Combinatorial Markov chains on linear extensions

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Errata and comments - I

- **Page 3, §2.1:** In "are labeled by integers in $[n] := \{1, 2, ..., n\}$", replace "integers" by "the integers", since otherwise it sounds as if one has the freedom to skip some of the integers or use them twice.

- **Page 3, §2.1:** In my opinion, the idea to label the vertices of P by the integers in $[n]$ is a fundamentally bad one. In the few parts of the paper where it is used (like Lemma 5.5 and all statements about poset derangements), it can be introduced locally. In all the rest of the paper, it is fully expendable and only confuses the reader by giving the integers $1, 2, ..., n$ a double role (once as the labels of the vertices of P, and again as the ordinal numbers of these vertices in a linear extension). Here are two examples of places where this leads to ambiguity:

  – In part (2) of the definition of promotion on page 4, does "labels covering i" mean "labels covering the vertex i" or "labels covering the vertex labelled i" ? I know it means the latter, but this is not really obvious from the wording. If you wouldn't label the vertices of P by $1, 2, ..., n$ by default, there would only be one possible meaning (namely, the correct one).

  – In the definition of P_j, I think you got yourself confused, because the definition that you give does not make (2.3) valid (I think). See below for how to correct this.

  As I said, in my opinion there is no reason to assume globally that the elements of P are labeled by the integers in $[n]$. I think the paper would become way more readable if you remove this assumption. Of course, with this change, linear extensions of P are no longer elements of S_n but are now bijections $[n] \to P$ [1]. "One-line notation" for elements π of S_n now means the obvious thing (just writing the images of $1, 2, ..., n$ under π in one line). The maps ∂_j and τ_j are still indexed by elements of $[n]$, but the maps $\widehat{\partial}_j$ are now indexed by elements of P (I believe this makes the difference between them clearer). The uniform transposition and the uniform promotion Markov chains still have their edge weights defined in the polynomial ring $\mathbb{Q}[x_1, x_2, ..., x_n]$, but the transposition and promotion Markov chains now have their edge weights defined in the polynomial ring $\mathbb{Q}[x_p \mid p \in P]$. In Theorem 4.5, the product $\prod_{i=1}^{n} \frac{x_1 + \cdots + x_i}{x_{\pi_1} + \cdots + x_{\pi_i}}$ no longer makes sense because $x_1 + \cdots + x_i$ is not defined; but

[1] But this is okay, since you never multiply or invert them. And if you ask me, it is a good thing as it saves me the hassle of remembering whether the label of a vertex i is π_i or π_i^{-1}: when the vertices of P are not identified with integers, only one of these two possibilities makes any sense.

you can just replace it by the simpler product $\prod_{i=1}^{n} \frac{1}{x_{\pi_1} + \cdots + x_{\pi_i}}$, which differs from it just by a factor independent on π (of course, you lose the $w(e) = 1$ property this way, but this is expected since you don't have a distinguished linear extension e anymore). Funnily, this simplifies both the proof of Theorem 4.5 (because the normalization you use there is precisely $\prod_{i=1}^{n} \frac{1}{x_{\pi_1} + \cdots + x_{\pi_i}}$) and the formulation of Theorem 5.1 (because now (5.1) gets replaced by the simple equality $Z_P = \prod_{i \in P} x_{\preceq i}$). Also, Theorem 4.7 needs to be adjusted (terms like $x_{\pi_i}^{i-\pi_i}$ make no sense anymore no matter if i is an element of P or of $[n]$), but again this adjustment actually makes it simpler (you can replace (4.7) by $w(\pi) = \prod_{i=1}^{n} x_{\pi_i}^{i}$, forgetting about $w(e) = 1$ since there is no e anymore).

- **Page 4:** You define P_j as "the natural (induced) subposet of P consisting of elements k such that $j \preceq k$". First of all, I don't think this is the definition you want to make. In order to make (2.3) true, it should be "the natural (induced) subposet of P consisting of elements k such that the element of P labelled j in π is $\preceq k$" (note that this depends on the choice of a linear extension π of P). But anyway, I don't see why you define P_j at all. You only ever use P_j in the definition of extended promotion ∂_j, where it is almost a red hering. I think that extended promotion is best defined along these lines: "Given a linear extension $\pi = \pi_1\pi_2...\pi_n$ of P, apply promotion to the linear extension $\pi_j\pi_{j+1}...\pi_n$ of the subposet $\{\pi_j, \pi_{j+1}, ..., \pi_n\}$ of P. Denoting the resulting linear extension by $\sigma_j\sigma_{j+1}...\sigma_n$, we then see that $\pi_1\pi_2...\pi_{j-1}\sigma_j\sigma_{j+1}...\sigma_n$ is a new linear extension of P, which we denote by $\pi\partial_j$."

- **Page 4, Example 2.1:** In my opinion, "namely 3 is in position 5 in π' " is a bit ambiguous, because π' can mean both a labelling of vertices of P and a word. It is best to say "namely 3 is in position 5 in the word π' ".

- **Page 5:** I know that you refer to Stanley's papers for notations, but I am not sure if every reader will follow the reference. For those who don't, it would help to point out that all maps of the form τ_j, ∂_j, $\widehat{\partial_j}$ and ∂ act on the right, and consequently their composition (as in (2.3)) is to be understood as "perform the leftmost map first, then the next one etc.".

  (It probably wouldn't harm to have an example for ∂_j, too.)

- **Page 6, Figure 2:** Replace "$\partial\pi$" by "$\pi\partial$" (one of you apparently doesn't like Stanley's notations very much – my sympathies).

- **Page 6, Lemma 2.3:** "each operator" → "the operator" (since j is already fixed).

- **Page 7:** Remove the comma before "define a directed graph".

- **Page 12, proof of Proposition 4.1:** I think the words "and label 1 is at position k of P" can be just removed, since you never use k.

- **Page 12, proof of Proposition 4.1:** At the very end, I believe "from Lemma 2.3" should be "from (2.3)".

- **Page 15, proof of Theorem 4.5:** Replace "hence (4.3) yields" by "hence the right hand side of (4.3) becomes".

- **Page 15, proof of Theorem 4.5:** Remove the word "set" from "In the first case, set $\widetilde{\pi} = \pi$".

- **Page 16, §5.1:** This is one of my trademark nitpicks: Add "disjoint" before "union" in "A **rooted forest** is a union of rooted trees."

- **Page 17, proof of Theorem 5.1:** Replace "of the sums over i" by "of the addends of the sum over i".

- **Page 19, Lemma 5.5:** It might be useful to point out explicitly that P is to be labeled consecutively within chains (as in Theorem 5.3).

- **Page 19, proof of Lemma 5.5:** In "define $N_i = n_1 + \cdots + n_{i-1}$ for all $2 \leq i \leq k$", replace the "k" by "$k+1$" (since you do use N_{k+1} in the very next sentence).

- **Page 21, proof of Lemma 5.5:** Replace "w'_{N_i+j}" by "$w'_{N'_i+j}$" on the first line of this page.

- **Page 23, Lemma 6.5:** It might help to point out that "reordering" means "reordering in such a way that the result is a linear extension".

- **Page 23, proof of Lemma 6.5:** I think the "$\widehat{\partial}_{\pi_i^{-1}}$" and the "$\widehat{\partial}_k$" on the second line of this proof should be "$\partial_{\pi_i^{-1}}$" and "∂_k", respectively.

- **Page 23, Example 6.6:** The "12345" at the very end should be in mathmode, not in textmode.

- **Page 24, proof of Lemma 6.7:** You write: "we hence must have $\widehat{\partial}_{\alpha_j} x = x$ for all $1 \leq j \leq m$". Why?

- **Page 24, proof of Lemma 6.7:** I don't see why "we have $\operatorname{Rfactor}(x) \subsetneq \operatorname{Rfactor}\left(x\widehat{\partial}_{\alpha_j}\right)$" either...

- Have you ever tried considering eigenvalues of the promotion and transposition chains for a Young diagram poset? I think an analogue of Theorem 5.2 has chances to hold in this case, and would be a very interesting result if it does.

  Let me explain what I mean by "analogue" first. It is not true that $\det(M - \lambda)$ factors into linear terms if P is the $[2] \times [2]$-rectangle poset and M is the transition matrix of the promotion graph of Section 3.4. Hence, Theorem 5.2 doesn't directly hold for Young tableaux. But I conjecture the following:

Conjecture. Let $\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots$ be a set of commuting indeterminates. If P is the northwest-oriented poset of a Young diagram λ (that is, the poset whose elements are the cells of λ and whose order is given by $(c \leq d) \iff (d$ lies weakly north and weakly west of $c))$, then $\det\left(\widetilde{M} - \lambda\right)$ factors into linear terms, where $\widetilde{M}$ is the result of substituting $y_{\operatorname{cont} c}$ for each x_c in the transition matrix of the promotion graph of Section 3.4. Here, $\operatorname{cont} p$ denotes the content of the cell p (that is, its row coordinate minus its column coordinate). Note that (as explained above) I am not identifying the elements of P with positive integers, so the variables x_c in the transition matrix of the promotion graph are indexed by cells of the Young diagram; thus, substituting $y_{\operatorname{cont} c}$ for x_c makes sense.

I have done some computations in Sage, checking that this conjecture holds whenever $|\lambda| \leq 6$. For example, when $\lambda = (3,3)$, the eigenvalues are $y_0, 0, y_0 + y_1, y_0 + y_{-1}, 2y_0 + 2y_1 + y_2 + y_{-1}$ if this tells you anything. There are counterexamples if λ is allowed to be a skew partition (for $\lambda = (3,3) \diagup (2)$, for instance). I can send you the code, but it is very hacky (the nice part is in patch #15428) and I don't have any hopes of running it for $|\lambda| = 7$ in reasonable time (multivariate polynomial factorization in high degrees isn't fun). It is not true that specializing one y_i to 1 and all the other y_i's to 0 yields matrices which pairwise commute, so at least from this viewpoint the conjecture is not trivial.

The idea that a polynomial identity depending on a forest can be made into a polynomial identity depending on a Young diagram by substituting $y_{\operatorname{cont} c}$ for x_c might appear mysterious at first sight, and I think it **is** mysterious. All I can say is that I have got this idea from a hook-length formula by Alex Postnikov which is very similar to your Theorem 5.1 in the same way as my above conjecture is to your Theorem 5.2. In some more detail:

Theorem 5.1 in your paper can be rewritten as follows: If P is a rooted forest, and x_p is a variable for every $p \in P$ (these variables should all commute), then

$$\sum_{\pi \in \mathcal{L}(P)} \prod_{i=1}^{|P|} \frac{1}{x_{\pi_1} + x_{\pi_2} + \cdots + x_{\pi_i}} = \prod_{p \in P} \frac{1}{\sum\limits_{q \in P;\ q \preceq p} x_q}.$$

This formula can be regarded as a kind of hook-length formula if you consider $\{q \in P \mid q \preceq p\}$ to be a "hook" of p in the forest (and setting all x_p to 1 actually yields the well-known hook-length formula for forests). There is a little-known analogue of this formula which was told to me by Alexander Postnikov: If λ is a Young diagram (not skew this time) and P is the northwest-oriented poset of λ, then

$$\sum_{\pi \in \mathcal{L}(P)} \prod_{i=1}^{|P|} \frac{1}{y_{\operatorname{cont}(\pi_1)} + y_{\operatorname{cont}(\pi_2)} + \cdots + y_{\operatorname{cont}(\pi_i)}} = \prod_{p \in P} \frac{1}{\sum\limits_{q \in \operatorname{hook} p} y_{\operatorname{cont} q}}.$$

Postnikov proved this using polyhedral combinatorics (I am trying to convince him to write up his proof); it specializes to the classical hook-length formula again by specializing all y_i to 1. It seems, however, that none of the proofs of the classical hook-length formula easily extend to this generalization (as far as I can tell). I have been told that this formula follows from Corollary 7.2 in Kento Nakada, *Colored hook formula for a generalized Young diagram*, Osaka J. Math. 45 (2008), but I don't understand root systems well enough to tell whether it does or just looks similar.

I don't know in how far $\mathcal{R}$-trivial monoids (or maybe a variant of them where $\mathcal{R}$-classes are somehow bounded rather than having cardinality 1) are useful to handle the conjecture. But there are weird things going on. The conjecture involves a matrix labelled by linear extensions of a Young diagram poset (i. e., by standard Young tableaux) whose eigenvalues are conjectured to be integer combinations of y_i's. This is reminiscent of the Reiner-Saliola-Welker conjecture (Conjecture 1.2 in arXiv:1102.2460v2) which involves certain operators on $\mathbb{Q}[S_n]$ which are suspected to have integer eigenvalues, but these operators break down into operators acting on every irreducible representation of S_n, and this means matrices labelled by standard Young tableaux. Of course, there is no guarantee of a connection, but I suspect at least some of the same methods can be used.