

A Determinantal Formula for the Exterior Powers of the Polynomial Ring*Dan Laksov & Anders Thorup*

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<http://dx.doi.org/10.1512/iumj.2007.56.2937>**Errata and addenda by Darij Grinberg (version of July 21, 2019)****Contents**

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I will refer to the results appearing in the paper “**A Determinantal Formula for the Exterior Powers of the Polynomial Ring**” by Dan Laksov and Anders Thorup by the numbers under which they appear in this preprint.

10. Errata

- **page 828, §0.3:** It is worth saying that if $g = b_N T^N + b_{N-1} T^{N-1} + b_{N-2} T^{N-2} + \dots$ is a Laurent series, and if i is an integer such that $i > N$, then you set $b_i = 0$. (This is used, for example, in the definition of $\text{Res}(g_1, \dots, g_n)$, since otherwise the entries $b_{i,-j}$ of the determinant are undefined if one of the series g_1, \dots, g_n begins with a very low negative power of T .)
- **page 829, Theorem 0.1:** In part (3), I’d replace “exponents” by “nonnegative exponents” for clarity.
- **page 829, Remark 0.5:** The reasoning for why “(2) is a consequence of (3)” can be simplified. All you need is the following: In order to prove (2) in general, it suffices to prove (2) in the case when f_1, \dots, f_n are monomials (since any polynomial is an A -linear combination of monomials). In other words, it suffices to prove (2) in the case when each $i \in \{1, 2, \dots, n\}$ satisfies $f_i = T^{q_i}$ for some $q_i \in \mathbb{N}$. So assume WLOG that we are in this case. Then, note that both the left hand side and the right hand side of (2) are alternating as functions in the parameters q_1, q_2, \dots, q_n (because the residue of n polynomials f_1, f_2, \dots, f_n vanishes when two of the polynomials are

equal, and switches sign if two of the polynomials are swapped). Thus, we can assume WLOG that $q_1 > q_2 > \cdots > q_n$. Assume this, and write the q_i in the form $q_i = h_i + n - i$ for some $h_i \in \mathbb{N}$; then, the inequalities $q_1 > q_2 > \cdots > q_n \geq 0$ lead to $h_1 \geq h_2 \geq \cdots \geq h_n \geq 0$. Thus, $f_1(X) \wedge \cdots \wedge f_n(X) = X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \cdots \wedge X^{h_n}$ (since each $i \in \{1, 2, \dots, n\}$ satisfies $f_i = T^{q_i} = T^{h_i+n-i}$ (since $q_i = h_i + n - i$)) and

$$\text{Res} \left(\frac{f_1}{P}, \dots, \frac{f_n}{P} \right) = s_{h_1, h_2, \dots, h_n}$$

(this has been proven in the previous paragraph). Thus, the claim of (2) is precisely the assertion (3).

Note that we did not need to assume (1) to make this argument.

- **page 831, proof of Lemma 1.1:** Before "To prove assertion (3)", add "Now assume I to be equipped with a total order."
- **page 832, §1.3:** When defining S here, you should perhaps say that this S is not the same S that was defined in §0.1, but rather a generalization thereof.
- **page 832, proof of Proposition 1.3:** Remove the four sentences that begin with "Then we have an equality" and end with "and hence f_y is in the kernel". (These four sentences merely repeat the four preceding sentences.)
- **page 832, proof of Proposition 1.3:** "Assume that M is A free" \rightarrow "Assume that M is A -free".
- **page 833, §2.1:** In "and it is clear that the action of S on $A[X_1, \dots, X_n]$ is determined by these equations", replace " $A[X_1, \dots, X_n]$ " by " $\wedge_A^n A[X]$ ". (Or perhaps remove these words altogether, since you don't seem to use them anywhere.)
- **page 833, proof of Proposition 2.1:** Replace "of (2.2)" by "of (2.1)".
- **page 833, proof of Proposition 2.1:** This proof is not complete. You implicitly build up an involution for each $k \in \{1, 2, \dots, n-1\}$ that pairs up cancelling addends on the right hand side of (2.1); but why don't these involutions for different values of k "snatch away" addends from one another? An addend may be paired up by more than one of the involutions. I suggest replacing the proof by a cleaner argument, which I show in the Appendix to these errata (Section 11.1):
- **page 834, proof of Corollary 2.2:** I find this proof somewhat harrowing to read; the combinatorics requires too much handwaving. I present a different proof (longer, but a lot less reliant on mental acrobatics) in the Appendix to these errata (Section 11.2).

- **page 834, proof of Corollary 2.2:** In the proof of Corollary 2.2 (your proof, not mine), replace "into $n + 1$ intervals" by "into $m + 1$ intervals".
- **page 835, §2.2:** "Note that the Schur function $s_{h_1 h_2 \dots h_m 0 \dots 0}$ " \rightarrow "Note that the Schur function $s_{h_1, h_2, \dots, h_m, 0, \dots, 0}$ ".
- **page 835, §2.2:** The sentence "For $m = 1$ Gatto's formula clearly holds" should be moved **after** the next sentence ("We prove Gatto's formula ..."), since at its current position it is unclear what the " m " stands for.
- **page 835, §2.2:** "on the positive integers m " \rightarrow "on the smallest positive integer m ".
- **page 835, §2.2:** Before "Development of the determinant", insert "Now assume that $(h_1, h_2, \dots, h_n) \in \mathbb{N}^n$ has $h_{m+1} = h_{m+2} = \dots = h_n = 0$ but $h_m > 0$ ".
- **page 835, §2.2:** In the first displayed equation of §2.2, replace the subscript " $h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_{m-1}, 0, \dots, 0$ " by " $h_1, \dots, h_{i-1}, h_{i+1} - 1, \dots, h_m - 1, 0, \dots, 0$ ".
- **page 835, proof of Lemma 2.3:** I cannot follow this proof at the point where you argue that " $f X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^0$ contains the term $X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \dots \wedge X^{h_n}$ ". A slightly different (but cleaner) proof of Lemma 2.3 is shown in the Appendix to these errata (Section 11.3).
- **page 836, §3:** On the first line of §3, I would replace "of Section 1" by "of (1.2)", just to be a bit more specific.
- **page 836, §3.1:** I suggest replacing "free A -module of rank 1 over S " by "free S -module of rank 1".
- **page 836, §3.1:** Replace " $\text{alt} \left(X_1^{h_1+n-1} \dots X_n^0 \right)$ " by " $\text{alt} \left(X_1^{h_1+n-1} \dots X_n^{h_n} \right)$ ".
- **page 836, §3.1:** I would simplify this whole paragraph, avoiding the first reference to [22], as follows:

"Let Δ be the Vandermonde determinant

$$\det \left(\left(X_j^{n-i} \right)_{1 \leq i \leq n, 1 \leq j \leq n} \right) = \text{alt} \left(X_1^{n-1} X_2^{n-2} \dots X_n^{n-n} \right).$$

Note that $\Delta = \prod_{1 \leq i < j \leq n} (X_i - X_j)$.

The Jacobi-Trudi formula ([22, I.3, (3.4) on p. 41]) shows that

$$s_{h_1, h_2, \dots, h_n} = \text{alt} \left(X_1^{h_1+n-1} X_2^{h_2+n-2} \dots X_n^{h_n+n-n} \right) / \Delta \quad (1)$$

for any $(h_1, h_2, \dots, h_n) \in \mathbb{N}^n$ satisfying $h_1 \geq h_2 \geq \dots \geq h_n$. (Keep in mind that what is called s_λ in [22, §I.3] corresponds to our

$\text{alt} \left(X_1^{h_1+n-1} X_2^{h_2+n-2} \cdots X_n^{h_n+n-n} \right) / \Delta$, where $(h_1, h_2, \dots, h_n) = (\lambda_1, \lambda_2, \dots, \lambda_n)$; meanwhile, what we call s_{h_1, h_2, \dots, h_n} corresponds to the determinant $\det (h_{\lambda_i - i + j})_{1 \leq i, j \leq n}$ in [22, §I.3].)

Let γ denote the S -linear isomorphism $\bigwedge_A^n A[X] \rightarrow A[X_1, \dots, X_n]^{\text{alt}}$ constructed in Proposition 3.1. Then, γ is bijective (since γ is an isomorphism) and thus injective. Now, for any $(h_1, h_2, \dots, h_n) \in \mathbb{N}^n$ satisfying $h_1 \geq h_2 \geq \dots \geq h_n$, we have

$$\begin{aligned} & \gamma \left(X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \cdots \wedge X^{h_n+n-n} \right) \\ &= \text{alt} \left(X_1^{h_1+n-1} X_2^{h_2+n-2} \cdots X_n^{h_n+n-n} \right) \quad (\text{by the definition of } \gamma) \\ &= s_{h_1, h_2, \dots, h_n} \Delta \quad (\text{by (1)}) \\ &= s_{h_1, h_2, \dots, h_n} \underbrace{\text{alt} \left(X_1^{n-1} X_2^{n-2} \cdots X_n^{n-n} \right)}_{=\gamma(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^{n-n})} \quad \left(\text{since } \Delta = \text{alt} \left(X_1^{n-1} X_2^{n-2} \cdots X_n^{n-n} \right) \right) \\ &= s_{h_1, h_2, \dots, h_n} \gamma \left(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^{n-n} \right) \\ &= \gamma \left(s_{h_1, h_2, \dots, h_n} \cdot X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^{n-n} \right) \quad (\text{since } \gamma \text{ is } S\text{-linear}) \end{aligned}$$

and therefore

$$X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \cdots \wedge X^{h_n+n-n} = s_{h_1, h_2, \dots, h_n} \cdot X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^{n-n}. \quad (2)$$

This proves part (3) of the Main Theorem. As we know from Remark 0.5, this entails that part (2) of the Main Theorem also holds. It remains to prove part (1).

The family

$$\left(X^{i_1} \wedge X^{i_2} \wedge \cdots \wedge X^{i_n} \right)_{(i_1, i_2, \dots, i_n) \in \mathbb{N}^n; i_1 > i_2 > \cdots > i_n}$$

and the family

$$\left(X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \cdots \wedge X^{h_n+n-n} \right)_{(h_1, h_2, \dots, h_n) \in \mathbb{N}^n; h_1 \geq h_2 \geq \cdots \geq h_n}$$

can be obtained from one another by relabelling¹. Hence, these two families have the same span. Since the first family spans the A -module $\bigwedge_A^n A[X]$

¹Indeed, there is a bijection from the set $\{(i_1, i_2, \dots, i_n) \in \mathbb{N}^n \mid i_1 > i_2 > \cdots > i_n\}$ to the set $\{(h_1, h_2, \dots, h_n) \in \mathbb{N}^n \mid h_1 \geq h_2 \geq \cdots \geq h_n\}$; this bijection sends each (i_1, i_2, \dots, i_n) to $(i_1 - n + 1, i_2 - n + 2, \dots, i_n - n + n)$. This bijection has the property that if it sends some n -tuple (i_1, i_2, \dots, i_n) to an n -tuple (h_1, h_2, \dots, h_n) , then $X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \cdots \wedge X^{h_n+n-n} = X^{i_1} \wedge X^{i_2} \wedge \cdots \wedge X^{i_n}$. Therefore, if we relabel the first of our two families using this bijection, then we obtain the second family.

(because the A -module $A[X]$ is spanned by X^0, X^1, X^2, \dots), we thus conclude that the second family spans the A -module $\bigwedge_A^n A[X]$ as well.

For any $(h_1, h_2, \dots, h_n) \in \mathbb{N}^n$ satisfying $h_1 \geq h_2 \geq \dots \geq h_n$, we have

$$\begin{aligned} & X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \dots \wedge X^{h_n+n-n} \\ &= \underbrace{s_{h_1, h_2, \dots, h_n}}_{\in S} \cdot X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n} \quad (\text{by (2)}) \\ &\in S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}). \end{aligned}$$

In other words, $f \in S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n})$ whenever f is an element of the family $(X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \dots \wedge X^{h_n+n-n})_{(h_1, h_2, \dots, h_n) \in \mathbb{N}^n; h_1 \geq h_2 \geq \dots \geq h_n}$. Hence, $f \in S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n})$ holds for each $f \in \bigwedge_A^n A[X]$ (since the family $(X^{h_1+n-1} \wedge X^{h_2+n-2} \wedge \dots \wedge X^{h_n+n-n})_{(h_1, h_2, \dots, h_n) \in \mathbb{N}^n; h_1 \geq h_2 \geq \dots \geq h_n}$ spans the A -module $\bigwedge_A^n A[X]$). In other words,

$$\bigwedge_A^n A[X] \subseteq S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}).$$

Combining this with $S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}) \subseteq \bigwedge_A^n A[X]$ (which is obvious), we obtain

$$\bigwedge_A^n A[X] = S \cdot (X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}).$$

Hence, the n -vector $X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}$ generates the S -module $\bigwedge_A^n A[X]$. Since the annihilator of this n -vector is zero (by Lemma 2.3), we thus conclude that the 1-tuple $(X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n})$ is a basis of the S -module $\bigwedge_A^n A[X]$. In other words, the S -module $\bigwedge_A^n A[X]$ is free of rank 1, generated by the n -vector $X^{n-1} \wedge X^{n-2} \wedge \dots \wedge X^{n-n}$. This proves part (1) of the Main Theorem. Thus, the Main Theorem is completely proven.

- **page 836, §4.1:** Before "The residue algebra $S[T]/P$ is freely generated", add "The polynomial $P \in S[T]$ is monic of degree n . Thus,".
- **page 836, §4.1:** After "well-known to be an isomorphism", add "(but this latter fact will not be used)".
- **page 836, §4.1:** After "and it is free of rank 1", add "with a basis consisting of the single element $\zeta^{n-1} \wedge \zeta^{n-2} \wedge \dots \wedge \zeta^0$ (since $S[\zeta]$ is a free S -module with basis $\zeta^{n-1}, \zeta^{n-2}, \dots, \zeta^0$)".
- **page 837, §4.1:** "exterior product" \rightarrow "exterior power".
- **page 837, proof of Lemma 4.1:** "from the equation $0 = P(\zeta) = \zeta^{n-1} - c_1 \zeta^{n-2} + \dots + (-1)^n c_n$ " \rightarrow "from the equation $0 = P(\zeta) = \zeta^n - c_1 \zeta^{n-1} + \dots + (-1)^n c_n$ ".

- **page 838, proof of Theorem 4.2:** "natural injection of S -algebras" \rightarrow "natural homomorphism of S -algebras". (It is true that this homomorphism is an injection, but this is not obvious at this point, and not needed for your argument; it is thus only distraction.)
- **page 838, proof of Theorem 4.2:** After "and equal to the identity on S ", add "(by the universal property of the residue algebra $S[\xi] = S[T]/P$, since $P(X_i) = 0$)".
- **page 838, proof of Theorem 4.2:** "with the natural surjection $A[X_1, \dots, X_n] \rightarrow \bigwedge_A^n A[X]$ " \rightarrow "with the natural surjection $A[X_1, \dots, X_n] = \bigotimes_A^n A[X] \rightarrow \bigwedge_A^n A[X]$ ".
- **page 838, proof of Theorem 4.2:** I suggest explaining somewhere what you mean by "alternating" when talking about maps out of a tensor power. (Namely, you say that an A -linear map $f : \bigotimes_A^n V \rightarrow W$ (for two A -modules V and W) is *alternating* if and only if the map

$$V^n \rightarrow W,$$

$$(v_1, v_2, \dots, v_n) \mapsto f(v_1 \otimes v_2 \otimes \dots \otimes v_n)$$

is alternating; equivalently, you say that $f : \bigotimes_A^n V \rightarrow W$ is alternating if and only if f factors through the canonical projection $\bigotimes_A^n V \rightarrow \bigwedge_A^n W$.)

- **page 838, §4.3:** After "is upper triangular with 1's on the diagonal", add "(by the equality $T^h/P(T) = \sum_{j=n-h}^{\infty} s_{h+j-n} T^{-j}$ in Remark 0.5, and because $s_0 = 1$)".
- **page 838, §4.3:** You write: "Therefore, since $\bigwedge_S^n S[\xi]$ is free of rank 1 over S with generator $\xi^{n-1} \wedge \dots \wedge \xi^0$, equation (4.5) is general". In my opinion, this can be explained better. Indeed, if $f_1, f_2, \dots, f_n \in S[T]$ are n polynomials, then the residue $\text{Res} \left(\frac{f_1}{P}, \frac{f_2}{P}, \dots, \frac{f_n}{P} \right)$ depends only on the residue classes of the polynomials f_1, f_2, \dots, f_n modulo P (but not on these polynomials themselves)². In other words, this residue can be depends only on the values $f_1(\xi), f_2(\xi), \dots, f_n(\xi)$ (since these values encode the same information as the residue classes of the polynomials f_1, f_2, \dots, f_n modulo

²This is because the residue $\text{Res} \left(\frac{f_1}{P}, \frac{f_2}{P}, \dots, \frac{f_n}{P} \right)$ does not change when a multiple of P is added to one of f_1, f_2, \dots, f_n (since this residue is S -multilinear in f_1, f_2, \dots, f_n and vanishes when one of f_1, f_2, \dots, f_n is divisible by P).

P). Hence, the map

$$\begin{aligned} \alpha : (S[\xi])^n &\rightarrow \bigwedge_S^n S[\xi], \\ (f_1(\xi), f_2(\xi), \dots, f_n(\xi)) &\mapsto \operatorname{Res} \left(\frac{f_1}{P}, \frac{f_2}{P}, \dots, \frac{f_n}{P} \right) \xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^0 \\ &\quad (\text{for } f_1, f_2, \dots, f_n \in S[T]) \end{aligned}$$

is well-defined. This map α is furthermore S -multilinear (since the residue is S -multilinear) and alternating (since the residue is alternating). Hence, it induces an S -linear map

$$\begin{aligned} \alpha' : \bigwedge_S^n S[\xi] &\rightarrow \bigwedge_S^n S[\xi], \\ f_1(\xi) \wedge f_2(\xi) \wedge \dots \wedge f_n(\xi) &\mapsto \operatorname{Res} \left(\frac{f_1}{P}, \frac{f_2}{P}, \dots, \frac{f_n}{P} \right) \xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^0 \\ &\quad (\text{for } f_1, f_2, \dots, f_n \in S[T]). \end{aligned}$$

But you have proved the equality (4.5) in the case when $f_i(T) = T^{n-i}$ for all $i \in \{1, 2, \dots, n\}$. In other words, you have shown that

$$\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n} = \operatorname{Res} \left(\frac{T^{n-1}}{P}, \frac{T^{n-2}}{P}, \dots, \frac{T^{n-n}}{P} \right) \xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^0.$$

But the definition of α' (applied to the polynomials $T^{n-1}, T^{n-2}, \dots, T^{n-n}$ instead of f_1, f_2, \dots, f_n) yields

$$\begin{aligned} &\alpha' \left(\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n} \right) \\ &= \operatorname{Res} \left(\frac{T^{n-1}}{P}, \frac{T^{n-2}}{P}, \dots, \frac{T^{n-n}}{P} \right) \xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^0. \end{aligned}$$

Comparing these two equalities, we find

$$\begin{aligned} \alpha' \left(\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n} \right) &= \xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n} \\ &= \operatorname{id} \left(\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n} \right). \end{aligned}$$

Hence, the two S -linear maps $\alpha' : \bigwedge_S^n S[\xi] \rightarrow \bigwedge_S^n S[\xi]$ and $\operatorname{id} : \bigwedge_S^n S[\xi] \rightarrow \bigwedge_S^n S[\xi]$ are equal to each other on the element $\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n}$. Since this element $\xi^{n-1} \wedge \xi^{n-2} \wedge \dots \wedge \xi^{n-n}$ generates the S -module $\bigwedge_S^n S[\xi]$, we can thus conclude that the two maps α' and id are identical (because if two S -linear maps are equal to each other on a given generating set of their domain, then they must be identical). In other words, $\alpha' = \operatorname{id}$. Hence, every $f_1, f_2, \dots, f_n \in S[T]$ satisfy

$$\begin{aligned} &\underbrace{\alpha'}_{=\operatorname{id}} (f_1(\xi) \wedge f_2(\xi) \wedge \dots \wedge f_n(\xi)) \\ &= \operatorname{id} (f_1(\xi) \wedge f_2(\xi) \wedge \dots \wedge f_n(\xi)) = f_1(\xi) \wedge f_2(\xi) \wedge \dots \wedge f_n(\xi). \end{aligned}$$

Hence,

$$\begin{aligned} & f_1(\xi) \wedge f_2(\xi) \wedge \cdots \wedge f_n(\xi) \\ &= \alpha' (f_1(\xi) \wedge f_2(\xi) \wedge \cdots \wedge f_n(\xi)) \\ &= \text{Res} \left(\frac{f_1}{P}, \frac{f_2}{P}, \dots, \frac{f_n}{P} \right) \xi^{n-1} \wedge \xi^{n-2} \wedge \cdots \wedge \xi^0 \end{aligned}$$

(by the definition of α'). Hence, (4.5) is proven.

- **page 839, §5.1:** Replace "isomorphism of S -algebras $S[\xi] \rightarrow S[X_i]$ " by "homomorphism of S -algebras $S[\xi] \rightarrow A[X_1, X_2, \dots, X_n]$ ". (It is true that this homomorphism restricts to an isomorphism of S -algebras $S[\xi] \rightarrow S[X_i]$, but this is neither needed nor easy to prove at this point.)
- **page 839, §5.1:** It is worth mentioning here that you are considering $\bigwedge_S^n S[\xi]$ to be equipped with its natural structure (not its symmetric structure) throughout §5.
- **page 839, §5.2:** Replace "when $F_i = T^{n-i}$ for $i = 1, 2, \dots, n$ " by "when $(F_i = T^{n-i}$ for $i = 1, 2, \dots, n$)". (The parentheses are meant to clarify the logical structure of this sentence:

"The function R is equal to 1 when $(F_i = T^{n-i}$ for $i = 1, 2, \dots, n$)",

not

"(The function R is equal to 1 when $F_i = T^{n-i}$) for $i = 1, 2, \dots, n$ ".
- **page 839, §5.2:** You write: "It follows immediately that". I don't find this obvious enough to deserve the word "immediately". The argument you are tacitly making here is essentially the argument you have done in §4.2 in order to prove (4.5); it is not in any way made unnecessary by the slight change of viewpoint done in §5.
- **page 839, §5.2:** "prove that the generator $\xi^{n-1} \wedge \cdots \wedge \xi^0$ has no S -torsion" \rightarrow "prove that the generator $X^{n-1} \wedge \cdots \wedge X^0$ has no S -torsion". (For the generator $\xi^{n-1} \wedge \cdots \wedge \xi^0$, this is obvious, but that is not the generator you need here.)
- **page 839, §5.2:** "Under the composition of the map (4.4) with the alternator" \rightarrow "Under the alternator". (The map (4.4) is not needed here.)
- **page 839, §5.2:** " $\bigwedge^n A[X]$ " \rightarrow " $\bigwedge_A^n A[X]$ ".
- **page 839, §5.2:** "the generator $\xi^{n-1} \wedge \cdots \wedge \xi^0$ is mapped" \rightarrow "the generator $X^{n-1} \wedge \cdots \wedge X^0$ is mapped".
- **page 839, §5.2:** After "the generator itself has no S -torsion", add "(since the alternator map alt is S -linear)".

- **page 839, §5.3:** "Therefore the target is a free S -module of rank 1" → "Therefore the target is generated (as an S -module) by the regular polynomial Δ . Hence, it is a free S -module of rank 1".
- **page 839, §5.3:** "It follows that both maps are isomorphisms." → "Hence the composite map in (5.2) is a surjective S -linear map between two free S -modules of rank 1, and thus is an isomorphism. It follows that both maps in (5.2) are isomorphisms (since they are surjective)."
- **page 839, §5.3:** I would replace "times Δ or, equivalently, an anti-symmetric" by "times Δ . Equivalently, an anti-symmetric".
- **page 840, §6.1:** After "see [2, VI 6.5], [24, 2.1], [6], or [25]", I would also add a reference to [LakTho12, §1.3] (where A_r is denoted by S_r).
- **page 840, §6.1:** In the sentence that defines $\partial^{1,\dots,r}$, replace "if $h_i = n - i$ for $i = 1, \dots, r$ " by "if $h_i = n - i$ for all $i = 1, \dots, r$ " (to prevent misunderstanding).
- **page 840, §6.1:** "we write $\partial = \partial^{1,\dots,n}$ " → "we write $\partial = \partial^{1,\dots,n-1}$ ".
- **page 840, §6.1:** Replace " $A_r [T] [[T^{-1}]] \rightarrow A [T] [[T^{-1}]]$ " by " $A_r [T] [[T^{-1}]] \rightarrow A [T] [[T^{-1}]]$ ".
- **page 841, proof of Lemma 6.1:** After "Now, $p(T)$ is an A -linear combination of monomials T^i ", add "with $i \leq n$, and in this combination the monomial T^n has coefficient 1".
- **page 841, proof of Lemma 6.1:** "we see, that" → "we see that".
- **page 841, Proposition 6.2:** This proposition is correct, but it is insufficient for what you want to use it for (namely, proving Proposition 6.3). In order to make it stronger, I suggest removing the words "of degree t " (so q can have any degree). This necessitates a minor tweak in the proof (see below).
- **page 842, (6.6):** On the right hand side of (6.6), add a comma before " $\frac{g^t}{q}$ ".
- **page 842, proof of Proposition 6.2:** The last paragraph of this proof is no longer correct now that I have generalized it. So let me suggest an alternative to this last paragraph:
 "Thus, we know that the left hand side of (6.6) vanishes if q divides some g_j , and is A -linear in each g_j . Hence, the left hand side of (6.6) does not change if we add a multiple of q to some g_j . The same holds for the right hand side (for the same reason). Thus, we can replace each polynomial g_j by its remainder modulo (q) . Hence, we can WLOG assume that all polynomials g_j have degree $< \deg q$. Assume this. Since both sides of (6.6)

are A -linear in each g_j , we can furthermore assume that each g_j is a single monomial: that is, there is a t -tuple $(m_1, m_2, \dots, m_t) \in \{0, 1, \dots, \deg q - 1\}^t$ such that each j satisfies $g_j = T^{m_j}$. Assume this, too. Furthermore, both sides of (6.6) vanish if two of the g_j are equal; thus, we can WLOG assume that m_1, m_2, \dots, m_t are distinct. Assume this. Finally, both sides of (6.6) are anti-symmetric in the g_j ; hence, we can WLOG assume that $m_1 \geq m_2 \geq \dots \geq m_t$ (since otherwise, we can just permute m_1, m_2, \dots, m_t so that this holds). Assume this. Combining $m_1 \geq m_2 \geq \dots \geq m_t$ with the fact that m_1, m_2, \dots, m_t are distinct, we obtain $m_1 > m_2 > \dots > m_t$. Therefore, $m_1 + 1 \geq m_2 + 2 \geq \dots \geq m_t + t$. Hence, each $j \in \{1, 2, \dots, t\}$ satisfies

$$m_j + j \leq m_1 + 1 \leq \deg q$$

$$\left(\begin{array}{l} \text{since } (m_1, m_2, \dots, m_t) \in \{0, 1, \dots, \deg q - 1\}^t \\ \text{and thus } m_1 \in \{0, 1, \dots, \deg q - 1\}, \text{ so that } m_1 \leq \deg q - 1 \end{array} \right)$$

and thus

$$m_j \leq \deg q - j.$$

Thus, for each $j \in \{1, 2, \dots, t\}$, the polynomial $g_j = T^{m_j}$ is monic of degree at most $\deg q - j$ (since $m_j \leq \deg q - j$), and therefore can be written in the form

$$g_j = c_j T^{\deg q - j} + (\text{lower order terms}), \quad (3)$$

where "(lower order terms)" means an A -linear combination of monomials T^u with $u < \deg q - j$, and where $c_j \in A$ is either 0 or 1 (depending on whether $m_j < \deg q - j$ or $m_j = \deg q - j$). Consider these c_j .

Thus, for each $j \in \{1, 2, \dots, t\}$, the Laurent series $\frac{g_j}{q} \in A[T][[T^{-1}]]$ has the form

$$\begin{aligned} \frac{g_j}{q} &= \frac{c_j T^{\deg q - j} + (\text{lower order terms})}{q} && \text{(by (3))} \\ &= c_j T^{-j} + (\text{lower order terms}), && (4) \end{aligned}$$

where "(lower order terms)" means an A -linear combination of monomials T^u with $u < -j$. (Here, we have used the fact that q is a monic polynomial of degree $\deg q$, and therefore division by q lowers the leading term of any Laurent series by the degree q .)

From (4), we see that the matrix used in defining the residue $\text{Res} \left(\frac{g_1}{q}, \dots, \frac{g_t}{q} \right)$ is upper-triangular, with diagonal entries c_1, c_2, \dots, c_t . Hence, its determinant is given by

$$\text{Res} \left(\frac{g_1}{q}, \dots, \frac{g_t}{q} \right) = c_1 c_2 \cdots c_t. \quad (5)$$

Also, $p_r q$ is a monic polynomial of degree $r + \deg q$ (since p_r and q are monic polynomials of degrees r and $\deg q$, respectively). Now, for each $j \in \{1, 2, \dots, t\}$, the Laurent series $\frac{g_j}{p_r q} \in A[T][[T^{-1}]]$ has the form

$$\begin{aligned} \frac{g_j}{p_r q} &= \frac{c_j T^{\deg q - j} + (\text{lower order terms})}{p_r q} && \text{(by (3))} \\ &= c_j T^{-r-j} + (\text{lower order terms}), \end{aligned} \tag{6}$$

where "(lower order terms)" means an A -linear combination of monomials T^u with $u < -r - j$. (Here, we have used the fact that $p_r q$ is a monic polynomial of degree $r + \deg q$, and therefore division by $p_r q$ lowers the leading term any Laurent series by the degree $r + \deg q$.)

Furthermore, for each $i \in \{1, 2, \dots, r\}$, the Laurent series $\frac{1}{p_i} \in A[T][[T^{-1}]]$ has the form

$$\frac{1}{p_i} = T^{-i} + (\text{lower order terms}), \tag{7}$$

where "(lower order terms)" means an A -linear combination of monomials T^u with $u < -i$. (This is simply because p_i is a monic polynomial of degree i .)

From (7) and (6), we see that the matrix used in defining the residue $\text{Res} \left(\frac{1}{p_1}, \dots, \frac{1}{p_r}, \frac{g_1}{p_r q}, \dots, \frac{g_t}{p_r q} \right)$ is upper-triangular, with diagonal entries $\underbrace{1, 1, \dots, 1}_{r \text{ times}}, c_1, c_2, \dots, c_t$. Hence, its determinant is given by

$$\text{Res} \left(\frac{1}{p_1}, \dots, \frac{1}{p_r}, \frac{g_1}{p_r q}, \dots, \frac{g_t}{p_r q} \right) = \underbrace{1 \cdot 1 \cdots 1}_{r \text{ times}} c_1 c_2 \cdots c_t = c_1 c_2 \cdots c_t.$$

Comparing this with (5), we obtain

$$\text{Res} \left(\frac{1}{p_1}, \dots, \frac{1}{p_r}, \frac{g_1}{p_r q}, \dots, \frac{g_t}{p_r q} \right) = \text{Res} \left(\frac{g_1}{q}, \dots, \frac{g_t}{q} \right).$$

This proves (6.6), and thus completes the proof of Proposition 6.2."

- **page 842, proof of Proposition 6.3:** Replace "the corresponding composition $\partial^{2, \dots, r}$ " by "the corresponding composition $\partial^{2, \dots, r} := \partial^2 \circ \dots \circ \partial^r$ ", since strictly speaking you have not defined $\partial^{2, \dots, r}$ yet (not that it isn't very obvious).
- **page 842, (6.8):** On the right hand side of the first line of (6.8), replace " $f(\xi_1)$ " by " $f_1(\xi_1)$ ".

- **page 842, proof of Proposition 6.3:** "follows from equation (6.7) applied with $r := 1$ and $q := q_1$ " \rightarrow "follows from equation (6.3) applied with $A := A_1$, $r := 1$, $a_1 := \xi_1$, $q := q_1$, $t := r - 1$ and $g_j := f_{j-1}$ (since $p_1 = T - \xi_1$ and $p = (T - \xi_1) \cdot q_1 = p_1 q_1$)".

Note that this relies on the generalized version of Proposition 6.2 suggested above, since q_1 has degree $n - 1$, not $r - 1$ (in general).

- **page 843, §7.1:** You write: "It is easy to check that the S -algebra $A[X_1, \dots, X_n]$ satisfies the universal properties of the splitting algebra of the *generic* polynomial $P = (T - X_1) \cdots (T - X_n) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$ over S with X_1, \dots, X_n as universal roots".

Let me spell out what this means and actually check that it is true.

First of all, the universal property of the splitting algebra of a polynomial has been stated in [LakTho12] (more precisely, [LakTho12, §1.2] defines factorization algebras through their universal property, and [LakTho12, §1.3] defines splitting algebras as a particular case of factorization algebras). Applying this property to the S -algebra $A[X_1, \dots, X_n]$, we see that the claim that "the S -algebra $A[X_1, \dots, X_n]$ satisfies the universal properties of the splitting algebra of the *generic* polynomial $P = (T - X_1) \cdots (T - X_n) = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$ over S with X_1, \dots, X_n as universal roots" boils down to the following statement:

Statement 1: Let B be any S -algebra. Let $p = \varphi_1 \varphi_2 \cdots \varphi_n$ be any factorization of p over $B[T]$ into monic linear polynomials $\varphi_1, \varphi_2, \dots, \varphi_n \in B[T]$. Then, there is a unique S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ such that the induced map³ $\gamma[T] : (A[X_1, \dots, X_n])[T] \rightarrow B[T]$ maps $T - X_i$ to φ_i for all $i \in \{1, 2, \dots, n\}$.

Thus, it suffices to prove Statement 1.

We shall prove it by showing a slightly nicer version of it first:

Statement 2: Let B be any S -algebra. Let $u_1, u_2, \dots, u_n \in B$ be any elements such that

$$P = (T - u_1)(T - u_2) \cdots (T - u_n) \quad \text{in } B[T]. \quad (8)$$

³Here and in the following, we are using the following notation: If P and Q are two rings, and if $\alpha : P \rightarrow Q$ is any ring homomorphism, then $\alpha[T]$ shall denote the ring homomorphism from $P[T]$ to $Q[T]$ that is defined by

$$(\alpha[T]) \left(\sum_{i=0}^m p_i T^i \right) = \sum_{i=0}^m \alpha(p_i) T^i \quad \text{for all } m \in \mathbb{N} \text{ and } p_0, p_1, \dots, p_m \in P.$$

(Thus, roughly speaking, $\alpha[T]$ is the map that transforms a polynomial $p \in P[T]$ by applying α to each coefficient of this polynomial.) This map $\alpha[T]$ is said to be *induced* by α .

If both rings P and Q are W -algebras for some commutative ring W , and if $\alpha : P \rightarrow Q$ is a W -algebra homomorphism, then the induced map $\alpha[T]$ is a $W[T]$ -algebra homomorphism.

Then, there is a unique S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ such that

$$(\gamma(X_i) = u_i \quad \text{for all } i \in \{1, 2, \dots, n\}). \quad (9)$$

[*Proof of Statement 2:* First of all, let us recall the universal property of the polynomial ring $A[X_1, \dots, X_n]$. This property shows that there is a unique A -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9). Denote this γ by η . Thus, $\eta : A[X_1, \dots, X_n] \rightarrow B$ is an A -algebra homomorphism and satisfies

$$(\eta(X_i) = u_i \quad \text{for all } i \in \{1, 2, \dots, n\}). \quad (10)$$

Next, we shall show that η is actually an S -algebra homomorphism. Let us consider the $A[T]$ -algebra homomorphism

$$\eta[T] : (A[X_1, \dots, X_n])[T] \rightarrow B[T]$$

induced by η . (This homomorphism $\eta[T]$ simply applies η to each coefficient of the polynomial it acts upon.)

For each $i \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} (\eta[T])(T - X_i) &= T - \underbrace{\eta(X_i)}_{\substack{=u_i \\ \text{(by (10))}}} && \text{(by the definition of } \eta[T]) \\ &= T - u_i. \end{aligned}$$

Multiplying these equalities for all $i \in \{1, 2, \dots, n\}$, we obtain

$$\begin{aligned} &(\eta[T])(T - X_1) \cdot (\eta[T])(T - X_2) \cdots (\eta[T])(T - X_n) \\ &= (T - u_1)(T - u_2) \cdots (T - u_n). \end{aligned} \quad (11)$$

On the other hand, B is an S -algebra. Thus, the map

$$\iota : S \rightarrow B, \quad s \mapsto s \cdot 1_B$$

is an S -algebra homomorphism, and therefore an A -algebra homomorphism. It thus induces an $A[T]$ -algebra homomorphism $\iota[T] : S[T] \rightarrow B[T]$. Note that the " P " on the left hand side of the equality (8) actually stands not for the polynomial $P \in S[T]$ itself, but rather for its image $(\iota[T])(P)$ under this homomorphism; thus, (8) rewrites as

$$(\iota[T])(P) = (T - u_1)(T - u_2) \cdots (T - u_n).$$

Comparing this with (11), we find

$$\begin{aligned} & (\eta [T]) (T - X_1) \cdot (\eta [T]) (T - X_2) \cdots (\eta [T]) (T - X_n) \\ &= (\iota [T]) (P). \end{aligned} \tag{12}$$

From $P = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n$, we obtain

$$(\eta [T]) (P) = T^n - \eta (c_1) T^{n-1} + \cdots + (-1)^n \eta (c_n)$$

(by the definition of $\eta [T]$). Hence,

$$\begin{aligned} & T^n - \eta (c_1) T^{n-1} + \cdots + (-1)^n \eta (c_n) \\ &= (\eta [T]) (P) = (\eta [T]) ((T - X_1) (T - X_2) \cdots (T - X_n)) \\ &\quad (\text{since } P = (T - X_1) (T - X_2) \cdots (T - X_n)) \\ &= (\eta [T]) (T - X_1) \cdot (\eta [T]) (T - X_2) \cdots (\eta [T]) (T - X_n) \\ &\quad (\text{since } \eta [T] \text{ is a ring homomorphism}) \\ &= (\iota [T]) (P) \quad (\text{by (12)}) \\ &= (\iota [T]) (T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n) \\ &\quad (\text{since } P = T^n - c_1 T^{n-1} + \cdots + (-1)^n c_n) \\ &= T^n - \iota (c_1) T^{n-1} + \cdots + (-1)^n \iota (c_n) \quad (\text{by the definition of } \iota [T]). \end{aligned}$$

This is an equality between two polynomials in $B [T]$. Comparing the coefficients on both sides of this equality, we find that

$$(-1)^i \eta (c_i) = (-1)^i \iota (c_i) \quad \text{for each } i \in \{1, 2, \dots, n\}.$$

In other words,

$$\eta (c_i) = \iota (c_i) \quad \text{for each } i \in \{1, 2, \dots, n\}. \tag{13}$$

Now, ι is an S -algebra homomorphism and thus an A -algebra homomorphism (since A is a subring of S). Hence, η and ι are two A -algebra homomorphisms. These two homomorphisms η and ι are equal on the n elements c_1, c_2, \dots, c_n (by (13)); thus, they are equal on a generating set of the A -algebra S (since the n elements c_1, c_2, \dots, c_n form a generating set of the A -algebra S ⁴). Therefore, these two homomorphisms must be identical⁵. In other words, $\eta = \iota$. Hence, η is an S -algebra homomorphism (since ι is an S -algebra homomorphism). Therefore, η is an S -algebra homomorphism $\gamma : A [X_1, \dots, X_n] \rightarrow B$ satisfying (9) (since $\eta (X_i) = u_i$ for all

⁴by the Fundamental Theorem on Symmetric Polynomials

⁵Here we are using the following fact: If two A -algebra homomorphisms have the same domain and the same codomain, and are equal on a generating set of their domain, then they must be identical.

$i \in \{1, 2, \dots, n\}$). Thus, there exists **at least one** S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9) (namely, $\gamma = \eta$).

On the other hand, it is easy to see that there exists **at most one** S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9)⁶.

Combining the previous two sentences, we conclude that there is **exactly one** S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9). This proves Statement 2.]

[*Proof of Statement 1:* For each $i \in \{1, 2, \dots, n\}$, we can write the polynomial $\varphi_i \in B[T]$ in the form $\varphi_i = T - u_i$ for some $u_i \in B$ (because φ_i is a monic linear polynomial over B , and since every monic linear polynomial over B can be written in this form). Consider this u_i . Thus, u_1, u_2, \dots, u_n are n elements of B . Now, in $B[T]$, we have

$$p = \varphi_1 \varphi_2 \cdots \varphi_n = (T - u_1)(T - u_2) \cdots (T - u_n)$$

(since $\varphi_i = T - u_i$ for each $i \in \{1, 2, \dots, n\}$). Hence, Statement 2 yields that there is a unique S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ such that

$$(\gamma(X_i) = u_i \quad \text{for all } i \in \{1, 2, \dots, n\}).$$

Now, for any S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$, we have the following chain of equivalences:

$$\begin{aligned} & \text{(the induced map } \gamma[T] : (A[X_1, \dots, X_n])[T] \rightarrow B[T] \\ & \quad \text{maps } T - X_i \text{ to } \varphi_i \text{ for all } i \in \{1, 2, \dots, n\}) \\ \iff & \left(\begin{array}{l} \underbrace{(\gamma[T])(T - X_i)}_{=T-\gamma(X_i)} = \underbrace{\varphi_i}_{=T-u_i} \text{ for all } i \in \{1, 2, \dots, n\} \\ \text{(by the definition of } \gamma[T]) \end{array} \right) \\ \iff & \left(\begin{array}{l} T - \gamma(X_i) = T - u_i \text{ for all } i \in \{1, 2, \dots, n\} \\ \iff (\gamma(X_i) = u_i) \end{array} \right) \\ \iff & (\gamma(X_i) = u_i \text{ for all } i \in \{1, 2, \dots, n\}). \end{aligned} \tag{14}$$

Now, recall that there is a unique S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ such that

$$(\gamma(X_i) = u_i \quad \text{for all } i \in \{1, 2, \dots, n\}).$$

⁶*Proof.* The universal property of the polynomial ring $A[X_1, \dots, X_n]$ shows that there exists exactly one A -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9). Hence, a fortiori, there exists at most one such A -algebra homomorphism. Thus, there exists at most one S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ satisfying (9) (because any S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ is automatically an A -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$). Qed.

In view of the equivalence (14), we can restate this as follows: There is a unique S -algebra homomorphism $\gamma : A[X_1, \dots, X_n] \rightarrow B$ such that the induced map $\gamma[T] : (A[X_1, \dots, X_n])[T] \rightarrow B[T]$ maps $T - X_i$ to φ_i for all $i \in \{1, 2, \dots, n\}$. This proves Statement 1.]

- **page 843, Proposition 7.1:** After "is an isomorphism", add "sending $X^{n-1} \wedge \dots \wedge X^0$ to 1" (since you end up using this later, in §7.2).
- **page 843, proof of Proposition 7.1:** After "So the S -module $\bigwedge_A^n A[X]$ is generated by $X^{h_1} \wedge \dots \wedge X^{h_n}$ for $0 \leq h_i \leq n - i$ ", add "(since this S -module is a quotient of $\bigotimes_A^n A[X] = A[X_1, \dots, X_n]$)".
- **page 843, §7.2:** Replace " $f_1(X_1) \wedge \dots \wedge f_n(X_n)$ " by " $f_1(X) \wedge \dots \wedge f_n(X)$ ".

11. Appendix: Some alternative proofs

11.1. An alternative proof of Proposition 2.1

Alternative proof of Proposition 2.1. Let $\mathbb{N} = \{0, 1, 2, \dots\}$, and let

$$\mathcal{K}_h = \{(j_1, j_2, \dots, j_n) \in \mathbb{N}^n \mid (j_i \geq h_i \text{ for all } i \in \{1, 2, \dots, n\}) \\ \text{and } j_1 + j_2 + \dots + j_n = h_1 + h_2 + \dots + h_n + h\}.$$

If $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$, then we say that (j_1, j_2, \dots, j_n) is *interlacing* if each $k \in \{2, 3, \dots, n\}$ satisfies $h_{k-1} > j_k$. Thus

$$\begin{aligned} & \{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \mid (j_1, j_2, \dots, j_n) \text{ is interlacing}\} \\ &= \{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \mid h_{k-1} > j_k \text{ for all } k \in \{2, 3, \dots, n\}\} \\ &= \{(j_1, j_2, \dots, j_n) \in \mathbb{N}^n \mid (h_{k-1} > j_k \text{ for all } k \in \{2, 3, \dots, n\}) \\ & \quad \text{and } (j_i \geq h_i \text{ for all } i \in \{1, 2, \dots, n\}) \\ & \quad \text{and } j_1 + j_2 + \dots + j_n = h_1 + h_2 + \dots + h_n + h \quad (\text{by the definition of } \mathcal{K}_h)\} \\ &= \{(j_1, j_2, \dots, j_n) \in \mathbb{N}^n \mid j_1 + j_2 + \dots + j_n = h_1 + h_2 + \dots + h_n + h \\ & \quad \text{and } \underbrace{(h_{k-1} > j_k \text{ for all } k \in \{2, 3, \dots, n\}) \text{ and } (j_i \geq h_i \text{ for all } i \in \{1, 2, \dots, n\})}_{\iff (j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_n \geq h_n)}\} \\ &= \{(j_1, j_2, \dots, j_n) \in \mathbb{N}^n \mid j_1 + j_2 + \dots + j_n = h_1 + h_2 + \dots + h_n + h \\ & \quad \text{and } j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_n \geq h_n\} \\ &= \mathcal{J}_h \end{aligned}$$

(by the definition of \mathcal{J}_h). Hence,

$$\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is interlacing}}} = \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} \quad (15)$$

(an equality of summation signs).

Also, the map

$$\begin{aligned} \mathcal{I}_h &\rightarrow \mathcal{K}_h, \\ (i_1, i_2, \dots, i_n) &\mapsto (h_1 + i_1, h_2 + i_2, \dots, h_n + i_n) \end{aligned}$$

is well-defined and a bijection (this follows easily from the definitions of \mathcal{I}_h and \mathcal{K}_h).

Hence, we can substitute (j_1, j_2, \dots, j_n) for $(h_1 + i_1, h_2 + i_2, \dots, h_n + i_n)$ in the sum

$\sum_{(i_1, i_2, \dots, i_n) \in \mathcal{I}_h} X^{h_1+i_1} \wedge X^{h_2+i_2} \wedge \dots \wedge X^{h_n+i_n}$. We thus obtain

$$\begin{aligned} &\sum_{(i_1, i_2, \dots, i_n) \in \mathcal{I}_h} X^{h_1+i_1} \wedge X^{h_2+i_2} \wedge \dots \wedge X^{h_n+i_n} \\ &= \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}. \end{aligned} \quad (16)$$

If $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$, then we say that (j_1, j_2, \dots, j_n) is *non-interlacing* if (j_1, j_2, \dots, j_n) is not interlacing.

Thus, if (j_1, j_2, \dots, j_n) is non-interlacing, then there exists some $k \in \{2, 3, \dots, n\}$ that satisfies $h_{k-1} \leq j_k$ (because otherwise, (j_1, j_2, \dots, j_n) would be interlacing). The **largest** such k will be called the *violation* of (j_1, j_2, \dots, j_n) .

If an n -tuple $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and has violation k , then:

- we say that (j_1, j_2, \dots, j_n) is *degenerate* if $j_k = j_{k-1}$;
- we say that (j_1, j_2, \dots, j_n) is *non-degenerate* if $j_k \neq j_{k-1}$.

Clearly, any non-interlacing n -tuple $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is either degenerate or non-degenerate (but not both).

If $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and degenerate, then

$$X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} = 0$$

(because if we let k denote the violation of (j_1, j_2, \dots, j_n) , then the degenerateness of (j_1, j_2, \dots, j_n) yields $j_k = j_{k-1}$, and therefore there are two equal elements among $X^{j_1}, X^{j_2}, \dots, X^{j_n}$). Thus,

$$\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing} \\ \text{and degenerate}}} \underbrace{X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}}_{=0} = 0. \quad (17)$$

If $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, and if k is the violation of (j_1, j_2, \dots, j_n) , then we have $j_k \geq h_{k-1}$ (since the definition of "violation" yields $h_{k-1} \leq j_k$) and $j_{k-1} \geq h_k$ (since $j_{k-1} \geq h_{k-1} \geq h_k$). Hence, in this case, the n -tuple $(j_1, j_2, \dots, j_{k-2}, j_k, j_{k-1}, j_{k+1}, j_{k+2}, \dots, j_n)$ (obtained from (j_1, j_2, \dots, j_n) by swapping the $(k-1)$ -st and k -th entries) still belongs to \mathcal{K}_h . Moreover, this n -tuple $(j_1, j_2, \dots, j_{k-2}, j_k, j_{k-1}, j_{k+1}, j_{k+2}, \dots, j_n)$ is non-interlacing (since $k \in \{2, 3, \dots, n\}$ satisfies $h_{k-1} \leq j_{k-1}$) and has violation k (because this k is still the largest such k ; indeed, no $i > k$ satisfies $h_{i-1} \leq j_i$), and thus is non-degenerate (since $j_k \neq j_{k-1}$ yields $j_{k-1} \neq j_k$).

Thus, we can define a map

$$\begin{aligned} \Phi : \{ \text{non-degenerate non-interlacing } (j_1, j_2, \dots, j_n) \in \mathcal{K}_h \} \\ \rightarrow \{ \text{non-degenerate non-interlacing } (j_1, j_2, \dots, j_n) \in \mathcal{K}_h \} \end{aligned}$$

as follows: If $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, and if k is the violation of (j_1, j_2, \dots, j_n) , then we set

$$\Phi(j_1, j_2, \dots, j_n) = (j_1, j_2, \dots, j_{k-2}, j_k, j_{k-1}, j_{k+1}, j_{k+2}, \dots, j_n)$$

(that is, Φ swaps the $(k-1)$ -st and k -th entries of (j_1, j_2, \dots, j_n) , while leaving all other entries unchanged). The previous paragraph shows that this map Φ is well-defined (i.e., if $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, then so is $\Phi(j_1, j_2, \dots, j_n)$) and preserves the violation (i.e., if $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, then the violation of (j_1, j_2, \dots, j_n) is also the violation of $\Phi(j_1, j_2, \dots, j_n)$). Thus Φ is an involution (that is, $\Phi \circ \Phi = \text{id}$), because applying Φ to $\Phi(j_1, j_2, \dots, j_n)$ will swap the same two entries that were swapped in the definition of $\Phi(j_1, j_2, \dots, j_n)$ and therefore recover the original n -tuple (j_1, j_2, \dots, j_n) . Moreover, if $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, then

$$\Phi(j_1, j_2, \dots, j_n) \neq (j_1, j_2, \dots, j_n).$$

⁷ In other words, the involution Φ has no fixed points. Finally, if $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is non-interlacing and non-degenerate, and if $(j'_1, j'_2, \dots, j'_n) = \Phi(j_1, j_2, \dots, j_n)$, then

$$X^{j'_1} \wedge X^{j'_2} \wedge \dots \wedge X^{j'_n} = -X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}$$

(because the n -tuple $(X^{j'_1}, X^{j'_2}, \dots, X^{j'_n})$ is obtained from the n -tuple $(X^{j_1}, X^{j_2}, \dots, X^{j_n})$ by swapping the $(k-1)$ -st and k -th entries). Thus, the fixed-point-free involution Φ pairs up the addends of the sum

$$\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing} \\ \text{and non-degenerate}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}$$

into pairs of mutually cancelling addends. Consequently, this sum is 0. In other words, we have

$$\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing} \\ \text{and non-degenerate}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} = 0. \quad (18)$$

Now, every non-interlacing $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is either degenerate or non-degenerate

⁷*Proof.* Let $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ be non-interlacing and non-degenerate. Let k be the violation of (j_1, j_2, \dots, j_n) . Thus, $j_k \neq j_{k-1}$ (since (j_1, j_2, \dots, j_n) is non-degenerate). Thus, the $(k-1)$ -st entry of $\Phi(j_1, j_2, \dots, j_n)$ is distinct from the $(k-1)$ -st entry of (j_1, j_2, \dots, j_n) (since the former entry is j_k , while the latter entry is j_{k-1}). Thus, $\Phi(j_1, j_2, \dots, j_n) \neq (j_1, j_2, \dots, j_n)$.

(but not both). Hence,

$$\begin{aligned}
& \sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} \\
&= \underbrace{\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing} \\ \text{and degenerate}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}}_{=0 \text{ (by (17))}} + \underbrace{\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing} \\ \text{and non-degenerate}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}}_{=0 \text{ (by (18))}} \\
&= 0. \tag{19}
\end{aligned}$$

But each $(j_1, j_2, \dots, j_n) \in \mathcal{K}_h$ is either interlacing or non-interlacing (but not both). Hence,

$$\begin{aligned}
& \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} \\
&= \sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is interlacing}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} + \underbrace{\sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is non-interlacing}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}}_{=0 \text{ (by (19))}} \\
&= \sum_{\substack{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h \\ \text{is interlacing}}} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} = \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}
\end{aligned}$$

(by (15)). Hence, the equality (2.1) becomes

$$\begin{aligned}
s_h \left(X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_n} \right) &= \sum_{(i_1, i_2, \dots, i_n) \in \mathcal{I}_h} X^{h_1+i_1} \wedge X^{h_2+i_2} \wedge \dots \wedge X^{h_n+i_n} \\
&= \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{K}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} \tag{by (16)} \\
&= \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}.
\end{aligned}$$

This proves Proposition 2.1. □

11.2. An alternative proof of Corollary 2.2

The following proof of Corollary 2.2 is not substantially different from the one in your paper, but it is a lot more explicit and requires less combinatorial skill to understand.

I will break the proof up into several lemmas. First, some definitions are needed:

Definition 11.1. Let $\mathbf{j} = (j_1, j_2, \dots, j_m) \in \mathbb{N}^m$ be an m -tuple of nonnegative integers. We say that \mathbf{j} is *nonincreasing* if $j_1 \geq j_2 \geq \dots \geq j_m$.

Definition 11.2. Fix $n \in \mathbb{N}$. We let \mathcal{NI} denote the set of all nonincreasing n -tuples of nonnegative integers.

Thus, $\mathcal{NI} \subseteq \mathbb{N}^n$ (since each element of \mathcal{NI} is an n -tuple of nonnegative integers, thus belongs to \mathbb{N}^n).

Definition 11.3. Let $\mathbf{j} = (j_1, j_2, \dots, j_m) \in \mathbb{N}^m$ be an m -tuple of nonnegative integers. We define $|\mathbf{j}|$ to be the nonnegative integer $j_1 + j_2 + \dots + j_m$.

Definition 11.4. Let $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$ be an n -tuple of nonnegative integers. We define $X^{\mathbf{j}}$ to be the n -vector $X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n} \in \bigwedge_A^n A[X]$.

Definition 11.5. Let $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$ and $\mathbf{h} = (h_1, h_2, \dots, h_n) \in \mathbb{N}^n$ be two n -tuples of nonnegative integers. We write $\mathbf{j} \circ \mathbf{h}$ if and only if we have $j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_n \geq h_n$ (that is, if and only if we have $j_i \geq h_i$ for each $i \in \{1, 2, \dots, n\}$ and $h_i > j_{i+1}$ for each $i \in \{1, 2, \dots, n-1\}$).

We can now rewrite Proposition 2.1 as follows:

Lemma 11.6. Let $\mathbf{h} \in \mathbb{N}^n$ be nonincreasing. Let $h \in \mathbb{Z}$. Then,

$$s_h X^{\mathbf{h}} = \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}}.$$

(Here, we are following the convention that $s_h = 0$ when $h < 0$.)

Proof of Lemma 11.6. Write the n -tuple $\mathbf{h} \in \mathbb{N}^n$ in the form $\mathbf{h} = (h_1, h_2, \dots, h_n)$. Hence, $|\mathbf{h}| = h_1 + h_2 + \dots + h_n$ (by the definition of $|\mathbf{h}|$) and $X^{\mathbf{h}} = X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_n}$ (by the definition of $X^{\mathbf{h}}$). Also, $(h_1, h_2, \dots, h_n) = \mathbf{h}$ is nonincreasing; in other words, $h_1 \geq h_2 \geq \dots \geq h_n$.

It is easy to see that Lemma 11.6 holds when $h < 0$ ⁸. Hence, for the rest of this proof, we WLOG assume that $h \geq 0$.

⁸*Proof.* Assume that $h < 0$. Thus, $s_h = 0$, so that $s_h X^{\mathbf{h}} = 0$.

On the other hand, we claim that there exists no $\mathbf{j} \in \mathcal{NI}$ satisfying $\mathbf{j} \circ \mathbf{h}$ and $|\mathbf{j}| = |\mathbf{h}| + h$.

Indeed, let $\mathbf{j} \in \mathcal{NI}$ be such that $\mathbf{j} \circ \mathbf{h}$ and $|\mathbf{j}| = |\mathbf{h}| + h$. Then, \mathbf{j} is a nonincreasing n -tuple of nonnegative integers (since $\mathbf{j} \in \mathcal{NI}$). Write this n -tuple \mathbf{j} in the form $\mathbf{j} = (j_1, j_2, \dots, j_n)$.

Recall that $\mathbf{j} \circ \mathbf{h}$; in other words, $j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_n \geq h_n$ (since this is how " $\mathbf{j} \circ \mathbf{h}$ " is defined). Hence, $j_i \geq h_i$ for each $i \in \{1, 2, \dots, n\}$. Adding up these n inequalities, we obtain $j_1 + j_2 + \dots + j_n \geq h_1 + h_2 + \dots + h_n$. The definition of $|\mathbf{j}|$ yields $|\mathbf{j}| = j_1 + j_2 + \dots + j_n \geq h_1 + h_2 + \dots + h_n = |\mathbf{h}|$. This contradicts $|\mathbf{j}| = |\mathbf{h}| + \underbrace{h}_{< 0} < |\mathbf{h}|$.

Now, forget that we fixed \mathbf{j} . We thus have found a contradiction for each $\mathbf{j} \in \mathcal{NI}$ satisfying $\mathbf{j} \circ \mathbf{h}$ and $|\mathbf{j}| = |\mathbf{h}| + h$. Hence, no such \mathbf{j} exists. Thus, the sum $\sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}}$ is empty, and thus

equals 0. Comparing this with $s_h X^{\mathbf{h}} = 0$, we find $s_h X^{\mathbf{h}} = \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}}$. Thus, Lemma 11.6 is

proven (under the assumption that $h < 0$).

Therefore, Proposition 2.1 yields

$$\begin{aligned} & s_h \left(X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_n} \right) \\ &= \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} X^{j_1} \wedge X^{j_2} \wedge \cdots \wedge X^{j_n}, \end{aligned} \quad (20)$$

where \mathcal{J}_h is the set of all n -tuples $(j_1, j_2, \dots, j_n) \in \mathbb{N}^n$ satisfying $j_1 + j_2 + \cdots + j_n = h_1 + h_2 + \cdots + h_n + h$ and $j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n$. Consider this set \mathcal{J}_h .

For any n -tuple $\mathbf{j} \in \mathbb{N}^n$, we have the logical equivalence

$$(\mathbf{j} \in \mathcal{J}_h) \iff (\mathbf{j} \in \mathcal{NI} \text{ and } \mathbf{j} \circ \mathbf{h} \text{ and } |\mathbf{j}| = |\mathbf{h}| + h) \quad (21)$$

9.

⁹*Proof of (21):* Let $\mathbf{j} \in \mathbb{N}^n$ be an n -tuple. Write \mathbf{j} in the form $\mathbf{j} = (j_1, j_2, \dots, j_n)$. Then, we have $|\mathbf{j}| = j_1 + j_2 + \cdots + j_n$ (by the definition of $|\mathbf{j}|$). Moreover, $\mathbf{j} = (j_1, j_2, \dots, j_n) \in \mathbb{N}^n$; hence, we have the chain of equivalences

$$\begin{aligned} (\mathbf{j} \in \mathcal{NI}) &\iff (\mathbf{j} \text{ is nonincreasing}) && \text{(by the definition of } \mathcal{NI}) \\ &\iff (j_1 \geq j_2 \geq \cdots \geq j_n) && \text{(by the definition of "nonincreasing")} \end{aligned}$$

and the equivalence

$$(\mathbf{j} \circ \mathbf{h}) \iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n)$$

(by the definition of " $\mathbf{j} \circ \mathbf{h}$ "), since $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h} = (h_1, h_2, \dots, h_n)$.

On the other hand, we know that $(j_1, j_2, \dots, j_n) \in \mathbb{N}^n$. Hence, we have the equivalence

$$\begin{aligned} & ((j_1, j_2, \dots, j_n) \in \mathcal{J}_h) \\ &\iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n \text{ and } j_1 + j_2 + \cdots + j_n = h_1 + h_2 + \cdots + h_n + h) \end{aligned} \quad (22)$$

(by the definition of \mathcal{J}_h).

Now, we have the following chain of equivalences:

$$\begin{aligned} & (\mathbf{j} \in \mathcal{NI} \text{ and } \mathbf{j} \circ \mathbf{h} \text{ and } |\mathbf{j}| = |\mathbf{h}| + h) \\ &\iff \underbrace{(\mathbf{j} \in \mathcal{NI})}_{\iff (j_1 \geq j_2 \geq \cdots \geq j_n)} \wedge \underbrace{(\mathbf{j} \circ \mathbf{h})}_{\iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n)} \wedge \left(\underbrace{|\mathbf{j}|}_{=j_1+j_2+\cdots+j_n} = \underbrace{|\mathbf{h}|}_{=h_1+h_2+\cdots+h_n} + h \right) \\ &\iff \underbrace{(j_1 \geq j_2 \geq \cdots \geq j_n) \wedge (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n)}_{\iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n)} \\ &\quad \text{(since the chain of inequalities } (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n) \text{ clearly implies } (j_1 \geq j_2 \geq \cdots \geq j_n)) \\ &\iff \wedge (j_1 + j_2 + \cdots + j_n = h_1 + h_2 + \cdots + h_n + h) \\ &\iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n) \wedge (j_1 + j_2 + \cdots + j_n = h_1 + h_2 + \cdots + h_n + h) \\ &\iff (j_1 \geq h_1 > j_2 \geq h_2 > \cdots > j_n \geq h_n \text{ and } j_1 + j_2 + \cdots + j_n = h_1 + h_2 + \cdots + h_n + h) \\ &\iff ((j_1, j_2, \dots, j_n) \in \mathcal{J}_h) \quad \text{(by (22))} \\ &\iff (\mathbf{j} \in \mathcal{J}_h) \quad \text{(since } (j_1, j_2, \dots, j_n) = \mathbf{j} \text{)}. \end{aligned}$$

This proves (21).

Now, recall that $\mathcal{J}_h \subseteq \mathbb{N}^n$. Hence, we have the following equality of summation signs:

$$\begin{aligned} \sum_{\mathbf{j} \in \mathcal{J}_h} &= \sum_{\substack{\mathbf{j} \in \mathbb{N}^n; \\ \mathbf{j} \in \mathcal{J}_h}} = \sum_{\substack{\mathbf{j} \in \mathbb{N}^n; \\ \mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} \quad \left(\begin{array}{l} \text{because for any } \mathbf{j} \in \mathbb{N}^n, \text{ we have the} \\ \text{logical equivalence (21)} \end{array} \right) \\ &= \sum_{\substack{\mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} \quad (\text{since } \mathcal{N}\mathcal{I} \subseteq \mathbb{N}^n). \end{aligned}$$

Hence,

$$\sum_{\mathbf{j} \in \mathcal{J}_h} X^{\mathbf{j}} = \sum_{\substack{\mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}},$$

so that

$$\begin{aligned} \sum_{\substack{\mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}} &= \sum_{\mathbf{j} \in \mathcal{J}_h} X^{\mathbf{j}} = \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} \underbrace{X^{(j_1, j_2, \dots, j_n)}}_{= X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}} \\ &\quad \text{(by the definition of } X^{(j_1, j_2, \dots, j_n)}) \\ &\quad \text{(here, we have renamed the summation index } \mathbf{j} \text{ as } (j_1, j_2, \dots, j_n)) \\ &= \sum_{(j_1, j_2, \dots, j_n) \in \mathcal{J}_h} X^{j_1} \wedge X^{j_2} \wedge \dots \wedge X^{j_n}. \end{aligned}$$

Comparing this with (20), we obtain

$$s_h \left(X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_n} \right) = \sum_{\substack{\mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}}.$$

In view of $X^{\mathbf{h}} = X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_n}$, this rewrites as

$$s_h X^{\mathbf{h}} = \sum_{\substack{\mathbf{j} \in \mathcal{N}\mathcal{I}; \\ \mathbf{j} \circledast \mathbf{h}; \\ |\mathbf{j}| = |\mathbf{h}| + h}} X^{\mathbf{j}}.$$

This proves Lemma 11.6. □

Next, we introduce another notation:

Definition 11.7. Let $\mathbf{h} = (h_1, h_2, \dots, h_m) \in \mathbb{N}^m$ be an m -tuple of nonnegative integers. Let $i \in \{1, 2, \dots, m\}$. Then, $h^{\sim i}$ is defined to be the $(m-1)$ -tuple $(h_1, h_2, \dots, h_{i-1}, h_{i+1}, \dots, h_m) \in \mathbb{N}^{m-1}$ of nonnegative integers. (This is obtained from h by removing the i -th entry.)

Lemma 11.8. Let $\mathbf{j} \in \mathbb{N}^n$ and $\mathbf{h} \in \mathbb{N}^{m+1}$ be two nonincreasing tuples such that $|\mathbf{j}| > |\mathbf{h}| - n$. Then,

$$\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circledast \mathbf{h}^{\sim i}}} (-1)^i = 0.$$

Proof of Lemma 11.8. Write the n -tuple $\mathbf{j} \in \mathbb{N}^n$ in the form $\mathbf{j} = (j_1, j_2, \dots, j_n)$. We extend the n -tuple $(j_1, j_2, \dots, j_n) \in \mathbb{N}^n \subseteq \mathbb{Z}^n$ to an $(n+1)$ -tuple $(j_1, j_2, \dots, j_{n+1}) \in \mathbb{Z}^{n+1}$ by setting $j_{n+1} = -1$.

Write the $(n+1)$ -tuple $\mathbf{h} \in \mathbb{N}^{n+1}$ in the form $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$. Thus, $h_1 \geq h_2 \geq \dots \geq h_{n+1}$ (since \mathbf{h} is nonincreasing).

Define the two sets

$$J = \left\{ i \in \{1, 2, \dots, n+1\} \mid \mathbf{j} \circ \mathbf{h}^{\sim i} \text{ and } j_i \geq h_i \right\} \quad \text{and} \quad (23)$$

$$H = \left\{ i \in \{1, 2, \dots, n+1\} \mid \mathbf{j} \circ \mathbf{h}^{\sim i} \text{ and } j_i < h_i \right\}. \quad (24)$$

Each $i \in \{1, 2, \dots, n+1\}$ satisfies either $j_i \geq h_i$ or $j_i < h_i$ (but not both at the same time). Hence, we can split the sum $\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}}} (-1)^i$ as follows:

$$\begin{aligned} \sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}}} (-1)^i &= \underbrace{\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ j_i \geq h_i}} (-1)^i}_{=\sum_{i \in J} \text{ (by (23))}} + \underbrace{\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ j_i < h_i}} (-1)^i}_{=\sum_{i \in H} \text{ (by (24))}} \\ &= \sum_{i \in J} (-1)^i + \sum_{i \in H} (-1)^i. \end{aligned} \quad (25)$$

Now, we claim that

$$u+1 \in H \quad \text{for each } u \in J. \quad (26)$$

[*Proof of (26):* Let $u \in J$. We must prove that $u+1 \in H$.

We have $u \in J$. In view of (23), this means that u is an element of $\{1, 2, \dots, n+1\}$ that satisfies $\mathbf{j} \circ \mathbf{h}^{\sim u}$ and $j_u \geq h_u$.

Hence, $j_u \geq h_u \geq 0$ (since $(h_1, h_2, \dots, h_{n+1}) = \mathbf{h} \in \mathbb{N}^{n+1}$). If we had $u = n+1$, then we would have $j_u = j_{n+1} = -1 < 0$, which would contradict $j_u \geq 0$. Hence, $u \neq n+1$. Combining this with $u \in \{1, 2, \dots, n+1\}$, we obtain $u \in \{1, 2, \dots, n\}$. Hence, $u+1 \in \{1, 2, \dots, n+1\}$. Thus, the n -tuple $\mathbf{h}^{\sim(u+1)}$ is well-defined.

Recall that $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$. Hence, the definitions of $\mathbf{h}^{\sim u}$ and $\mathbf{h}^{\sim(u+1)}$ yield

$$\begin{aligned} \mathbf{h}^{\sim u} &= (h_1, h_2, \dots, h_{u-1}, h_{u+1}, \dots, h_{n+1}) \quad \text{and} \\ \mathbf{h}^{\sim(u+1)} &= (h_1, h_2, \dots, h_u, h_{u+2}, \dots, h_{n+1}). \end{aligned}$$

We have $\mathbf{j} \circ \mathbf{h}^{\sim u}$. In view of $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h}^{\sim u} = (h_1, h_2, \dots, h_{u-1}, h_{u+1}, \dots, h_{n+1})$, this rewrites as

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_{u-1} \geq h_{u-1} > j_u \geq h_{u+1} > j_{u+1} \geq h_{u+2} > \dots > j_n \geq h_{n+1}$$

(by the definition of the notation " $\mathbf{j} \circ \mathbf{h}^{\sim u}$ "). We can split this chain of inequalities into three pieces as follows:

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_{u-1} \geq h_{u-1} > j_u \quad (27)$$

$$j_u \geq h_{u+1} > j_{u+1};$$

$$j_{u+1} \geq h_{u+2} > j_{u+2} \geq h_{u+3} > \dots > j_n \geq h_{n+1}. \quad (28)$$

From $h_1 \geq h_2 \geq \dots \geq h_{n+1}$, we obtain $h_u \geq h_{u+1}$, so that $h_u \geq h_{u+1} > j_{u+1}$. Hence,

$$j_u \geq h_u > j_{u+1}. \quad (29)$$

We can splice the three chains of inequalities (27), (29) and (28) together into one long chain:

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_u \geq h_u > j_{u+1} \geq h_{u+2} > j_{u+2} \geq h_{u+3} > \dots > j_n \geq h_{n+1}.$$

This rewrites as $\mathbf{j} \circ \mathbf{h}^{\sim(u+1)}$ (by the definition of the notation " $\mathbf{j} \circ \mathbf{h}^{\sim(u+1)}$ ", since $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h}^{\sim(u+1)} = (h_1, h_2, \dots, h_u, h_{u+2}, \dots, h_{n+1})$).

Also, $j_{u+1} < h_{u+1}$ (since $h_{u+1} > j_{u+1}$). Now, we know that $u+1$ is an element of $\{1, 2, \dots, n+1\}$ and satisfies $\mathbf{j} \circ \mathbf{h}^{\sim(u+1)}$ and $j_{u+1} < h_{u+1}$. In view of (24), this rewrites as $u+1 \in H$. This proves (26).]

Next, we claim that

$$u-1 \in J \quad \text{for each } u \in H. \quad (30)$$

[Proof of (30): Let $u \in H$. We must prove that $u-1 \in J$.

We have $u \in H$. In view of (24), this means that u is an element of $\{1, 2, \dots, n+1\}$ and satisfies $\mathbf{j} \circ \mathbf{h}^{\sim u}$ and $j_u < h_u$.

It is not hard to see that $u \neq 1$ ¹⁰. Combining this with $u \in \{1, 2, \dots, n+1\}$, we obtain $u \in \{2, 3, \dots, n+1\}$, so that $u-1 \in \{1, 2, \dots, n+1\}$. Hence, the n -tuple $\mathbf{h}^{\sim(u-1)}$ is well-defined.

¹⁰Proof. Assume the contrary. Thus, $u = 1$. Hence, $\mathbf{h}^{\sim u} = \mathbf{h}^{\sim 1} = (h_2, h_3, \dots, h_{n+1})$ (by the definition of $\mathbf{h}^{\sim 1}$, because $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$). Also, $j_u < h_u$; this rewrites as $j_1 < h_1$ (since $u = 1$). In other words, $h_1 > j_1$.

But recall that $\mathbf{j} \circ \mathbf{h}^{\sim u}$. In view of $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h}^{\sim u} = (h_2, h_3, \dots, h_{n+1})$, this rewrites as

$$j_1 \geq h_2 > j_2 \geq h_3 > \dots > j_n \geq h_{n+1}$$

(by the definition of the notation " $\mathbf{j} \circ \mathbf{h}^{\sim u}$ "). Thus, in particular, we have $h_i > j_i$ for each $i \in \{2, 3, \dots, n\}$. This inequality also holds for $i = 1$ (since $h_1 > j_1$), and thus holds for all $i \in \{1, 2, \dots, n\}$. Hence, for each $i \in \{1, 2, \dots, n\}$, we have $h_i \geq j_i + 1$ (because $h_i > j_i$, but both h_i and j_i are integers). Hence,

$$\sum_{i=1}^n h_i \geq \sum_{i=1}^n (j_i + 1) = \underbrace{\sum_{i=1}^n j_i}_{=j_1+j_2+\dots+j_n} + \underbrace{\sum_{i=1}^n 1}_{=n} = (j_1 + j_2 + \dots + j_n) + n.$$

But $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and thus $|\mathbf{j}| = j_1 + j_2 + \dots + j_n$ (by the definition of $|\mathbf{j}|$). Hence,

$$\begin{aligned} \sum_{i=1}^n h_i &\geq \underbrace{(j_1 + j_2 + \dots + j_n)}_{=|\mathbf{j}| > |\mathbf{h}| - n} + n > |\mathbf{h}| - n + n = |\mathbf{h}| = h_1 + h_2 + \dots + h_{n+1} \\ &\quad \text{(by the definition of } |\mathbf{h}|, \text{ since } \mathbf{h} = (h_1, h_2, \dots, h_{n+1})) \\ &= \underbrace{(h_1 + h_2 + \dots + h_n)}_{=\sum_{i=1}^n h_i} + \underbrace{h_{n+1}}_{\geq 0} \geq \sum_{i=1}^n h_i. \\ &\quad \text{(since } (h_1, h_2, \dots, h_{n+1}) = \mathbf{h} \in \mathbb{N}^{n+1}) \end{aligned}$$

This is absurd. This contradiction shows that our assumption was false. Qed.

Recall that $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$. Hence, the definitions of $\mathbf{h}^{\sim u}$ and $\mathbf{h}^{\sim(u-1)}$ yield

$$\begin{aligned}\mathbf{h}^{\sim u} &= (h_1, h_2, \dots, h_{u-1}, h_{u+1}, \dots, h_{n+1}) && \text{and} \\ \mathbf{h}^{\sim(u-1)} &= (h_1, h_2, \dots, h_{u-2}, h_u, \dots, h_{n+1}).\end{aligned}$$

We have $\mathbf{j} \otimes \mathbf{h}^{\sim u}$. In view of $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h}^{\sim u} = (h_1, h_2, \dots, h_{u-1}, h_{u+1}, \dots, h_{n+1})$, this rewrites as

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_{u-1} \geq h_{u-1} > j_u \geq h_{u+1} > j_{u+1} \geq h_{u+2} > \dots > j_n \geq h_{n+1}$$

(by the definition of the notation " $\mathbf{j} \otimes \mathbf{h}^{\sim u}$ "). We can split this chain of inequalities into three pieces as follows:¹¹

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_{u-2} \geq h_{u-2} > j_{u-1}; \tag{31}$$

$$j_{u-1} \geq h_{u-1} > j_u;$$

$$j_u \geq h_{u+1} > j_{u+1} \geq h_{u+2} > \dots > j_n \geq h_{n+1}. \tag{32}$$

From $h_1 \geq h_2 \geq \dots \geq h_{n+1}$, we obtain $h_{u-1} \geq h_u$, so that $j_{u-1} \geq h_{u-1} \geq h_u$. Hence,

$$j_{u-1} \geq h_u > j_u \quad (\text{since } j_u < h_u). \tag{33}$$

We can splice the three chains of inequalities (31), (33) and (32) together into one long chain:

$$j_1 \geq h_1 > j_2 \geq h_2 > \dots > j_{u-1} \geq h_u > j_u \geq h_{u+1} > j_{u+1} \geq h_{u+2} > \dots > j_n \geq h_{n+1}.$$

This rewrites as $\mathbf{j} \otimes \mathbf{h}^{\sim(u-1)}$ (by the definition of the notation " $\mathbf{j} \otimes \mathbf{h}^{\sim(u-1)}$ ", since $\mathbf{j} = (j_1, j_2, \dots, j_n)$ and $\mathbf{h}^{\sim(u-1)} = (h_1, h_2, \dots, h_{u-2}, h_u, \dots, h_{n+1})$).

Also, $j_{u-1} \geq h_{u-1}$ (as we have seen). Now, we know that $u-1$ is an element of $\{1, 2, \dots, n+1\}$ and satisfies $\mathbf{j} \otimes \mathbf{h}^{\sim(u-1)}$ and $j_{u-1} \geq h_{u-1}$. In view of (23), this rewrites as $u-1 \in J$. This proves (30).]

Now, define a map

$$\alpha : J \rightarrow H, \quad u \mapsto u + 1.$$

(This map is well-defined, due to (26).)

Also, define a map

$$\beta : H \rightarrow J, \quad u \mapsto u - 1.$$

(This map is well-defined, due to (30).)

Clearly, the maps α and β are mutually inverse. Thus, the map α is invertible, i.e., is a bijection. Now, (25) becomes

$$\begin{aligned}\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \otimes \mathbf{h}^{\sim i}}} (-1)^i &= \sum_{i \in J} (-1)^i + \underbrace{\sum_{i \in H} (-1)^i}_{= \sum_{i \in J} (-1)^{\alpha(i)}} &&= \sum_{i \in J} (-1)^i + \sum_{i \in J} \underbrace{(-1)^{\alpha(i)}}_{= (-1)^{i+1}} \\ &&& \text{(since } \alpha(i) = i+1 \text{ (by the definition of } \alpha)) \\ &&& \text{(here, we have substituted } \alpha(i) \text{ for } i \text{ in the sum, since the map } \alpha: J \rightarrow H \text{ is a bijection)} \\ &= \sum_{i \in J} (-1)^i + \sum_{i \in J} \underbrace{(-1)^{i+1}}_{= -(-1)^i} = \sum_{i \in J} (-1)^i - \sum_{i \in J} (-1)^i = 0.\end{aligned}$$

This proves Lemma 11.8. □

¹¹We are using $u \in \{2, 3, \dots, n+1\}$ here.

Lemma 11.9. Let $\mathbf{h} = (h_1, h_2, \dots, h_{n+1}) \in \mathbb{N}^{n+1}$ be a nonincreasing tuple. Let $q \in \mathbb{Z}$ be such that $q < n$. Then,

$$\sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}} = 0.$$

Proof of Lemma 11.9. The $(n+1)$ -tuple $(h_1, h_2, \dots, h_{n+1})$ is nonincreasing. In other words, $h_1 \geq h_2 \geq \dots \geq h_{n+1}$.

Let $i \in \{1, 2, \dots, n+1\}$. Then, from $h_1 \geq h_2 \geq \dots \geq h_{n+1}$, we obtain $h_1 \geq h_2 \geq \dots \geq h_{i-1} \geq h_{i+1} \geq \dots \geq h_{n+1}$.

In other words, the n -tuple $\mathbf{h}^{\sim i}$ is nonincreasing (since $\mathbf{h}^{\sim i} = (h_1, h_2, \dots, h_{i-1}, h_{i+1}, \dots, h_{n+1})$). Moreover, it is easy to see that $|\mathbf{h}^{\sim i}| + h_i = |\mathbf{h}|$.

Thus, Lemma 11.6 (applied to $\mathbf{h}^{\sim i}$ and $h_i - q$ instead of \mathbf{h} and h) yields

$$s_{h_i-q} X^{\mathbf{h}^{\sim i}} = \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ |\mathbf{j}| = |\mathbf{h}^{\sim i}| + h_i - q}} X^{\mathbf{j}} = \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ |\mathbf{j}| = |\mathbf{h}| - q}} X^{\mathbf{j}} \quad (34)$$

(since $|\mathbf{h}^{\sim i}| + h_i = |\mathbf{h}|$).

Now, forget that we fixed i . We thus have proven the equality (34) for each $i \in \{1, 2, \dots, n+1\}$.

Recall that $q < n$; thus, $|\mathbf{h}| - q > |\mathbf{h}| - n$. Now,

$$\begin{aligned} & \sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}} \\ &= \sum_{i \in \{1, 2, \dots, n+1\}} \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ |\mathbf{j}| = |\mathbf{h}| - q \\ \text{(by (34))}}} X^{\mathbf{j}} \\ &= \sum_{i \in \{1, 2, \dots, n+1\}} (-1)^i \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ |\mathbf{j}| = |\mathbf{h}| - q}} X^{\mathbf{j}} = \sum_{i \in \{1, 2, \dots, n+1\}} \underbrace{\sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}; \\ |\mathbf{j}| = |\mathbf{h}| - q}}}_{\substack{\sum_{\mathbf{j} \in \mathcal{NI}; \\ |\mathbf{j}| = |\mathbf{h}| - q} \\ \sum_{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}}} (-1)^i X^{\mathbf{j}} \\ &= \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ |\mathbf{j}| = |\mathbf{h}| - q}} \sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}}} (-1)^i X^{\mathbf{j}} = \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ |\mathbf{j}| = |\mathbf{h}| - q}} \underbrace{\left(\sum_{\substack{i \in \{1, 2, \dots, n+1\}; \\ \mathbf{j} \circ \mathbf{h}^{\sim i}}} (-1)^i \right)}_{\substack{=0 \\ \text{(by Lemma 11.8} \\ \text{(since } |\mathbf{j}| = |\mathbf{h}| - q > |\mathbf{h}| - n))}} X^{\mathbf{j}} \\ &= \sum_{\substack{\mathbf{j} \in \mathcal{NI}; \\ |\mathbf{j}| = |\mathbf{h}| - q}} 0 X^{\mathbf{j}} = 0. \end{aligned}$$

This proves Lemma 11.9. □

Lemma 11.10. Let $\mathbf{h} = (h_1, h_2, \dots, h_{n+1}) \in \mathbb{N}^{n+1}$ be any $(n+1)$ -tuple. Let $q \in \mathbb{Z}$ be such that $q < n$. Then,

$$\sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}} = 0.$$

Proof of Lemma 11.10. If we swap two adjacent entries of the $(n+1)$ -tuple \mathbf{h} (say, the r -th and the $(r+1)$ -st entry, where r is some element of $\{1, 2, \dots, n\}$), then the sum $\sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}}$ flips its sign (indeed, all addends of this sum except for the r -th and $(r+1)$ -st one flip their sign, whereas the r -th and the $(r+1)$ -th addends trade places and also flip their signs). Therefore, the sum $\sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}}$ is anti-symmetric in the h_i (since any permutation of the entries of the $(n+1)$ -tuple \mathbf{h} can be achieved by repeatedly swapping adjacent entries). Hence, in proving that this sum equals 0, we can WLOG assume that $h_1 \geq h_2 \geq \dots \geq h_{n+1}$. Assume this. In other words, the $(n+1)$ -tuple \mathbf{h} is nonincreasing (since $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$). Hence, Lemma 11.9 yields $\sum_{i=1}^{n+1} (-1)^i s_{h_i-q} X^{\mathbf{h}^{\sim i}} = 0$. This proves Lemma 11.10. \square

Alternative proof of Corollary 2.2. Let $m \in \{1, 2, \dots, n\}$ and $(h_1, h_2, \dots, h_m) \in \mathbb{N}^m$. Let us extend the m -tuple $(h_1, h_2, \dots, h_m) \in \mathbb{N}^m$ to an $(n+1)$ -tuple $(h_1, h_2, \dots, h_{n+1}) \in \mathbb{Z}^{n+1}$ by setting

$$(h_i = n+1-i \quad \text{for each } i > m). \quad (35)$$

Thus,

$$(h_1, h_2, \dots, h_{n+1}) = (h_1, h_2, \dots, h_m, n-m, n-m-1, \dots, 0) \in \mathbb{N}^{n+1}.$$

Denote this $(n+1)$ -tuple $(h_1, h_2, \dots, h_{n+1}) \in \mathbb{N}^{n+1}$ by \mathbf{h} . Thus, $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$. From (35), we also obtain

$$\begin{aligned} (h_{m+1}, h_{m+2}, \dots, h_{n+1}) &= (n-m, n-m-1, \dots, 0) \quad \text{and thus} \\ (h_{m+2}, h_{m+3}, \dots, h_{n+1}) &= (n-m-1, n-m-2, \dots, 0). \end{aligned}$$

We have $m \leq n \leq n+1$. Furthermore, $m-n \leq 0$ (since $m \leq n$), so that $0 \in \{m-n, m-n+1, \dots, 0\}$.

Also, $n-m < n$ (since $m > 0$). Hence, Lemma 11.10 (applied to $q = n-m$) yields

$$\sum_{i=1}^{n+1} (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}} = 0.$$

Hence,

$$0 = \sum_{i=1}^{n+1} (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}} = \sum_{i=1}^m (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}} + \sum_{i=m+1}^{n+1} (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}}$$

(here, we have split the sum at $i = m+1$, since $0 \leq m \leq n+1$). Thus,

$$\sum_{i=m+1}^{n+1} (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}} = - \sum_{i=1}^m (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}}. \quad (36)$$

But every $i \in \{m+1, m+2, \dots, n+1\}$ satisfies $i > m$ and thus $h_i = n+1-i$ (by (35)) and therefore

$$\underbrace{h_i}_{=n+1-i} - (n-m) = (n+1-i) - (n-m) = m+1-i. \quad (37)$$

Hence,

$$\begin{aligned} & \sum_{i=m+1}^{n+1} (-1)^i \underbrace{s_{h_i-(n-m)}}_{\substack{=s_{m+1-i} \\ \text{(by (37))}}} X^{\mathbf{h}^{\sim i}} \\ &= \sum_{i=m+1}^{n+1} (-1)^i s_{m+1-i} X^{\mathbf{h}^{\sim i}} = \sum_{i=m-n}^0 (-1)^{m+1-i} s_i X^{\mathbf{h}^{\sim(m+1-i)}} \\ & \quad \text{(here, we have substituted } m+1-i \text{ for } i \text{ in the sum)} \\ &= \underbrace{(-1)^{m+1-0}}_{=(-1)^{m+1}} \underbrace{s_0}_{=1} X^{\mathbf{h}^{\sim(m+1-0)}} + \sum_{i=m-n}^{-1} (-1)^{m+1-i} \underbrace{s_i}_{\substack{=0 \\ \text{(since } i \leq -1 < 0)}} X^{\mathbf{h}^{\sim(m+1-i)}} \\ & \quad \left(\text{here, we have split off the addend for } i=0 \text{ from the sum,} \right. \\ & \quad \left. \text{since } 0 \in \{m-n, m-n+1, \dots, 0\} \right) \\ &= (-1)^{m+1} X^{\mathbf{h}^{\sim(m+1-0)}} + \underbrace{\sum_{i=m-n}^{-1} (-1)^{m+1-i} 0 X^{\mathbf{h}^{\sim(m+1-i)}}}_{=0} = (-1)^{m+1} X^{\mathbf{h}^{\sim(m+1-0)}}. \end{aligned}$$

Comparing this equality with (36), we find

$$(-1)^{m+1} X^{\mathbf{h}^{\sim(m+1-0)}} = - \sum_{i=1}^m (-1)^i s_{h_i-(n-m)} X^{\mathbf{h}^{\sim i}}.$$

Multiplying both sides of this equality with $(-1)^{m+1}$, we find

$$\begin{aligned} X^{\mathbf{h}^{\sim(m+1-0)}} &= \underbrace{-(-1)^{m+1}}_{=(-1)^m} \sum_{i=1}^m (-1)^i \underbrace{s_{h_i-(n-m)}}_{=s_{h_i-n+m}} X^{\mathbf{h}^{\sim i}} = (-1)^m \sum_{i=1}^m (-1)^i s_{h_i-n+m} X^{\mathbf{h}^{\sim i}} \\ &= \sum_{i=1}^m (-1)^{m+i} s_{h_i-n+m} X^{\mathbf{h}^{\sim i}}. \end{aligned} \quad (38)$$

But

$$\mathbf{h}^{\sim(m+1-0)} = \mathbf{h}^{\sim(m+1)} = (h_1, h_2, \dots, h_m, h_{m+2}, h_{m+3}, \dots, h_{n+1})$$

(since $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$). Hence, the definition of $X^{\mathbf{h}^{\sim(m+1-0)}}$ yields

$$\begin{aligned} X^{\mathbf{h}^{\sim(m+1-0)}} &= X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_m} \wedge \underbrace{X^{h_{m+2}} \wedge X^{h_{m+3}} \wedge \dots \wedge X^{h_{n+1}}}_{\substack{=X^{n-m-1} \wedge X^{n-m-2} \wedge \dots \wedge X^0 \\ \text{(since } (h_{m+2}, h_{m+3}, \dots, h_{n+1}) = (n-m-1, n-m-2, \dots, 0))}} \\ &= X^{h_1} \wedge X^{h_2} \wedge \dots \wedge X^{h_m} \wedge X^{n-m-1} \wedge X^{n-m-2} \wedge \dots \wedge X^0. \end{aligned}$$

Comparing this with (38), we obtain

$$\begin{aligned} & X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_m} \wedge X^{n-m-1} \wedge X^{n-m-2} \wedge \cdots \wedge X^0 \\ &= \sum_{i=1}^m (-1)^{m+i} s_{h_i-n+m} X^{\mathbf{h}^{\sim i}}. \end{aligned} \quad (39)$$

Also, every $i \in \{1, 2, \dots, m\}$ satisfies

$$\mathbf{h}^{\sim i} = (h_1, h_2, \dots, h_{i-1}, h_{i+1}, \dots, h_{n+1})$$

(since $\mathbf{h} = (h_1, h_2, \dots, h_{n+1})$) and therefore

$$\begin{aligned} X^{\mathbf{h}^{\sim i}} &= X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_{i-1}} \wedge X^{h_{i+1}} \wedge \cdots \wedge X^{h_{n+1}} \\ &= X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_{i-1}} \wedge X^{h_{i+1}} \wedge \cdots \wedge X^{h_m} \wedge \underbrace{X^{h_{m+1}} \wedge X^{h_{m+2}} \wedge \cdots \wedge X^{h_{n+1}}}_{=X^{n-m} \wedge X^{n-m-1} \wedge \cdots \wedge X^0} \\ &\quad \text{(since } (h_{m+1}, h_{m+2}, \dots, h_{n+1}) = (n-m, n-m-1, \dots, 0)) \\ &\quad \text{(since } i \leq m) \\ &= X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_{i-1}} \wedge X^{h_{i+1}} \wedge \cdots \wedge X^{h_m} \wedge X^{n-m} \wedge X^{n-m-1} \wedge \cdots \wedge X^0. \end{aligned}$$

Hence, (39) rewrites as

$$\begin{aligned} & X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_m} \wedge X^{n-m-1} \wedge X^{n-m-2} \wedge \cdots \wedge X^0 \\ &= \sum_{i=1}^m (-1)^{m+i} s_{h_i-n+m} \\ &\quad \cdot \left(X^{h_1} \wedge X^{h_2} \wedge \cdots \wedge X^{h_{i-1}} \wedge X^{h_{i+1}} \wedge \cdots \wedge X^{h_m} \wedge X^{n-m} \wedge X^{n-m-1} \wedge \cdots \wedge X^0 \right). \end{aligned}$$

This proves Corollary 2.2. □

11.3. An alternative proof of Lemma 2.3

Alternative proof of Lemma 2.3. We must prove that each $f \in S$ satisfying $f \cdot (X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) = 0$ satisfies $f = 0$. So let $f \in S$ be such that $f \cdot (X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) = 0$. We must prove that $f = 0$.

Consider the alternator map $\text{alt} : \Lambda_A^n A[X] \rightarrow A[X_1, X_2, \dots, X_n]$. This map alt is S -linear (by Proposition 1.3). Hence,

$$\begin{aligned} f \cdot \text{alt} \left(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0 \right) &= \text{alt} \left(\underbrace{f \cdot (X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)}_{=0} \right) \\ &= \text{alt} 0 = 0. \end{aligned} \quad (40)$$

But the definition of alt yields¹²

$$\begin{aligned} & \text{alt} \left(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0 \right) \\ &= \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma X_{\sigma(1)}^{n-1} X_{\sigma(2)}^{n-2} \cdots X_{\sigma(n)}^0. \end{aligned} \quad (41)$$

¹²Here, we are using the notation $(-1)^\sigma$ for the sign of a permutation σ .

Now, we claim that this element $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ of $A[X_1, X_2, \dots, X_n]$ is regular (i.e., not a zero-divisor). Here are two ways to prove this:

[First proof of the fact that $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is regular: Equip the set \mathbb{N}^n with the lexicographic order; this is the total order in which

$$(h_1, h_2, \dots, h_n) > (k_1, k_2, \dots, k_n) \quad \text{if and only if} \\ \text{the first non-zero term in the sequence } h_1 - k_1, h_2 - k_2, \dots, h_n - k_n \text{ is positive.}$$

Now, (41) yields

$$\begin{aligned} & \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) \\ &= \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma X_{\sigma(1)}^{n-1} X_{\sigma(2)}^{n-2} \cdots X_{\sigma(n)}^{n-n} \\ &= X_1^{n-1} X_2^{n-2} \cdots X_n^{n-n} + (\text{lower order terms}), \end{aligned} \quad (42)$$

where "(lower order terms)" means an A -linear combination of monomials $X_1^{v_1} X_2^{v_2} \cdots X_n^{v_n}$ with $(v_1, v_2, \dots, v_n) < (n-1, n-2, \dots, n-n)$. (Of course, the " $<$ " sign here refers to the lexicographic order on \mathbb{N}^n .)

On the other hand, let $g \in A[X_1, X_2, \dots, X_n]$ be nonzero. Then, g has at least one nonzero coefficient. Hence, we can find some nonzero $c \in A$ and some $(g_1, g_2, \dots, g_n) \in \mathbb{N}^n$ such that

$$g = cX_1^{g_1} X_2^{g_2} \cdots X_n^{g_n} + (\text{lower order terms}), \quad (43)$$

where "(lower order terms)" means an A -linear combination of monomials $X_1^{u_1} X_2^{u_2} \cdots X_n^{u_n}$ with $(u_1, u_2, \dots, u_n) < (g_1, g_2, \dots, g_n)$. Consider this c and this (g_1, g_2, \dots, g_n) .

But it is easy to see that the lexicographic order on \mathbb{N}^n respects entrywise addition of n -tuples in \mathbb{N}^n (which, of course, corresponds to multiplication of monomials in $A[X_1, X_2, \dots, X_n]$). To be more precise: If four n -tuples

$$(u_1, u_2, \dots, u_n), (p_1, p_2, \dots, p_n), (v_1, v_2, \dots, v_n) \text{ and } (q_1, q_2, \dots, q_n)$$

in \mathbb{N}^n satisfy

$$(u_1, u_2, \dots, u_n) \leq (p_1, p_2, \dots, p_n) \text{ and} \quad (44)$$

$$(v_1, v_2, \dots, v_n) \leq (q_1, q_2, \dots, q_n), \quad (45)$$

then we have

$$(u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \leq (p_1, p_2, \dots, p_n) + (q_1, q_2, \dots, q_n)$$

(where the addition of n -tuples is entrywise), and this inequality becomes an equality only when both (44) and (45) become equalities.

Hence, if we multiply the equalities (43) and (42), then we find

$$\begin{aligned} & g \cdot \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) \\ &= cX_1^{g_1+(n-1)} X_2^{g_2+(n-2)} \cdots X_n^{g_n+(n-n)} + (\text{lower order terms}), \end{aligned}$$

where "(lower order terms)" means an A -linear combination of monomials $X_1^{u_1} X_2^{u_2} \cdots X_n^{u_n}$ with $(u_1, u_2, \dots, u_n) < (g_1 + (n-1), g_2 + (n-2), \dots, g_n + (n-2))$. Thus, the coefficient of the monomial $X_1^{g_1+(n-1)} X_2^{g_2+(n-2)} \cdots X_n^{g_n+(n-n)}$ in the polynomial $g \cdot \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is c , which is nonzero. Hence, the polynomial $g \cdot \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ has at least one nonzero coefficient. Thus, this polynomial $g \cdot \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is nonzero.

Now, forget that we fixed g . We thus have proven that $g \cdot \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is nonzero whenever $g \in A[X_1, X_2, \dots, X_n]$ is nonzero. In other words, the element $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ of $A[X_1, X_2, \dots, X_n]$ is regular. Qed.]

[Second proof of the fact that $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is regular: From (41), we obtain

$$\begin{aligned} & \text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) \\ &= \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma X_{\sigma(1)}^{n-1} X_{\sigma(2)}^{n-2} \cdots X_{\sigma(n)}^{n-n} \\ &= \det \left(\left(X_j^{n-i} \right)_{1 \leq i \leq n, 1 \leq j \leq n} \right) \quad (\text{by the definition of a determinant}) \\ &= \prod_{1 \leq i < j \leq n} (X_i - X_j) \end{aligned} \tag{46}$$

(by the well-known formula for the Vandermonde determinant). But the polynomial $\prod_{1 \leq i < j \leq n} (X_i - X_j)$ is a regular element of $A[X_1, X_2, \dots, X_n]$ (this is well-known; see, e.g., [Grinbe19, Corollary 4.4]). In view of (46), this rewrites as follows: The polynomial $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ is a regular element of $A[X_1, X_2, \dots, X_n]$. Qed.]

Either way, we have now shown that the element $\text{alt}(X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0)$ of $A[X_1, X_2, \dots, X_n]$ is regular. Hence, from (40), we obtain $f = 0$ (since $f \in S \subseteq A[X_1, X_2, \dots, X_n]$).

Now, forget that we fixed f . We thus have shown that each $f \in S$ satisfying $f \cdot (X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0) = 0$ satisfies $f = 0$. In other words, the annihilator of $X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0$ in S is zero. In other words, the annihilator of $X^{n-1} \wedge X^{n-2} \wedge \cdots \wedge X^0$ in $A[X_1, X_2, \dots, X_n]^{\text{sym}}$ is zero (since $S = A[X_1, X_2, \dots, X_n]^{\text{sym}}$). This proves Lemma 2.3. \square

References

- [Grinbe19] Darij Grinberg, *Regular elements of a ring, monic polynomials and "lcm-coprimality"*, 16 July 2019.
<http://www.cip.ifi.lmu.de/~grinberg/algebra/regpol.pdf>
- [LakTho12] Dan Laksov, Anders Thorup, *Splitting Algebras and Schubert Calculus*, Indiana University Mathematics Journal, Vol. 61, No. 3 (2012).