Invariant Theory with Applications

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`http://www.win.tue.nl/~jdraisma/teaching/invtheory0910/lecturenotes12.pdf`

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Errata and addenda by Darij Grinberg

The following is a haphazard list of errors I found in "Invariant Theory with Applications" by Jan Draisma and Dion Gijswijt.

16. Errata

- **Page 5, §1.1:** Replace "Clearly, the elements of V^* are regular of degree" by "Clearly, the elements of V^* are regular functions and are homogeneous of degree".
- **Page 7, §1.3:** "discribed" $\to$ "described".
- **Page 8, Example 1.3.2:** "althought" $\to$ "although".
- **Page 8, Example 1.3.3:** "with the same exponent" $\to$ "with the same coefficient".
- **Page 9, proof of Proposition 1.4.1:** Replace "To each $c = (c_1, \ldots, c_n \in \mathbb{C}^n$" by "To each $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$".
- **Page 9, proof of Proposition 1.4.1:** In (1.8), the entries in the last column should be $-c_n, -c_{n-1}, \ldots, -c_2, -c_1$ (not $-c_n, -c_{n-1}, \ldots, c_2, c_1$).
- **Page 9, proof of Proposition 1.4.1:** Replace "shows that $\chi_{A_c}(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$" by "shows that $\chi_{A_c}(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-1} t + c_n$".
- **Page 9, Exercise 1.4.2:** Replace "that $\chi_{A_c}(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$" by "that $\chi_{A_c}(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-1} t + c_n$".
- **Page 9, proof of Proposition 1.4.1:** In (1.9), replace "$(s_1(A_c), s_2(A_c), \ldots, s_n(A_c)$" by "$(s_1(A_c), s_2(A_c), \ldots, s_n(A_c))$".
- **Page 9, proof of Proposition 1.4.1:** Replace "dense in $\mathcal{O}(\mathrm{Mat}_n(\mathbb{C}))$" by "dense in $\mathrm{Mat}_n(\mathbb{C})$". (This mistake appears twice.)
- **Page 9, Exercise 1.4.3:** Replace "nonzero eigenvalues" by "eigenvalues".
- **Page 10, Exercise 1.4.3:** Replace "distinct and nonzero" by "nonzero".

- **Page 10, Exercise 1.4.3:** It might be worth noticing that "the fact" you are mentioning about the Vandermonde determinant is a consequence of Lemma 2.2.4 below (using the well-known fact that the determinant of a square matrix equals the determinant of its transpose).

- **Page 15, Theorem 2.2.9:** You misspell "Sylvester" as "Sylverster".

- **Page 15, proof of Theorem 2.2.9:** Remove the comma in "Since, $\widetilde{A}$ contains".

- **Page 15, proof of Theorem 2.2.9:** You write: "it follows that $\operatorname{Bez}(f)$ has rank $2k+r$". How does this follow? I only see that $\operatorname{Bez}(f)$ has rank $\leq 2k+r$.

- **Page 20:** Replace "every element $T \in U \otimes V$" by "every element $t \in U \otimes V$".

- **Page 20:** Replace "for T to zero" by "for t to zero".

- **Page 21:** "with of k-tensors" $\rightarrow$ "with k-tensors".

- **Page 23:** "so that the v^α, $|\alpha| = k$ a basis of V" should be "so that the v^α with $|\alpha| = k$ form a basis of $S^k V$".

- **Page 23:** Replace "$\pi(v_1 \otimes \cdots v_k)$" by "$\pi(v_1 \otimes \cdots \otimes v_k)$".

- **Page 23, Exercise 3.0.13:** Replace "$v \otimes v \cdots \otimes v$" by "$v \otimes v \otimes \cdots \otimes v$".

- **Page 24, Exercise 3.1.4:** You should require that at least one of U and V is finite-dimensional.

- **Page 24, Exercise 3.1.4:** Replace "isomorhism" by "isomorphism".

- **Page 25:** Replace "so that $g(hf) = (hg)f$" by "so that $g(hf) = (gh)f$".

- **Page 25, Example 4.0.8:** Replace "G module" by "G-module".

- **Page 26, Example 4.0.9:** Replace "G module" by "G-module".

- **Page 27, proof of Proposition 4.0.7:** "$(v \mid v) = \sum\limits_{g \in G} (gv \mid gv)$" should be "$(v \mid v) = \sum\limits_{g \in G} (gv \mid gv)'$".

- **Page 28, Lemma 4.1.1:** Replace "G modules" by "G-modules".

- **Page 28, §4.1:** "of the isomorphism classes of G-modules" should be "of the isomorphism classes of irreducible G-modules".

- **Page 29, Exercise 4.1.2:** Remove the superscript "^{G}".

- **Page 31, Lemma 5.0.9:** "Dixon's" → "Dickson's".
- **Page 32, proof of Hilbert's Basis Theorem:** "Dixon's" → "Dickson's".
- **Page 32:** In (5.2), add a whitespace before "for all $f \in V_1$".
- **Page 32:** "This a G-module morphism" → "This is a G-module morphism".
- **Page 33, Exercise 5.0.13:** "with zero coefficient" should be "with constant coefficient equal to 0".
- **Page 33, Exercise 5.0.13:** I am wondering whether you really mean "subalgebra" here and not "graded subalgebra".
- **Page 34, proof of Theorem 5.1.1:** Replace "$|\beta| <= |G|$" by "$|\beta| \leq |G|$".
- **Page 34, proof of Theorem 5.1.1:** Replace "$p_j = \sum_{|\alpha|=j} f_\alpha z_1^{\alpha_1} \cdots x_n^{\alpha_n}$" by "$p_j = \sum_{|\alpha|=j} f_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$".
- **Page 34, proof of Theorem 5.1.1:** You write: "Recall that p_j is a polynomial in $p_1, \ldots, p_{|G|}$". Did you actually prove this anywhere? (This is a particular case of the following fact: In the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$, each S_n-invariant polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n}$ can be written as a polynomial in the Newton polynomials $p_1, p_2, \ldots, p_n$. [1] This is probably worth stating as an exercise in Chapter 2.
- **Page 37, proof of the weak Nullstellensatz:** Replace "$f_{k,\xi} := (x_1, \ldots, x_{n-1}, \xi)$" by "$f_{k,\xi} := f_k(x_1, \ldots, x_{n-1}, \xi)$".
- **Page 37, proof of the weak Nullstellensatz:** Replace all three "$\sum_{i=1}^{k}$" signs by "$\sum_{j=1}^{k}$" signs.
- **Page 37:** "Nulstellensatz" → "Nullstellensatz".
- **Page 39, proof of Theorem 6.1.10:** Replace the "$\sum_{i=1}^{k}$" sign by a "$\sum_{j=1}^{k}$" sign.
- **Page 41, Lemma 6.2.6:** Replace "from $\mathbb{C}[Y]$ $\mathbb{C}[X]$" by "from $\mathbb{C}[Y]$ to $\mathbb{C}[X]$".

[1] The *proof* of this fact is easy: By Theorem 2.1.1, it suffices to show that the $s_1, s_2, \ldots, s_n$ are polynomials in $p_1, p_2, \ldots, p_n$. In other words, it suffices to show that s_k is a polynomial in $p_1, p_2, \ldots, p_n$ for each $k \in \{1, 2, \ldots, n\}$. But this easily follows by strong induction over k (indeed, (2.18) gives a way to write each s_k for $k \in \{1, 2, \ldots, n\}$ as a polynomial in $p_1, p_2, \ldots, p_n$, provided that $s_1, s_2, \ldots, s_{k-1}$ have already been written in this form).

- **Page 41, proof of Lemma 6.2.8:** "are a regular maps" $\to$ "are regular maps".
- **Page 42, Example 6.3.3:** Replace "act on the W" by "act on the vector space W".
- **Page 43, Theorem 6.3.4:** In property 4, replace "$\phi : Z \mapsto \mathbb{C}^m$" by "$\phi : Z \to \mathbb{C}^m$".
- **Page 43, proof of Theorem 6.3.4:** In the proof of property 3, replace "$\phi : Z \mapsto U$" by "$\phi : Z \to \mathbb{C}^m$".
- **Page 47, proof of Theorem 7.0.14:** Replace "Hence w is in the null-cone N_V" by "Hence w is in the null-cone N_W".
- **Page 49:** Replace "Let $W \bigoplus_{d=0}^{\infty} W_d$ be a direct sum" by "Let $W = \bigoplus_{d=0}^{\infty} W_d$ be a direct sum".
- **Page 49:** In (8.1), replace "V" and "V_d" by "W" and "W_d", respectively.
- **Page 49, Example 8.0.18:** Replace "$H(\mathbb{C}[x_1, \ldots, x_n])$" by "$H(\mathbb{C}[x_1, \ldots, x_n], t)$".
- **Page 50, Theorem 8.1.1:** Replace "of a finite group" by "of a finite group G".
- **Page 50, proof of Theorem 8.1.1:** In (8.5), replace "$\operatorname{tr}(L_d(g))$" by "$t^d \operatorname{tr}(L_d(g))$".
- **Page 50, proof of Theorem 8.1.1:** Replace "lets fix" by "let's fix".
- **Page 50, proof of Theorem 8.1.1:** Replace "the inner sum $\sum\limits_{d=0}^{\infty} \operatorname{tr}(L_d(g))$" by "the inner sum $\sum\limits_{d=0}^{\infty} t^d \operatorname{tr}(L_d(g))$".
- **Page 50, proof of Theorem 8.1.1:** You write: "Pick a basis $x_1, \ldots, x_n$ of V^* that is a system of eigenvectors for $L_1(g)$". It is worth justifying why such a basis exists. (Namely, you are using the apocryphal theorem from linear algebra that says that if U is a finite-dimensional $\mathbb{C}$-vector space, and if α is an element of $\operatorname{GL}(U)$ having finite order, then α is diagonalizable. You are applying this theorem to $U = V^*$ and $\alpha = L_1(g)$, which is allowed because the element $L_1(g)$ of $\operatorname{GL}(V^*)$ has finite order (since the element g of G has finite order). This is not a difficult argument, but I don't think it is obvious enough to be entirely left to the reader.)
- **Page 50, proof of Theorem 8.1.1:** Replace "for a system" by "form a system".
- **Page 50, proof of Theorem 8.1.1:** On the first line of the computation (8.7), replace "$(1 + \lambda_n t + \lambda_n t^2 + \cdots)$" by "$(1 + \lambda_n t + \lambda_n^2 t^2 + \cdots)$".

- **Page 50, proof of Theorem 8.1.1:** On the first line of the computation (8.8), replace "$\operatorname{tr}(L_d(g))$" by "$t^d \operatorname{tr}(L_d(g))$".

- **Page 50, proof of Theorem 8.1.1:** On the third line of the computation (8.8), replace "$\det(I - \rho(g)\, t$" by "$\det(I - \rho(g)\, t)$".

- **Page 51, §8.2:** Replace "which u and v, differ" by "which u and v differ".

- **Page 51, §8.1:** It is worth pointing out that you use the word "code" to mean "linear code".

- **Page 52:** "Furhermore" → "Furthermore".

- **Page 53, Theorem 8.2.6:** Replace "$(x^4 - y^4)$" by "$(x^4 - y^4)^4$".

- **Page 57, Example 9.1.8:** Replace "$\prod_k \prod_l$" by "$\sum_k \sum_l$".

- **Page 59, §9.2:** Replace "Consider the map $\lambda : G \to \operatorname{GL}[\mathbb{C}[x_{ij}, 1/\det(x)])$" by "Consider the map $\lambda : G \to \operatorname{GL}(\mathbb{C}[x_{ij}, 1/\det(x)])$".

- **Page 65, Exercise 10.0.12:** Replace "larger enough" by "large enough".

- **Page 65, proof of Proposition 10.0.13:** Replace "standard basis $\mathbb{C}^2$" by "standard basis of $\mathbb{C}^2$".

- **Page 65, proof of Proposition 10.0.13:** Replace "induced basis of $S^d(V)$" by "induced basis of $S^k(V)$".

- **Page 65, proof of Proposition 10.0.13:** Replace "$\sum_i d(\lambda)\, x^i y^{k-i}$" by "$\sum_i d_i(\lambda)\, x^i y^{k-i}$".

- **Page 65, proof of Proposition 10.0.13:** You claim that "d_0 and every d_i with $c_i \neq 0$ are nonzero polynomials with λ". I would suggest explaining why they are nonzero. (Namely, the polynomial d_0 is nonzero because $d_0 = \sum_i c_i \lambda^i y^k$ (and because not all c_i are 0); meanwhile, the polynomials d_i with $c_i \neq 0$ are nonzero because they satisfy $d_i(0) = c_i \neq 0$.)

- **Page 66, proof of Proposition 10.0.13:** Replace "Then for every i the vector $\mu^k \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} u = \sum_i \lambda^i c_i x^i y^{k-i}$ belongs to U" by "Then for every $\mu \in \{\mu_0, \mu_1, \ldots, \mu_k\}$ the vector $\mu^k \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} u = \sum_i \lambda^i c_i x^i y^{k-i}$ (with $\lambda = \mu^2$) belongs to U".

- **Page 66, proof of Proposition 10.0.13:** In (10.4), replace "$S^d(\operatorname{End}(S^2(V) \oplus \mathbb{C}))$" by "$S^d(S^2(V) \oplus \mathbb{C})$".

- **Page 66, Exercise 10.0.14:** Replace "$\mathrm{SL}_2(\mathbb{C})$ module" by "$\mathrm{SL}_2(\mathbb{C})$-module".

- **Page 68, §11.1:** Replace "it identifies the space $\operatorname{End}\left(V^{\otimes k}\right)^{S_k}$ with $\left(\operatorname{End}(V)^{\otimes k}\right)^{S_k}$ of symmetric tensors" by "it identifies the space $\operatorname{End}\left(V^{\otimes k}\right)^{S_k}$ with the space $\left(\operatorname{End}(V)^{\otimes k}\right)^{S_k}$ of symmetric tensors".

- **Page 68, §11.1:** Replace "Applying the following theorem to $H = S_n$" by "Applying the following theorem to $H = S_k$".

- **Page 69, proof of Theorem 11.2.1:** "represations" → "representations".

- **Page 69, proof of Theorem 11.2.1:** You write: "By complete reducibility, the map $\left((U^*)^{\otimes d}\right)^G \to \left(S^d U^*\right)^G$ is surjective". Actually, you don't need to use complete reducibility here: The projection map

  $$\pi : (U^*)^{\otimes d} \to S^d U^*, \qquad u_1 \otimes u_2 \otimes \cdots \otimes u_d \mapsto u_1 u_2 \cdots u_d$$

  has a G-equivariant section – namely, the linear map

  $$\psi : S^d U^* \to (U^*)^{\otimes d}, \qquad u_1 u_2 \cdots u_d \mapsto \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(d)}.$$

  Hence, the restriction $\left((U^*)^{\otimes d}\right)^G \to \left(S^d U^*\right)^G$ of the map π to the G-invariants has a section as well (namely, the restriction of the section $\psi : S^d U^* \to (U^*)^{\otimes d}$ to the G-invariants). Therefore, this restriction is surjective.

  I like this argument more not just because it avoids the use of complete reducibility, but also because it is more general (it works for any subgroup G of GL_n, including those for which the representations involved fail to be completely reducible).

- **Page 70, proof of Theorem 11.2.1:** "If $d = k$" should be "If $k = d - k$".

- **Page 74:** "a fix a stochastic" → "we fix a stochastic".

- **Page 75, §12.3:** Replace "formal linear combinations of the alphabet V" by "formal linear combinations of the alphabet B".

- **Page 75, §12.3:** Replace "Next we define a polynomial map $\psi_T : \operatorname{rep}(T) \to \bigotimes_{p \in \operatorname{leaf}(T)}$" by "Next we define a polynomial map $\Psi_T : \operatorname{rep}(T) \to \bigotimes_{p \in \operatorname{leaf}(T)} V_p$". (There were two typos here: "ψ_T" should be "Ψ_T", and the "V_p" was missing.)

- **Page 76, §12.3:** I suppose that "$\bigotimes_{p \in \operatorname{leaf}(T) f(p)}$" should be "$\bigotimes_{p \in \operatorname{leaf}(T)} f(p)$".