 3.4) false. Instead, I believe that you want to define a different notion of weight, namely the following one: "The *weight* of a word w (on the symbols $1, 2, 3, \ldots$) is defined as the sequence $(n_1, n_2, n_3, \ldots)$, where n_i is the number of i's in w. When w is a lattice permutation (but also in some other cases), this sequence is a partition. For example, the weight of the word 11122132 is $(4, 3, 1, 0, 0, 0, \ldots) = (4, 3, 1)$."

- **Page 9, §3.2:** Replace "*w fits into π*" by "*$a_1 a_2 \ldots a_n$ fits into π*".
- **Page 9, §3.2:** Replace "$f(\omega(1)) \ldots f(\omega(n))$" by "$f\left(\omega^{-1}(1)\right) \ldots f\left(\omega^{-1}(n)\right)$".
- **Page 9, §3.2:** Replace "we have that τ fits into π" by "we say that τ fits into π".
- **Page 9, §3.2:** Replace "and where $<_2$ is given on the elements of E_ν by $(x, y) <_2 (x', y')$ if and only if either $y > y'$, or $y = y'$ and $x < x'$" by "and where $<_2$ is given on the elements of E_ν by $(x, y) <_2 (x', y')$ if and only if either $x > x'$, or $x = x'$ and $y < y'$ (in other words, $<_2$ compares two cells of E_ν by decreasing row number, or, when the rows are equal, by increasing column number)". This definition is needed to ensure that the reading word in the example on page 10 is the right one.
- **Page 9, §3.2:** Let's be self-contained: After "Recall that there is a well-known bijection between standard tableaux of shape ν and lattice permutations of weight ν, see [S1] Prop.7.10.3 (d).", add: "Explicitly, this bijection sends a standard tableau T with n entries to the lattice permutation $r_1 r_2 \ldots r_n$, where r_k is the number of the row in which the entry k lies in T.".
- **Page 10, Theorem 3.4:** Replace "(π, π_ν)" by "$\langle \pi, \pi_\nu \rangle$".
- **Page 10:** Replace "[Ma] (9.2)" by "[Ma] (Chapter 1, (9.2))". In the same sentence, replace "[S1] Th.A.1.3.3" by "[S1] Theorem A1.5.3".

  Another reference for the same result is [Gashar98, Theorem 1.2] for $\theta = \varnothing$.
- **Page 10:** You claim that part (ii) of Theorem 3.4 "is the classical formulation of the Littlewood-Richardson rule". I find this misleading, since Theorem 3.4 comes nowhere close to **proving** the Littlewood-Richardson rule (about the Hall inner product of a skew Schur functions with a Schur function). What is true is that Theorem 3.4 **looks like** the Littlewood-Richardson rule, but it differs from it in that it gives an expression for a scalar product $\langle \pi, \pi_\nu \rangle$ on $\mathbb{Z}\mathbf{D}$, while the Littlewood-Richardson rule gives an expression for a Hall inner product $\langle s_{\lambda/\mu}, s_\nu \rangle$ on the ring of symmetric functions. That

the expressions are equal (when π is a skew Ferrers diagram $\pi_{\lambda/\mu}$) makes the two results similar, but I do not think it allows to derive one from the other. Or am I missing something simple?

- **Page 10, proof of Lemma 3.2:** Replace "τ fits into π" by "a permutation τ fits into π".
- **Page 10:** You write that Proposition 3.1 "is equivalent to a result of Stanley, see [G] Th.1 or [S1] Th.7.19.14". There is no Theorem 7.19.14 in [S1]; I suspect that you mean Theorem 7.19.4 instead. (Besides, the equivalence of your Proposition 3.1 to the result of Stanley that you mention is not completely obvious. See below for a different proof of Proposition 3.1, which in my opinion is simpler than deriving it from Stanley's result.)
- **Page 10, example below Proposition 3.2:** "whose inverse is 5 3 9 2 6 10 11 1 4 6 7" should be "whose inverse is 5 3 9 2 6 10 11 1 4 7 8" (the last two letters were wrong).
- **Page 11, Lemma 3.3:** Replace "$\nu = (\nu_1 > ... > \nu_k > 0)$" by "$\nu = (\nu_1 \geq ... \geq \nu_k > 0)$".
- **Page 11, proof of proposition:** "Proof of proposition" should be "Proof of proposition 3.2" (it is not the only proposition around).
- **Page 11, proof of proposition:** Replace "Lemma 5.2" by "Lemma 3.3" twice.
- **Page 11, proof of proposition:** Replace "done in(i)" by "done in (i)".
- **Page 12, proof of theorem:** "complements of lattice permutation" → "complements of lattice permutations".

6. Additional details and proofs

Note: The below addenda were written in approx. 2014, as an exercise in understanding and formalizing some arguments in this paper and in the theory of combinatorial Hopf algebras in general. I was neither trying to be concise nor good at mathematical writing; instead I was attempting to prepare the arguments for formal verification (a project that has yet to materialize). –DG, 2024

6.1. Page 5, §2.3: associativity of the internal product

Let me add an additional fact: The internal product $\circ$ is associative. In other words:

Proposition 6.1. Any three elements E, F and G of $\mathbb{Z}\mathbf{D}$ satisfy $(E \circ F) \circ G = E \circ (F \circ G)$.

Before we prove this, let us first establish some notation.

Definition 6.2. Let A and B be two sets. Let $<_B$ denote the smaller relation of a partial order on B. (We write this relation $<_B$ in infix notation; this means that, if b and b' are two elements of B, then we write $b <_B b'$ for $(b, b') \in (<_B)$.) Let $f : A \to B$ be a map.

We define a new binary relation $(<_B)^f$ on the set A as follows: For any two elements a and a' of A, we set $a \, (<_B)^f \, a'$ if and only if $f(a) <_B f(a')$. (Here, we again write the relation $(<_B)^f$ in infix notation.)

Proposition 6.3. Let A and B be two sets. Let $<_B$ denote the smaller relation of a partial order on B. Let $f : A \to B$ be a map. Then, the binary relation $(<_B)^f$ is the smaller relation of a partial order on A.

Proof of Proposition 6.3. Let us write both relations $<_B$ and $(<_B)^f$ in infix notation.

We know that the relation $<_B$ is the smaller relation of a partial order. In other words, the relation $<_B$ is irreflexive, transitive and asymmetric.

We now notice that the relation $(<_B)^f$ is irreflexive[1], transitive[2] and asymmetric[3]. In other words, the relation $(<_B)^f$ is the smaller relation of a partial order

[1] *Proof.* Let $a \in A$ be such that $a \, (<_B)^f \, a$. We recall that $a \, (<_B)^f \, a$ holds if and only if $f(a) <_B f(a)$ (according to the definition of "$a \, (<_B)^f \, a$"). Hence, $f(a) <_B f(a)$ must hold (because $a \, (<_B)^f \, a$ holds). But this contradicts the fact that the relation $<_B$ is irreflexive. We thus have found a contradiction.

Now, let us forget that we fixed a. We thus have found a contradiction for each $a \in A$ satisfying $a \, (<_B)^f \, a$. Therefore, there exists no $a \in A$ satisfying $a \, (<_B)^f \, a$. In other words, the relation $(<_B)^f$ is irreflexive. Qed.

[2] *Proof.* Let a, a' and a'' be three elements of A such that $a \, (<_B)^f \, a'$ and $a' \, (<_B)^f \, a''$. We shall prove that $a \, (<_B)^f \, a''$.

We recall that $a \, (<_B)^f \, a'$ holds if and only if $f(a) <_B f(a')$ (according to the definition of "$a \, (<_B)^f \, a'$"). Thus, $f(a) <_B f(a')$ must hold (since $a \, (<_B)^f \, a'$ holds).

We recall that $a' \, (<_B)^f \, a''$ holds if and only if $f(a') <_B f(a'')$ (according to the definition of "$a' \, (<_B)^f \, a''$"). Thus, $f(a') <_B f(a'')$ must hold (since $a' \, (<_B)^f \, a''$ holds).

From $f(a) <_B f(a')$ and $f(a') <_B f(a'')$, we obtain $f(a) <_B f(a'')$ (since the relation $<_B$ is transitive).

We recall that $a \, (<_B)^f \, a''$ holds if and only if $f(a) <_B f(a'')$ (according to the definition of "$a \, (<_B)^f \, a''$"). Thus, $a \, (<_B)^f \, a''$ must hold (since $f(a) <_B f(a'')$ holds).

Now, let us forget that we fixed a, a' and a''. We thus have shown that if a, a' and a'' are three elements of A such that $a \, (<_B)^f \, a'$ and $a' \, (<_B)^f \, a''$, then $a \, (<_B)^f \, a''$. In other words, the relation $(<_B)^f$ is transitive. Qed.

[3] *Proof.* It is known that every irreflexive and transitive relation is asymmetric. Applying this

on A (because the smaller relations of partial orders are characterized by being irreflexive, transitive and asymmetric). This proves Proposition 6.3. $\square$

The following proposition, while not needed in the proof of Proposition 6.1, will be useful much later:

Proposition 6.4. Let A and B be two sets. Let $<_B$ denote the smaller relation of a total order on B. Let $f : A \to B$ be an injective map. Then, the binary relation $(<_B)^f$ is the smaller relation of a total order on A.

Proof of Proposition 6.4. Let us write both relations $<_B$ and $(<_B)^f$ in infix notation.

The relation $<_B$ is the smaller relation of a total order on B, and therefore also the smaller relation of a partial order on B. Hence, Proposition 6.3 shows that the binary relation $(<_B)^f$ is the smaller relation of a partial order on A. It now remains to prove that this partial order is total.

Every two distinct elements a and a' of A satisfy either $a \,(<_B)^f\, a'$ or $a' \,(<_B)^f\, a$ [4]. Thus, the binary relation $(<_B)^f$ is the smaller relation of a total order on A (since we already know that the binary relation $(<_B)^f$ is the smaller relation of a partial order on A). This proves Proposition 6.4. $\square$

to the relation $(<_B)^f$, we conclude that the relation $(<_B)^f$ is asymmetric (since $(<_B)^f$ is irreflexive and transitive).

[4] *Proof.* Let a and a' be two distinct elements of A. Then, $a \neq a'$ (since a and a' are distinct). If we had $f(a) = f(a')$, then we would have $a = a'$ (since the map f is injective), which would contradict $a \neq a'$. Hence, we cannot have $f(a) = f(a')$. Thus, we have $f(a) \neq f(a')$. In other words, the elements $f(a)$ and $f(a')$ of B are distinct.

But we know that $<_B$ is the smaller relation of a total order on B. Thus, if b and b' are two distinct elements of B, then either $b <_B b'$ or $b' <_B b$. Applying this to $b = f(a)$ and $b' = f(a')$, we conclude that either $f(a) <_B f(a')$ or $f(a') <_B f(a)$. We thus are in one of the following two cases:

Case 1: We have $f(a) <_B f(a')$.

Case 2: We have $f(a') <_B f(a)$.

Let us first consider Case 1. In this case, we have $f(a) <_B f(a')$. Thus, $a \,(<_B)^f\, a'$ (because we have $a \,(<_B)^f\, a'$ if and only if $f(a) <_B f(a')$ (according to the definition of "$a \,(<_B)^f\, a'$")). Hence, either $a \,(<_B)^f\, a'$ or $a' \,(<_B)^f\, a$. We thus have shown that $\left(\text{either } a \,(<_B)^f\, a' \text{ or } a' \,(<_B)^f\, a\right)$ in Case 1.

Let us now consider Case 2. In this case, we have $f(a') <_B f(a)$. Thus, $a' \,(<_B)^f\, a$ (because we have $a' \,(<_B)^f\, a$ if and only if $f(a') <_B f(a)$ (according to the definition of "$a' \,(<_B)^f\, a$")). Hence, either $a \,(<_B)^f\, a'$ or $a' \,(<_B)^f\, a$. We thus have shown that $\left(\text{either } a \,(<_B)^f\, a' \text{ or } a' \,(<_B)^f\, a\right)$ in Case 2.

Now, we have proven $\left(\text{either } a \,(<_B)^f\, a' \text{ or } a' \,(<_B)^f\, a\right)$ in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, this yields that $\left(\text{either } a \,(<_B)^f\, a' \text{ or } a' \,(<_B)^f\, a\right)$ always holds. Qed.

We now show the main tool with which we will work in our proof of Proposition 6.1:

Lemma 6.5. Let $(E, <_{E1}, <_{E2})$ and $(F, <_{F1}, <_{F2})$ be two double posets.

(a) For every bijection $\phi : E \to F$, we have

$$(E, <_{E1}, <_{E2}) \times_\phi (F, <_{F1}, <_{F2}) \cong \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \qquad \text{as double posets.}$$

(b) We have

$$(E, <_{E1}, <_{E2}) \circ (F, <_{F1}, <_{F2}) = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E, <_{E1}) \to (F, <_{F2})}} \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$$

in $\mathbb{Z}\mathbf{D}$.

(c) For every bijection $\phi : E \to F$, we have

$$(E, <_{E1}, <_{E2}) \times_\phi (F, <_{F1}, <_{F2}) \cong \left(E, (<_{F1})^{\phi}, <_{E2}\right) \qquad \text{as double posets.}$$

(d) We have

$$(E, <_{E1}, <_{E2}) \circ (F, <_{F1}, <_{F2}) = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E, <_{E1}) \to (F, <_{F2})}} \left(E, (<_{F1})^{\phi}, <_{E2}\right)$$

in $\mathbb{Z}\mathbf{D}$.

Proof of Lemma 6.5. We shall use the abbreviation E for $(E, <_{E1}, <_{E2})$, and we shall use the abbreviation F for $(F, <_{F1}, <_{F2})$.

The definition of the set $E \times_\phi F$ yields

$$\begin{aligned} E \times_\phi F &= \{(e, f) \in E \times F \mid \phi(e) = f\} \\ &= \{(u, v) \in E \times F \mid \phi(u) = v\} \end{aligned} \tag{1}$$

(here, we renamed the indices e and f as u and v). Clearly, this shows that $E \times_\phi F \subseteq E \times F$. The definition of the double poset $E \times_\phi F$ shows that $E \times_\phi F = (E \times_\phi F, <_1, <_2)$, where any two elements $(e, f) \in E \times_\phi F$ and $(e', f') \in E \times_\phi F$ satisfy

$$\left((e, f) <_1 (e', f') \text{ if and only if } f <_{F1} f'\right) \tag{2}$$

and

$$\left((e, f) <_2 (e', f') \text{ if and only if } e <_{E2} e'\right). \tag{3}$$

(a) Let ϕ be a bijection $E \to F$. Hence, ϕ^{-1} is a bijection $F \to E$. Thus, Proposition 6.3 (applied to F, E, $<_{E2}$ and ϕ^{-1} instead of A, B, $<_B$ and f) shows that the binary relation $(<_{E2})^{\phi^{-1}}$ is the smaller relation of a partial order on F. Hence, $\left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$ is a well-defined double poset.

Define a map $p_2 : E \times_\phi F \to F$ by $\left(p_2(e,f) = f \text{ for every } (e,f) \in E \times_\phi F\right)$. Thus, p_2 is the projection on the second component. On the other hand, consider the map $\left(\phi^{-1}, \operatorname{id}\right) : F \to E \times_\phi F$ which sends every $f \in F$ to $\left(\phi^{-1}(f), \underbrace{\operatorname{id}(f)}_{=f}\right) = \left(\phi^{-1}(f), f\right) \in E \times_\phi F$ [5]. The maps p_2 and $\left(\phi^{-1}, \operatorname{id}\right)$ are mutually inverse[6]. Hence, these two maps p_2 and $\left(\phi^{-1}, \operatorname{id}\right)$ are invertible and satisfy $p_2^{-1} = \left(\phi^{-1}, \operatorname{id}\right)$.

The map p_2 is a poset homomorphism $\left(E \times_\phi F, <_1\right) \to \left(F, <_{F1}\right)$ [7] and a

[5] Let us check that this map $\left(\phi^{-1}, \operatorname{id}\right)$ is well-defined:

Fix $f \in F$. Then, $\left(\phi^{-1}(f), f\right)$ is an element of $E \times F$ satisfying $\phi\left(\phi^{-1}(f)\right) = f$. In other words, $\left(\phi^{-1}(f), f\right)$ is an element (u,v) of $E \times F$ satisfying $\phi(u) = v$. In other words,

$$\left(\phi^{-1}(f), f\right) \in \{(u,v) \in E \times F \mid \phi(u) = v\} = E \times_\phi F \qquad \text{(by (1))}.$$

Now, let us forget that we fixed f. We thus have shown that $\left(\phi^{-1}(f), f\right) \in E \times_\phi F$ for every $f \in F$. Thus, the map $\left(\phi^{-1}, \operatorname{id}\right)$ is well-defined, qed.

[6] *Proof.* Every $f \in F$ satisfies

$$\left(p_2 \circ \left(\phi^{-1}, \operatorname{id}\right)\right)(f) = p_2\left(\underbrace{\left(\phi^{-1}, \operatorname{id}\right)(f)}_{\substack{=\left(\phi^{-1}(f), \operatorname{id}(f)\right) \\ \text{(by the definition of } \left(\phi^{-1}, \operatorname{id}\right))}}\right) = p_2\left(\phi^{-1}(f), \operatorname{id}(f)\right) = \operatorname{id}(f)$$
$$\text{(by the definition of } p_2).$$

In other words, $p_2 \circ \left(\phi^{-1}, \operatorname{id}\right) = \operatorname{id}$.

On the other hand, let $(e,f) \in E \times_\phi F$. Then, $(e,f) \in E \times_\phi F = \{(u,v) \in E \times F \mid \phi(u) = v\}$ (according to (1)). In other words, (e,f) is an element (u,v) of $E \times F$ satisfying $\phi(u) = v$. In other words, (e,f) is an element of $E \times F$ and satisfies $\phi(e) = f$. Now, $p_2(e,f) = f$ (by the definition of p_2) and $\phi^{-1}(f) = e$ (since $\phi(e) = f$). Now,

$$\left(\left(\phi^{-1}, \operatorname{id}\right) \circ p_2\right)(e,f) = \left(\phi^{-1}, \operatorname{id}\right)\left(\underbrace{p_2(e,f)}_{=f}\right) = \left(\phi^{-1}, \operatorname{id}\right)(f)$$
$$= \left(\underbrace{\phi^{-1}(f)}_{=e}, \underbrace{\operatorname{id}(f)}_{=f}\right) = (e,f) = \operatorname{id}(e,f).$$

Let us now forget that we fixed (e,f). We thus have shown that $\left(\left(\phi^{-1}, \operatorname{id}\right) \circ p_2\right)(e,f) = \operatorname{id}(e,f)$ for every $(e,f) \in E \times_\phi F$. In other words, $\left(\phi^{-1}, \operatorname{id}\right) \circ p_2 = \operatorname{id}$. Combined with $p_2 \circ \left(\phi^{-1}, \operatorname{id}\right) = \operatorname{id}$, this yields that the maps p_2 and $\left(\phi^{-1}, \operatorname{id}\right)$ are mutually inverse. Qed.

[7] *Proof.* Let g and g' be two elements of $E \times_\phi F$ satisfying $g <_1 g'$. We shall show that $p_2(g) <_{F1} p_2(g')$.

Indeed, $g \in E \times_\phi F = \{(u,v) \in E \times F \mid \phi(u) = v\}$. In other words, g has the form (u,v) for some $(u,v) \in E \times F$ satisfying $\phi(u) = v$. Let us denote this (u,v) by (e,f). Thus, (e,f) is poset homomorphism $\left(E\times_{\phi}F,<_{2}\right)\to\left(F,\left(<_{E2}\right)^{\phi^{-1}}\right)$ [8]. Hence, the map p_2 is a homomorphism of double posets $\left(E\times_{\phi}F,<_{1},<_{2}\right)\to\left(F,<_{F1},\left(<_{E2}\right)^{\phi^{-1}}\right)$.

On the other hand, the map p_2^{-1} is a poset homomorphism $(F,<_{F1})\to\left(E\times_{\phi}F,<_{1}\right)$

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an element of $E\times F$ and satisfies $\phi(e)=f$ and $g=(e,f)$. Applying the map p_2 to both sides of the equality $g=(e,f)$, we obtain $p_2(g)=p_2(e,f)=f$ (by the definition of p_2).

Also, $g'\in E\times_{\phi}F=\{(u,v)\in E\times F\mid\phi(u)=v\}$. In other words, g' has the form (u,v) for some $(u,v)\in E\times F$ satisfying $\phi(u)=v$. Let us denote this (u,v) by (e',f'). Thus, (e',f') is an element of $E\times F$ and satisfies $\phi(e')=f'$ and $g'=(e',f')$. Applying the map p_2 to both sides of the equality $g'=(e',f')$, we obtain $p_2(g')=p_2(e',f')=f'$ (by the definition of p_2).

We have $(e,f)=g<_1 g'=(e',f')$. Due to (2), this yields $f<_{F1}f'$. In other words, $p_2(g)<_{F1}p_2(g')$ (since $p_2(g)=f$ and $p_2(g')=f'$).

Now, let us forget that we fixed g and g'. We thus have proven that if g and g' are two elements of $E\times_{\phi}F$ satisfying $g<_1 g'$, then $p_2(g)<_{F1}p_2(g')$. In other words, the map p_2 is a strictly order-preserving map $\left(E\times_{\phi}F,<_{1}\right)\to(F,<_{F1})$. As a consequence, this map p_2 is a poset homomorphism $\left(E\times_{\phi}F,<_{1}\right)\to(F,<_{F1})$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

[8]*Proof.* Let g and g' be two elements of $E\times_{\phi}F$ satisfying $g<_2 g'$. We shall show that $p_2(g)\left(<_{E2}\right)^{\phi^{-1}}p_2(g')$.

Indeed, $g\in E\times_{\phi}F=\{(u,v)\in E\times F\mid\phi(u)=v\}$. In other words, g has the form (u,v) for some $(u,v)\in E\times F$ satisfying $\phi(u)=v$. Let us denote this (u,v) by (e,f). Thus, (e,f) is an element of $E\times F$ and satisfies $\phi(e)=f$ and $g=(e,f)$. Applying the map p_2 to both sides of the equality $g=(e,f)$, we obtain $p_2(g)=p_2(e,f)=f$ (by the definition of p_2).

Also, $g'\in E\times_{\phi}F=\{(u,v)\in E\times F\mid\phi(u)=v\}$. In other words, g' has the form (u,v) for some $(u,v)\in E\times F$ satisfying $\phi(u)=v$. Let us denote this (u,v) by (e',f'). Thus, (e',f') is an element of $E\times F$ and satisfies $\phi(e')=f'$ and $g'=(e',f')$. Applying the map p_2 to both sides of the equality $g'=(e',f')$, we obtain $p_2(g')=p_2(e',f')=f'$ (by the definition of p_2).

We have $(e,f)=g<_2 g'=(e',f')$. Due to (3), this yields $e<_{E2}e'$. Since $\phi(e)=f$, we have $e=\phi^{-1}(f)$, so that $\phi^{-1}(f)=e<_{E2}e'=\phi^{-1}(f')$ (since $\phi(e')=f'$).

But $f\left(<_{E2}\right)^{\phi^{-1}}f'$ holds if and only if $\phi^{-1}(f)<_{E2}\phi^{-1}(f')$ (due to the definition of "$f\left(<_{E2}\right)^{\phi^{-1}}f'$"). Thus, $f\left(<_{E2}\right)^{\phi^{-1}}f'$ holds (since $\phi^{-1}(f)<_{E2}\phi^{-1}(f')$ holds). In other words, $p_2(g)\left(<_{E2}\right)^{\phi^{-1}}p_2(g')$ holds (since $p_2(g)=f$ and $p_2(g')=f'$).

Now, let us forget that we fixed g and g'. We thus have proven that if g and g' are two elements of $E\times_{\phi}F$ satisfying $g<_2 g'$, then $p_2(g)\left(<_{E2}\right)^{\phi^{-1}}p_2(g')$. In other words, the map p_2 is a strictly order-preserving map $\left(E\times_{\phi}F,<_{2}\right)\to\left(F,\left(<_{E2}\right)^{\phi^{-1}}\right)$. As a consequence, this map p_2 is a poset homomorphism $\left(E\times_{\phi}F,<_{2}\right)\to\left(F,\left(<_{E2}\right)^{\phi^{-1}}\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

[9] and a poset homomorphism $\left(F, (<_{E2})^{\phi^{-1}}\right) \to (E \times_\phi F, <_2)$ [10]. Hence, the map p_2^{-1} is a homomorphism of double posets $\left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \to (E \times_\phi F, <_1, <_2)$.

Now, we know that the map p_2 is a homomorphism of double posets

[9] *Proof.* Let f and f' be two elements of F satisfying $f <_{F1} f'$. We shall prove that $p_2^{-1}(f) <_1 p_2^{-1}(f')$.

We have $\underbrace{p_2^{-1}}_{=(\phi^{-1}, \mathrm{id})}(f) = (\phi^{-1}, \mathrm{id})(f) = \left(\phi^{-1}(f), \underbrace{\mathrm{id}(f)}_{=f}\right) = (\phi^{-1}(f), f)$, so that $(\phi^{-1}(f), f) = p_2^{-1}(f) \in E \times_\phi F$.

Also, $\underbrace{p_2^{-1}}_{=(\phi^{-1}, \mathrm{id})}(f') = (\phi^{-1}, \mathrm{id})(f') = \left(\phi^{-1}(f'), \underbrace{\mathrm{id}(f')}_{=f'}\right) = (\phi^{-1}(f'), f')$, so that $(\phi^{-1}(f'), f') = p_2^{-1}(f') \in E \times_\phi F$.

Now, (2) (applied to $\phi^{-1}(f)$ and $\phi^{-1}(f')$ instead of e and e') yields $\left((\phi^{-1}(f), f) <_1 (\phi^{-1}(f'), f') \text{ if and only if } f <_{F1} f'\right)$ (since $(\phi^{-1}(f), f) \in E \times_\phi F$ and $(\phi^{-1}(f'), f') \in E \times_\phi F$). Thus, we have $(\phi^{-1}(f), f) <_1 (\phi^{-1}(f'), f')$ (since $f <_{F1} f'$). In other words, $p_2^{-1}(f) <_1 p_2^{-1}(f')$ (since $p_2^{-1}(f) = (\phi^{-1}(f), f)$ and $p_2^{-1}(f') = (\phi^{-1}(f'), f')$).

Now, let us forget that we fixed f and f'. We thus have shown that if f and f' are two elements of F satisfying $f <_{F1} f'$, then $p_2^{-1}(f) <_1 p_2^{-1}(f')$. In other words, the map p_2^{-1} is a strictly order-preserving map $(F, <_{F1}) \to (E \times_\phi F, <_1)$. As a consequence, this map p_2^{-1} is a poset homomorphism $(F, <_{F1}) \to (E \times_\phi F, <_1)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

[10] *Proof.* Let f and f' be two elements of F satisfying $f\,(<_{E2})^{\phi^{-1}} f'$. We shall prove that $p_2^{-1}(f) <_2 p_2^{-1}(f')$.

We have $f\,(<_{E2})^{\phi^{-1}} f'$ if and only if $\phi^{-1}(f) <_{E2} \phi^{-1}(f')$ (due to the definition of "$f\,(<_{E2})^{\phi^{-1}} f'$"). Thus, we have $\phi^{-1}(f) <_{E2} \phi^{-1}(f')$ (since we have $f\,(<_{E2})^{\phi^{-1}} f'$).

We have $\underbrace{p_2^{-1}}_{=(\phi^{-1}, \mathrm{id})}(f) = (\phi^{-1}, \mathrm{id})(f) = \left(\phi^{-1}(f), \underbrace{\mathrm{id}(f)}_{=f}\right) = (\phi^{-1}(f), f)$, so that $(\phi^{-1}(f), f) = p_2^{-1}(f) \in E \times_\phi F$.

Also, $\underbrace{p_2^{-1}}_{=(\phi^{-1}, \mathrm{id})}(f') = (\phi^{-1}, \mathrm{id})(f') = \left(\phi^{-1}(f'), \underbrace{\mathrm{id}(f')}_{=f'}\right) = (\phi^{-1}(f'), f')$, so that $(\phi^{-1}(f'), f') = p_2^{-1}(f') \in E \times_\phi F$.

Now, (3) (applied to $\phi^{-1}(f)$ and $\phi^{-1}(f')$ instead of e and e') yields $\left((\phi^{-1}(f), f) <_2 (\phi^{-1}(f'), f') \text{ if and only if } \phi^{-1}(f) <_{E2} \phi^{-1}(f')\right)$ (since $(\phi^{-1}(f), f) \in E \times_\phi F$ and $(\phi^{-1}(f'), f') \in E \times_\phi F$). Thus, we have $(\phi^{-1}(f), f) <_2 (\phi^{-1}(f'), f')$ (since $\phi^{-1}(f) <_{E2} \phi^{-1}(f')$). In other words, $p_2^{-1}(f) <_2 p_2^{-1}(f')$ (since $p_2^{-1}(f) = (\phi^{-1}(f), f)$ and $p_2^{-1}(f') = (\phi^{-1}(f'), f')$).

Now, let us forget that we fixed f and f'. We thus have shown that if f and f' are two elements of F satisfying $f\,(<_{E2})^{\phi^{-1}} f'$, then $p_2^{-1}(f) <_2 p_2^{-1}(f')$. In other words, the map p_2^{-1} is a strictly order-preserving map $\left(F, (<_{E2})^{\phi^{-1}}\right) \to (E \times_\phi F, <_2)$. As a consequence,

$\left(E \times_\phi F, <_1, <_2\right) \to \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$, while its inverse p_2^{-1} is a homomorphism of double posets $\left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \to \left(E \times_\phi F, <_1, <_2\right)$. In other words, p_2 is an isomorphism of double posets $\left(E \times_\phi F, <_1, <_2\right) \to \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$. Thus, the double posets $\left(E \times_\phi F, <_1, <_2\right)$ and $\left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$ are isomorphic. That is, we have $\left(E \times_\phi F, <_1, <_2\right) \cong \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$ as double posets. Hence,

$$E \times_\phi F = \left(E \times_\phi F, <_1, <_2\right) \cong \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \tag{4}$$

as double posets. Thus,

$$\underbrace{(E, <_{E1}, <_{E2})}_{=E} \times_\phi \underbrace{(F, <_{F1}, <_{F2})}_{=F} = E \times_\phi F \cong \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$$

as double posets. This proves Lemma 6.5 **(a)**.

(b) The definition of $E \circ F$ shows that $E \circ F$ is the sum of $E \times_\phi F$ for all increasing bijections $\phi : (E, <_{E1}) \to (F, <_{F2})$. In other words,

$$\begin{aligned} E \circ F &= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1}) \to (F,<_{F2})}} \underbrace{E \times_\phi F}_{\substack{= \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \\ \text{(according to (4))}}} \\ &= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1}) \to (F,<_{F2})}} \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right) \end{aligned}$$

in $\mathbb{Z}\mathbf{D}$. Thus,

$$\underbrace{(E, <_{E1}, <_{E2})}_{=E} \circ \underbrace{(F, <_{F1}, <_{F2})}_{=F} = E \circ F = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1}) \to (F,<_{F2})}} \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$$

in $\mathbb{Z}\mathbf{D}$. This proves Lemma 6.5 **(b)**.

(c) Let ϕ be a bijection $E \to F$. Proposition 6.3 (applied to E, F, $<_{F1}$ and ϕ instead of A, B, $<_B$ and f) shows that the binary relation $(<_{F1})^\phi$ is the smaller relation of a partial order on E. Hence, $\left(E, (<_{F1})^\phi, <_{E2}\right)$ is a well-defined double poset.

Define a map $p_1 : E \times_\phi F \to E$ by $\left(p_1(e, f) = e \text{ for every } (e, f) \in E \times_\phi F\right)$. Thus, p_1 is the projection on the first component. On the other hand, consider the map $(\mathrm{id}, \phi) : E \to E \times_\phi F$ which sends every $e \in E$ to $\left(\underbrace{\mathrm{id}(e)}_{=e}, \phi(e)\right) =$

this map p_2^{-1} is a poset homomorphism $\left(F, (<_{E2})^{\phi^{-1}}\right) \to \left(E \times_\phi F, <_2\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

$(e, \phi(e)) \in E \times_\phi F$ [11]. The maps p_1 and (id, ϕ) are mutually inverse[12]. Hence, these two maps p_1 and (id, ϕ) are invertible and satisfy $p_1^{-1} = (\mathrm{id}, \phi)$.

The map p_1 is a poset homomorphism $\left(E \times_\phi F, <_1\right) \to \left(E, (<_{F1})^\phi\right)$ [13] and

[11]Let us check that this map (id, ϕ) is well-defined:

Fix $e \in E$. Then, $(e, \phi(e))$ is an element of $E \times F$ satisfying $\phi(e) = \phi(e)$. In other words, $(e, \phi(e))$ is an element (u, v) of $E \times F$ satisfying $\phi(u) = v$. In other words,

$$(e, \phi(e)) \in \{(u, v) \in E \times F \mid \phi(u) = v\} = E \times_\phi F \qquad \text{(by (1))}.$$

Now, let us forget that we fixed e. We thus have shown that $(e, \phi(e)) \in E \times_\phi F$ for every $e \in E$. Thus, the map (id, ϕ) is well-defined, qed.

[12]*Proof.* Every $e \in E$ satisfies

$$(p_1 \circ (\mathrm{id}, \phi))(e) = p_1 \left(\underbrace{(\mathrm{id}, \phi)(e)}_{\substack{=(\mathrm{id}(e), \phi(e)) \\ \text{(by the definition of } (\mathrm{id}, \phi))}} \right) = p_1(\mathrm{id}(e), \phi(e)) = \mathrm{id}(e) \qquad \text{(by the definition of } p_1).$$

In other words, $p_1 \circ (\mathrm{id}, \phi) = \mathrm{id}$.

On the other hand, let $(e, f) \in E \times_\phi F$. Then, $(e, f) \in E \times_\phi F = \{(u, v) \in E \times F \mid \phi(u) = v\}$ (according to (1)). In other words, (e, f) is an element (u, v) of $E \times F$ satisfying $\phi(u) = v$. In other words, (e, f) is an element of $E \times F$ and satisfies $\phi(e) = f$. Now, $p_1(e, f) = e$ (by the definition of p_1). Now,

$$((\mathrm{id}, \phi) \circ p_1)(e, f) = (\mathrm{id}, \phi) \left(\underbrace{p_1(e, f)}_{=e} \right) = (\mathrm{id}, \phi)(e) = \left(\underbrace{\mathrm{id}(e)}_{=e}, \underbrace{\phi(e)}_{=f} \right) = (e, f) = \mathrm{id}(e, f).$$

Let us now forget that we fixed (e, f). We thus have shown that $((\mathrm{id}, \phi) \circ p_1)(e, f) = \mathrm{id}(e, f)$ for every $(e, f) \in E \times_\phi F$. In other words, $(\mathrm{id}, \phi) \circ p_1 = \mathrm{id}$. Combined with $p_1 \circ (\mathrm{id}, \phi) = \mathrm{id}$, this yields that the maps p_1 and (id, ϕ) are mutually inverse. Qed.

[13]*Proof.* Let g and g' be two elements of $E \times_\phi F$ satisfying $g <_1 g'$. We shall show that $p_1(g) (<_{F1})^\phi p_1(g')$.

Indeed, $g \in E \times_\phi F = \{(u, v) \in E \times F \mid \phi(u) = v\}$. In other words, g has the form (u, v) for some $(u, v) \in E \times F$ satisfying $\phi(u) = v$. Let us denote this (u, v) by (e, f). Thus, (e, f) is an element of $E \times F$ and satisfies $\phi(e) = f$ and $g = (e, f)$. Applying the map p_1 to both sides of the equality $g = (e, f)$, we obtain $p_1(g) = p_1(e, f) = e$ (by the definition of p_1).

Also, $g' \in E \times_\phi F = \{(u, v) \in E \times F \mid \phi(u) = v\}$. In other words, g' has the form (u, v) for some $(u, v) \in E \times F$ satisfying $\phi(u) = v$. Let us denote this (u, v) by (e', f'). Thus, (e', f') is an element of $E \times F$ and satisfies $\phi(e') = f'$ and $g' = (e', f')$. Applying the map p_1 to both sides of the equality $g' = (e', f')$, we obtain $p_1(g') = p_1(e', f') = e'$ (by the definition of p_1).

We have $(e, f) = g <_1 g' = (e', f')$. Due to (2), this yields $f <_{F1} f'$. Thus, $\phi(e) = f <_{F1} f' = \phi(e')$ (since $\phi(e') = f'$).

But $e (<_{F1})^\phi e'$ holds if and only if $\phi(e) <_{F1} \phi(e')$ (due to the definition of "$e (<_{F1})^\phi e'$"). Thus, $e (<_{F1})^\phi e'$ holds (since $\phi(e) <_{F1} \phi(e')$ holds). In other words, $p_1(g) (<_{F1})^\phi p_1(g')$ holds (since $p_1(g) = e$ and $p_1(g') = e'$).

Now, let us forget that we fixed g and g'. We thus have proven that if g and g' are two elements of $E \times_\phi F$ satisfying $g <_1 g'$, then $p_1(g) (<_{F1})^\phi p_1(g')$. In other words, the map p_1 is a poset homomorphism $\left(E \times_{\phi} F, <_2\right) \to \left(E, <_{E2}\right)$ [14]. Hence, the map p_1 is a homomorphism of double posets $\left(E \times_{\phi} F, <_1, <_2\right) \to \left(E, \left(<_{F1}\right)^{\phi}, <_{E2}\right)$.

On the other hand, the map p_1^{-1} is a poset homomorphism $\left(E, \left(<_{F1}\right)^{\phi}\right) \to$

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a strictly order-preserving map $\left(E \times_{\phi} F, <_1\right) \to \left(E, \left(<_{F1}\right)^{\phi}\right)$. As a consequence, this map p_1 is a poset homomorphism $\left(E \times_{\phi} F, <_1\right) \to \left(E, \left(<_{F1}\right)^{\phi}\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

[14] *Proof.* Let g and g' be two elements of $E \times_{\phi} F$ satisfying $g <_2 g'$. We shall show that $p_1(g) <_{E2} p_1(g')$.

Indeed, $g \in E \times_{\phi} F = \{(u, v) \in E \times F \mid \phi(u) = v\}$. In other words, g has the form (u, v) for some $(u, v) \in E \times F$ satisfying $\phi(u) = v$. Let us denote this (u, v) by (e, f). Thus, (e, f) is an element of $E \times F$ and satisfies $\phi(e) = f$ and $g = (e, f)$. Applying the map p_1 to both sides of the equality $g = (e, f)$, we obtain $p_1(g) = p_1(e, f) = e$ (by the definition of p_1).

Also, $g' \in E \times_{\phi} F = \{(u, v) \in E \times F \mid \phi(u) = v\}$. In other words, g' has the form (u, v) for some $(u, v) \in E \times F$ satisfying $\phi(u) = v$. Let us denote this (u, v) by (e', f'). Thus, (e', f') is an element of $E \times F$ and satisfies $\phi(e') = f'$ and $g' = (e', f')$. Applying the map p_1 to both sides of the equality $g' = (e', f')$, we obtain $p_1(g') = p_1(e', f') = e'$ (by the definition of p_1).

We have $(e, f) = g <_2 g' = (e', f')$. Due to (3), this yields $e <_{E2} e'$. In other words, $p_1(g) <_{E2} p_1(g')$ (since $p_1(g) = e$ and $p_1(g') = e'$).

Now, let us forget that we fixed g and g'. We thus have proven that if g and g' are two elements of $E \times_{\phi} F$ satisfying $g <_2 g'$, then $p_1(g) <_{E2} p_1(g')$. In other words, the map p_1 is a strictly order-preserving map $\left(E \times_{\phi} F, <_2\right) \to \left(E, <_{E2}\right)$. As a consequence, this map p_1 is a poset homomorphism $\left(E \times_{\phi} F, <_2\right) \to \left(E, <_{E2}\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

$\left(E \times_\phi F, <_1\right)$ [15] and a poset homomorphism $(E, <_{E2}) \to \left(E \times_\phi F, <_2\right)$ [16]. Hence, the map p_1^{-1} is a homomorphism of double posets $\left(E, (<_{F1})^\phi, <_{E2}\right) \to \left(E \times_\phi F, <_1, <_2\right)$.

Now, we know that the map p_1 is a homomorphism of double posets

[15] *Proof.* Let e and e' be two elements of E satisfying $e\,(<_{F1})^\phi\,e'$. We shall prove that $p_1^{-1}(e) <_1 p_1^{-1}(e')$.

We have $e\,(<_{F1})^\phi\,e'$ if and only if $\phi(e) <_{F1} \phi(e')$ (due to the definition of "$e\,(<_{F1})^\phi\,e'$"). Thus, we have $\phi(e) <_{F1} \phi(e')$ (since we have $e\,(<_{F1})^\phi\,e'$).

We have $\underbrace{p_1^{-1}}_{=(\mathrm{id},\phi)}(e) = (\mathrm{id},\phi)(e) = \left(\underbrace{\mathrm{id}(e)}_{=e}, \phi(e)\right) = (e, \phi(e))$, so that $(e,\phi(e)) = p_1^{-1}(e) \in E \times_\phi F$.

Also, $\underbrace{p_1^{-1}}_{=(\mathrm{id},\phi)}(e') = (\mathrm{id},\phi)(e') = \left(\underbrace{\mathrm{id}(e')}_{=e'}, \phi(e')\right) = (e', \phi(e'))$, so that $(e',\phi(e')) = p_1^{-1}(e') \in E \times_\phi F$.

Now, (2) (applied to $\phi(e)$ and $\phi(e')$ instead of f and f') yields $((e,\phi(e)) <_1 (e',\phi(e'))$ if and only if $\phi(e) <_{F1} \phi(e'))$ (since $(e,\phi(e)) \in E \times_\phi F$ and $(e',\phi(e')) \in E \times_\phi F$). Thus, we have $(e,\phi(e)) <_1 (e',\phi(e'))$ (since $\phi(e) <_{F1} \phi(e')$). In other words, $p_1^{-1}(e) <_1 p_1^{-1}(e')$ (since $p_1^{-1}(e) = (e,\phi(e))$ and $p_1^{-1}(e') = (e',\phi(e'))$).

Now, let us forget that we fixed e and e'. We thus have shown that if e and e' are two elements of E satisfying $e\,(<_{F1})^\phi\,e'$, then $p_1^{-1}(e) <_1 p_1^{-1}(e')$. In other words, the map p_1^{-1} is a strictly order-preserving map $\left(E, (<_{F1})^\phi\right) \to \left(E \times_\phi F, <_1\right)$. As a consequence, this map p_1^{-1} is a poset homomorphism $\left(E, (<_{F1})^\phi\right) \to \left(E \times_\phi F, <_1\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

[16] *Proof.* Let e and e' be two elements of E satisfying $e <_{E2} e'$. We shall prove that $p_1^{-1}(e) <_2 p_1^{-1}(e')$.

We have $\underbrace{p_1^{-1}}_{=(\mathrm{id},\phi)}(e) = (\mathrm{id},\phi)(e) = \left(\underbrace{\mathrm{id}(e)}_{=e}, \phi(e)\right) = (e, \phi(e))$, so that $(e,\phi(e)) = p_1^{-1}(e) \in E \times_\phi F$.

Also, $\underbrace{p_1^{-1}}_{=(\mathrm{id},\phi)}(e') = (\mathrm{id},\phi)(e') = \left(\underbrace{\mathrm{id}(e')}_{=e'}, \phi(e')\right) = (e', \phi(e'))$, so that $(e',\phi(e')) = p_1^{-1}(e') \in E \times_\phi F$.

Now, (3) (applied to $\phi(e)$ and $\phi(e')$ instead of f and f') yields $((e,\phi(e)) <_2 (e',\phi(e'))$ if and only if $e <_{E2} e')$ (since $(e,\phi(e)) \in E \times_\phi F$ and $(e',\phi(e')) \in E \times_\phi F$). Thus, we have $(e,\phi(e)) <_2 (e',\phi(e'))$ (since $e <_{E2} e'$). In other words, $p_1^{-1}(e) <_2 p_1^{-1}(e')$ (since $p_1^{-1}(e) = (e,\phi(e))$ and $p_1^{-1}(e') = (e',\phi(e'))$).

Now, let us forget that we fixed e and e'. We thus have shown that if e and e' are two elements of E satisfying $e <_{E2} e'$, then $p_1^{-1}(e) <_2 p_1^{-1}(e')$. In other words, the map p_1^{-1} is a strictly order-preserving map $(E, <_{E2}) \to \left(E \times_\phi F, <_2\right)$. As a consequence, this map p_1^{-1} is a poset homomorphism $(E, <_{E2}) \to \left(E \times_\phi F, <_2\right)$ (since any strictly order-preserving map between two posets is a poset homomorphism). Qed.

$\left(E \times_\phi F, <_1, <_2\right) \to \left(E, (<_{F1})^\phi, <_{E2}\right)$, while its inverse p_1^{-1} is a homomorphism of double posets $\left(E, (<_{F1})^\phi, <_{E2}\right) \to \left(E \times_\phi F, <_1, <_2\right)$. In other words, p_1 is an isomorphism of double posets $\left(E \times_\phi F, <_1, <_2\right) \to \left(E, (<_{F1})^\phi, <_{E2}\right)$. Thus, the double posets $\left(E \times_\phi F, <_1, <_2\right)$ and $\left(E, (<_{F1})^\phi, <_{E2}\right)$ are isomorphic. That is, we have $\left(E \times_\phi F, <_1, <_2\right) \cong \left(E, (<_{F1})^\phi, <_{E2}\right)$ as double posets. Hence,

$$E \times_\phi F = \left(E \times_\phi F, <_1, <_2\right) \cong \left(E, (<_{F1})^\phi, <_{E2}\right) \tag{5}$$

as double posets. Thus,

$$\underbrace{(E, <_{E1}, <_{E2})}_{=E} \times_\phi \underbrace{(F, <_{F1}, <_{F2})}_{=F} = E \times_\phi F \cong \left(E, (<_{F1})^\phi, <_{E2}\right)$$

as double posets. This proves Lemma 6.5 **(c)**.

(d) The definition of $E \circ F$ shows that $E \circ F$ is the sum of $E \times_\phi F$ for all increasing bijections $\phi : (E, <_{E1}) \to (F, <_{F2})$. In other words,

$$\begin{aligned} E \circ F &= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \underbrace{E \times_\phi F}_{\substack{=\left(E,(<_{F1})^\phi,<_{E2}\right) \\ \text{(according to (5))}}} \\ &= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(E, (<_{F1})^\phi, <_{E2}\right) \end{aligned}$$

in $\mathbb{Z}\mathbf{D}$. Thus,

$$\underbrace{(E, <_{E1}, <_{E2})}_{=E} \circ \underbrace{(F, <_{F1}, <_{F2})}_{=F} = E \circ F = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(E, (<_{F1})^\phi, <_{E2}\right)$$

in $\mathbb{Z}\mathbf{D}$. This proves Lemma 6.5 **(d)**. □

Now, we can finally prove Proposition 6.1:

Proof of Proposition 6.1. Let E, F and G be three elements of $\mathbb{Z}\mathbf{D}$. We have to prove the equality $(E \circ F) \circ G = E \circ (F \circ G)$. Since this equality is $\mathbb{Z}$-linear in each of E, F and G, we can WLOG assume that E, F and G are (isomorphism classes of) double posets (since (isomorphism classes of) double posets span the $\mathbb{Z}$-module $\mathbb{Z}\mathbf{D}$). Assume this. Let us write the double poset E in the form $E = (E, <_{E1}, <_{E2})$. Let us write the double poset F in the form $F = (F, <_{F1}, <_{F2})$. Let us write the double poset G in the form $(G, <_{G1}, <_{G2})$.

We have

$$\underbrace{E}_{=(E,<_{E1},<_{E2})} \circ \underbrace{F}_{=(F,<_{F1},<_{F2})}$$
$$= (E, <_{E1}, <_{E2}) \circ (F, <_{F1}, <_{F2}) = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(F, <_{F1}, (<_{E2})^{\phi^{-1}}\right)$$
$$\text{(by Lemma 6.5 (b))}$$
$$= \sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(F, <_{F1}, (<_{E2})^{\alpha^{-1}}\right)$$
$$\text{(here, we renamed the summation index } \phi \text{ as } \alpha\text{)}.$$

Thus,

$$\underbrace{(E \circ F)}_{= \sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(F,<_{F1},(<_{E2})^{\alpha^{-1}}\right)} \circ \underbrace{G}_{=(G,<_{G1},<_{G2})}$$
$$= \left(\sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(F, <_{F1}, (<_{E2})^{\alpha^{-1}}\right) \right) \circ (G, <_{G1}, <_{G2})$$
$$= \sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \underbrace{\left(F, <_{F1}, (<_{E2})^{\alpha^{-1}}\right) \circ (G, <_{G1}, <_{G2})}_{\substack{= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\phi},(<_{E2})^{\alpha^{-1}}\right) \\ \text{(by Lemma 6.5 (d), applied to } F,\ <_{F1},\ (<_{E2})^{\alpha^{-1}},\ G,\ <_{G1} \text{ and } <_{G2} \\ \text{instead of } E,\ <_{E1},\ <_{E2},\ F,\ <_{F1} \text{ and } <_{F2})}}$$
$$\text{(since the operation } \circ \text{ is } \mathbb{Z}\text{-bilinear)}$$
$$= \sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F, (<_{G1})^{\phi}, (<_{E2})^{\alpha^{-1}}\right)$$
$$= \sum_{\substack{\alpha \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F, (<_{G1})^{\beta}, (<_{E2})^{\alpha^{-1}}\right)$$
$$\text{(here, we renamed the summation index } \phi \text{ as } \beta \text{ in the inner sum)}$$
$$= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F, (<_{G1})^{\beta}, (<_{E2})^{\phi^{-1}}\right) \qquad (6)$$
$$\text{(here, we renamed the summation index } \alpha \text{ as } \phi \text{ in the outer sum)}.$$

On the other hand,

$$\underbrace{F}_{=(F,<_{F1},<_{F2})} \circ \underbrace{G}_{=(G,<_{G1},<_{G2})}$$
$$= (F,<_{F1},<_{F2}) \circ (G,<_{G1},<_{G2}) = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\phi},<_{F2}\right)$$
$$\left(\begin{array}{c}\text{by Lemma 6.5 \textbf{(d)}, applied to} \\ F,\ <_{F1},\ <_{F2},\ G,\ <_{G1} \text{ and } <_{G2} \\ \text{instead of } E,\ <_{E1},\ <_{E2},\ F,\ <_{F1} \text{ and } <_{F2}\end{array}\right)$$
$$= \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\beta},<_{F2}\right)$$
$$(\text{here, we renamed the summation index } \phi \text{ as } \beta).$$

Thus,

$$\underbrace{E}_{=(E,<_{E1},<_{E2})} \circ \underbrace{(F \circ G)}_{=\sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\beta},<_{F2}\right)}$$
$$= (E,<_{E1},<_{E2}) \circ \left(\sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\beta},<_{F2}\right)\right)$$
$$= \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \underbrace{(E,<_{E1},<_{E2}) \circ \left(F,(<_{G1})^{\beta},<_{F2}\right)}_{\substack{=\sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \left(F,(<_{G1})^{\beta},(<_{E2})^{\phi^{-1}}\right) \\ \text{(by Lemma 6.5 \textbf{(b)}, applied to } (<_{G1})^{\beta} \\ \text{instead of } <_{F1})}}$$
$$(\text{since the operation } \circ \text{ is } \mathbb{Z}\text{-bilinear})$$
$$= \underbrace{\sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}}}_{=\sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}}} \left(F,(<_{G1})^{\beta},(<_{E2})^{\phi^{-1}}\right)$$
$$= \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E,<_{E1})\to(F,<_{F2})}} \sum_{\substack{\beta \text{ is an increasing} \\ \text{bijection } (F,<_{F1})\to(G,<_{G2})}} \left(F,(<_{G1})^{\beta},(<_{E2})^{\phi^{-1}}\right).$$

Compared with (6), this yields $(E \circ F) \circ G = E \circ (F \circ G)$. This proves Proposition 6.1. □

6.2. Page 5, proof of Lemma 2.1: an alternative argument

In the proof of Lemma 2.1, you take several lines (starting with "Moreover, let $(f,g), (f',g') \in F \times_\psi G$" and ending with "Finally $e <_2 e'$") in order to prove that the map $\beta^{-1} : (F \times_\psi G, <_1) \to (E, <_2)$ is increasing. Here is a different proof of this: We have $\beta = (\mathrm{id}, \psi) \circ \phi$. Thus, β is a composition of two bijections (since (id, ψ) and ϕ are bijections), thus itself a bijection. In other words, β is invertible. For every $e \in E$, we have

$$\underbrace{\beta}_{=(\mathrm{id},\psi)\circ\phi}(e) = ((\mathrm{id},\psi)\circ\phi)(e) = (\mathrm{id},\psi)(\phi(e))$$
$$= \left(\underbrace{\mathrm{id}(\phi(e))}_{=\phi(e)}, \psi(\phi(e))\right) = (\phi(e), \psi(\phi(e))).$$

Thus, for every $e \in E$, we have

$$\begin{aligned}
&\left(\left(p_1 \circ \alpha^{-1} \circ p_2\right) \circ \beta\right)(e) \\
&= \left(p_1 \circ \alpha^{-1} \circ p_2\right)\left(\underbrace{\beta(e)}_{=(\phi(e),\psi(\phi(e)))}\right) = \left(p_1 \circ \alpha^{-1} \circ p_2\right)(\phi(e), \psi(\phi(e))) \\
&= p_1\left(\alpha^{-1}\left(\underbrace{p_2(\phi(e), \psi(\phi(e)))}_{\substack{=\psi(\phi(e)) \\ \text{(by the definition of } p_2)}}\right)\right) = p_1\left(\underbrace{\alpha^{-1}(\psi(\phi(e)))}_{=\left(\alpha^{-1}\circ\psi\right)(\phi(e))}\right) \\
&= p_1\left(\left(\alpha^{-1}\circ\psi\right)(\phi(e))\right) = p_1\left(\underbrace{\left(\phi^{-1}, \mathrm{id}\right)(\phi(e))}_{=\left(\phi^{-1}(\phi(e)), \mathrm{id}(\phi(e))\right)}\right) \\
&\qquad \left(\text{since } \alpha^{-1} \circ \underbrace{\psi}_{=\alpha\circ\left(\phi^{-1},\mathrm{id}\right)} = \alpha^{-1}\circ\alpha\circ\left(\phi^{-1},\mathrm{id}\right) = \left(\phi^{-1},\mathrm{id}\right)\right) \\
&= p_1\left(\phi^{-1}(\phi(e)), \mathrm{id}(\phi(e))\right) = \phi^{-1}(\phi(e)) \qquad \text{(by the definition of } p_1) \\
&= e = \mathrm{id}(e).
\end{aligned}$$

In other words, $\left(p_1 \circ \alpha^{-1} \circ p_2\right) \circ \beta = \mathrm{id}$. Since β is invertible, this yields that

$$\beta^{-1} = p_1 \circ \alpha^{-1} \circ p_2. \tag{7}$$

But α is a picture, and thus α^{-1} is increasing as a map from $(G, <_1)$ to $(E \times_\phi F, <_2)$. Now, we know that:

- the map p_2 is increasing as a map from $(F \times_\psi G, <_1)$ to $(G, <_1)$;
- the map α^{-1} is increasing as a map from $(G, <_1)$ to $(E \times_\phi F, <_2)$;
- the map p_1 is increasing as a map from $(E \times_\phi F, <_2)$ to $(E, <_2)$.

Combining these, we conclude that the composition $p_1 \circ \alpha^{-1} \circ p_2$ is increasing as a map from $(F \times_\psi G, <_1)$ to $(E, <_2)$. In other words, β^{-1} is increasing as a map from $(F \times_\psi G, <_1)$ to $(E, <_2)$ (since $\beta^{-1} = p_1 \circ \alpha^{-1} \circ p_2$).

6.3. Page 6, proof of Lemma 2.1: an alternative argument

In the proof of Lemma 2.1, you spend several lines to prove that the map $\alpha^{-1} : (G, <_1) \to (E \times_\phi F, <_2)$ is increasing. Here is a different proof of this:

We have $\psi = \alpha \circ (\phi^{-1}, \mathrm{id})$ [17] and $\beta = (\mathrm{id}, \psi) \circ \phi$ [18]. Thus, we can prove that (7) holds (similarly to how we did it above). In other words, $\beta^{-1} = p_1 \circ \alpha^{-1} \circ p_2$. Thus, $p_1^{-1} \circ \underbrace{\beta^{-1}}_{= p_1 \circ \alpha^{-1} \circ p_2} \circ p_2^{-1} = \underbrace{p_1^{-1} \circ p_1}_{= \mathrm{id}} \circ \alpha^{-1} \circ \underbrace{p_2 \circ p_2^{-1}}_{= \mathrm{id}} = \alpha^{-1}$, so that $\alpha^{-1} = p_1^{-1} \circ \beta^{-1} \circ p_2^{-1}$.

But β is a picture, and thus β^{-1} is increasing as a map from $(F \times_\psi G, <_1)$ to $(E, <_2)$. Now, we know that:

- the map p_2^{-1} is increasing as a map from $(G, <_1)$ to $(F \times_\psi G, <_1)$;
- the map β^{-1} is increasing as a map from $(F \times_\psi G, <_1)$ to $(E, <_2)$;
- the map p_1^{-1} is increasing as a map from $(E, <_2)$ to $(E \times_\phi F, <_2)$.

Combining these, we conclude that the composition $p_1^{-1} \circ \beta^{-1} \circ p_2^{-1}$ is increasing as a map from $(G, <_1)$ to $(E \times_\phi F, <_2)$. In other words, α^{-1} is increasing as a map from $(G, <_1)$ to $(E \times_\phi F, <_2)$ (since $\alpha^{-1} = p_1^{-1} \circ \beta^{-1} \circ p_2^{-1}$).

[17] *Proof.* We know that the inverse p_2^{-1} of the bijection $p_2 : E \times_\psi F \to F$ is given by $p_2^{-1} = (\phi^{-1}, \mathrm{id})$. Thus, $\underbrace{\alpha}_{= \psi \circ p_2} \circ \underbrace{(\phi^{-1}, \mathrm{id})}_{= p_2^{-1}} = \psi \circ \underbrace{p_2 \circ p_2^{-1}}_{= \mathrm{id}} = \psi$, qed.

[18] *Proof.* We know that the inverse p_1^{-1} of the bijection $p_1 : F \times_\psi G \to G$ is given by $p_1^{-1} = (\mathrm{id}, \psi)$. Thus, $\underbrace{(\mathrm{id}, \psi)}_{= p_1^{-1}} \circ \underbrace{\phi}_{= p_1 \circ \beta} = \underbrace{p_1^{-1} \circ p_1}_{= \mathrm{id}} \circ \beta = \beta$, qed.

6.4. Page 9, §3.1: descent compositions vs. descent sets

On page 9, you define the descent composition of a permutation $\sigma \in S_n$ in terms of its ascending runs. Let me give an alternative description of descent compositions:

Proposition 6.6. For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we denote the number $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$ by $|\alpha|$ and call it the *size* of the composition α, and we say that α is a *composition of* $|\alpha|$.

For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, define a set $D(\alpha)$ by

$$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \tag{8}$$
$$= \{\alpha_1 + \alpha_2 + \cdots + \alpha_p \mid p \in \{1, 2, \ldots, \ell - 1\}\}. \tag{9}$$

(Here, the set $\{1, 2, \ldots, q\}$ is to be understood as the empty set whenever $q < 1$. In particular, $\{1, 2, \ldots, 0 - 1\} = \varnothing$. Thus, if α is the empty composition $()$, then $D(\alpha) = \varnothing$.)

(a) For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we have

$$D(\alpha) \subseteq \{1, 2, \ldots, |\alpha| - 1\}, \tag{10}$$

$$|D(\alpha)| = \max\{\ell - 1, 0\} \tag{11}$$

and

$$(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}) = (\text{the increasing list of } D(\alpha)). \tag{12}$$

(Here, if S is any finite set of integers, then the *increasing list* of S is defined to be the list of all elements of S in increasing order (with each element of S appearing precisely once). This list is uniquely determined.)

(b) For every $n \in \mathbb{N}$ and every subset I of $\{1, 2, \ldots, n - 1\}$, there exists a unique composition α of n such that $D(\alpha) = I$.

(c) Let $n \in \mathbb{N}$. Let $\sigma \in S_n$ be a permutation. Let $\operatorname{Des} \sigma$ denote the subset

$$\{i \in \{1, 2, \ldots, n - 1\} \mid \sigma(i) > \sigma(i + 1)\}$$

of $\{1, 2, \ldots, n - 1\}$. Proposition 6.6 **(b)** (applied to $I = \operatorname{Des} \sigma$) yields that there exists a unique composition α of n such that $D(\alpha) = \operatorname{Des} \sigma$. Consider this α. Then, $\alpha = C(\sigma)$.

Proof of Proposition 6.6. **(a)** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Then, α_1, α_2, ..., α_ℓ are positive integers (since α is a composition). Now, every $p \in \{1, 2, \ldots, \ell - 1\}$ satisfies

$\alpha_1 + \alpha_2 + \cdots + \alpha_p \in \{1, 2, \ldots, |\alpha| - 1\}$ [19]. Hence,

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_p \mid p \in \{1, 2, \ldots, \ell - 1\}\} \subseteq \{1, 2, \ldots, |\alpha| - 1\}.$$

Thus, (9) becomes

$$D(\alpha) = \{\alpha_1 + \alpha_2 + \cdots + \alpha_p \mid p \in \{1, 2, \ldots, \ell - 1\}\} \subseteq \{1, 2, \ldots, |\alpha| - 1\}.$$

This proves (10).

Next, let us prove (12). Indeed, every $i \in \{1, 2, \ldots, \ell - 2\}$ satisfies $\alpha_1 + \alpha_2 + \cdots + \alpha_i < \alpha_1 + \alpha_2 + \cdots + \alpha_{i+1}$ [20]. In other words,

$$\alpha_1 < \alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 + \alpha_3 < \cdots < \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}.$$

Thus, the list $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1})$ is strictly increasing. Hence, all entries of this list are pairwise distinct.

But $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. Therefore, the list $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1})$ is a list of all elements of $D(\alpha)$. Thus, this list is a list of all elements of $D(\alpha)$ in increasing order (since it is strictly increasing). Moreover, each element of $D(\alpha)$ appears precisely once in this list (since each element of $D(\alpha)$ must appear at least once in this list (because this list is a list of all elements of $D(\alpha)$) and must appear at most once in this list (since all entries of this list are pairwise distinct)). Hence, the list $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1})$ is the list of all elements of $D(\alpha)$ in increasing order (with each element of $D(\alpha)$ appearing precisely once). In other words, the list $(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1})$ is the increasing list of $D(\alpha)$ (since the increasing list of $D(\alpha)$ is the list of all elements

[19] *Proof.* Let $p \in \{1, 2, \ldots, \ell - 1\}$. Then, $p \geq 1$ and $p \leq \ell - 1$, so that $p \geq 1 > 0$ and $\ell - \underbrace{p}_{\leq \ell - 1 < \ell} > \ell - \ell = 0$. Now, $\alpha_1, \alpha_2, \ldots, \alpha_p$ are positive integers (since $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers). Thus, $\alpha_1 + \alpha_2 + \cdots + \alpha_p$ is a nonempty sum of positive integers (in fact, it is nonempty because the number of its addends is $p > 0$), and thus a positive integer. Also, $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_\ell$ are positive integers (since $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers). Thus, $\alpha_{p+1} + \alpha_{p+2} + \cdots + \alpha_\ell$ is a nonempty sum of positive integers (in fact, it is nonempty because the number of its addends is $\ell - p > 0$), and thus a positive integer. Hence, $\alpha_{p+1} + \alpha_{p+2} + \cdots + \alpha_\ell > 0$.

Now, the definition of $|\alpha|$ yields

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = (\alpha_1 + \alpha_2 + \cdots + \alpha_p) + \underbrace{(\alpha_{p+1} + \alpha_{p+2} + \cdots + \alpha_\ell)}_{>0} > \alpha_1 + \alpha_2 + \cdots + \alpha_p.$$

Hence, $\alpha_1 + \alpha_2 + \cdots + \alpha_p < |\alpha|$. Since $\alpha_1 + \alpha_2 + \cdots + \alpha_p$ is a positive integer, this shows that $\alpha_1 + \alpha_2 + \cdots + \alpha_p$ is a positive integer smaller than $|\alpha|$. Thus, $\alpha_1 + \alpha_2 + \cdots + \alpha_p \in \{1, 2, \ldots, |\alpha| - 1\}$, qed.

[20] *Proof.* Let $i \in \{1, 2, \ldots, \ell - 2\}$. Then, α_{i+1} is a positive integer (since $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers), so that $\alpha_{i+1} > 0$. Now, $\alpha_1 + \alpha_2 + \cdots + \alpha_{i+1} = (\alpha_1 + \alpha_2 + \cdots + \alpha_i) + \underbrace{\alpha_{i+1}}_{>0} > \alpha_1 + \alpha_2 + \cdots + \alpha_i$. In other words, $\alpha_1 + \alpha_2 + \cdots + \alpha_i < \alpha_1 + \alpha_2 + \cdots + \alpha_{i+1}$, qed.

of $D(\alpha)$ in increasing order (with each element of $D(\alpha)$ appearing precisely once) (due to the definition of "increasing list")). This proves (12).

It is known that every finite subset S of $\mathbb{Z}$ satisfies

$$|S| = (\text{the length of the increasing list of } S).$$

Applying this to $S = D(\alpha)$, we obtain

$$\begin{aligned} |D(\alpha)| &= \left(\text{the length of } \underbrace{\text{the increasing list of } D(\alpha)}_{\substack{=(\alpha_1,\alpha_1+\alpha_2,\alpha_1+\alpha_2+\alpha_3,\ldots,\alpha_1+\alpha_2+\cdots+\alpha_{\ell-1}) \\ \text{(by (12))}}}\right) \\ &= (\text{the length of the list } (\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \ldots, \alpha_1+\alpha_2+\cdots+\alpha_{\ell-1})) \\ &= |\{1,2,\ldots,\ell-1\}| = \max\{\ell-1, 0\}. \end{aligned}$$

This proves (11). Thus, Proposition 6.6 **(a)** is proven.

(b) Let $n \in \mathbb{N}$. Let I be a subset of $\{1,2,\ldots,n-1\}$.

If $n = 0$, then Proposition 6.6 **(b)** is easy to prove[21]. Hence, for the rest of this proof, we can WLOG assume that we don't have $n = 0$. Assume this.

We have $n \neq 0$ (since we don't have $n = 0$) and thus $n > 0$ (since $n \in \mathbb{N}$).

Let $(i_1, i_2, \ldots, i_k)$ be the increasing list of the set I. Then, $(i_1, i_2, \ldots, i_k)$ is the list of all elements of I in increasing order (with each element of I appearing precisely once) (according to the definition of the "increasing list"). As a consequence, we have $I = \{i_1, i_2, \ldots, i_k\}$ and $i_1 < i_2 < \cdots < i_k$. Let us extend this list $(i_1, i_2, \ldots, i_k)$ to a list $(i_0, i_1, i_2, \ldots, i_k, i_{k+1})$ by defining two additional elements i_0 and i_{k+1} by $i_0 = 0$ and $i_{k+1} = n$. Notice that $i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$ [22]. Thus, $i_{p-1} < i_p$ for every $p \in \{1,2,\ldots,k+1\}$. In other words, $i_p - i_{p-1} > 0$

[21] *Proof.* Assume that $n = 0$. Then, $I \subseteq \{1,2,\ldots,n-1\} = \varnothing$, so that $I = \varnothing$. Now, there exists a composition α of 0 such that $D(\alpha) = \varnothing$ (namely, $\alpha = ()$). Also, there exists at most one composition α of 0 such that $D(\alpha) = \varnothing$ (in fact, there exists at most one composition α of 0 (namely, $()$)). Hence, there exists a unique composition α of 0 such that $D(\alpha) = \varnothing$ (since there exists a composition α of 0 such that $D(\alpha) = \varnothing$). In other words, there exists a unique composition α of n such that $D(\alpha) = I$ (since $n = 0$ and $I = \varnothing$). In other words, Proposition 6.6 **(b)** is proven.

[22] *Proof.* Let us first assume that $k \neq 0$. Then, $k \geq 1$ (since $k \in \mathbb{N}$), and thus the integers i_1 and i_k are well-defined. Now, $i_1 \in \{i_1, i_2, \ldots, i_k\} = I \subseteq \{1,2,\ldots,n-1\}$, so that $i_1 \geq 1$ and thus $i_1 > 0$. Hence, $i_0 = 0 < i_1$. Furthermore, $k \geq 1$, so that $i_k \in \{i_1, i_2, \ldots, i_k\} = I \subseteq \{1,2,\ldots,n-1\}$, thus $i_k \leq n-1 < n = i_{k+1}$.

Combining $i_0 < i_1$ with $i_1 < i_2 < \cdots < i_k$, we obtain $i_0 < i_1 < i_2 < \cdots < i_k$. Combining this with $i_k < i_{k+1}$, we obtain $i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$.

Let us now forget that we assumed that $k \neq 0$. We thus have proven $i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$ under the assumption that $k \neq 0$. Thus, for the rest of this proof, we can WLOG assume that we don't have $k \neq 0$.

We have $k = 0$ (since we don't have $k \neq 0$). Thus, $i_{k+1} = i_{0+1} = i_1$, so that $i_1 = i_{k+1} = n > 0 = i_0$, so that $i_0 < i_1$. This rewrites as $i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$ (since $k = 0$). Thus, $i_0 < i_1 < i_2 < \cdots < i_k < i_{k+1}$ is proven.

for every $p \in \{1, 2, \ldots, k+1\}$. Hence, $i_p - i_{p-1}$ is a positive integer for every $p \in \{1, 2, \ldots, k+1\}$. In other words, $i_1 - i_0$, $i_2 - i_1$, ..., $i_{k+1} - i_k$ are positive integers. Hence, $(i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)$ is a composition. Let us denote this composition by τ. We then have $\tau = (i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)$, so that τ is a composition of n [23]. Moreover, $D(\tau) = I$ [24]. Hence, τ is a composition

[23] *Proof.* From $\tau = (i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)$, we obtain

$$|\tau| = |(i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)| = (i_1 - i_0) + (i_2 - i_1) + \cdots + (i_{k+1} - i_k) = \sum_{p=1}^{k+1} (i_p - i_{p-1})$$

$$= \sum_{p=1}^{k+1} i_p - \sum_{p=1}^{k+1} i_{p-1} = \underbrace{\sum_{p=1}^{k+1} i_p}_{= \sum_{p=1}^{k} i_p + i_{k+1}} - \underbrace{\sum_{p=0}^{k} i_p}_{= i_0 + \sum_{p=1}^{k} i_p}$$

(here, we have substituted p for $p-1$ in the second sum)

$$= \left(\sum_{p=1}^{k} i_p + i_{k+1} \right) - \left(i_0 + \sum_{p=1}^{k} i_p \right) = i_{k+1} - \underbrace{i_0}_{=0} = i_{k+1} = n.$$

Thus, τ is a composition of n, qed.

[24] *Proof.* Every $p \in \{1, 2, \ldots, k\}$ satisfies

$$(i_1 - i_0) + (i_2 - i_1) + \cdots + (i_p - i_{p-1})$$

$$= \sum_{q=1}^{p} (i_q - i_{q-1}) = \sum_{q=1}^{p} i_q - \sum_{q=1}^{p} i_{q-1} = \underbrace{\sum_{q=1}^{p} i_q}_{= \sum_{q=1}^{p-1} i_q + i_p} - \underbrace{\sum_{q=0}^{p-1} i_q}_{= i_0 + \sum_{q=1}^{p-1} i_q}$$

(here, we substituted q for $q-1$ in the second sum)

$$= \left(\sum_{q=1}^{p-1} i_q + i_p \right) + \left(i_0 + \sum_{q=1}^{p-1} i_q \right) = i_p - \underbrace{i_0}_{=0} = i_p. \qquad (13)$$

Now, from (9) (applied to τ, $k+1$ and $(i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k)$ instead of α, ℓ and $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$), we obtain

$$D(\tau) = \left\{ (i_1 - i_0) + (i_2 - i_1) + \cdots + (i_p - i_{p-1}) \ \mid \ p \in \left\{ 1, 2, \ldots, \underbrace{(k+1) - 1}_{=k} \right\} \right\}$$

$$= \left\{ \underbrace{(i_1 - i_0) + (i_2 - i_1) + \cdots + (i_p - i_{p-1})}_{\substack{= i_p \\ \text{(by (13))}}} \ \mid \ p \in \{1, 2, \ldots, k\} \right\}$$

$$= \{ i_p \ \mid \ p \in \{1, 2, \ldots, k\} \} = \{ i_1, i_2, \ldots, i_k \} = I,$$

qed.

of n satisfying $D(\tau) = I$. In other words, τ is a composition α of n such that $D(\alpha) = I$.

On the other hand, there exists at most one composition α of n such that $D(\alpha) = I$. [25] Since we know that such an α exists (namely, τ), we can thus

[25] *Proof.* Let β and γ be two compositions α of n such that $D(\alpha) = I$. We shall show that $\beta = \gamma$.

We know that β is a composition α of n such that $D(\alpha) = I$. In other words, β is a composition of n and satisfies $D(\beta) = I$. Let us write the composition β in the form $(\beta_1, \beta_2, \ldots, \beta_\ell)$. Then, $|\beta| = n$ (since β is a composition of n). Since $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_\ell$ (by the definition of $|\beta|$), this rewrites as $\beta_1 + \beta_2 + \cdots + \beta_\ell = n$. If $\ell = 0$, then $\beta_1 + \beta_2 + \cdots + \beta_\ell = (\text{empty sum}) = 0$, which contradicts $\beta_1 + \beta_2 + \cdots + \beta_\ell = n \neq 0$. Hence, we cannot have $\ell = 0$. Thus, $\ell > 0$ (since $\ell \in \mathbb{N}$).

The equality (12) (applied to β and $(\beta_1, \beta_2, \ldots, \beta_\ell)$ instead of α and $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$) yields

$$\begin{aligned}
&(\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}) \\
&= \left(\text{the increasing list of } \underbrace{D(\beta)}_{=I}\right) = (\text{the increasing list of } I) \\
&= (i_1, i_2, \ldots, i_k) \qquad (\text{since } (i_1, i_2, \ldots, i_k) \text{ is the increasing list of } I).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\underbrace{0}_{=i_0}, \underbrace{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}}_{=(i_1, i_2, \ldots, i_k)}, \underbrace{\beta_1 + \beta_2 + \cdots + \beta_\ell}_{=n=i_{k+1}}\right) \\
&= (i_0, i_1, i_2, \ldots, i_k, i_{k+1}).
\end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}
&\left(\text{the length of the list } \underbrace{(0, \beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}, \beta_1 + \beta_2 + \cdots + \beta_\ell)}_{=(i_0, i_1, i_2, \ldots, i_k, i_{k+1})}\right) \\
&= (\text{the length of the list } (i_0, i_1, i_2, \ldots, i_k, i_{k+1})) = k + 2,
\end{aligned}$$

so that

$$\begin{aligned}
&k + 2 \\
&= (\text{the length of the list } (0, \beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{\ell-1}, \beta_1 + \beta_2 + \cdots + \beta_\ell)) \\
&= \ell + 1
\end{aligned}$$

and therefore $k = \ell - 1$. Now, (14) yields

$$\beta_1 + \beta_2 + \cdots + \beta_p = i_p \qquad \text{for every } p \in \{0, 1, \ldots, \ell\}. \tag{15}$$

Now, let us fix $p \in \{1, 2, \ldots, \ell\}$. Then, (15) yields $\beta_1 + \beta_2 + \cdots + \beta_p = i_p$. Meanwhile, (15) (applied to $p - 1$ instead of p) yields $\beta_1 + \beta_2 + \cdots + \beta_{p-1} = i_{p-1}$. Now, $\underbrace{(\beta_1 + \beta_2 + \cdots + \beta_p)}_{=i_p} - \underbrace{(\beta_1 + \beta_2 + \cdots + \beta_{p-1})}_{=i_{p-1}} = i_p - i_{p-1}$, so that

$$i_p - i_{p-1} = (\beta_1 + \beta_2 + \cdots + \beta_p) - (\beta_1 + \beta_2 + \cdots + \beta_{p-1}) = \beta_p.$$

conclude that there exists a unique composition α of n such that $D(\alpha) = I$. This proves Proposition 6.6 **(b)**.

(c) We know that α is the unique composition of n such that $D(\alpha) = \operatorname{Des}\sigma$. Thus, α is a composition of n and satisfies $D(\alpha) = \operatorname{Des}\sigma$.

Let us write the composition α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Then, $|\alpha| = |(\alpha_1, \alpha_2, \ldots, \alpha_\ell)| = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$, so that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = |\alpha| = n$ (since α is a composition of n).

From (12), we obtain

$$\begin{aligned}
&(\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}) \\
&= \left(\text{the increasing list of } \underbrace{D(\alpha)}_{=\operatorname{Des}\sigma}\right) = (\text{the increasing list of } \operatorname{Des}\sigma).
\end{aligned}$$

Let us first prove that the word $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ can be split into ℓ consecutive ascending runs of lengths $\alpha_1, \alpha_2, \ldots, \alpha_\ell$.

For every $p \in \{0, 1, \ldots, \ell\}$, let i_p denote the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_p$. Then, the definition of i_0 yields $i_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_0 = (\text{empty sum}) = 0$. Also, the definition of i_ℓ yields $i_\ell = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell = n$. Every $p \in \{1, 2, \ldots, \ell\}$ satisfies

$$i_p - i_{p-1} = \alpha_p. \tag{16}$$

[26] Moreover, $i_0 < i_1 < \cdots < i_\ell$ [27].

Thus, $\beta_p = i_p - i_{p-1}$.

Let us now forget that we fixed p. We thus have shown that $\beta_p = i_p - i_{p-1}$ for every $p \in \{1, 2, \ldots, \ell\}$. Thus,

$$\begin{aligned}
(\beta_1, \beta_2, \ldots, \beta_\ell) &= (i_1 - i_0, i_2 - i_1, \ldots, i_\ell - i_{\ell-1}) = \left(i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - \underbrace{i_{(k+1)-1}}_{=i_k}\right) \\
&\qquad (\text{since } \ell = k+1 \text{ (since } k = \ell - 1)) \\
&= (i_1 - i_0, i_2 - i_1, \ldots, i_{k+1} - i_k) = \tau,
\end{aligned}$$

so that $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell) = \tau$.

We thus have shown that $\beta = \tau$. The same argument (but applied to γ instead of β) shows that $\gamma = \tau$. Thus, $\beta = \gamma$.

Let us now forget that we fixed β and γ. We thus have shown that if β and γ are two compositions α of n such that $D(\alpha) = I$, then $\beta = \gamma$. In other words, there exists at most one composition α of n such that $D(\alpha) = I$. This completes the proof.

[26] *Proof of (16):* Let $p \in \{1, 2, \ldots, \ell\}$. Then, the definition of i_p yields $i_p = \alpha_1 + \alpha_2 + \cdots + \alpha_p$. But the definition of i_{p-1} yields $i_{p-1} = \alpha_1 + \alpha_2 + \cdots + \alpha_{p-1}$. Hence,

$$i_p - i_{p-1} = (\alpha_1 + \alpha_2 + \cdots + \alpha_p) - (\alpha_1 + \alpha_2 + \cdots + \alpha_{p-1}) = \alpha_p.$$

This proves (16).

[27] *Proof.* Let $p \in \{1, 2, \ldots, \ell\}$. Then, α_p is a positive integer (since $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are positive integers (since $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition)). Now, (16) yields $i_p - i_{p-1} = \alpha_p > 0$ (since

But for every $q \in \{1, 2, \ldots, \ell\}$, the list $\left(\sigma\left(i_{q-1}+1\right), \sigma\left(i_{q-1}+2\right), \ldots, \sigma\left(i_q\right)\right)$ is strictly increasing[28]. Thus, for every $q \in \{1, 2, \ldots, \ell\}$, the list $\left(\sigma\left(i_{q-1}+1\right), \sigma\left(i_{q-1}+2\right), \ldots, \sigma\left(i_q\right)\right)$ is an ascending run of the word σ. This

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α_p is a positive integer). In other words, $i_{p-1} < i_p$.

Let us now forget that we fixed p. We thus have proven that $i_{p-1} < i_p$ for every $p \in \{1, 2, \ldots, \ell\}$. In other words, $i_0 < i_1 < \cdots < i_\ell$, qed.

[28] *Proof.* Let $q \in \{1, 2, \ldots, \ell\}$. Since $q \leq \ell$ and $i_0 < i_1 < \cdots < i_\ell$, we have $i_q \leq i_\ell = n$. Because of this, and because of $\underbrace{i_{q-1}}_{\geq 0} + 1 \geq 1$, it is clear that each of the integers $i_{q-1}+1$, $i_{q-1}+2$, $\ldots$, i_q belongs to $\{1, 2, \ldots, n\}$. Hence, the list $\left(\sigma\left(i_{q-1}+1\right), \sigma\left(i_{q-1}+2\right), \ldots, \sigma\left(i_q\right)\right)$ is well-defined.

Now, let $g \in \left\{1, 2, \ldots, i_q - i_{q-1} - 1\right\}$. We shall now show that $\sigma\left(i_{q-1}+g\right) < \sigma\left(i_{q-1}+(g+1)\right)$.

Indeed, assume the contrary. Then, $\sigma\left(i_{q-1}+g\right) \geq \sigma\left(i_{q-1}+(g+1)\right)$. Let $h = i_{q-1}+g$. Then,

$$\sigma\left(\underbrace{h}_{=i_{q-1}+g}\right) = \sigma\left(i_{q-1}+g\right) \geq \sigma\left(i_{q-1}+(g+1)\right) = \sigma\left(\underbrace{i_{q-1}+g}_{=h}+1\right) = \sigma(h+1).$$

Moreover, recall that $g \in \left\{1, 2, \ldots, i_q - i_{q-1} - 1\right\}$. Thus, $g \geq 1$ and $g \leq i_q - i_{q-1} - 1$. Combining $h = i_{q-1} + \underbrace{g}_{\geq 1} \geq \underbrace{i_{q-1}}_{\geq 0} + 1 \geq 1$ with $h = i_{q-1} + \underbrace{g}_{\leq i_q - i_{q-1} - 1} \leq i_{q-1} + \left(i_q - i_{q-1} - 1\right) = \underbrace{i_q}_{\leq n} - 1 \leq n-1$, we obtain $1 \leq h \leq n-1$, so that $h \in \{1, 2, \ldots, n-1\}$.

The map σ is a permutation, thus bijective, thus injective. Hence, $\sigma(u) \neq \sigma(v)$ for any two distinct elements u and v of $\{1, 2, \ldots, n\}$. Applying this to $u = h$ and $v = h+1$, we obtain $\sigma(h) \neq \sigma(h+1)$. Combined with $\sigma(h) \geq \sigma(h+1)$, we obtain $\sigma(h) > \sigma(h+1)$. Now, we know that h is an element of $\{1, 2, \ldots, n-1\}$ satisfying $\sigma(h) > \sigma(h+1)$. In other words,

$$\begin{aligned} h &\in \{i \in \{1, 2, \ldots, n-1\} \mid \sigma(i) > \sigma(i+1)\} = \operatorname{Des}\sigma \\ &\qquad \left(\begin{array}{c}\text{since } \operatorname{Des}\sigma = \{i \in \{1, 2, \ldots, n-1\} \mid \sigma(i) > \sigma(i+1)\} \\ \text{(by the definition of } \operatorname{Des}\sigma)\end{array}\right) \\ &= D(\alpha) = \left\{\alpha_1 + \alpha_2 + \cdots + \alpha_p \mid p \in \{1, 2, \ldots, \ell-1\}\right\} \qquad \text{(by (9))}. \end{aligned}$$

In other words, there exists some $p \in \{1, 2, \ldots, \ell-1\}$ such that $h = \alpha_1 + \alpha_2 + \cdots + \alpha_p$. Let us consider this p.

The definition of i_p yields $i_p = \alpha_1 + \alpha_2 + \cdots + \alpha_p = h$ (since $h = \alpha_1 + \alpha_2 + \cdots + \alpha_p$).

Now, let us assume (for the sake of contradiction) that $p \leq q-1$. Then, $i_p \leq i_{q-1}$ (since $i_0 < i_1 < \cdots < i_\ell$); but this contradicts $i_p = h \geq i_{q-1} + 1 > i_{q-1}$. This contradiction shows that our assumption (that $p \leq q-1$) was wrong. Hence, we cannot have $p \leq q-1$. We thus must have $p > q-1$. Since p and $q-1$ are integers, this yields $p \geq (q-1)+1 = q$. Thus, $q \leq p$, so that $i_q \leq i_p$ (since $i_0 < i_1 < \cdots < i_\ell$), so that $i_q \leq i_p = h \leq i_q - 1 < i_q$. But this is absurd. This contradiction shows that our assumption was wrong. Hence, $\sigma\left(i_{q-1}+g\right) < \sigma\left(i_{q-1}+(g+1)\right)$ is proven.

Let us now forget that we fixed g. Thus we have proven that $\sigma\left(i_{q-1}+g\right) < \sigma\left(i_{q-1}+(g+1)\right)$ for every $g \in \left\{1, 2, \ldots, i_q - i_{q-1} - 1\right\}$. In other words, the list $\left(\sigma\left(i_{q-1}+1\right), \sigma\left(i_{q-1}+2\right), \ldots, \sigma\left(i_q\right)\right)$ is strictly increasing. Qed.

ascending run has length α_q [29]. Hence, for every $q \in \{1, 2, \ldots, \ell\}$, the list $(\sigma(i_{q-1}+1), \sigma(i_{q-1}+2), \ldots, \sigma(i_q))$ is an ascending run of the word σ having length α_q. Thus, for q ranging over the set $\{1, 2, \ldots, \ell\}$, these lists $(\sigma(i_{q-1}+1), \sigma(i_{q-1}+2), \ldots, \sigma(i_q))$ are ℓ ascending runs of the word σ having lengths $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. These ℓ ascending runs are clearly consecutive, and cover the whole word σ (since $i_0 = 0$ and $i_\ell = n$). Hence, the word σ can be split into ℓ consecutive ascending runs of lengths $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ (namely, into the ascending runs $(\sigma(i_{q-1}+1), \sigma(i_{q-1}+2), \ldots, \sigma(i_q))$ for $q \in \{1, 2, \ldots, \ell\}$). In particular, the word σ has ℓ consecutive ascending runs of lengths $\alpha_1, \alpha_2, \ldots, \alpha_\ell$.

Let now $(c_1, c_2, \ldots, c_k)$ be any composition of n such that the word σ has k consecutive ascending runs of lengths $c_1, c_2, \ldots, c_k$, and such that $k \leq \ell$. We shall prove that $(c_1, c_2, \ldots, c_k) = \alpha$.

This is obvious if $n = 0$ [30]. Hence, for the rest of the proof of $(c_1, c_2, \ldots, c_k) = \alpha$, we can WLOG assume that we don't have $n = 0$. Assume this. We have $n \geq 1$ (since $n \in \mathbb{N}$ and since we don't have $n = 0$). Hence, $\ell \geq 1$ [31] and $k \geq 1$ [32]. Thus, $\ell - 1 \geq 0$ (since $\ell \geq 1$) and $k - 1 \geq 0$ (and $k \geq 1$).

We know that $(c_1, c_2, \ldots, c_k)$ is a composition of n. Thus, $|(c_1, c_2, \ldots, c_k)| = n$, so that $n = |(c_1, c_2, \ldots, c_k)| = c_1 + c_2 + \cdots + c_k$.

The word σ has k consecutive ascending runs of lengths $c_1, c_2, \ldots, c_k$. In other words, there exist $k+1$ elements $j_0, j_1, \ldots, j_k$ of $\{0, 1, \ldots, n\}$ satisfying the following properties:

- We have $j_0 \leq j_1 \leq \cdots \leq j_k$.
- For every $q \in \{1, 2, \ldots, k\}$, the list $(\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))$ is an ascending run of σ having length c_q.

[29] *Proof.* Fix $q \in \{1, 2, \ldots, \ell\}$. We need to show that the list $(\sigma(i_{q-1}+1), \sigma(i_{q-1}+2), \ldots, \sigma(i_q))$ has length α_q.

But we have $i_{q-1} < i_q$ (since $i_0 < i_1 < \cdots < i_\ell$). Thus, the list $(\sigma(i_{q-1}+1), \sigma(i_{q-1}+2), \ldots, \sigma(i_q))$ has length $i_q - i_{q-1} = \alpha_q$ (by (16), applied to $p = q$). This completes our proof.

[30] *Proof.* Assume that $n = 0$. Then, $(c_1, c_2, \ldots, c_k)$ is a composition of n. In other words, $(c_1, c_2, \ldots, c_k)$ is a composition of 0 (since $n = 0$). Hence, $(c_1, c_2, \ldots, c_k) = \varnothing$ (since $\varnothing$ is the only composition of 0). On the other hand, α is a composition of n. In other words, α is a composition of 0 (since $n = 0$). Hence, $\alpha = \varnothing$ (since $\varnothing$ is the only composition of 0). Thus, $(c_1, c_2, \ldots, c_k) = \varnothing = \alpha$, qed.

[31] *Proof.* We have $i_\ell = n \geq 1 > 0 = i_0$, so that $i_\ell \neq i_0$ and thus $\ell \neq 0$. Since $\ell \in \mathbb{N}$, this yields $\ell \geq 1$, qed.

[32] *Proof.* Assume the contrary. Then, $k < 1$. Since $k \in \mathbb{N}$, this shows that $k = 0$. Hence, $|(c_1, c_2, \ldots, c_k)| = \Big|\underbrace{(c_1, c_2, \ldots, c_0)}_{=()=\varnothing}\Big| = |\varnothing| = 0$. Compared with $|(c_1, c_2, \ldots, c_k)| = n$ (since $(c_1, c_2, \ldots, c_k)$ is a composition of n), this yields $n = 0$, so that $0 = n \geq 1$. But this is absurd. This contradiction shows that our assumption was wrong, qed.

Consider these elements $j_0, j_1, \ldots, j_k$. [33] It is easy to see that

$$c_q = j_q - j_{q-1} \qquad \text{for every } q \in \{1, 2, \ldots, k\}. \tag{17}$$

[34] It is now easy to see that $(j_0 = 0 \text{ and } j_k = n)$ [35]. Furthermore,

$$\operatorname{Des} \sigma \subseteq \{j_1, j_2, \ldots, j_{k-1}\} \tag{19}$$

[36]. But $\left| \underbrace{\operatorname{Des} \sigma}_{= D(\alpha)} \right| = |D(\alpha)| = \max\{\ell - 1, 0\}$ (by (11)). But $\ell - 1 \geq 0$ and thus $\max\{\ell - 1, 0\} = \ell - 1$. Hence, $|\operatorname{Des} \sigma| = \max\{\ell - 1, 0\} = \ell - 1$. But

[33] Visually speaking, these elements $j_0, j_1, \ldots, j_k$ are the borders of the k ascending runs.

[34] *Proof of (17):* Let $q \in \{1, 2, \ldots, k\}$. Then, one of the two above properties of the elements $j_0, j_1, \ldots, j_k$ shows that the list $(\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))$ is an ascending run of σ having length c_q. Hence, this list $(\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))$ has length c_q. In other words,

$$(\text{the length of the list } (\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))) = c_q. \tag{18}$$

But $j_{q-1} \leq j_q$ (since $j_0 \leq j_1 \leq \cdots \leq j_k$). Hence, the list $(\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))$ has length $j_q - j_{q-1}$. In other words,

$$(\text{the length of the list } (\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))) = j_q - j_{q-1}.$$

Compared with (18), this yields $c_q = j_q - j_{q-1}$. This proves (17).

[35] *Proof.* We know that $j_0, j_1, \ldots, j_k$ are elements of $\{0, 1, \ldots, n\}$. In particular, j_0 and j_k are elements of $\{0, 1, \ldots, n\}$. Since j_k is an element of $\{0, 1, \ldots, n\}$, we have $j_k \leq n$, so that $n \geq j_k$. Combined with

$$\begin{aligned} n &= c_1 + c_2 + \cdots + c_k = \sum_{q=1}^{k} \underbrace{c_q}_{\substack{= j_q - j_{q-1} \\ \text{(by (17))}}} = \sum_{q=1}^{k} (j_q - j_{q-1}) = \sum_{q=1}^{k} j_q - \sum_{q=1}^{k} j_{q-1} \\ &= \underbrace{\sum_{q=1}^{k} j_q}_{= \sum_{q=1}^{k-1} j_q + j_k} - \underbrace{\sum_{q=0}^{k-1} j_q}_{= j_0 + \sum_{q=1}^{k-1} j_q} \qquad \text{(here, we substituted } q \text{ for } q-1 \text{ in the second sum)} \\ &= \left(\sum_{q=1}^{k-1} j_q + j_k \right) - \left(j_0 + \sum_{q=1}^{k-1} j_q \right) = j_k - \underbrace{j_0}_{\substack{\geq 0 \\ \text{(since } j_0 \in \{0,1,\ldots,n\})}} \leq j_k, \end{aligned}$$

this yields $n = j_k$. Thus, $j_k = n$. Now, $n = \underbrace{j_k}_{=n} - j_0 = n - j_0$, so that $0 = -j_0$ and thus $j_0 = 0$.

Hence, $(j_0 = 0 \text{ and } j_k = n)$, qed.

[36] *Proof of (19):* Let $h \in \operatorname{Des} \sigma$. Then,

$$h \in \operatorname{Des} \sigma = \{i \in \{1, 2, \ldots, n-1\} \mid \sigma(i) > \sigma(i+1)\}$$

(by the definition of $\operatorname{Des} \sigma$). In other words, h is an element of $\{1, 2, \ldots, n-1\}$ and satisfies

$|\{j_1, j_2, \ldots, j_{k-1}\}| \leq k-1$ (since $j_1, j_2, \ldots, j_{k-1}$ are $k-1$ integers (since $k-1 \geq 0$)). Now,

$$|\operatorname{Des}\sigma| = \underbrace{\ell}_{\substack{\geq k \\ (\text{since } k \leq \ell)}} - 1 \geq k-1 \geq |\{j_1, j_2, \ldots, j_{k-1}\}|$$

(since $|\{j_1, j_2, \ldots, j_{k-1}\}| \leq k-1$). But it is clear that if S is a finite set, and if T is a subset of S satisfying $|T| \geq |S|$, then we must have $T = S$. Applying this to $S = \{j_1, j_2, \ldots, j_{k-1}\}$ and $T = \operatorname{Des}\sigma$, we obtain

$$\operatorname{Des}\sigma = \{j_1, j_2, \ldots, j_{k-1}\} \tag{21}$$

(since $\operatorname{Des}\sigma \subseteq \{j_1, j_2, \ldots, j_{k-1}\}$ and $|\operatorname{Des}\sigma| \geq |\{j_1, j_2, \ldots, j_{k-1}\}|$).

$\sigma(h) > \sigma(h+1)$.

We have $h \in \{1, 2, \ldots, n-1\}$, so that $1 \leq h \leq n-1$. Now, $h \leq n-1 < n = j_k$ (since $j_k = n$). Thus, there exists at least one $q \in \{0, 1, \ldots, k\}$ satisfying $h \leq j_q$ (namely, $q = k$). Hence, there exists the **smallest** $q \in \{0, 1, \ldots, k\}$ satisfying $h \leq j_q$. Let us denote this smallest q by r. Thus, r is an element $q \in \{0, 1, \ldots, k\}$ satisfying $h \leq j_q$. In other words, r is an element of $\{0, 1, \ldots, k\}$ satisfying $h \leq j_r$. We have $j_0 = 0 < 1 \leq h \leq j_r$, so that $j_0 \neq j_r$ and thus $0 \neq r$. Hence, $r \neq 0$. Since $r \in \{0, 1, \ldots, k\}$ and $r \neq 0$, we have $r \in \{0, 1, \ldots, k\} \setminus \{0\} = \{1, 2, \ldots, k\}$. Thus, $r-1 \in \{0, 1, \ldots, k-1\} \subseteq \{0, 1, \ldots, k\}$.

But

$$\text{every } q \in \{0, 1, \ldots, k\} \text{ satisfying } h \leq j_q \text{ must satisfy } q \geq r \tag{20}$$

(because r is the **smallest** $q \in \{0, 1, \ldots, k\}$ satisfying $h \leq j_q$).

Let us assume (for the sake of contradiction) that $h \leq j_{r-1}$. Then, (20) (applied to $q = r-1$) yields $r-1 \geq r$ (since $r-1 \in \{0, 1, \ldots, k\}$). But this is absurd. This contradiction shows that our assumption (that $h \leq j_{r-1}$) is false. Thus, we do not have $h \leq j_{r-1}$. Hence, we have $h > j_{r-1}$. Thus, $h \geq j_{r-1} + 1$ (since h and j_{r-1} are integers), so that $h - j_{r-1} \geq 1$.

Let us assume (for the sake of contradiction) that $h \neq j_r$. Combined with $h \leq j_r$, this yields $h < j_r$. Thus, $h \leq j_r - 1$ (since h and j_r are integers), so that $\underbrace{h}_{\leq j_r - 1} - j_{r-1} \leq j_r - 1 - j_{r-1} = j_r - j_{r-1} - 1$. Combined with $h - j_{r-1} \geq 1$, this yields $1 \leq h - j_{r-1} \leq j_r - j_{r-1} - 1$, so that $h - j_{r-1} \in \{1, 2, \ldots, j_r - j_{r-1} - 1\}$.

But recall that, for every $q \in \{1, 2, \ldots, k\}$, the list $(\sigma(j_{q-1}+1), \sigma(j_{q-1}+2), \ldots, \sigma(j_q))$ is an ascending run of σ having length c_q. Applying this to $q = r$, we conclude that the list $(\sigma(j_{r-1}+1), \sigma(j_{r-1}+2), \ldots, \sigma(j_r))$ is an ascending run of σ having length c_r. In particular, this list is an ascending run, and therefore increasing. In other words, every $g \in \{1, 2, \ldots, j_r - j_{r-1} - 1\}$ satisfies $\sigma(j_{r-1} + g) \leq \sigma(j_{r-1} + (g+1))$. Applying this to $g = h - j_{r-1}$, we obtain

$$\sigma(j_{r-1} + (h - j_{r-1})) \leq \sigma(j_{r-1} + ((h - j_{r-1}) + 1))$$

(since $h - j_{r-1} \in \{1, 2, \ldots, j_r - j_{r-1} - 1\}$). Since $j_{r-1} + (h - j_{r-1}) = h$ and $j_{r-1} + ((h - j_{r-1}) + 1) = h+1$, this rewrites as $\sigma(h) \leq \sigma(h+1)$. This contradicts $\sigma(h) > \sigma(h+1)$.

This contradiction shows that our assumption (that $h \neq j_r$) was wrong. Hence, we must have $h = j_r$. Thus, $j_r = h < j_k$, so that $j_r \neq j_k$ and thus $r \neq k$. Combining this with $r \in \{1, 2, \ldots, k\}$, we obtain $r \in \{1, 2, \ldots, k\} \setminus \{k\} = \{1, 2, \ldots, k-1\}$. Now, $h = j_r \in \{j_1, j_2, \ldots, j_{k-1}\}$ (since $r \in \{1, 2, \ldots, k-1\}$).

Now, let us forget that we fixed h. We thus have shown that every $h \in \operatorname{Des}\sigma$ satisfies $h \in \{j_1, j_2, \ldots, j_{k-1}\}$. In other words, $\operatorname{Des}\sigma \subseteq \{j_1, j_2, \ldots, j_{k-1}\}$. This proves (19).

On the other hand, $D\left(\left(c_1, c_2, \ldots, c_k\right)\right) = \left\{j_1, j_2, \ldots, j_{k-1}\right\}$ [37]. Compared with (21), this yields $D\left(\left(c_1, c_2, \ldots, c_k\right)\right) = \operatorname{Des} \sigma$.

Now, recall that α was defined as the unique composition of n such that $D(\alpha) = \operatorname{Des} \sigma$. In other words, α is the unique composition β of n such that $D(\beta) = \operatorname{Des} \sigma$. The "uniqueness" part of this statement shows that every composition β of n such that $D(\beta) = \operatorname{Des} \sigma$ must satisfy $\beta = \alpha$. Applying this to $\beta = \left(c_1, c_2, \ldots, c_k\right)$, we obtain $\left(c_1, c_2, \ldots, c_k\right) = \alpha$ (since $\left(c_1, c_2, \ldots, c_k\right)$ is a composition of n and satisfies $D\left(\left(c_1, c_2, \ldots, c_k\right)\right) = \operatorname{Des} \sigma$). Thus, $\left(c_1, c_2, \ldots, c_k\right) = \alpha$ is proven.

Now, let us forget that we fixed $\left(c_1, c_2, \ldots, c_k\right)$. We thus have shown the following fact:

> *Fact A:* If $\left(c_1, c_2, \ldots, c_k\right)$ is any composition of n such that the word σ has k consecutive ascending runs of lengths c_1, c_2, ..., c_k, and such that $k \leq \ell$, then $\left(c_1, c_2, \ldots, c_k\right) = \alpha$.

We also know that $\alpha = \left(\alpha_1, \alpha_2, \ldots, \alpha_\ell\right)$ is a composition of n with the property that the word σ has ℓ consecutive ascending runs of lengths α_1, α_2, ..., α_ℓ. In other words, α is a composition $\left(c_1, c_2, \ldots, c_k\right)$ of n such that the word σ has k consecutive ascending runs of lengths c_1, c_2, ..., c_k. Moreover, among all such compositions $\left(c_1, c_2, \ldots, c_k\right)$, the composition α is the one with the minimum k (because of Fact A, and since α has $k = \ell$). In other words, we have proven the following fact:

[37] *Proof.* For every $p \in \{1, 2, \ldots, k-1\}$, we have

$$\begin{aligned} c_1 + c_2 + \cdots + c_p &= \sum_{q=1}^{p} \underbrace{c_q}_{\substack{= j_q - j_{q-1} \\ \text{(by (17))}}} = \sum_{q=1}^{p} \left(j_q - j_{q-1}\right) = \sum_{q=1}^{p} j_q - \sum_{q=1}^{p} j_{q-1} \\ &= \underbrace{\sum_{q=1}^{p} j_q}_{= \sum_{q=1}^{p-1} j_q + j_p} - \underbrace{\sum_{q=0}^{p-1} j_q}_{= j_0 + \sum_{q=1}^{p-1} j_q} \qquad \text{(here, we substituted } q \text{ for } q-1 \text{ in the second sum)} \\ &= \left(\sum_{q=1}^{p-1} j_q + j_p\right) - \left(j_0 + \sum_{q=1}^{p-1} j_q\right) = j_p - \underbrace{j_0}_{=0} = j_p. \end{aligned} \qquad (22)$$

Now, applying (9) to $\left(c_1, c_2, \ldots, c_k\right)$, k and c_i instead of α, ℓ and α_i, we obtain

$$\begin{aligned} D\left(\left(c_1, c_2, \ldots, c_k\right)\right) &= \left\{ \underbrace{c_1 + c_2 + \cdots + c_p}_{\substack{= j_p \\ \text{(by (22))}}} \mid p \in \{1, 2, \ldots, k-1\} \right\} \\ &= \left\{ j_p \mid p \in \{1, 2, \ldots, k-1\} \right\} = \left\{ j_1, j_2, \ldots, j_{k-1} \right\}. \end{aligned}$$

This proves (21).

> *Fact B:* Among all compositions $(c_1, c_2, \ldots, c_k)$ of n such that the word σ has k consecutive ascending runs of lengths c_1, c_2, ..., c_k, the composition α is the one with the minimum k.

Recall the definition of $C(\sigma)$. This definition says that, among all compositions $(c_1, c_2, \ldots, c_k)$ of n such that the word σ has k consecutive ascending runs of lengths c_1, c_2, ..., c_k, the composition $C(\sigma)$ is the one with the minimum k. Comparing this description of $C(\sigma)$ with the description of α given in Fact B, we obtain that $\alpha = C(\sigma)$. This proves Proposition 6.6 **(c)**. □

6.5. Page 9, Corollary 3.3: a proof

For the sake of completeness, let me give a proof of Corollary 3.3 using [GriRei15, Theorem 5.2.11]. But first, let me recall your definition of the notion of a π-partition:

Definition 6.7. Let $\pi = (E, <_1, <_2)$ be a double poset. Let X be a totally ordered set. A *π-partition* into X means a map $x : E \to X$ such that any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ implies } x(e) \leq x(e')\right)$$

and

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } x(e) < x(e')\right).$$

Proof of Corollary 3.3. Let π be a special double poset. We shall show that $\Gamma(\pi) = (F \circ L)(\pi)$.

We use the definition of **QSym** given in [GriRei15, Definition 5.1.5] (where it is denoted by QSym). The definition of $\Gamma(\pi)$ then rewrites as follows:

$$\Gamma(\pi) = \sum_{\substack{f \text{ is a } \pi\text{-partition} \\ \text{into } \{1,2,3,\ldots\}}} \prod_{e \in E} x_{f(e)}. \tag{23}$$

Write π in the form $\pi = (E, <_1, <_2)$. Then, $<_2$ is a total order (since π is special). Hence, we can WLOG assume that $E = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, and that the second order $<_2$ of π is the natural order on $\{1, 2, \ldots, n\}$ (since we can achieve this by relabelling the elements of E). Assume this. Then, $(E, <_1)$ becomes a labelled poset in the sense of [GriRei15, Definition 5.2.1] (since $E \subseteq \mathbb{Z}$). Also, $<_2$ is the natural order on $\{1, 2, \ldots, n\}$. In other words, $<_2$ is the restriction of $<_{\mathbb{Z}}$ to E (where $<_{\mathbb{Z}}$ is the natural order on $\mathbb{Z}$). Thus, $\geq_2$ is the restriction of $\geq_{\mathbb{Z}}$ to E.

We recall the definition of a P-partition (for P being a labelled poset) given in [GriRei15, Definition 5.2.1]. It is easy to see that the $(E, <_1)$-partitions are the same as the π-partitions into $\{1, 2, 3, \ldots\}$ [38]. But now recall the definition of

[38]*Proof.* Let x be a π-partition into $\{1, 2, 3, \ldots\}$. By the definition of a π-partition, we then know that x is a map $E \to \{1, 2, 3, \ldots\}$ such that any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ implies } x(e) \leq x(e')\right) \quad (24)$$

and

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } x(e) < x(e')\right) \quad (25)$$

(since x is a π-partition). Thus, any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ and } e \geq_{\mathbb{Z}} e' \text{ implies } x(e) < x(e')\right). \quad (26)$$

(*Proof of (26):* Let e and e' be two elements of E. Assume that $e <_1 e'$ and $e \geq_{\mathbb{Z}} e'$. We have $e \geq_{\mathbb{Z}} e'$. In other words, $e \geq_2 e'$ (since $\geq_2$ is the restriction of $\geq_{\mathbb{Z}}$ to E). Thus, (25) shows that $x(e) < x(e')$. This proves (26).)

Now, if i and j are two elements of E, then

$$(i <_1 j \text{ and } i <_{\mathbb{Z}} j \text{ implies } x(i) \leq x(j)) \quad (27)$$

(according to (24), applied to $e = i$ and $e' = j$) and

$$(i <_1 j \text{ and } i >_{\mathbb{Z}} j \text{ implies } x(i) < x(j)) \quad (28)$$

(according to (26), applied to $e = i$ and $e' = j$). In other words, x is an $(E, <_1)$-partition (due to the definition of an $(E, <_1)$-partition).

Now, let us forget that we fixed x. We thus have shown that if x is a π-partition into $\{1, 2, 3, \ldots\}$, then x is an $(E, <_1)$-partition. In other words,

$$\text{every } \pi\text{-partition into } \{1, 2, 3, \ldots\} \text{ is an } (E, <_1)\text{-partition.} \quad (29)$$

Let now y be an $(E, <_1)$-partition. According to the definition of an $(E, <_1)$-partition, we thus know that y is a map $E \to \{1, 2, 3, \ldots\}$ such that any two elements i and j of E satisfy

$$(i <_1 j \text{ and } i <_{\mathbb{Z}} j \text{ implies } y(i) \leq y(j)) \quad (30)$$

and

$$(i <_1 j \text{ and } i >_{\mathbb{Z}} j \text{ implies } y(i) < y(j)) \quad (31)$$

(since y is an $(E, <_1)$-partition).

Now, let e and e' be two elements of E.

Assume that $e <_1 e'$. We shall now prove that $y(e) \leq y(e')$. If $e <_{\mathbb{Z}} e'$, then this follows immediately from (30) (applied to $e = i$ and $e' = j$). Hence, for the rest of this proof, we can WLOG assume that we don't have $e <_{\mathbb{Z}} e'$. Assume this. We have $e \geq_{\mathbb{Z}} e'$ (since we don't have $e <_{\mathbb{Z}} e'$). But $e \neq e'$ (since $e <_1 e'$, and since $<_1$ is a partial order). Combined with $e \geq_{\mathbb{Z}} e'$, this yields $e >_{\mathbb{Z}} e'$. Thus, $y(e) < y(e')$ (by (31)), so that $y(e) \leq y(e')$. Thus, $y(e) \leq y(e')$ is proven.

Let us now forget that we assumed that $e <_1 e'$. We thus have shown that

$$\left(e <_1 e' \text{ implies } y(e) \leq y(e')\right). \quad (32)$$

Now, assume that $e <_1 e'$ and $e \geq_2 e'$. We have $e \geq_2 e'$. In other words, $e \geq_{\mathbb{Z}} e'$ (since $\geq_2$ is the restriction of $\geq_{\mathbb{Z}}$ to E). But $e <_1 e'$, so that $e \neq e'$ (since $<_1$ is a partial order). Combined with $e \geq_{\mathbb{Z}} e'$, this yields $e >_{\mathbb{Z}} e'$. Thus, (31) (applied to $i = e$ and $j = e'$) yields $y(e) < y(e')$.

F_P (for P being a labelled poset) given in [GriRei15, Definition 5.2.1]. This yields

$$F_{(E,<_1)} = \sum_{f \in \mathcal{A}((E,<_1))} \mathbf{x}_f,$$

where $\mathcal{A}((E,<_1))$ denotes the set of all $(E,<_1)$-partitions and where $\mathbf{x}_f$ is defined as $\prod_{i \in E} x_{f(i)}$. Thus,

$$F_{(E,<_1)} = \underbrace{\sum_{f \in \mathcal{A}((E,<_1))}}_{\substack{= \sum_{f \text{ is a } (E,<_1)\text{-partition}} \\ \text{(since } \mathcal{A}((E,<_1)) \text{ is the set} \\ \text{of all } (E,<_1)\text{-partitions)}}} \underbrace{\mathbf{x}_f}_{= \prod_{i \in E} x_{f(i)}} = \underbrace{\sum_{f \text{ is a } (E,<_1)\text{-partition}}}_{\substack{= \sum_{f \text{ is a } \pi\text{-partition into } \{1,2,3,...\}} \\ \text{(since the } (E,<_1)\text{-partitions are} \\ \text{the same as the } \pi\text{-partitions} \\ \text{into } \{1,2,3,...\})}} \underbrace{\prod_{i \in E} x_{f(i)}}_{\substack{= \prod_{e \in E} x_{f(e)} \\ \text{(here, we renamed} \\ \text{the index } i \text{ as } e)}}$$

$$= \sum_{\substack{f \text{ is a } \pi\text{-partition} \\ \text{into } \{1,2,3,...\}}} \prod_{e \in E} x_{f(e)} = \Gamma(\pi) \qquad \text{(by (23))}. \tag{34}$$

Let us now compute $F(L(\pi))$.

Recall that you have defined the notion of a "linear extension" of a special double poset; in particular, this yields the notion of a "linear extension" of π. On the other hand, in [GriRei15, §5.2], the notion of a "linear extension" of a labelled poset is defined; in particular, this yields the notion of a "linear extension" of $(E,<_1)$. Now, we shall see that these two notions agree: Namely,

the linear extensions of $(E,<_1)$ are precisely the linear extensions of π. (35)

[39]

Let us now forget that we assumed that $e <_1 e'$ and $e \geq_2 e'$. We thus have shown that

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } y(e) < y(e')\right). \tag{33}$$

Now, let us forget that we fixed e and e'. We thus have proven that any two elements e and e' of E satisfy (32) and (33). In other words, y is a π-partition into $\{1,2,3,\ldots\}$ (according to the definition of a π-partition).

Let us now forget that we fixed y. We thus have proven that if y is an $(E,<_1)$-partition, then y is a π-partition into $\{1,2,3,\ldots\}$. In other words,

$$\text{every } (E,<_1)\text{-partition is a } \pi\text{-partition into } \{1,2,3,\ldots\}.$$

Combining this with (29), we conclude that the $(E,<_1)$-partitions are the same as the π-partitions into $\{1,2,3,\ldots\}$. Qed.

[39] *Proof of (35):* A linear extension of π is the same as a total order on the set E which extends the first order $<_1$ of π (according to the definition of a "linear extension" of π). In other words, a linear extension of π is the same as a total order on the set E which extends the partial order $<_1$. In other words, a linear extension of π is the same as a total order on the set E which extends $(E,<_1)$ as a poset.

You identify linear extensions of π with a certain kind of words, and also with permutations in S_n. Similarly, in [GriRei15, §5.2], linear extensions of $(E, <_1)$ are identified with a certain kind of words. It is clear that these identifications are the same – i.e., if σ is a linear extension of π, then the word with which the linear extension σ of π is identified is identical with the word with which the linear extension σ of $(E, <_1)$ is identified.[40] Following [GriRei15, §5.2], we use the notation $\mathcal{L}(P)$ for the set of all linear extensions of a labelled poset P. Then, [GriRei15, Theorem 5.2.11] says that $F_P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} F_w(\mathbf{x})$ for every labelled poset P. Applying this to $P = (E, <_1)$, we obtain

$$F_{(E,<_1)}(\mathbf{x}) = \underbrace{\sum_{w \in \mathcal{L}((E,<_1))}}_{\substack{= \sum_{\substack{w \text{ is a linear} \\ \text{extension of } (E,<_1)}} \\ \text{(since } \mathcal{L}((E,<_1)) \text{ is the set of all} \\ \text{linear extensions of } (E,<_1))}} F_w(\mathbf{x}) = \underbrace{\sum_{\substack{w \text{ is a linear} \\ \text{extension of } (E,<_1)}}}_{\substack{= \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} \\ \text{(since the linear extensions} \\ \text{of } (E,<_1) \text{ are precisely the} \\ \text{linear extensions of } \pi)}} F_w(\mathbf{x})$$

$$= \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} F_w(\mathbf{x}).$$

Now, (34) yields

$$\Gamma(\pi) = F_{(E,<_1)} = F_{(E,<_1)}(\mathbf{x}) = \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} F_w(\mathbf{x}). \tag{36}$$

On the other hand, $L(\pi)$ is the sum of all linear extensions of π (according to the definition of L). In other words,

$$L(\pi) = \sum_{\substack{\sigma \text{ is a linear} \\ \text{extension of } \pi}} \sigma.$$

But a linear extension of $(E, <_1)$ is the same as a linear order on the set E which extends $(E, <_1)$ as a poset (according to the definition of "linear extension" given in [GriRei15, §5.2]). Since "linear order" is synonymous for "total order", this rewrites as follows: A linear extension of $(E, <_1)$ is the same as a total order on the set E which extends $(E, <_1)$ as a poset. In other words, a linear extension of $(E, <_1)$ is the same as a linear extension of π (since a linear extension of π is the same as a total order on the set E which extends $(E, <_1)$ as a poset). In other words, the linear extensions of $(E, <_1)$ are precisely the linear extensions of π. This proves (35).

[40] This is because both of these words are defined in the same way (namely, as the list of all elements of E in σ-increasing order).

Applying the map F to both sides of this equality, we obtain

$$\begin{aligned} F\left(L\left(\pi\right)\right) &= F\left(\sum_{\substack{\sigma \text{ is a linear} \\ \text{extension of } \pi}} \sigma\right) = \sum_{\substack{\sigma \text{ is a linear} \\ \text{extension of } \pi}} \underbrace{F\left(\sigma\right)}_{\substack{=F_{C(\sigma)} \\ \text{(by the definition of } F\text{)}}} \\ &\qquad \left(\text{since } F \text{ is } \mathbb{Z}\text{-linear}\right) \\ &= \sum_{\substack{\sigma \text{ is a linear} \\ \text{extension of } \pi}} F_{C(\sigma)} = \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} F_{C(w)} \qquad (37) \\ &\qquad \left(\text{here, we renamed the summation index } \sigma \text{ as } w\right). \end{aligned}$$

Now, let us fix a linear extension w of π. We recall that w is thus identified with a word and with a permutation in S_n. Proposition 6.6 **(b)** (applied to $I = \operatorname{Des} w$) yields that there exists a unique composition α of n such that $D\left(\alpha\right) = \operatorname{Des} w$. Consider this α. Then, $\alpha = C\left(w\right)$ (according to Proposition 6.6 **(c)**, applied to $\sigma = w$). Notice that the fundamental quasisymmetric function F_α is denoted by L_α in [GriRei15, §5.2].

Now, in [GriRei15, Proposition 5.2.10], it is shown that $F_w\left(\mathbf{x}\right)$ equals the fundamental quasisymmetric function L_α. Since F_α is denoted by L_α in [GriRei15, §5.2], this result rewrites as follows: $F_w\left(\mathbf{x}\right)$ equals the fundamental quasisymmetric function F_α. In other words, $F_w\left(\mathbf{x}\right) = F_\alpha = F_{C(w)}$ (since $\alpha = C\left(w\right)$).

Let us now forget that we fixed w. We thus have shown that $F_w\left(\mathbf{x}\right) = F_{C(w)}$ for every linear extension w of π. Thus, (36) becomes

$$\begin{aligned} \Gamma\left(\pi\right) &= \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} \underbrace{F_w\left(\mathbf{x}\right)}_{=F_{C(w)}} = \sum_{\substack{w \text{ is a linear} \\ \text{extension of } \pi}} F_{C(w)} = F\left(L\left(\pi\right)\right) \qquad (\text{by } (37)) \\ &= \left(F \circ L\right)\left(\pi\right). \end{aligned}$$

Let us now forget that we fixed π. We thus have proven that $\Gamma\left(\pi\right) = \left(F \circ L\right)\left(\pi\right)$ for every special double poset π. Thus, $\left(\Gamma\mid_{\mathbb{Z}\mathbf{DS}}\right)\left(\pi\right) = \Gamma\left(\pi\right) = \left(F \circ L\right)\left(\pi\right)$ for every special double poset π. In other words, the two maps $\Gamma\mid_{\mathbb{Z}\mathbf{DS}}$ and $F \circ L$ are equal to each other on the basis of the $\mathbb{Z}$-module $\mathbb{Z}\mathbf{DS}$ consisting of the (isomorphism classes of) special double posets. Since these two maps $\Gamma\mid_{\mathbb{Z}\mathbf{DS}}$ and $F \circ L$ are $\mathbb{Z}$-linear, this shows that these two maps $\Gamma\mid_{\mathbb{Z}\mathbf{DS}}$ and $F \circ L$ are identical (because if two $\mathbb{Z}$-linear maps from the same domain are equal to each other on a basis of the domain, then these two maps must be identical). In other words, $\Gamma\mid_{\mathbb{Z}\mathbf{DS}} = F \circ L$. This proves Corollary 3.3.

Let me also prove the claim that "the linear function $F : \mathbb{Z}S \to \mathbf{QSym}$ defined by $\sigma \mapsto F_{C(\sigma)}$ is a homomorphism of bialgebras". You refer to "[MR] Th.3.3", but as far as I can tell this fact is never explicitly stated in [MR]; it takes some work to derive it from the results of [MR]. Here is a simple way to derive it from [GriRei15, Corollary 8.1.14]:

Proof of the fact that the linear function $F : \mathbb{Z}S \to \mathbf{QSym}$ defined by $\sigma \mapsto F_{C(\sigma)}$ is a homomorphism of bialgebras: For every $n \in \mathbb{N}$ and $w \in S_n$, we let $\gamma\left(w\right)$ denote the unique composition α of n such that $D(\alpha) = \operatorname{Des} w$ (where $\operatorname{Des} w$ is defined as in Proposition 6.6 **(c)**). (This unique composition exists according to Proposition 6.6 **(b)** (applied to $\sigma = w$ and $I = \operatorname{Des} w$).)

Part of [GriRei15, Corollary 8.1.14(a)] (applied to $\mathbf{k} = \mathbb{Z}$) states (in the notations of [GriRei15, §8]) that the $\mathbb{Z}$-linear map

$$\begin{aligned} \mathrm{FQSym} &\to \mathrm{QSym}, \\ F_w &\mapsto L_{\gamma(w)} \end{aligned}$$

is a Hopf morphism. Translating this into our notations, we obtain the following: The $\mathbb{Z}$-linear map

$$\begin{aligned} \mathbb{Z}S &\to \mathbf{QSym}, \\ w &\mapsto F_{\gamma(w)} \end{aligned}$$

is a Hopf morphism[41]. Renaming the index w as σ in this fact, we rewrite it as follows: The $\mathbb{Z}$-linear map

$$\begin{aligned} \mathbb{Z}S &\to \mathbf{QSym}, \\ \sigma &\mapsto F_{\gamma(\sigma)} \end{aligned}$$

is a Hopf morphism. In other words, the $\mathbb{Z}$-linear map

$$\begin{aligned} \mathbb{Z}S &\to \mathbf{QSym}, \\ \sigma &\mapsto F_{C(\sigma)} \end{aligned}$$

is a Hopf morphism (because every $n \in \mathbb{N}$ and every $\sigma \in S_n$ satisfy $\gamma(\sigma) = C(\sigma)$ [42]). In particular, this map is a homomorphism of bialgebras. This completes our proof that the linear function $F : \mathbb{Z}S \to \mathbf{QSym}$ defined by $\sigma \mapsto F_{C(\sigma)}$ is a homomorphism of bialgebras.

6.6. Page 10, Proposition 3.1: properties of standard permutations

Let me give a self-contained proof of Proposition 3.1.

First, I am going to give a formal definition of the standard permutation of a word. Let me start with the following simple fact:

[41] In our translation, we used the following dictionary:

- What is called FQSym in [GriRei15] is what we call $\mathbb{Z}S$.
- What is called QSym in [GriRei15] is what we call **QSym**.
- What is called F_w in [GriRei15] (for w being a permutation) is what we call w.
- What is called L_α in [GriRei15] (for α being a composition) is what we call F_α.

[42] *Proof.* Let $n \in \mathbb{N}$ and $\sigma \in S_n$. Proposition 6.6 **(c)** (applied to $\alpha = \gamma(\sigma)$) yields that $\gamma(\sigma) = C(\sigma)$ (since $\gamma(\sigma)$ is the unique composition α of n such that $D(\alpha) = \operatorname{Des} \sigma$ (due to the definition of $\gamma(\sigma)$)). Qed.

Proposition 6.8. Let P be a poset. Let $w = a_1 a_2 \cdots a_n$ be a word over the poset P (that is, a finite list of elements of P). We define a binary relation $\prec_w$ on the set $\{1, 2, \ldots, n\}$ as follows: For any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$, we set $i \prec_w j$ if and only if

$$\left(\text{either } a_i < a_j \text{ or } \left(a_i = a_j \text{ and } i < j\right)\right).$$

(a) This binary relation $\prec_w$ is the smaller relation of a partial order on the set $\{1, 2, \ldots, n\}$.

(b) Assume that the poset P is totally ordered. Then, the binary relation $\prec_w$ is the smaller relation of a total order on the set $\{1, 2, \ldots, n\}$.

Proof of Proposition 6.8. **(a)** Any two elements p and q of $\{1, 2, \ldots, n\}$ satisfying $p \prec_w q$ must satisfy

$$a_p \leq a_q. \tag{38}$$

[43]

[43] *Proof of (38):* Let p and q be two elements of $\{1, 2, \ldots, n\}$ satisfying $p \prec_w q$.

We have $p \prec_w q$ if and only if $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (since we have $p \prec_w q$). In other words, we are in one of the following two cases:

Case 1: We have $a_p < a_q$.

Case 2: We have $\left(a_p = a_q \text{ and } p < q\right)$.

Let us first consider Case 1. In this case, we have $a_p < a_q$. Thus, $a_p \leq a_q$. Hence, (38) is proven in Case 1.

Let us now consider Case 1. In this case, we have $\left(a_p = a_q \text{ and } p < q\right)$. Thus, $a_p = a_q$, so that $a_p \leq a_q$. Hence, (38) is proven in Case 2.

We have now proven (38) in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, this yields that (38) always holds. Qed.

The binary relation $\prec_w$ is irreflexive[44], transitive[45] and antisymmetric[46]. Therefore, this binary relation $\prec_w$ is the smaller relation of a partial order on the set $\{1, 2, \ldots, n\}$ (since every irreflexive, transitive and antisymmetric relation on a set S must be the smaller relation of a partial order on S). This proves Proposition 6.8 **(a)**.

(b) Proposition 6.8 **(a)** shows that the binary relation $\prec_w$ is the smaller relation of a partial order on the set $\{1, 2, \ldots, n\}$. In order to prove Proposition 6.8 **(b)**,

[44] *Proof.* Let p be an element of $\{1, 2, \ldots, n\}$ such that $p \prec_w p$.

We have $p \prec_w p$ if and only if $\left(\text{either } a_p < a_p \text{ or } \left(a_p = a_p \text{ and } p < p\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $\left(\text{either } a_p < a_p \text{ or } \left(a_p = a_p \text{ and } p < p\right)\right)$ (since we have $p \prec_w p$). Since $a_p < a_p$ is impossible, we thus have $\left(a_p = a_p \text{ and } p < p\right)$. Thus, $p < p$, which is absurd.

Now, let us forget that we fixed p. We thus have obtained a contradiction for every $p \in \{1, 2, \ldots, n\}$ satisfying $p \prec_w p$. Hence, there exists no $p \in \{1, 2, \ldots, n\}$ satisfying $p \prec_w p$. In other words, the relation $\prec_w$ is irreflexive, qed.

[45] *Proof.* Let p, q and r be three elements of $\{1, 2, \ldots, n\}$ such that $p \prec_w q$ and $q \prec_w r$.

We are going to show that $p \prec_w r$.

We have $p \prec_w q$ and thus $a_p \leq a_q$ (by (38)). Also, we have $q \prec_w r$ and thus $a_q \leq a_r$ (by (38), applied to q and r instead of p and q). Thus, $a_p \leq a_q \leq a_r$. Hence, either $a_p = a_r$ or $a_p < a_r$. In other words, we are in one of the following two cases:

Case 1: We have $a_p = a_r$.

Case 2: We have $a_p < a_r$.

Let us first consider Case 1. In this case, we have $a_p = a_r$. Thus, $a_r = a_p \leq a_q$. Combined with $a_q \leq a_r$, this yields $a_r = a_q$, so that $a_q = a_r$.

We have $p \prec_w q$ if and only if $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (since we have $p \prec_w q$). Since $a_p < a_q$ is impossible (because $a_p = a_r = a_q$), this yields that we have $\left(a_p = a_q \text{ and } p < q\right)$. Hence, $p < q$.

We have $q \prec_w r$ if and only if $\left(\text{either } a_q < a_r \text{ or } \left(a_q = a_r \text{ and } q < r\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $\left(\text{either } a_q < a_r \text{ or } \left(a_q = a_r \text{ and } q < r\right)\right)$ (since we have $q \prec_w r$). Since $a_q < a_r$ is impossible (since $a_q = a_r$), this shows that we have $\left(a_q = a_r \text{ and } q < r\right)$. Hence, $q < r$.

Now, $a_p = a_r$ and $p < q < r$. Hence, $\left(a_p = a_r \text{ and } p < r\right)$, and therefore $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$.

Now, we have $p \prec_w r$ if and only if $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $p \prec_w r$ (since we have $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$). Hence, $p \prec_w r$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $a_p < a_r$. Thus, $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$.

Now, we have $p \prec_w r$ if and only if $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $p \prec_w r$ (since we have $\left(\text{either } a_p < a_r \text{ or } \left(a_p = a_r \text{ and } p < r\right)\right)$). Hence, $p \prec_w r$ is proven in Case 2.

Now, we have proven $p \prec_w r$ in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we can thus conclude that $p \prec_w r$ always holds.

Now, let us forget that we fixed p, q and r. We thus have shown that if p, q and r are any three elements of $\{1, 2, \ldots, n\}$ such that $p \prec_w q$ and $q \prec_w r$, then $p \prec_w r$. In other words, the relation $\prec_w$ is transitive. Qed.

[46] *Proof.* It is known that any irreflexive transitive binary relation is antisymmetric. Thus, the binary relation $\prec_w$ is antisymmetric (since it is irreflexive and transitive). Qed.

it is clearly enough to show that this order is total. In other words, it is clearly enough to prove that if p and q are two distinct elements of $\{1, 2, \ldots, n\}$, then either $p \prec_w q$ or $q \prec_w p$. Let us prove this now.

Let p and q be two distinct elements of $\{1, 2, \ldots, n\}$. We need to show that either $p \prec_w q$ or $q \prec_w p$.

We can WLOG assume that $p \leq q$ (since our situation is symmetric in p and q, and thus we can interchange p with q to obtain $p \leq q$). Assume this. Combining $p \leq q$ with $p \neq q$ (since p and q are distinct), we obtain $p < q$.

The poset P is totally ordered. Hence, we have either $a_p < a_q$ or $a_p = a_q$ or $a_p > a_q$. In other words, we must be in one of the following three cases:

Case 1: We have $a_p < a_q$.

Case 2: We have $a_p = a_q$.

Case 3: We have $a_p > a_q$.

Let us first consider Case 1. In this case, we have $a_p < a_q$. Hence, $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$.

Now, we have $p \prec_w q$ if and only if $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $p \prec_w q$ (since we have $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$). Consequently, we have either $p \prec_w q$ or $q \prec_w p$. Thus, (either $p \prec_w q$ or $q \prec_w p$) is proven in Case 1.

Let us now consider Case 2. In this case, we have $a_p = a_q$. Hence, $\left(a_p = a_q \text{ and } p < q\right)$, so that $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$.

Now, we have $p \prec_w q$ if and only if $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $p \prec_w q$ (since we have $\left(\text{either } a_p < a_q \text{ or } \left(a_p = a_q \text{ and } p < q\right)\right)$). Consequently, we have either $p \prec_w q$ or $q \prec_w p$. Thus, (either $p \prec_w q$ or $q \prec_w p$) is proven in Case 2.

Let us finally consider Case 3. In this case, we have $a_p > a_q$. Hence, $a_q < a_p$, so that $\left(\text{either } a_q < a_p \text{ or } \left(a_q = a_p \text{ and } q < p\right)\right)$.

Now, we have $q \prec_w p$ if and only if $\left(\text{either } a_q < a_p \text{ or } \left(a_q = a_p \text{ and } q < p\right)\right)$ (according to the definition of the relation $\prec_w$). Hence, we must have $q \prec_w p$ (since we have $\left(\text{either } a_q < a_p \text{ or } \left(a_q = a_p \text{ and } q < p\right)\right)$). Consequently, we have either $p \prec_w q$ or $q \prec_w p$. Thus, (either $p \prec_w q$ or $q \prec_w p$) is proven in Case 3.

Thus, (either $p \prec_w q$ or $q \prec_w p$) is proven in each of the three Cases 1, 2 and 3. Since these three Cases cover all possibilities, this yields that (either $p \prec_w q$ or $q \prec_w p$) always holds.

Now, let us forget that we fixed p and q. We thus have shown that if p and q are any two distinct elements of $\{1, 2, \ldots, n\}$, then either $p \prec_w q$ or $q \prec_w p$. Thus, the binary relation $\prec_w$ is the smaller relation of a total order on the set $\{1, 2, \ldots, n\}$ (since we already know that the binary relation $\prec_w$ is the smaller relation of a partial order on the set $\{1, 2, \ldots, n\}$). This proves Proposition 6.8 **(b)**. □

Definition 6.9. Let P be a totally ordered poset. Let $w = a_1 a_2 \cdots a_n$ be a word over the poset P. We define the *standardization* of the word w as follows:

Construct a binary relation $\prec_w$ on the set $\{1, 2, \ldots, n\}$ as in Proposition 6.8. Then, Proposition 6.8 **(b)** shows that this relation is the smaller relation of a total order on the set $\{1, 2, \ldots, n\}$. In other words, $(\{1, 2, \ldots, n\}, \prec_w)$ is a totally ordered set. Thus, there is a unique order isomorphism $(\{1, 2, \ldots, n\}, \prec_w) \to (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$. We define the *standard permutation* of w as this order isomorphism, regarded as a permutation of $\{1, 2, \ldots, n\}$. This standard permutation is also denoted as the *standardization* of w.

Example 6.10. Let P be a three-element totally ordered poset $\{x, y, z\}$ with $x < y < z$. Let w be the word $yzxy$ over the poset P. Then, the binary relation $\prec_w$ is the total order given by $3 \prec_w 1 \prec_w 4 \prec_w 2$. Hence, the unique order isomorphism $(\{1, 2, \ldots, n\}, \prec_w) \to (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$ sends 3 to 1, sends 1 to 2, sends 4 to 3, and sends 2 to 4. In other words, this unique order isomorphism is the permutation $(2, 4, 1, 3)$ (written in one-line notation). Thus, the standard permutation of the word w is this permutation $(2, 4, 1, 3)$.

Proposition 6.11. Let P be a totally ordered poset. Let $w = a_1 a_2 \cdots a_n$ be a word over the poset P. Let s be the standard permutation of w. Let $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$.

(a) We have the following logical equivalence:

$$(s(i) < s(j)) \iff \left(\text{either } a_i < a_j \text{ or } (a_i = a_j \text{ and } i < j)\right).$$

(b) If $i < j$, then we have the following logical equivalence:

$$(s(i) < s(j)) \iff (a_i \leq a_j).$$

(c) If $i \geq j$, then we have the following logical equivalence:

$$(s(i) < s(j)) \iff (a_i < a_j).$$

(d) If $s(i) < s(j)$, then $a_i \leq a_j$.

Proof of Proposition 6.11. Recall a very basic property of order isomorphisms: If $(U, <_U)$ and $(V, <_V)$ are two posets, if f is an order isomorphism $(U, <_U) \to (V, <_V)$, and if p and q are two elements of U, then we have the following logical equivalence:

$$(p <_U q) \iff (f(p) <_V f(q)). \tag{39}$$

(a) Construct a binary relation $\prec_w$ on the set $\{1, 2, \ldots, n\}$ as in Proposition 6.8. We recall that the standard permutation of w is the unique order isomorphism $(\{1, 2, \ldots, n\}, \prec_w) \to (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$ (indeed, this is how it was defined). In other words, s is the unique order isomorphism $(\{1, 2, \ldots, n\}, \prec_w) \to (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$ (since s is the standard permutation of w). In particular, s is an order isomorphism $(\{1, 2, \ldots, n\}, \prec_w) \to (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$. Hence, for every two elements p and q of $\{1, 2, \ldots, n\}$, we have the following logical equivalence:

$$(p \prec_w q) \iff (s(p) <_{\mathbb{Z}} s(q)). \tag{40}$$

(Indeed, this follows from (39), applied to $(U, <_U) = (\{1, 2, \ldots, n\}, \prec_w)$, $(V, <_V) = (\{1, 2, \ldots, n\}, <_{\mathbb{Z}})$ and $f = s$).

Applying (40) to $p = i$ and $q = j$, we obtain the logical equivalence

$$(i \prec_w j) \iff (s(i) <_{\mathbb{Z}} s(j)) \iff (s(i) < s(j)) \quad \left(\begin{array}{c}\text{since the relation } <_{\mathbb{Z}} \text{ on } \{1, 2, \ldots, n\} \\ \text{is a restriction of the relation } < \text{ on } \mathbb{Z}\end{array}\right).$$

Hence, we have the logical equivalence

$$(s(i) < s(j)) \iff (i \prec_w j) \iff \left(\text{either } a_i < a_j \text{ or } \left(a_i = a_j \text{ and } i < j\right)\right)$$

(because $i \prec_w j$ is equivalent to $\left(\text{either } a_i < a_j \text{ or } \left(a_i = a_j \text{ and } i < j\right)\right)$ (by the definition of the relation $\prec_w$)). This proves Proposition 6.11 **(a)**.

(b) Assume that $i < j$. Proposition 6.11 **(a)** shows that we have the following logical equivalence:

$$\begin{aligned}(s(i) < s(j)) &\iff \left(\text{either } a_i < a_j \text{ or } \underbrace{\left(a_i = a_j \text{ and } i < j\right)}_{\substack{\text{this is equivalent to } (a_i = a_j) \\ \text{(since } i<j \text{ is true)}}}\right) \\ &\iff \left(\text{either } a_i < a_j \text{ or } a_i = a_j\right) \iff \left(a_i \leq a_j\right).\end{aligned}$$

This proves Proposition 6.11 **(b)**.

(c) Assume that $i \geq j$. Then, $i < j$ is false. Proposition 6.11 **(a)** shows that we have the following logical equivalence:

$$\begin{aligned}(s(i) < s(j)) &\iff \left(\text{either } a_i < a_j \text{ or } \underbrace{\left(a_i = a_j \text{ and } i < j\right)}_{\substack{\text{this is false} \\ \text{(since } i<j \text{ is false)}}}\right) \\ &\iff \left(\text{either } a_i < a_j \text{ or } (\text{false})\right) \iff \left(a_i < a_j\right).\end{aligned}$$

This proves Proposition 6.11 **(c)**.

(d) We have the following logical implication:

$$\begin{aligned}(s(i) < s(j)) &\iff \left(\text{either } a_i < a_j \text{ or } \underbrace{\left(a_i = a_j \text{ and } i < j\right)}_{\text{this implies } a_i = a_j}\right) \\ &\qquad (\text{by Proposition 6.11 } \textbf{(a)}) \\ &\Longrightarrow \left(\text{either } a_i < a_j \text{ or } a_i = a_j\right) \iff \left(a_i \leq a_j\right).\end{aligned}$$

In other words, if $s(i) < s(j)$, then $a_i \leq a_j$. This proves Proposition 6.11 **(d)**. □

Next, let us show another simple property of standardization:

Proposition 6.12. Let P be a totally ordered poset. Let $w = a_1 a_2 \cdots a_n$ be a word over the poset P. Let s be the standard permutation of w. We regard the permutation s as a word over the poset $\{1, 2, \ldots, n\}$ (where the partial order on this poset is just the usual order inherited from $\mathbb{Z}$) by writing s in one-line notation. Then, the standard permutation of this word s is s again.

Before we prove this, let us prove a basic fact:

Lemma 6.13. Let $n \in \mathbb{N}$. Let s and t be two permutations of $\{1, 2, \ldots, n\}$. Assume that every two elements i and j of $\{1, 2, \ldots, n\}$ satisfying $i < j$ satisfy the following logical equivalence:

$$(s(i) < s(j)) \iff (t(i) < t(j)). \tag{41}$$

Then, $s = t$.

Lemma 6.13 is a restatement of the well-known fact that any permutation of $\{1, 2, \ldots, n\}$ is uniquely determined by its set of inversions. However, for the sake of completeness, let us give a self-contained proof of Lemma 6.13.

Proof of Lemma 6.13. Let u be the permutation $s \circ t^{-1}$ of $\{1, 2, \ldots, n\}$. Then, u is a bijection (since u is a permutation) and therefore an injection. It is easy to see that every two elements i and j of $\{1, 2, \ldots, n\}$ satisfy the following logical equivalence:

$$(i < j) \iff (u(i) < u(j)). \tag{42}$$

[47]

Now, we are going to prove that $u(k) = k$ for every $k \in \{1, 2, \ldots, n\}$.

[47] *Proof of (42):* Let i and j be two elements of $\{1, 2, \ldots, n\}$. We want to prove the equivalence (42).

We must be in one of the following three cases:

Case 1: We have $t^{-1}(i) < t^{-1}(j)$.

Case 2: We have $t^{-1}(i) = t^{-1}(j)$.

Case 3: We have $t^{-1}(i) > t^{-1}(j)$.

Let us first consider Case 1. In this case, we have $t^{-1}(i) < t^{-1}(j)$. Thus, (41) (applied to $t^{-1}(i)$ and $t^{-1}(j)$ instead of i and j) yields that we have the following logical equivalence: $\left(s\left(t^{-1}(i)\right) < s\left(t^{-1}(j)\right)\right) \iff \left(t\left(t^{-1}(i)\right) < t\left(t^{-1}(j)\right)\right)$. Thus, we have the following logical equivalence:

$$\left(s\left(t^{-1}(i)\right) < s\left(t^{-1}(j)\right)\right) \iff \left(\underbrace{t\left(t^{-1}(i)\right)}_{=i} < \underbrace{t\left(t^{-1}(j)\right)}_{=j}\right) \iff (i < j).$$

Indeed, assume the contrary. Then, we don't have $(u(k) = k \text{ for every } k \in \{1,2,\ldots,n\})$. Thus, there exists a $k \in \{1,2,\ldots,n\}$ such that $u(k) \neq k$. Let p be the smallest such k.

We know that p is the **smallest** $k \in \{1,2,\ldots,n\}$ such that $u(k) \neq k$. Hence, if $k \in \{1,2,\ldots,n\}$ is such that $u(k) \neq k$, then

$$k \geq p. \tag{43}$$

On the other hand, p is the smallest $k \in \{1,2,\ldots,n\}$ such that $u(k) \neq k$. Hence, p is a $k \in \{1,2,\ldots,n\}$ such that $u(k) \neq k$. In other words, $p \in$

Hence, we have the following logical equivalence:

$$(i < j) \iff \left(\underbrace{s\left(t^{-1}(i)\right)}_{=\left(s\circ t^{-1}\right)(i)} < \underbrace{s\left(t^{-1}(j)\right)}_{=\left(s\circ t^{-1}\right)(j)} \right) \iff \left(\underbrace{\left(s \circ t^{-1}\right)}_{=u}(i) < \underbrace{\left(s \circ t^{-1}\right)}_{=u}(j) \right)$$
$$\iff (u(i) < u(j)).$$

Thus, (42) is proven in Case 1.

Let us now consider Case 2. In this case, we have $t^{-1}(i) = t^{-1}(j)$. Hence, $i = t\left(\underbrace{t^{-1}(i)}_{=t^{-1}(j)}\right) = t\left(t^{-1}(j)\right) = j$. Therefore, we do not have $i < j$. Also, $u\left(\underbrace{i}_{=j}\right) = u(j)$. Thus, we do not have $u(i) < u(j)$. Hence, we have neither $i < j$ nor $u(i) < u(j)$. Therefore, we have the logical equivalence $(i < j) \iff (u(i) < u(j))$. Thus, (42) is proven in Case 2.

Let us first consider Case 3. In this case, we have $t^{-1}(i) > t^{-1}(j)$. Hence, if we had $i = j$, then we would have $t^{-1}\left(\underbrace{i}_{=j}\right) = t^{-1}(j)$, which would contradict $t^{-1}(i) > t^{-1}(j)$. Thus, we cannot have $i = j$. Moreover, u is an injection. Hence, if we had $u(i) = u(j)$, then we would have $i = j$, which would contradict the fact that we cannot have $i = j$. Hence, we cannot have $u(i) = u(j)$.

From $t^{-1}(i) > t^{-1}(j)$, we obtain $t^{-1}(j) < t^{-1}(i)$. Consequently, (41) (applied to $t^{-1}(j)$ and $t^{-1}(i)$ instead of i and j) yields that we have the following logical equivalence: $\left(s\left(t^{-1}(j)\right) < s\left(t^{-1}(i)\right)\right) \iff \left(t\left(t^{-1}(j)\right) < t\left(t^{-1}(i)\right)\right)$. Thus, we have the following logical equivalence:

$$\left(s\left(t^{-1}(j)\right) < s\left(t^{-1}(i)\right)\right) \iff \left(\underbrace{t\left(t^{-1}(j)\right)}_{=j} < \underbrace{t\left(t^{-1}(i)\right)}_{=i} \right) \iff (j < i).$$

Hence, we have the following logical equivalence:

$$(j < i) \iff \left(\underbrace{s\left(t^{-1}(j)\right)}_{=\left(s\circ t^{-1}\right)(j)} < \underbrace{s\left(t^{-1}(i)\right)}_{=\left(s\circ t^{-1}\right)(i)} \right) \iff \left(\underbrace{\left(s \circ t^{-1}\right)}_{=u}(j) < \underbrace{\left(s \circ t^{-1}\right)}_{=u}(i) \right)$$
$$\iff (u(j) < u(i)).$$

$\{1, 2, \ldots, n\}$ and $u(p) \neq p$. Let $q = u^{-1}(p)$. Then, $u(q) = p$. Also, $p \neq q$ [48]. Thus, $u(q) = p \neq q$. Hence, $q \geq p$ (by (43), applied to $k = q$). Combined with $q \neq p$, this yields $q > p$. Thus, $p < q$. But (42) (applied to $i = p$ and $j = q$) yields that we have the following logical equivalence: $(p < q) \iff (u(p) < u(q))$. Thus, $u(p) < u(q)$ holds (since $p < q$ holds). Hence, $u(p) < u(q) = p$.

If we had $u(u(p)) \neq u(p)$, then we would have $u(p) \geq p$ (by (43), applied to $k = u(p)$), which would contradict $u(p) < p$. Hence, we do not have $u(u(p)) \neq u(p)$. In other words, we have $u(u(p)) = u(p)$. Since u is an injection, this yields $u(p) = p$. This contradicts $u(p) < p$.

This contradiction proves that our assumption was wrong. Hence, we have shown that $u(k) = k$ for every $k \in \{1, 2, \ldots, n\}$. Thus, every $k \in \{1, 2, \ldots, n\}$ satisfies $u(k) = k = \mathrm{id}(k)$. Hence, $u = \mathrm{id}$. Since $u = s \circ t^{-1}$, this rewrites as $s \circ t^{-1} = \mathrm{id}$. Thus, $s = t$. Lemma 6.13 is thus proven. $\square$

Proof of Proposition 6.12. Let t denote the standard permutation of s (where we regard s as a word by writing s in one-line notation). Regarded as a word, the

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Hence, we have the following logical equivalence:

$$\begin{aligned}
&\Big(\text{not } \underbrace{j < i}_{\text{this is equivalent to } (u(j)<u(i))}\Big) \\
&\iff (\text{not } u(j) < u(i)) \iff (u(j) \geq u(i)) \\
&\iff (u(i) \leq u(j)) \iff \Big(u(i) < u(j) \text{ or } \underbrace{u(i) = u(j)}_{\substack{\text{this is equivalent to (false)} \\ \text{(since we cannot have } u(i)=u(j))}}\Big) \\
&\iff (u(i) < u(j) \text{ or (false)}) \iff (u(i) < u(j)).
\end{aligned}$$

Thus, we have the following logical equivalence:

$$\begin{aligned}
&(u(i) < u(j)) \\
&\iff (\text{not } j < i) \iff (j \geq i) \iff (i \leq j) \\
&\iff \Big(i < j \text{ or } \underbrace{i = j}_{\substack{\text{this is equivalent to (false)} \\ \text{(since we cannot have } i=j)}}\Big) \iff (i < j \text{ or (false)}) \iff (i < j).
\end{aligned}$$

In other words, we have the logical equivalence $(i < j) \iff (u(i) < u(j))$. Thus, (42) is proven in Case 3.

We have now proven (42) in each of the three Cases 1, 2 and 3. Since these three Cases cover all possibilities, this yields that (42) always holds. Qed.

[48] *Proof.* Assume the contrary. Then, $p = q$, so that $u\Big(\underbrace{p}_{=q}\Big) = u(q) = p$. This contradicts $u(p) \neq p$. This contradiction shows that our assumption was wrong, qed.

permutation s has the form $s = s(1)s(2)\cdots s(n)$.

Let i and j be two elements of $\{1,2,\ldots,n\}$ such that $i < j$. Then, Proposition 6.11 **(b)** (applied to $\{1,2,\ldots,n\}$, s, $s(k)$ and t instead of P, w, a_k and s) yields that we have the following logical equivalence:

$$(t(i) < t(j)) \iff (s(i) < s(j)).$$

Now, let us forget that we fixed i and j. We thus have proven that every two elements i and j of $\{1,2,\ldots,n\}$ satisfying $i < j$ satisfy the following logical equivalence:

$$(t(i) < t(j)) \iff (s(i) < s(j)).$$

Lemma 6.13 (applied to t and s instead of s and t) thus shows that $t = s$. In other words, t is s. In other words, the standard permutation of s is s (since t is the standard permutation of s). This proves Proposition 6.12. □

We can now prove Proposition 3.1. Indeed, let us prove a slightly stronger claim:

Proposition 6.14. Let π be a special double poset. Let n be the size of the special double poset π (that is, the size of the underlying set of π). Let $w = a_1a_2\cdots a_n$ be a word whose letters belong to a totally ordered alphabet A.

(a) The word w fits into π if and only if the standard permutation of w fits into π.

(b) Write the special double poset π in the form $(E, <_1, <_2)$. Let ω be the labelling of the special double poset π. Let s be the standard permutation of w. The word w fits into π if and only if the map $s \circ \omega : E \to \{1,2,\ldots,n\}$ is an order homomorphism $(E, <_1) \to \{1,2,\ldots,n\}$. (Here, $\{1,2,\ldots,n\}$ denotes the poset $(\{1,2,\ldots,n\}, <)$, where the relation $<$ is the smaller relation of $\mathbb{Z}$. In other words, $\{1,2,\ldots,n\} = (\{1,2,\ldots,n\}, <_{\mathbb{Z}})$.)

Proposition 6.14 **(a)** is precisely Proposition 3.1.

Proof of Proposition 6.14. Write the special double poset π in the form $(E, <_1, <_2)$. Let ω be the labelling of the special double poset π.

The map ω is the labelling of the special double poset $\pi = (E, <_1, <_2)$. In other words, ω is the unique order isomorphism $(E, <_2) \to \{1,2,\ldots,n\}$ (because the labelling of the special double poset $(E, <_1, <_2)$ is the unique order isomorphism $(E, <_2) \to \{1,2,\ldots,n\}$ (by the definition of the labelling of the special double poset $(E, <_1, <_2)$)). In particular, ω is an order isomorphism $(E, <_2) \to \{1,2,\ldots,n\}$. In other words, ω is an order isomorphism $(E, <_2) \to (\{1,2,\ldots,n\}, <)$. Hence, for any two elements p and q of E, we have the following logical equivalence:

$$(p <_2 q) \iff (\omega(p) < \omega(q)) \tag{44}$$

(by (39), applied to $(U, <_U) = (E, <_2)$, $(V, <_V) = (\{1,2,\ldots,n\}, <)$ and $f = \omega$).

The map ω is an order isomorphism, thus a bijection, and therefore an injection.

Let s be the standard permutation of w. Then, s is a permutation, hence a bijection. Thus, the map $s \circ \omega$ is a bijection (since it is the composition of the two bijections s and ω). In particular, this yields that the map $s \circ \omega$ is an injection.

Let η denote the map

$$E \to A, \qquad e \mapsto a_{\omega(e)}.$$

The definition of the notion of "the word $a_1 a_2 \cdots a_n$ fits into π" yields the following: The word $a_1 a_2 \cdots a_n$ fits into π if and only if the map

$$E \to A, \qquad e \mapsto a_{\omega(e)}$$

is a π-partition. In other words, the word $a_1 a_2 \cdots a_n$ fits into π if and only if the map η is a π-partition[49]. In other words, the word w fits into π if and only if the map η is a π-partition (since $w = a_1 a_2 \cdots a_n$).

Definition 6.7 (applied to A and η instead of X and x) yields that η is a π-partition if and only if any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ implies } \eta(e) \leq \eta(e')\right) \tag{45}$$

and

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } \eta(e) < \eta(e')\right). \tag{46}$$

(b) We are going to prove the following two logical implications:

$$\begin{aligned} &(\text{the word } w \text{ fits into } \pi) \\ &\Longrightarrow \left(\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}\right) \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\left(\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}\right) \\ &\Longrightarrow (\text{the word } w \text{ fits into } \pi). \end{aligned} \tag{48}$$

Proof of (47): We assume that the word w fits into π. We then want to show that the map $s \circ \omega$ is an order homomorphism $(E, <_1) \to \{1, 2, \ldots, n\}$.

Recall that the word w fits into π if and only if the map η is a π-partition. Hence, the map η is a π-partition (since we know that the word w fits into π). In other words, any two elements e and e' of E satisfy (45) and (46)[50].

Now, let e and e' be two elements of E such that $e \leq_1 e'$. We are going to show that $(s \circ \omega)(e) \leq (s \circ \omega)(e')$. This is obvious when $e = e'$ (because when

[49] since the map

$$E \to A, \qquad e \mapsto a_{\omega(e)}$$

is the map η

[50] since we know that η is a π-partition if and only if any two elements e and e' of E satisfy (45) and (46)

$e = e'$, then $(s \circ \omega) \left(\underbrace{e}_{=e'} \right) = (s \circ \omega)(e')$). Hence, for the rest of this proof, we can WLOG assume that we don't have $e = e'$. Assume this. Then, $e \neq e'$ (since not $e = e'$) and thus $e <_1 e'$ (since $e \leq_1 e'$ and $e \neq e'$) and thus $\eta(e) \leq \eta(e')$ (by (45)). The definition of η yields $\eta(e) = a_{\omega(e)}$ and $\eta(e') = a_{\omega(e')}$. Thus, $a_{\omega(e)} = \eta(e) \leq \eta(e') = a_{\omega(e')}$.

We must be in one of the following two cases:

Case 1: We have $\omega(e) < \omega(e')$.

Case 2: We have $\omega(e) \geq \omega(e')$.

Let us first consider Case 1. In this case, we have $\omega(e) < \omega(e')$. Hence, Proposition 6.11 **(b)** (applied to $i = \omega(e)$ and $j = \omega(e')$) yields that we have the following logical equivalence:

$$\left(s(\omega(e)) < s\left(\omega(e')\right)\right) \iff \left(a_{\omega(e)} \leq a_{\omega(e')}\right).$$

Thus, we have $s(\omega(e)) < s(\omega(e'))$ (since we know that we have $a_{\omega(e)} \leq a_{\omega(e')}$). Hence, $(s \circ \omega)(e) = s(\omega(e)) < s(\omega(e')) = (s \circ \omega)(e')$. Thus, $(s \circ \omega)(e) \leq (s \circ \omega)(e')$. We have thus proven $(s \circ \omega)(e) \leq (s \circ \omega)(e')$ in Case 1.

Let us now consider Case 2. In this case, we have $\omega(e) \geq \omega(e')$. If we had $\omega(e) = \omega(e')$, then we would have $e = e'$ (since the map ω is an injection), which would contradict $e \neq e'$. Hence, we cannot have $\omega(e) = \omega(e')$. We thus have $\omega(e) \neq \omega(e')$. Combined with $\omega(e) \geq \omega(e')$, this yields $\omega(e) > \omega(e')$. Thus, $\omega(e') < \omega(e)$. But (44) (applied to $p = e'$ and $q = e$) yields that we have the following logical equivalence: $(e' <_2 e) \iff (\omega(e') < \omega(e))$. Hence, $e' <_2 e$ holds (since $\omega(e') < \omega(e)$ holds). Thus, $e >_2 e'$, so that $e \geq_2 e'$. Therefore, (46) yields $\eta(e) < \eta(e')$. Hence, $a_{\omega(e)} = \eta(e) < \eta(e') = a_{\omega(e')}$.

But $\omega(e) \geq \omega(e')$. Hence, Proposition 6.11 **(c)** (applied to $i = \omega(e)$ and $j = \omega(e')$) yields that we have the following logical equivalence:

$$\left(s(\omega(e)) < s\left(\omega(e')\right)\right) \iff \left(a_{\omega(e)} < a_{\omega(e')}\right).$$

Thus, we have $s(\omega(e)) < s(\omega(e'))$ (since we know that we have $a_{\omega(e)} < a_{\omega(e')}$). Hence, $(s \circ \omega)(e) = s(\omega(e)) < s(\omega(e')) = (s \circ \omega)(e')$. Thus, $(s \circ \omega)(e) \leq (s \circ \omega)(e')$. We have thus proven $(s \circ \omega)(e) \leq (s \circ \omega)(e')$ in Case 2.

Hence, we have proven $(s \circ \omega)(e) \leq (s \circ \omega)(e')$ in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that $(s \circ \omega)(e) \leq (s \circ \omega)(e')$ always holds.

Now, let us forget that we fixed e and e'. We thus have proven that if e and e' are any two elements of E such that $e \leq_1 e'$, then $(s \circ \omega)(e) \leq (s \circ \omega)(e')$. In other words, the map $s \circ \omega$ is an order homomorphism $(E, <_1) \to \{1, 2, \ldots, n\}$. Thus, (47) is proven.

Proof of (48): We assume that the map $s \circ \omega$ is an order homomorphism $(E, <_1) \to \{1, 2, \ldots, n\}$. We are going to show that the word w fits into π.

Let now e and e' be any two elements of E. We will prove (45) and (46).

Proof of (45): Assume that $e <_1 e'$. We are going to show that $\eta(e) \leq \eta(e')$.

We have $e <_1 e'$, so that $e \leq_1 e'$. Thus, $(s \circ \omega)(e) \leq (s \circ \omega)(e')$ (since the map $s \circ \omega$ is an order homomorphism $(E, <_1) \to \{1, 2, \ldots, n\}$). But if we had $(s \circ \omega)(e) = (s \circ \omega)(e')$, then we would have $e = e'$ (since the map $s \circ \omega$ is an injection), which would contradict $e <_1 e'$. Thus, we do not have $(s \circ \omega)(e) = (s \circ \omega)(e')$. In other words, we have $(s \circ \omega)(e) \neq (s \circ \omega)(e')$. Combined with $(s \circ \omega)(e) \leq (s \circ \omega)(e')$, this yields $(s \circ \omega)(e) < (s \circ \omega)(e')$. Thus,

$$s(\omega(e)) = (s \circ \omega)(e) < (s \circ \omega)(e') = s(\omega(e')). \tag{49}$$

Thus, Proposition 6.11 **(d)** (applied to $i = \omega(e)$ and $j = \omega(e')$) yields $a_{\omega(e)} \leq a_{\omega(e')}$. But the definition of η yields $\eta(e) = a_{\omega(e)}$ and $\eta(e') = a_{\omega(e')}$. Hence, $\eta(e) = a_{\omega(e)} \leq a_{\omega(e')} = \eta(e')$. Hence, $\eta(e) \leq \eta(e')$ is shown. Thus, we have proven (45).

Proof of (46): Assume that $e <_1 e'$ and $e \geq_2 e'$. We are going to show that $\eta(e) < \eta(e')$.

We have $e <_1 e'$. Hence, we can prove (49) (in the same way as we have proved it in our above proof of (45)). Thus, we have $s(\omega(e)) < s(\omega(e'))$.

We have $e \neq e'$ (since $e <_1 e'$). Combined with $e \geq_2 e'$, this yields $e >_2 e'$, so that $e' <_2 e$. But (44) (applied to $p = e'$ and $q = e$) yields that we have the following logical equivalence: $(e' <_2 e) \iff (\omega(e') < \omega(e))$. Thus, we have $\omega(e') < \omega(e)$ (since we have $e' <_2 e$). Hence, $\omega(e) > \omega(e')$, so that $\omega(e) \geq \omega(e')$. Thus, Proposition 6.11 **(c)** (applied to $i = \omega(e)$ and $j = \omega(e')$) yields that we have the following logical equivalence:

$$(s(\omega(e)) < s(\omega(e'))) \iff \left(a_{\omega(e)} < a_{\omega(e')}\right).$$

Hence, we have $a_{\omega(e)} < a_{\omega(e')}$ (since we have $s(\omega(e)) < s(\omega(e'))$). But the definition of η yields $\eta(e) = a_{\omega(e)}$ and $\eta(e') = a_{\omega(e')}$. Hence, $\eta(e) = a_{\omega(e)} < a_{\omega(e')} = \eta(e')$. Hence, $\eta(e) < \eta(e')$ is shown. Thus, we have proven (46).

Now, we have proven both (45) and (46).

Let us now forget that we fixed e and e'. We thus have shown that any two elements e and e' of E satisfy (45) and (46). In other words, the map η is a π-partition[51]. In other words, the word w fits into π (since we know that the word w fits into π if and only if the map η is a π-partition). This proves (48).

We now have proven both implications (47) and (48). Combining these two implications, we obtain the logical equivalence

$$\begin{aligned} &(\text{the word } w \text{ fits into } \pi) \\ \iff &(\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}). \end{aligned}$$

[51] since we know that η is a π-partition if and only if any two elements e and e' of E satisfy (45) and (46)

In other words, Proposition 6.14 **(b)** is proven.

(a) Proposition 6.14 **(b)** yields that the word w fits into π if and only if the map $s \circ \omega : E \to \{1, 2, \ldots, n\}$ is an order homomorphism $(E, <_1) \to \{1, 2, \ldots, n\}$. In other words, we have the following logical equivalence:

$$\begin{aligned} & (\text{the word } w \text{ fits into } \pi) \\ \iff & (\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}) . \end{aligned} \tag{50}$$

But Proposition 6.12 shows that the standard permutation of the word s is s again. Hence, we can also apply (50) to $\{1, 2, \ldots, n\}$, s and $s(k)$ instead of A, w and a_k. As a result, we obtain the following logical equivalence:

$$\begin{aligned} & (\text{the word } s \text{ fits into } \pi) \\ \iff & (\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}) . \end{aligned} \tag{51}$$

Hence, we have the following logical equivalence:

$$\begin{aligned} & (\text{the word } w \text{ fits into } \pi) \\ \iff & (\text{the map } s \circ \omega \text{ is an order homomorphism } (E, <_1) \to \{1, 2, \ldots, n\}) \\ & \qquad (\text{by (50)}) \\ \iff & (\text{the word } s \text{ fits into } \pi) \qquad (\text{by (51)}) \\ \iff & (\text{the standard permutation of } w \text{ fits into } \pi) \\ & \qquad (\text{since } s \text{ is the standard permutation of } w) . \end{aligned}$$

In other words, Proposition 6.14 **(a)** is proven. □

Now that we are already studying π-partitions, let us also prove Theorem 1 in [G]. Let us first give our own version of it (whose equivalence to Theorem 1 in [G], however, is rather obvious, unlike that of Proposition 3.1):

Theorem 6.15. Let X be a totally ordered alphabet. For every double poset D, we let $\mathcal{A}(D)$ denote the set of all D-partitions into X.

Let $\pi = (E, <_1, <_2)$ be a special double poset. Then, the sets $\mathcal{A}((E, \prec, <_2))$, where $\prec$ runs over all linear extensions of π, are pairwise disjoint. The union of these sets is $\mathcal{A}(\pi)$.

Theorem 6.15 is not only easily seen to be equivalent to Theorem 1 in [G], but also quickly yields [GriRei15, Theorem 5.2.11].

Before we prove Theorem 6.15, let us state an auxiliary fact which itself might be of some interest:

Proposition 6.16. Let X be a totally ordered alphabet. For every double poset D, we let $\mathcal{A}(D)$ denote the set of all D-partitions into X.

Let $\pi = (E, <_1, <_2)$ be a special double poset. Let n be the size of the special double poset π (that is, the size of the underlying set E of π). Let ω be the labelling of the special double poset π. Thus, ω is the unique order isomorphism $(E, <_2) \to \{1, 2, \ldots, n\}$ (by the definition of the labelling of the special double poset $(E, <_1, <_2)$).

For every $f : E \to X$, we let $\mathbf{w}(f)$ denote the word $f(\omega^{-1}(1)) f(\omega^{-1}(2)) \cdots f(\omega^{-1}(n))$. This word $\mathbf{w}(f)$ gives rise to a binary relation $\prec_{\mathbf{w}(f)}$ on the set $\{1, 2, \ldots, n\}$ (defined according to Definition 6.9), and this relation, in turn, gives rise to a binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ on the set E.

(a) If $f \in \mathcal{A}(\pi)$, then the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is a linear extension of π.

(b) If $\prec$ is any linear extension of π, then

$$\mathcal{A}((E, \prec, <_2)) = \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}.$$

Proof of Proposition 6.16. We know that the map ω is an order isomorphism $(E, <_2) \to (\{1, 2, \ldots, n\}, <)$. Hence, for any two elements p and q of E, we have the following logical equivalence:

$$(p <_2 q) \iff (\omega(p) < \omega(q)) \tag{52}$$

(by (39), applied to $(U, <_U) = (E, <_2)$, $(V, <_V) = (\{1, 2, \ldots, n\}, <)$ and $f = \omega$).

The map ω is an order isomorphism, thus a bijection, thus an injection.

(a) Let $f \in \mathcal{A}(\pi)$. Then, f is an element of $\mathcal{A}(\pi)$. In other words, f is a π-partition into X (since $\mathcal{A}(\pi)$ is the set of all π-partitions into X (by the definition of $\mathcal{A}(\pi)$)).

Definition 6.7 (applied to $x = f$) yields that the map f is a π-partition if and only if any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ implies } f(e) \leq f(e')\right) \tag{53}$$

and

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } f(e) < f(e')\right). \tag{54}$$

Hence, any two elements e and e' of E satisfy (53) and (54) (since f is a π-partition).

Proposition 6.8 **(b)** (applied to $\mathbf{w}(f)$, $f(\omega^{-1}(k))$ and X instead of w, a_k and P) yields that the binary relation $\prec_{\mathbf{w}(f)}$ is the smaller relation of a total order on the set $\{1, 2, \ldots, n\}$. Thus, Proposition 6.4 (applied to E, $\{1, 2, \ldots, n\}$, $\prec_{\mathbf{w}(f)}$ and ω instead of A, B, $<_B$ and f) yields that the relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is the smaller relation of a total order on E.

We shall now show that this relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ extends the first order $<_1$ on E.

For any two elements e and e' of E, we have the following logical equivalence:

$$\left(e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'\right) \iff \left(\text{either } f(e) < f(e') \text{ or } \left(f(e) = f(e') \text{ and } \omega(e) < \omega(e')\right)\right). \qquad (55)$$

[52]

Now, let e and e' be two elements of E such that $e <_1 e'$. We are going to show that $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$.

We now recall that $<_2$ is a total order (since $(E, <_1, <_2)$ is a special double poset). Hence, we have either $e <_2 e'$ or $e \geq_2 e'$. In other words, we must be in

[52] *Proof of (55):* Let e and e' be two elements of E. We need to prove the logical equivalence (55).

Define n elements $a_1, a_2, \ldots, a_n$ of X by $\left(a_k = f\left(\omega^{-1}(k)\right) \text{ for every } k \in \{1, 2, \ldots, n\}\right)$. Then, $a_1 a_2 \cdots a_n = f\left(\omega^{-1}(1)\right) f\left(\omega^{-1}(2)\right) \cdots f\left(\omega^{-1}(n)\right) = \mathbf{w}(f)$.

Let $i = \omega(e)$ and $j = \omega(e')$. Then, the definition of a_i yields $a_i = f\left(\underbrace{\omega^{-1}(i)}_{\substack{=e \\ (\text{since } i=\omega(e))}}\right) = f(e)$, and the definition of a_j yields $a_j = f\left(\underbrace{\omega^{-1}(j)}_{\substack{=e' \\ (\text{since } j=\omega(e'))}}\right) = f(e')$.

Recall that $\mathbf{w}(f) = a_1 a_2 \cdots a_n$. Hence, we have $i \prec_{\mathbf{w}(f)} j$ if and only if $\left(\text{either } a_i < a_j \text{ or } \left(a_i = a_j \text{ and } i < j\right)\right)$ (due to the definition of "$i \prec_{\mathbf{w}(f)} j$"). In other words, we have the following logical equivalence:

$$\left(i \prec_{\mathbf{w}(f)} j\right) \iff \left(\text{either } a_i < a_j \text{ or } \left(a_i = a_j \text{ and } i < j\right)\right).$$

We have $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ if and only if $\omega(e) \prec_{\mathbf{w}(f)} \omega(e')$ (due to the definition of "$e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$"). Hence, we have the following logical equivalence:

$$\begin{aligned}
&\left(e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'\right) \\
\iff &\left(\underbrace{\omega(e)}_{=i} \prec_{\mathbf{w}(f)} \underbrace{\omega(e')}_{=j}\right) \iff \left(i \prec_{\mathbf{w}(f)} j\right) \\
\iff &\left(\text{either } \underbrace{a_i}_{=f(e)} < \underbrace{a_j}_{=f(e')} \text{ or } \left(\underbrace{a_i}_{=f(e)} = \underbrace{a_j}_{=f(e')} \text{ and } \underbrace{i}_{=\omega(e)} < \underbrace{j}_{=\omega(e')}\right)\right) \\
\iff &\left(\text{either } f(e) < f(e') \text{ or } \left(f(e) = f(e') \text{ and } \omega(e) < \omega(e')\right)\right).
\end{aligned}$$

Thus, (55) is proven.

one of the following two cases:

Case 1: We have $e <_2 e'$.

Case 2: We have $e \geq_2 e'$.

Let us first consider Case 1. In this case, we have $e <_2 e'$. Applying (52) to $p = e$ and $q = e'$, we obtain the following logical equivalence: $(e <_2 e') \iff (\omega(e) < \omega(e'))$. Hence, we have $\omega(e) < \omega(e')$ (since $e <_2 e'$). But (53) shows that $f(e) \leq f(e')$ (since $e <_1 e'$). Thus, either $f(e) < f(e')$ or $f(e) = f(e')$. Hence, either $f(e) < f(e')$ or $(f(e) = f(e')$ and $\omega(e) < \omega(e'))$ (because we have $i < j$). Thus, we have $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ (because of the equivalence (55)). Hence, $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $e \geq_2 e'$. Hence, (54) shows that $f(e) < f(e')$ (since $e <_1 e'$). Hence, either $f(e) < f(e')$ or $(f(e) = f(e')$ and $\omega(e) < \omega(e'))$. Thus, we have $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ (because of the equivalence (55)). Hence, $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ is proven in Case 2.

Thus, $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ is proven in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, this yields that $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$ always holds.

Let us now forget that we fixed e and e'. We thus have shown that if e and e' are any two elements of E such that $e <_1 e'$, then $e \left(\prec_{\mathbf{w}(f)}\right)^{\omega} e'$. Hence, the relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ extends the relation $<_1$. Combining this with the fact that $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is the smaller relation of a total order on E, we conclude that $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is a total order on E which extends the first order $<_1$ of E. In other words, $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is a linear extension of π (because this is how a linear extension of π is defined). This proves Proposition 6.16 **(a)**.

(b) Let $\prec$ be any linear extension of π. In other words, $\prec$ is a total order on E which extends the first order $<_1$ of E (because this is how a linear extension of π is defined).

We are going to prove that

$$\mathcal{A}((E, \prec, <_2)) \subseteq \left\{ f \in \mathcal{A}(\pi) \ \mid \ \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\} \quad (56)$$

and

$$\left\{ f \in \mathcal{A}(\pi) \ \mid \ \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\} \subseteq \mathcal{A}((E, \prec, <_2)). \quad (57)$$

Proof of (56): Let $g \in \mathcal{A}((E, \prec, <_2))$. We shall show that $g \in \mathcal{A}(\pi)$ and that the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ equals $\prec$.

We know that g is an element of $\mathcal{A}((E, \prec, <_2))$. In other words, f is a $(E, \prec, <_2)$-partition into X (since $\mathcal{A}((E, \prec, <_2))$ is the set of all $(E, \prec, <_2)$-partitions into X (by the definition of $\mathcal{A}\left(\left(E, \prec, <_2\right)\right)$)).

Definition 6.7 (applied to g, $\prec$ and $\left(E, \prec, <_2\right)$ instead of x, $<_1$ and π) yields that the map g is a $\left(E, \prec, <_2\right)$-partition if and only if any two elements e and e' of E satisfy

$$\left(e \prec e' \text{ implies } g\left(e\right) \leq g\left(e'\right)\right) \tag{58}$$

and

$$\left(e \prec e' \text{ and } e \geq_2 e' \text{ implies } g\left(e\right) < g\left(e'\right)\right). \tag{59}$$

Hence, any two elements e and e' of E satisfy (58) and (59) (since g is a $\left(E, \prec, <_2\right)$-partition). Now, it is easy to see that any two elements e and e' of E satisfy

$$\left(e <_1 e' \text{ implies } g\left(e\right) \leq g\left(e'\right)\right) \tag{60}$$

[53] and

$$\left(e <_1 e' \text{ and } e \geq_2 e' \text{ implies } g\left(e\right) < g\left(e'\right)\right) \tag{61}$$

[54]. But Definition 6.7 (applied to $x = g$) yields that the map g is a π-partition if and only if any two elements e and e' of E satisfy (60) and (61). Hence, the map g is a π-partition (since any two elements e and e' of E satisfy (60) and (61)). More precisely, g is a π-partition into X. In other words, g is an element of $\mathcal{A}\left(\pi\right)$ (since $\mathcal{A}\left(\pi\right)$ is the set of all π-partitions into X (by the definition of $\mathcal{A}\left(\pi\right)$)). That is, we have $g \in \mathcal{A}\left(\pi\right)$.

It is easy to see that any two elements e and e' of E satisfying $e \prec e'$ satisfy

$$\left(\text{either } g\left(e\right) < g\left(e'\right) \text{ or } \left(g\left(e\right) = g\left(e'\right) \text{ and } \omega\left(e\right) < \omega\left(e'\right)\right)\right) \tag{62}$$

[55].

[53] *Proof of (60):* Let e and e' be two elements of E. We need to prove (60). Assume that $e <_1 e'$. Then, $e \prec e'$ (since the relation $\prec$ extends the first order $<_1$ of E). Hence, $g\left(e\right) \leq g\left(e'\right)$ (according to (58)). This proves (60).

[54] *Proof of (61):* Let e and e' be two elements of E. We need to prove (61). Assume that $e <_1 e'$ and $e \geq_2 e'$. We have $e <_1 e'$; therefore, $e \prec e'$ (since the relation $\prec$ extends the first order $<_1$ of E). Hence, $g\left(e\right) < g\left(e'\right)$ (according to (59)). This proves (61).

[55] *Proof of (62):* Let e and e' be two elements of E satisfying $e \prec e'$. We need to prove (62).

We distinguish between two cases:

Case 1: We have $\omega\left(e\right) \leq \omega\left(e'\right)$.

Case 2: We don't have $\omega\left(e\right) \leq \omega\left(e'\right)$.

Let us first consider Case 1. In this case, we have $\omega\left(e\right) \leq \omega\left(e'\right)$. If we have $\omega\left(e\right) = \omega\left(e'\right)$, then we have $e = e'$ (since ω is an injection), which contradicts $e \prec e'$. Hence, we cannot have $\omega\left(e\right) = \omega\left(e'\right)$. We thus have $\omega\left(e\right) \neq \omega\left(e'\right)$. Combined with $\omega\left(e\right) \leq \omega\left(e'\right)$, this yields $\omega\left(e\right) < \omega\left(e'\right)$. Also, $e \prec e'$, and thus $g\left(e\right) \leq g\left(e'\right)$ (due to (58)). In other words, either $g\left(e\right) < g\left(e'\right)$ or $g\left(e\right) = g\left(e'\right)$. Hence, either $g\left(e\right) < g\left(e'\right)$ or $\left(g\left(e\right) = g\left(e'\right) \text{ and } \omega\left(e\right) < \omega\left(e'\right)\right)$ (because we have $\omega\left(e\right) < \omega\left(e'\right)$). Thus, (62) is proven in Case 1.

Let us now consider Case 2. In this case, we don't have $\omega\left(e\right) \leq \omega\left(e'\right)$. In other words, we have $\omega\left(e\right) > \omega\left(e'\right)$. In other words, $\omega\left(e'\right) < \omega\left(e\right)$. Applying (52) to $p = e'$ and $q = e$, we obtain the logical equivalence $\left(e' <_2 e\right) \iff \left(\omega\left(e'\right) < \omega\left(e\right)\right)$. Thus, we have $e' <_2 e$ (since we have $\omega\left(e'\right) < \omega\left(e\right)$). In other words, $e >_2 e'$, so that $e \geq_2 e'$. Recall also that $e \prec e'$. Thus, (59) shows that $g\left(e\right) < g\left(e'\right)$. Hence, either $g\left(e\right) < g\left(e'\right)$ or $\left(g\left(e\right) = g\left(e'\right) \text{ and } \omega\left(e\right) < \omega\left(e'\right)\right)$. Thus, (62) is proven in Case 2.

We have now proven (62) in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, this shows that (62) always holds, qed.

Now, let i and j be any two elements of E. We are going to prove that we have the logical equivalence $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right) \iff (i \prec j)$.

In the proof of Proposition 6.16 **(a)**, we have shown that for every $f \in \mathcal{A}(\pi)$ and every two elements e and e' of E, the logical equivalence (55) holds. Thus, (55) (applied to $f = g$, $e = i$ and $e' = j$) shows that we have the following logical equivalence:

$$\begin{aligned}&\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right)\\ &\iff (\text{either } g(i) < g(j) \text{ or } (g(i) = g(j) \text{ and } \omega(i) < \omega(j))) \end{aligned} \tag{63}$$

(since we have $g \in \mathcal{A}(\pi)$). Also, (55) (applied to $f = g$, $e = j$ and $e' = i$) shows that we have the following logical equivalence:

$$\begin{aligned}&\left(j\left(\prec_{\mathbf{w}(g)}\right)^{\omega}i\right)\\ &\iff (\text{either } g(j) < g(i) \text{ or } (g(j) = g(i) \text{ and } \omega(j) < \omega(i))). \end{aligned} \tag{64}$$

Now, let us recall that $\prec$ is a total order. Thus, we have either $i \prec j$ or $i = j$ or $j \prec i$. In other words, we must be in one of the following three cases:

Case 1: We have $i \prec j$.

Case 2: We have $i = j$.

Case 3: We have $j \prec i$.

Let us first consider Case 1. In this case, we have $i \prec j$. Hence, (62) (applied to $e = i$ and $e' = j$) yields

$$(\text{either } g(i) < g(j) \text{ or } (g(i) = g(j) \text{ and } \omega(i) < \omega(j))).$$

Therefore, we have $i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j$ (according to the equivalence (63)). Thus, both statements $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right)$ and $(i \prec j)$ hold. Therefore, we have the equivalence $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right) \iff (i \prec j)$. Hence, the equivalence $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right) \iff (i \prec j)$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $i = j$. Hence, we have neither $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right)$ nor $(i \prec j)$ (because both $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ and $\prec$ are total orders). Therefore, we have the equivalence $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right) \iff (i \prec j)$. Hence, the equivalence $\left(i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j\right) \iff (i \prec j)$ is proven in Case 2.

Let us finally consider Case 3. In this case, we have $j \prec i$. Hence, (62) (applied to $e = j$ and $e' = i$) yields

$$(\text{either } g(j) < g(i) \text{ or } (g(j) = g(i) \text{ and } \omega(j) < \omega(i))).$$

Thus, we have $j\left(\prec_{\mathbf{w}(g)}\right)^{\omega}i$ (according to the equivalence (64)). Therefore, we do not have $i\left(\prec_{\mathbf{w}(g)}\right)^{\omega}j$ (since the relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ is asymmetric (because

$\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ is a total order)). Also, we have $j \prec i$, and thus we do not have $i \prec j$ (since the relation $\prec$ is asymmetric (because $\prec$ is a total order)). Hence, we have neither $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right)$ nor $(i \prec j)$ (because both $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ and $\prec$ are total orders). Therefore, we have the equivalence $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right) \iff (i \prec j)$. Hence, the equivalence $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right) \iff (i \prec j)$ is proven in Case 3.

We thus have proven the equivalence $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right) \iff (i \prec j)$ in each of the three Cases 1, 2 and 3. Since these three Cases cover all possibilities, this yields that the equivalence $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right) \iff (i \prec j)$ always holds.

Now, let us forget that we fixed i and j. We thus have proven the logical equivalence $\left(i \left(\prec_{\mathbf{w}(g)}\right)^{\omega} j\right) \iff (i \prec j)$ for all $i \in E$ and $j \in E$. In other words, the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\prec$.

So we know that $g \in \mathcal{A}(\pi)$, and that the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\prec$. In other words, g is an element f of $\mathcal{A}(\pi)$ such that the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ equals $\prec$. In other words,

$$g \in \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}.$$

Let us now forget that we fixed g. We thus have shown that
$g \in \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}$ for every
$g \in \mathcal{A}((E, \prec, <_2))$. In other words,

$$\mathcal{A}((E, \prec, <_2)) \subseteq \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}.$$

This proves (56).

Proof of (57): Let $g \in \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}$. Then, g is an element f of $\mathcal{A}(\pi)$ such that the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ equals $\prec$. In other words, g is an element of $\mathcal{A}(\pi)$, and the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\prec$.

In the proof of Proposition 6.16 **(a)**, we have shown that for every $f \in \mathcal{A}(\pi)$ and every two elements e and e' of E, the logical equivalence (55) holds. We are going to use this in the following.

We are going to prove that $g \in \mathcal{A}((E, \prec, <_2))$. In order to do so, we shall show that any two elements e and e' of E satisfy (58) and (59).

Let e and e' be two elements of E. Let us first prove (58).

Indeed, we assume that $e \prec e'$. Then, we have $e \prec e'$. In other words, $e \left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'$ (since the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\prec$). But recall that

$g \in \mathcal{A}(\pi)$. Thus, we can apply (55) to $f = g$. As a result, we conclude that the following logical equivalence holds:

$$\left(e\left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'\right)$$
$$\Longleftrightarrow \left(\text{either } g(e) < g(e') \text{ or } \left(g(e) = g(e') \text{ and } \omega(e) < \omega(e')\right)\right).$$

Therefore, we have

$$\left(\text{either } g(e) < g(e') \text{ or } \left(g(e) = g(e') \text{ and } \omega(e) < \omega(e')\right)\right)$$

(since we have $e\left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'$). Consequently, we have

$$\left(\text{either } g(e) < g(e') \text{ or } g(e) = g(e')\right)$$

(because of the logical implication
$(g(e) = g(e') \text{ and } \omega(e) < \omega(e')) \Longrightarrow (g(e) = g(e'))$). In other words, $g(e) \leq g(e')$. This proves (58).

Let us next prove (59).

Indeed, we assume that $e \prec e'$ and $e \geq_2 e'$. Then, it is easy to see that $\omega(e) < \omega(e')$ is false[56]. Now, recall that $e \prec e'$. In other words, $e\left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'$ (since the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\prec$). But recall that $g \in \mathcal{A}(\pi)$. Thus, we can apply (55) to $f = g$. As a result, we conclude that the following logical equivalence holds:

$$\left(e\left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'\right)$$
$$\Longleftrightarrow \left(\text{either } g(e) < g(e') \text{ or } \left(g(e) = g(e') \text{ and } \underbrace{\omega(e) < \omega(e')}_{\substack{\text{this is false} \\ \text{(since } \omega(e)<\omega(e') \text{ is false)}}}\right)\right)$$
$$\Longleftrightarrow \left(\text{either } g(e) < g(e') \text{ or } \underbrace{\left(g(e) = g(e') \text{ and } (\text{false})\right)}_{\text{this is false}}\right)$$
$$\Longleftrightarrow \left(\text{either } g(e) < g(e') \text{ or } (\text{false})\right) \Longleftrightarrow \left(g(e) < g(e')\right).$$

Hence, we have $g(e) < g(e')$ (since we have $e\left(\prec_{\mathbf{w}(g)}\right)^{\omega} e'$). This proves (59).

[56] *Proof.* Assume the contrary. Then, we have $\omega(e) < \omega(e')$. Applying (52) to $p = e$ and $q = e'$, we obtain the logical equivalence $(e <_2 e') \Longleftrightarrow (\omega(e) < \omega(e'))$. Hence, we have $e <_2 e'$ (since we have $\omega(e) < \omega(e')$). But this contradicts $e \geq_2 e'$. This contradiction shows that our assumption was wrong, qed.

Now, let us forget that we fixed e and e'. We thus have proven that any two elements e and e' of E satisfy (58) and (59).

But Definition 6.7 (applied to g, $\prec$ and $(E, \prec, <_2)$ instead of x, $<_1$ and π) yields that the map g is a $(E, \prec, <_2)$-partition if and only if any two elements e and e' of E satisfy (58) and (59). Hence, g is a $(E, \prec, <_2)$-partition (since we know that any two elements e and e' of E satisfy (58) and (59)). More precisely, g is a $(E, \prec, <_2)$-partition into X. In other words, g is an element of $\mathcal{A}((E, \prec, <_2))$ (since $\mathcal{A}((E, \prec, <_2))$ is the set of all $(E, \prec, <_2)$-partitions into X (by the definition of $\mathcal{A}((E, \prec, <_2))$)). Thus, $g \in \mathcal{A}((E, \prec, <_2))$.

Let us now forget that we fixed g. We thus have shown that $g \in \mathcal{A}((E, \prec, <_2))$ for every
$g \in \left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\}$. In other words,

$$\left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\} \subseteq \mathcal{A}((E, \prec, <_2)).$$

This proves (57).

We thus have proven the inclusions (56) and (57). Combining these two inclusions, we obtain

$$\mathcal{A}((E, \prec, <_2)) = \left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\}.$$

This proves Proposition 6.16 **(b)**. □

Proof of Theorem 6.15. We have

$$\mathcal{A}((E, \prec, <_2)) = \left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\}$$

for every linear extension $\prec$ of π (due to Proposition 6.16 **(b)**).

The sets $\left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\}$, where $\prec$ runs over all linear extensions of π, are pairwise disjoint (since the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ is uniquely determined by f). In other words, the sets $\mathcal{A}((E, \prec, <_2))$, where $\prec$ runs over all linear extensions of π, are pairwise disjoint (because $\mathcal{A}((E, \prec, <_2)) = \left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\}$ for every linear extension $\prec$ of π). Thus, in order to prove Theorem 6.15, it remains to show that the union of these sets is $\mathcal{A}(\pi)$. In other words, it remains to show that

$$\bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec\right\} = \mathcal{A}(\pi). \tag{65}$$

But we have

$$\mathcal{A}(\pi) \subseteq \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}$$

[57]. Combining this with

$$\bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \underbrace{\left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}}_{\subseteq \mathcal{A}(\pi)}$$
$$\subseteq \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \mathcal{A}(\pi) \subseteq \mathcal{A}(\pi),$$

we obtain

$$\bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\} = \mathcal{A}(\pi).$$

This proves (65). Thus, the proof of Theorem 6.15 is complete. ☐

[57] *Proof.* Let $g \in \mathcal{A}(\pi)$. Then, the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ is a linear extension of π (according to Proposition 6.16 **(a)**, applied to $f = g$). Thus, $\left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \left(\prec_{\mathbf{w}(g)}\right)^{\omega} \right\}$ is a summand in the union $\bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}$. As a consequence,

$$\left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \left(\prec_{\mathbf{w}(g)}\right)^{\omega} \right\}$$
$$\subseteq \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}.$$

But g is an element f of $\mathcal{A}(\pi)$ such that the binary relation $\left(\prec_{\mathbf{w}(f)}\right)^{\omega}$ equals $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ (since g is an element of $\mathcal{A}(\pi)$, and since the binary relation $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$ equals $\left(\prec_{\mathbf{w}(g)}\right)^{\omega}$). In other words,

$$g \in \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \left(\prec_{\mathbf{w}(g)}\right)^{\omega} \right\}$$
$$\subseteq \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}.$$

Let us now forget that we fixed g. We thus have shown that every $g \in \mathcal{A}(\pi)$ satisfies $g \in \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\}$. In other words,

$$\mathcal{A}(\pi) \subseteq \bigcup_{\substack{\prec \text{ is a linear} \\ \text{extension of } \pi}} \left\{ f \in \mathcal{A}(\pi) \mid \text{the binary relation } \left(\prec_{\mathbf{w}(f)}\right)^{\omega} \text{ equals } \prec \right\},$$

qed.

7. Some brief remarks

7.1. How nondegenerate is the form?

As mentioned in the errata above, the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}\mathbf{D}$ gives rise to a Hopf algebra homomorphism $\mathbb{Z}\mathbf{D} \to (\mathbb{Z}\mathbf{D})^*$ which is **not** bijective. However, [Foissy11, Theorem 36 1.] shows that this homomorphism is at least injective; it becomes bijective upon base change from $\mathbb{Z}$ to a characteristic-0 field.

We can in fact say a bit more. For any double poset $E = (E, <_1, <_2)$, we define its *relational weight* $\operatorname{rlw}(E)$ to be the integer

$$\begin{aligned} \operatorname{rlw}(E) := &(\# \text{ of pairs } (i,j) \in E \times E \text{ such that } i <_1 j) \\ &- (\# \text{ of pairs } (i,j) \in E \times E \text{ such that } i <_2 j). \end{aligned}$$

It is easy to see the following (essentially a result of Foissy in [Foissy11, proof of Theorem 36 1.]):

Proposition 7.1. Let E and F be two double posets. Let $\widetilde{F}$ be the double poset obtained from F by swapping its two orders (i.e.: if $F = (F, <_1, <_2)$, then $\widetilde{F} = (F, <_2, <_1)$). Then:

(a) If $\operatorname{rlw}(E) > \operatorname{rlw}\left(\widetilde{F}\right)$, then there exists no picture from E to F, and thus we have $\langle E, F \rangle = 0$.

(b) If $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$, then the pictures from E to F are precisely the double poset isomorphisms from E to $\widetilde{F}$.

(c) If $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$ but not $E \cong \widetilde{F}$ as double posets, then $\langle E, F \rangle = 0$.

(d) If $E \cong \widetilde{F}$ as double posets, then $\langle E, F \rangle = |\operatorname{Aut} E|$, where $\operatorname{Aut} E$ is the group of automorphisms of the double poset E.

Proof sketch. Write the double posets E and F as $E = (E, <_1, <_2)$ and $F = (F, <_1, <_2)$. (Do not worry about the double meaning of the notations $<_1$ and $<_2$; the context will always make clear on which poset we are working on.)

(a) Let $\phi : E \to F$ be a picture. Consider the map

$$\begin{aligned} &\text{from } \{\text{pairs } (i,j) \in E \times E \text{ such that } i <_1 j\} \\ &\text{to } \{\text{pairs } (u,v) \in F \times F \text{ such that } u <_2 v\} \end{aligned}$$

that sends each pair (i,j) to $(\phi(i), \phi(j))$. This map is well-defined (since ϕ is a picture, so that $i <_1 j$ entails $\phi(i) <_2 \phi(j)$) and injective (since ϕ is bijective). Hence,

$$\begin{aligned} &(\# \text{ of pairs } (i,j) \in E \times E \text{ such that } i <_1 j) \\ &\leq (\# \text{ of pairs } (u,v) \in F \times F \text{ such that } u <_2 v). \end{aligned}$$

Similarly, using ϕ^{-1} instead of ϕ, we obtain

$$\begin{aligned}&(\text{\# of pairs } (i,j) \in F \times F \text{ such that } i <_1 j) \\ &\leq (\text{\# of pairs } (u,v) \in E \times E \text{ such that } u <_2 v).\end{aligned}$$

Adding these two inequalities together, we find

$$\begin{aligned}&(\text{\# of pairs } (i,j) \in E \times E \text{ such that } i <_1 j) \\ &\qquad + (\text{\# of pairs } (i,j) \in F \times F \text{ such that } i <_1 j) \\ &\leq (\text{\# of pairs } (u,v) \in F \times F \text{ such that } u <_2 v) \\ &\qquad + (\text{\# of pairs } (u,v) \in E \times E \text{ such that } u <_2 v).\end{aligned}$$

In other words,

$$\begin{aligned}&(\text{\# of pairs } (i,j) \in E \times E \text{ such that } i <_1 j) \\ &\qquad - (\text{\# of pairs } (u,v) \in E \times E \text{ such that } u <_2 v) \\ &\leq (\text{\# of pairs } (u,v) \in F \times F \text{ such that } u <_2 v) \\ &\qquad - (\text{\# of pairs } (i,j) \in F \times F \text{ such that } i <_1 j).\end{aligned}$$

This inequality can be rewritten as $\operatorname{rlw}(E) \leq \operatorname{rlw}\left(\widetilde{F}\right)$ (since its left hand side is $\operatorname{rlw}(E)$, while its right hand side is $\operatorname{rlw}\left(\widetilde{F}\right)$).

Forget that we fixed ϕ. We thus have shown that if $\phi : E \to F$ is any picture, then $\operatorname{rlw}(E) \leq \operatorname{rlw}\left(\widetilde{F}\right)$. Hence, if $\operatorname{rlw}(E) > \operatorname{rlw}\left(\widetilde{F}\right)$, then there exists no picture from E to F, and thus we have $\langle E, F \rangle = 0$ by the definition of the pairing $\langle , \rangle$. This proves Proposition 7.1 **(a)**.

(b) Assume that $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$. Let $\phi : E \to F$ be a picture. Revisit the above proof of Proposition 7.1 **(a)**, and note that the new assumption $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$ means that equality must hold in all relevant inequalities. Hence, in particular, the map

$$\begin{aligned}&\text{from } \{\text{pairs } (i,j) \in E \times E \text{ such that } i <_1 j\} \\ &\text{to } \{\text{pairs } (u,v) \in F \times F \text{ such that } u <_2 v\}\end{aligned}$$

that sends each pair (i,j) to $(\phi(i), \phi(j))$ must not only be injective, but also be surjective (otherwise, the inequality that it leads to would be a strict inequality), and hence bijective. In other words, for any pair $(i,j) \in E \times E$, we have $i <_1 j$ if and only if $\phi(i) <_2 \phi(j)$ (since ϕ is a bijection). In other words, the map ϕ is an order isomorphism from $(E, <_1)$ to $(F, <_2)$. Similarly, the map ϕ^{-1} is an order isomorphism from $(F, <_1)$ to $(E, <_2)$. In other words, ϕ is an order isomorphism from $(E, <_2)$ to $(F, <_1)$.

Now we know that ϕ is both an order isomorphism from $(E, <_1)$ to $(F, <_2)$ and an order isomorphism from $(E, <_2)$ to $(F, <_1)$. In other words, ϕ is a double poset isomorphism from $(E, <_1, <_2)$ to $(F, <_2, <_1)$. In other words, ϕ is a double poset isomorphism from E to $\widetilde{F}$ (since $E = (E, <_1, <_2)$ and $\widetilde{F} = (F, <_2, <_1)$).

Forget that we fixed ϕ. We thus have shown that each picture $\phi : E \to F$ is a double poset isomorphism from E to $\widetilde{F}$. Conversely, it is easy to see that any double poset isomorphism from E to $\widetilde{F}$ is a picture $\phi : E \to F$. Combining these two facts, we conclude that the pictures $\phi : E \to F$ are precisely the double poset isomorphisms from E to $\widetilde{F}$. This proves Proposition 7.1 **(b)**.

(c) Assume that $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$ but not $E \cong \widetilde{F}$ as double posets. As we recall, $\langle E, F \rangle$ is defined as the number of all pictures $\phi : E \to F$. But these pictures are precisely the double poset isomorphisms from E to $\widetilde{F}$ (by Proposition 7.1 **(b)**), and the number of these isomorphisms is 0 (since we don't have $E \cong \widetilde{F}$ as double posets). Hence, we conclude that $\langle E, F \rangle = 0$. This proves Proposition 7.1 **(c)**.

(d) Assume that $E \cong \widetilde{F}$ as double posets. Then, of course, $\operatorname{rlw}(E) = \operatorname{rlw}\left(\widetilde{F}\right)$. As we recall, $\langle E, F \rangle$ is defined as the number of all pictures $\phi : E \to F$. But these pictures are precisely the double poset isomorphisms from E to $\widetilde{F}$ (by Proposition 7.1 **(b)**), and these are clearly in bijection with the double poset isomorphisms from E to E (since $E \cong \widetilde{F}$ as double posets), i.e., with the automorphisms of the double poset E. Hence, we conclude that $\langle E, F \rangle$ is the number of all automorphisms of the double poset E. In other words, $\langle E, F \rangle = |\operatorname{Aut} E|$, where $\operatorname{Aut} E$ is the group of automorphisms of the double poset E. This proves Proposition 7.1 **(d)**. □

Proposition 7.1 tells us something about the Gram matrix[58] of the bilinear form $\langle , \rangle$ on the basis $\mathbf{D}$ of $\mathbb{Z}\mathbf{D}$: Namely, this matrix is upper-triangular up to a permutation of its rows (or columns), and its diagonal entries (after this permutation) are the numbers $|\operatorname{Aut} E|$ for all double posets E. Since these numbers are not all invertible in $\mathbb{Z}$ (in fact, some of them are larger than 1), we conclude that the form $\langle , \rangle$ fails to be nondegenerate; however, it becomes nondegenerate if we change our base ring from $\mathbb{Z}$ to a characteristic-0 field.

We can actually rescale the form $\langle , \rangle$ to make it nondegenerate: Namely, we define a new bilinear form $\langle , \rangle' : \mathbb{Z}\mathbf{D} \times \mathbb{Z}\mathbf{D} \to \mathbb{Z}$ by

$$\langle E, F \rangle' := \frac{|\{\text{pictures } \alpha : E \to F\}|}{|\operatorname{Aut} E|} \tag{66}$$

for all double posets E and F. We claim the following:

[58] If $f : V \times V \to \mathbf{k}$ is a bilinear form on a $\mathbf{k}$-module V, then its *Gram matrix* on a given basis $(b_1, b_2, \ldots, b_m)$ of V is the matrix $\left(f\left(b_i, b_j\right)\right)_{i,j \in \{1,2,\ldots,m\}} \in \mathbf{k}^{m \times m}$. Of course, this definition applies (with the obvious changes) to infinite bases as well. One of the main uses of Gram matrices is in determining whether a bilinear form is nondegenerate: A bilinear form $f : V \times V \to \mathbf{k}$ on a finite-dimensional $\mathbf{k}$-vector space V over a field $\mathbf{k}$ is nondegenerate if and only if its Gram matrix (on any given basis) is invertible. We shall use a variant of this result here, in which V is infinite-dimensional but graded, and we use the graded version of nondegeneracy.

Proposition 7.2. **(a)** This form $\langle , \rangle'$ is well-defined (i.e., the fraction on the right hand side of (66) really is an integer).

(b) However, this new bilinear form $\langle , \rangle'$ is no longer symmetric.

(c) Nevertheless, it still satisfies

$$\langle EF, G \rangle' = \langle E \otimes F, \delta G \rangle'$$

for any three double posets E, F, G.

(d) But it does **not** satisfy $\langle G, EF \rangle' = \langle \delta G, E \otimes F \rangle'$, so it does not make the Hopf algebra $\mathbb{Z}\mathbf{D}$ self-dual.

Proof idea. **(a)** Let E and F be two double posets. Then, the automorphism group $\operatorname{Aut} E$ of the double poset E acts freely on the set of all pictures $\alpha : E \to F$. (The action is a right action by composition: $\alpha\varphi = \alpha \circ \varphi$ for any picture $\alpha : E \to F$ and any automorphism $\varphi \in \operatorname{Aut} E$.) Hence, $|\{\text{pictures } \alpha : E \to F\}|$ is a multiple of $|\operatorname{Aut} E|$. Thus, the fraction on the right hand side of (66) really is an integer. This proves Proposition 7.2 **(a)**.

(b) This is easy to see: Let $E = (\{1,2\}, <_\varnothing, <_\varnothing)$ and $F = (\{1,2\}, <_\varnothing, <_\mathbb{Z})$, where $<_\varnothing$ is the antichain partial order (i.e., no two distinct elements are comparable). Then, $|\operatorname{Aut} E| = 2$ but $|\operatorname{Aut} F| = 1$ (since the second order of F distinguishes the two elements). But $\langle E, F \rangle = 2$ (indeed, since the first orders on both E and F are antichains, it is clear that the pictures from E to F are just the bijections from E to F). Hence, $\langle E, F \rangle' = \dfrac{2}{2}$ but $\langle F, E \rangle' = \dfrac{2}{1}$, which of course is not the same. Thus, the form $\langle , \rangle'$ is not symmetric. This proves Proposition 7.2 **(b)**.

(c) This follows from $\langle EF, G \rangle = \langle E \otimes F, \delta G \rangle$, which is proved on page 3, once we show that $\operatorname{Aut}(EF) \cong \operatorname{Aut} E \times \operatorname{Aut} F$. But the latter is a nice exercise (show that the second order on EF forces any automorphism of EF to send E to E and F to F).

(d) Easy counterexample omitted. ☐

7.2. What is the kernel of Γ ?

What is the kernel of the morphism $\Gamma : \mathbb{Z}\mathbf{D} \to \mathbf{QSym}$ from Theorem 2.2?

7.3. The kernel of L

Note that it is not hard to describe the kernel of the map $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ (defined on page 7). Namely:

Proposition 7.3. The kernel of the map $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ is spanned by all elements of the form

$$[E + (a <_1 b)] + [E + (b <_1 a)] - E \qquad (67)$$

where $\mathbf{E} = (E, <_1, <_2)$ is a special double poset and a and b are two incomparable elements of $(E, <_1)$. Here, the notation "$[\mathbf{E} + (a <_1 b)]$" means the double poset $\mathbf{E}$ with its first order $<_1$ extended so that a becomes smaller than b (and thus all elements that are $\leq_1 a$ become smaller than all elements that are $\geq_1 b$). Likewise, the notation "$[\mathbf{E} + (b <_1 a)]$" is to be understood. (The second orders of all three posets are the same.)

Proof idea. It is easy to see that any element of the form (67) lies in the kernel of L, since any linear extension of $\mathbf{E}$ is either a linear extension of $[\mathbf{E} + (a <_1 b)]$ or a linear extension of $[\mathbf{E} + (b <_1 a)]$ (but not both).

It remains to show that these elements span $\operatorname{Ker} L$. For this, it suffices to show that any isomorphism class $\mathbf{E}$ of a special double poset can be reduced to a linear combination of double total orders (i.e., double posets whose **both** orders are total) by adding a few elements of the form (67). This can be proved by induction on the number of incomparable pairs in the first order of $\mathbf{E}$ (that is, pairs $(a, b) \in E \times E$ that satisfy neither $a <_1 b$ nor $b <_1 a$). In fact, each of the two double posets $[\mathbf{E} + (a <_1 b)]$ and $[\mathbf{E} + (b <_1 a)]$ has fewer incomparable pairs in its first order than $\mathbf{E}$ does, and so we can apply the induction hypothesis to them. $\square$

The elements of the form (67) have another peculiar property:

Proposition 7.4. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset, and let $\mathbf{F}$ be a special double poset. Let a and b be two elements of E that are incomparable in the poset $(E, <_1)$. Then,

$$([\mathbf{E} + (a <_1 b)] + [\mathbf{E} + (b <_1 a)] - \mathbf{E}) \circ \mathbf{F} = 0.$$

(See Proposition 7.3 for the meaning of $[\mathbf{E} + (a <_1 b)]$ and $[\mathbf{E} + (b <_1 a)]$.)

Proof idea. Write $\mathbf{E}$ and $\mathbf{F}$ as $\mathbf{E} = (E, <_{E1}, <_{E2})$ and $\mathbf{F} = (F, <_{F1}, <_{F2})$. Then, Lemma 6.5 **(d)** yields

$$\mathbf{E} \circ \mathbf{F} = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } (E, <_{E1}) \to (F, <_{F2})}} \left(E, (<_{F1})^{\phi}, <_{E2}\right);$$

$$[\mathbf{E} + (a <_1 b)] \circ \mathbf{F} = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } \left(E, <_{[\mathbf{E}+(a<_1 b)]}\right) \to (F, <_{F2})}} \left(E, (<_{F1})^{\phi}, <_{E2}\right);$$

$$[\mathbf{E} + (b <_1 a)] \circ \mathbf{F} = \sum_{\substack{\phi \text{ is an increasing} \\ \text{bijection } \left(E, <_{[\mathbf{E}+(b<_1 a)]}\right) \to (F, <_{F2})}} \left(E, (<_{F1})^{\phi}, <_{E2}\right).$$

But an increasing bijection $\phi : (E, <_{E1}) \to (F, <_{F2})$ is either an increasing bijection from $([\mathbf{E} + (a <_1 b)], <_1)$ to $(F, <_{F2})$ or an increasing bijection from

$([E + (b <_1 a)], <_1)$ to $(F, <_{F2})$ (depending on whether it satisfies $\phi(a) <_{F2} \phi(b)$ or satisfies $\phi(b) <_{F2} \phi(a)$ (in fact, one of these two inequalities must hold, since F is special)). Hence, the above equalities yield

$$[E + (a <_1 b)] \circ F + [E + (b <_1 a)] \circ F = E \circ F.$$

This proves Proposition 7.4. □

Corollary 7.5. The kernel of the map $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ is the Jacobson radical of the nonunital algebra $(\mathbb{Z}\mathbf{DS}, \circ)$.

Proof idea. Consider the internal multiplication $\circ$ on $\mathbb{Z}S$ given by multiplying permutations of the same size: i.e., by

$$\sigma \circ \tau = \begin{cases} \sigma\tau, & \text{if } \sigma, \tau \in S_n \text{ for the same } n; \\ 0, & \text{if } \sigma \in S_n \text{ and } \tau \in S_m \text{ for } n \neq m. \end{cases}$$

The algebra $(\mathbb{Z}S, \circ)$ is just the direct product of the group rings of the symmetric groups S_n for all $n \geq 0$. As such, it is "almost unital" (its unity would be the sum of all identity permutations, if this sum was not infinite); it is a nonunital Frobenius algebra (with the Jöllenbeck scalar product being the Frobenius form), and its Jacobson radical is 0 (by Maschke's theorem). The map $L : \mathbb{Z}\mathbf{DS} \to \mathbb{Z}S$ is a surjective ring morphism from the nonunital ring $(\mathbb{Z}\mathbf{DS}, \circ)$ to the nonunital ring $(\mathbb{Z}S, \circ)$.

If $a \in \operatorname{Ker} L$, then Proposition 7.3 shows that a is a linear combination of elements of the form (67), and therefore Proposition 7.4 shows that $a \circ F = 0$ for each special double poset F; hence, $a \circ b = 0$ for each $b \in \mathbb{Z}\mathbf{DS}$. In particular, we thus have $a \circ b = 0$ for each $a \in \operatorname{Ker} L$. This shows that the ideal $\operatorname{Ker} L$ of $\mathbb{Z}\mathbf{DS}$ is nilpotent. Hence, $\operatorname{Ker} L$ is contained in the Jacobson radical of $(\mathbb{Z}\mathbf{DS}, \circ)$.

Conversely, any element of the Jacobson radical of $(\mathbb{Z}\mathbf{DS}, \circ)$ must be mapped to an element of the Jacobson radical of $(\mathbb{Z}S, \circ)$ under the map L (since L is a surjective ring morphism). But the Jacobson radical of $(\mathbb{Z}S, \circ)$ is 0. Thus, any element of the Jacobson radical of $(\mathbb{Z}\mathbf{DS}, \circ)$ must be mapped to 0 under the map L. In other words, the Jacobson radical of $(\mathbb{Z}\mathbf{DS}, \circ)$ is contained in $\operatorname{Ker} L$. Altogether, Corollary 7.5 is proved. □

Corollary 7.5 is somewhat reminiscent of the projection map NSym $\to$ Sym, whose kernel is (at least in characteristic 0) the Jacobson radical of the nonunital algebra that is NSym under the internal product. (This was originally proved by Solomon in the language of descent algebras; it served as a foundation for the Garsia-Reutenauer study of the descent algebra [GarReu89, Theorem 1.1].) Can we do the Garsia-Reutenauer descent-algebra theory for $\mathbb{Z}\mathbf{DS}$ instead of the descent algebra?

7.4. Weirder questions

Here are two open-ended questions that crossed my mind:

- Can double posets be seen as bimodules over some sort of algebra, and the internal product correspond to tensoring these bimodules? Or what other categorical reason is there for the associativity of the internal product?
- When do two double posets $(E, <_1, <_2)$ and $(F, <_1, <_2)$ satisfy

  $$\begin{aligned} &(\text{number of pictures from } (E, <_1, <_2) \text{ to } (F, <_1, <_2)) \\ &= (\text{number of pictures from } (E, <_1, >_2) \text{ to } (F, <_1, >_2)) \ ? \end{aligned}$$

  This would generalize a symmetry of the Littlewood–Richardson coefficients.

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