On blocks of Deligne's category $\underline{\mathrm{Rep}}(S_t)$.

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arXiv:0910.5695v2, version of 15 Oct 2010

`0910.5695v2.pdf`

Errata I

This list refers to the version of the paper on arXiv, not the (currently) older version on Jonny Comes's website.

- **Page 2, §1.1:** Terminological question: is "F-linear bifunctor $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$" really the right terminology, or should "F-linear" be "F-bilinear" here? The latter seems to give more google hits, but this isn't exactly an argument...

- **Page 3:** You write: "By a *partition* π of a finite set S we mean a collection $\pi_1, \ldots, \pi_n$ of disjoint subsets of S with $S = \bigcup_i \pi_i$." I think you should add "nonempty" before "subsets" here.

- **Page 5:** When you write

$$g(v_{\boldsymbol{i}}) := \sum_{\boldsymbol{i}' \in [m,d]} g_{\boldsymbol{i}'}^{\boldsymbol{i}} v_{\boldsymbol{i}'} \qquad (\boldsymbol{i} \in [n,d]),$$

  I think the ":=" should be an "=:", since it is the $g_{\boldsymbol{i}'}^{\boldsymbol{i}}$'s that you are defining herein.

- **Pages 6, 7 and 8:** The notation "not among" appears once on each of these three pages, and I fear it is slightly ambiguous. When you say that some objects are not among some other objects, you mean that the set of the former objects is disjoint from the set of the latter objects. However, one could misunderstand this notation as meaning that the set of the former objects is not contained in the set of the latter objects...

- **Page 11, Proposition 2.20:** Replace "are two decomposition" by "are two decompositions".

- **Page 11, 2.3:** I am wondering why you write "(not equal to id_0)" here; I don't see how the $\pi = \mathrm{id}_0$ case is any different from the rest...

- **Page 12, Example 2.22:** In (ii), you can replace "positive" by "nonnegative" and nothing goes wrong (but I guess you don't care).

- **Page 12, proof of Lemma 3.1:** I don't think the claim that "$FS_n \cap (\zeta) = 0$" is completely obvious. The simplest proof of $FS_n \cap (\zeta) = 0$ that I know of uses a nontrivial idea which does not appear in your paper (that of the propagating number of a partition)[1].

[1] The proof goes as follows:

For any partition $\pi \in P_{m,\ell}$, define the *propagating number* of π as the number of all parts S of π satisfying both $S \cap \{1, 2, ..., m\} \neq \varnothing$ and $S \cap \{1', 2', ..., \ell'\} \neq \varnothing$. Denote this number by $\operatorname{prop} \pi$.

- **Page 12, proof of Lemma 3.1:** You write: "Indeed, $\pi_{j,k} = \sigma\zeta\sigma$ where $\sigma \in S_n \subset P_{n,n}$ is the product of transpositions $(j, n-1)(k, n)$." This does not always work (e. g., it fails when $j = n-2$ and $k = n-1$). A construction that does prove that $\pi_{j,k} \in (\zeta)$ in all cases is by letting σ be any permutation in S_n sending j and k to $n-1$ and n, respectively; then, $\pi_{j,k} = \sigma^{-1}\zeta\sigma$ (not $\sigma\zeta\sigma$, but that's fine).

- **Page 12, proof of Lemma 3.1:** Replace "or there exist" by "or there exist distinct".

- **Page 13, before Lemma 3.3:** You refer to [Ben91] for the proof of Lemma 3.3; I was not able to follow that reference. Do you have a page or section?

- **Page 13, Lemma 3.3:** Replace "is a primitive idempotents" by "is a primitive idempotent".

- **Page 18, proof of Lemma 3.13:** You write: "If we let $x_1, \ldots, x_r \in F$ denote the eigenvalues of ϕ_a". Do you mean $\overline{F}$ rather than F here?

- **Page 19, Theorem 3.15:** I don't see how you get that t is an algebraic **integer** if $\underline{\operatorname{Rep}}(S_t; F)$ is not semisimple – do you mean by any chance that t is an algebraic **number**? (The polynomial $\det M_n(t)$ isn't monic, and I don't see a quick reason why its leading coefficient factors out. Of course, later results will show that t is an algebraic integer and even a honest integer, but this isn't obvious to me at the stage of Section 3.)

- **Page 20:** On the first line of this page, you write: "will show $\operatorname{Rep}(S_d; F)$ is equivalent to the quotient of $\underline{\operatorname{Rep}}(S_t; F)$ by the so-called "negligible morphisms."". I think you want $\underline{\operatorname{Rep}}(S_d; F)$ instead of $\underline{\operatorname{Rep}}(S_t; F)$ here.

- **Page 21, proof of Proposition 3.23:** You write: "$\operatorname{tr}((f \otimes \operatorname{id}_C) \circ h) = \operatorname{tr}(f \circ (\operatorname{id}_A \otimes ev_C) \circ (h \otimes \operatorname{id}_C) \circ (\operatorname{id}_B \otimes coev_C))$". I think "$(h \otimes \operatorname{id}_C)$" should be "$(h \otimes \operatorname{id}_{C^\vee})$".

- **Page 21, Proposition 3.25:** Have you ever defined the partition $\lambda(d)$? I know that it means $(d - |\lambda|, \lambda_1, \lambda_2, ...)$ (which indeed is a partition in the case when $d - |\lambda| \geq \lambda_1$), but I guess that's a definition better not left to the Deligne paper (particularly since the Deligne paper denotes it differently).

- **Page 23, proof of Proposition 3.31:** In the proof of (1), replace "must be in column $\mu_i + 1$" by "must be in column $\lambda_i + 1$".

- **Page 23, proof of Proposition 3.31:** In the proof of (1), replace "to the upper right corner in a $(|\lambda| - i) \times (\lambda_i + 1)$ grid" by "to the upper right corner in a $(|\lambda| - i + 1) \times (\lambda_i + 1)$ grid".

Then, it is easy to see that any two partitions $\pi \in P_{m,\ell}$ and $\chi \in P_{\ell,k}$ satisfy $\operatorname{prop}(\chi \star \pi) \leq \operatorname{prop}\chi$ and $\operatorname{prop}(\chi \star \pi) \leq \operatorname{prop}\pi$. As a consequence, if we denote by L_n the F-vector subspace of $FP_n(t)$ spanned by all π with $\operatorname{prop}\pi < n$, then L_n is an ideal of $FP_n(t)$. Thus, $(\zeta) \subseteq L_n$ (since $\zeta \in L_n$). Moreover, it is clear that $FS_n \cap L_n = 0$. Hence, $FS_n \cap \underbrace{(\zeta)}_{\subseteq L_n} \subseteq FS_n \cap L_n = 0$, hence $FS_n \cap (\zeta) = 0$, qed.

- **Page 23, proof of Proposition 3.31:** In the proof of (2), replace "to the upper right corner in a $(|\lambda| - c_i) \times i$ grid" by "to the upper right corner in a $(|\lambda| - c_i + 1) \times i$ grid".

- **Page 24, Proposition 4.3:** The formula (4.5) is false in the case $r = 1$; indeed, it says that $S(\pi, 1, d) = d - a$ in this case, which is wrong, because in truth (I believe)

  $$S(\pi, 1, d) = \begin{cases} 1, & \text{if } \pi \geq \mathrm{id}_n; \\ 0, & \text{otherwise} \end{cases}$$

  (where id_n is the identity permutation in S_n considered as an element of $P_{n,n}$, and $\geq$ is the partial order defined shortly before (2.1)).

  This error does not surprise me, because r-cycles for $r > 1$ have several properties that 1-cycles don't have. For example, given an r-cycle $(k_1, k_2, ..., k_r)$, we can uniquely reconstruct the elements $k_1, k_2, ..., k_r$ as being the elements not fixed under the r-cycle. This breaks down for $r = 1$, and the proof breaks here.

  So the formula (4.5) needs to be corrected to $S(\pi, r, d) = 1$ in the case when $r = 1$. The interesting thing is that this seems to give two different versions of $\omega_n^1(t)$ depending on whether you modify the definition of $q_{\pi,r,t}$ accordingly (namely, by setting $q_{\pi,1,t} = 1$ whenever $S(\pi, 1, d) \neq 0$) or leave it as it is. The one obtained by modifying the definition $q_{\pi,r,t}$ is simply $\mathrm{id}_n \in FP_n(t)$. The one obtained by leaving the definition of $q_{\pi,r,t}$ as it is, unfortunately, does not satisfy Proposition 4.6 (not even its part (ii), so it does not induce an endomorphism of the identity functor).

- **Page 24, proof of Proposition 4.3:** Remove the "in" from "Now, an r-cycle in $\sigma \in S_d$".

- **Page 26, proof of Proposition 4.6:** Here you write: "On the other hand, (2.2) shows that $f(\omega_n^r(t))$ maps $v_{\boldsymbol{i}} \mapsto \sum_{\boldsymbol{i}' \in [n,d]} q_{\pi(\boldsymbol{i},\boldsymbol{i}'),r,d} v_{\boldsymbol{i}'}$." The $f(\omega_n^r(t))$ should be an $f(\omega_n^r(d))$ here.

- **Page 26, one line above Proposition 4.9:** Replace comma by semicolon in "$\underline{\mathrm{Rep}}(S_t, F)$".

- **Page 27, Theorem 4.8:** Numerous things are slightly wrong here.

  First, in (4.7), the product $(\mu_i + k - 1)(\mu_i + k - 2) \cdots (\mu_i + k - r)$ should be $(\mu_i + k)(\mu_i + k - 1) \cdots (\mu_i + k - r + 1)$. Check on λ being a 1-row partition (and $r \neq 1$, see below).

  Second, the formula, even corrected this way, does not hold for $r = 1$; instead, we have $\xi_{1,k}^\lambda = 1$ (for obvious reasons).

  Finally, this is not a real error, but I am pretty sure that "k is any positive integer" should be "k is any nonnegative integer".

- **Page 27, after Theorem 4.8:** I think that a good reference for Theorem 4.8 (not in the form it appears in your text, but in a form very close to the one in Fulton-Harris) is: Tullio Ceccherini-Silberstein, Fabio Scarabotti, Filippo Tolli, *Representation Theory of the Symmetric Groups – The Okounkov-Vershik*

Approach, Character Formulas, and Partition Algebras, CUP 2010, Proposition 4.2.11.

- **Page 27, Proposition 4.9:** Replace comma by semicolon in "$\underline{\operatorname{Rep}}(S_t, F)$".

- **Page 28, proof of Lemma 5.4:** There is a minor hole in the proof. The formula

$$\xi_{r,k}^{\lambda(t)} = \frac{1}{r}\sum_{i=0}^{k}\mu_i^r + (\text{terms of total degree less than } r)$$

holds only for $r > 1$, so you need an extra argument to get the first power sum of the $\mu_0, ..., \mu_k$ equal to the first power sum of the $\mu_0', ..., \mu_k'$. Fortunately, this argument is very simple: Since

$$\sum_{i=0}^{k}\mu_i^1 = \sum_{i=0}^{k}\mu_i = \underbrace{\sum_{i=0}^{k}\lambda_i}_{=t-|\lambda|+|\lambda|=t} - \frac{k(k+1)}{2} = t - \frac{k(k+1)}{2}$$

and (similarly) $\sum\limits_{i=0}^{k}\mu_i'^1 = t - \dfrac{k(k+1)}{2}$, we have $\sum\limits_{i=0}^{k}\mu_i^1 = \sum\limits_{i=0}^{k}\mu_i'^1$.

- **Page 29, Proposition 5.5:** Replace "$\lambda_\mu(t)$" by "$\mu_\lambda(t)$".

- **Page 29, proof of Proposition 5.5:** Replace "suppose $\mu_i' > \mu_{i+1}'$ for all $i > 0$" by "suppose $\mu_i' \in \mathbb{Z}$ and $\mu_i' > \mu_{i+1}'$ for all $i > 0$".

- **Page 29, proof of Proposition 5.5:** Replace "Set $\lambda_i' = \mu_i + i$" by "Set $\lambda_i' = \mu_i' + i$".

- **Page 29, proof of Proposition 5.5:** Replace "Moreover, $\mu_i > \mu_{i+1}$ for $i > 0$" by "Moreover, $\mu_i' > \mu_{i+1}'$ for $i > 0$".

- **Page 29, Example 5.7:** In part (2), replace "λ" by "$\varnothing$".

- **Page 30, proof of Corollary 5.9:** After "$\lambda_1^{(i)} = \mu_0 + 1 = d - |\lambda^{(0)}| + 1$", add "(for $i > 0$)".

- **Page 32, Corollary 5.17:** Remove a superfluous period from "a nonnegative integer..".

- **Page 35, proof of Lemma 5.20:** Replace "as in as in" by "as in".

- **Page 37, proof of Proposition 6.1:** Replace "$\tau\mu = \pi$" by "$\tau\mu = \mu$". Similarly, replace "$-s_n\pi s_n$" by "$-s_n\mu s_n$".

- **Page 38, proof of Lemma 6.2:** You write: "notice that for $\sigma \in S_{n+1}$ there are exactly $(n-1)!$ pairs (τ_1, τ_2) with $\tau_1, \tau_2 \in S_n$ such that $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1}\sigma x_{n+1}^n$". This is not precisely true. What you mean is: "notice that for $\sigma \in S_{n+1}$ satisfying $\sigma(n+1) \neq n+1$ there are exactly $(n-1)!$ pairs (τ_1, τ_2) with $\tau_1, \tau_2 \in S_n$ such that $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1}\sigma x_{n+1}^n$ (whereas those $\sigma \in S_{n+1}$ which satisfy $\sigma(n+1) = n+1$ satisfy $x_n^{n+1}\sigma x_{n+1}^n = 0$ in $\underline{\operatorname{Rep}}(S_0; F)$)".

- **Page 38, proof of Lemma 6.2:** You write: "Conversely, given any $\tau_1, \tau_2 \in S_n$, either $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = 0$ or there exists a unique $\sigma \in S_{n+1}$ with $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1} \sigma x_{n+1}^n$." The first of these two alternatives cannot happen. I think you want to say "Conversely, given any $\tau_1, \tau_2 \in S_n$, there exists a unique $\sigma \in S_{n+1}$ with $\sigma(n+1) \neq n+1$ and $\tau_1 x_n^{n-1} x_{n-1}^n \tau_2 = x_n^{n+1} \sigma x_{n+1}^n$."

- **Page 38, Lemma 6.3:** I think the "γ_n" in (6) and in (7) should be a "$-\gamma_n$", and the proof should be modified accordingly.

- **Page 38, proof of Lemma 6.3:** In the proof of (3), you write: "$\gamma_n = \sum_{\pi \in P_{n+1,n}} c_\pi \pi$". The sum should run over $P_{n,n}$, not $P_{n+1,n}$.

- **Page 39, Proof of Proposition 6.6:** You write: "First, notice $d \neq 0$ as we are assuming a nontrivial block exists in $\underline{\text{Rep}}(S_d; F)$." Don't you mean "a non-minimal nontrivial block" rather than "a nontrivial block" here? After all, $\underline{\text{Rep}}(S_0; F)$ has a non-trivial block, too.

- **Page 41, Appendix A:** You write: "A is a free $F[[u]]$-algebra of finite rank". What exactly is a free $F[[u]]$-algebra? Do you mean an $F[[u]]$-algebra which is free as an $F[[u]]$-module? (I think so...)

- **Page 41, proof of Lemma A.1:** Replace "$ab\left(1 - \sum_{i=1}^\infty x^i u^i\right) = 1$" by "$ab\left(1 + \sum_{i=1}^\infty (-x)^i u^i\right) = 1$".

- **Page 42, proof of Theorem A.2:** Replace "$a_i^2 - 2(2a_i - 1)b_i$" by "$a_i^2 - 2(2a_i - 1)a_i b_i$" in the long equation.

- **References, [Fro68]:** The word "Bände" is German for "volumes". I think you want the singular form "Band" ("volume").