A primer of Hopf algebras

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Septembre 2006

IHES/M/06/40

Errata and questions - I

- **Page 9, §2.1:** Here, Cartier claims that "by invariant theory, Ω_p for $p > 2n$ is decomposable as a product of forms of degree $\leq 2n-1$". I don't know what results from invariant theory yield this; however, I think the Amitsur-Levitzki theorem yields that $\Omega_p = 0$ for $p > 2n$ (and, even stronger, the antisymmetrization of $A_1 A_2 ... A_p$ (and not only of $\operatorname{Tr}(A_1 A_2 ... A_p)$) is 0 for $p > 2n$).

- **Page 9, §2.1:** Here, Cartier claims that "It follows that the algebra $\mathcal{T}^{\cdot}(U(n)) = \bigoplus_{p \geq 0} \mathcal{T}^p(U(n))$ possesses a basis of the form

$$\Omega_{p_1} \wedge ... \wedge \Omega_{p_r}, \qquad 1 \leq p_1 < \cdots < p_r < 2n, \qquad p_i \text{ odd.}$$

" I don't see why this is a basis. It is clear from the above that it is a spanning set, but why is it linearly independent?

- **Page 18, §2.4:** I don't understand the proof of Lemma 2.4.1. Why can we "select the term of the form $u \otimes t_r$" and be sure that it vanishes? This sounds reasonable only if we already know that all products $t_{i_1} ... t_{i_s}$ for $1 \leq i_1 < \cdots < i_s \leq r$ are linearly independent.

- **Page 19, §2.5:** Here it is written that "Then there is a natural duality between $P_{\cdot}$ and $P^{\cdot}$ and more precisely between the homogeneous components P_n and P^n."

  I don't think this is true. Take the tensor Hopf algebra TV of a finite-dimensional vector space V in characteristic 0. Then, the set of primitive elements of TV is (isomorphic to) the free Lie algebra over V, whereas the set of primitive elements of the graded dual of TV is V^* (this is easily seen since the graded dual of TV is isomorphic to the shuffle Hopf algebra of V). The free Lie algebra over V has a totally different Hilbert series than V^*, so there cannot be a natural duality between the homogeneous components P_n and P^n in this case.

  Maybe Cartier is speaking of the case when the conditions of **D.** are satisfied.

- **Page 20, §2.5:** Here it is claimed that "Moreover $A^{\cdot}$ is the free graded-commutative algebra over $P^{\cdot}$". I think this again requires the conditions of **D.** to be true.

- **Page 22, §3.2:** The formulae (26), (27) and (28) contradict each other. In fact, using the formulae (26) and (27), we have

$$\begin{aligned}
\left(\left(\Delta \otimes 1_V\right) \circ \Pi\right)(e_j) &= \left(\Delta \otimes 1_V\right)\left(\Pi\left(e_j\right)\right) = \left(\Delta \otimes 1_V\right)\left(\sum_{i=1}^{d(\pi)} u_{ij,\pi} \otimes e_i\right) \\
&= \sum_{i=1}^{d(\pi)} \underbrace{\Delta\left(u_{ij,\pi}\right)}_{=\sum_{\ell=1}^{d(\pi)} u_{i\ell,\pi} \otimes u_{\ell j,\pi}} \otimes e_i \\
&= \sum_{i=1}^{d(\pi)} \sum_{\ell=1}^{d(\pi)} u_{i\ell,\pi} \otimes u_{\ell j,\pi} \otimes e_i
\end{aligned}$$

and

$$\begin{aligned}
\left(\left(1_{\mathcal{O}(G)} \otimes \Pi\right) \circ \Pi\right)(e_j) &= \left(1_{\mathcal{O}(G)} \otimes \Pi\right)\left(\Pi\left(e_j\right)\right) = \left(1_{\mathcal{O}(G)} \otimes \Pi\right)\left(\sum_{i=1}^{d(\pi)} u_{ij,\pi} \otimes e_i\right) \\
&= \sum_{i=1}^{d(\pi)} u_{ij,\pi} \otimes \Pi\left(e_i\right) = \sum_{\ell=1}^{d(\pi)} u_{\ell j,\pi} \otimes \underbrace{\Pi\left(e_\ell\right)}_{=\sum_{i=1}^{d(\pi)} u_{i\ell,\pi} \otimes e_i} \\
&= \sum_{\ell=1}^{d(\pi)} \sum_{i=1}^{d(\pi)} u_{\ell j,\pi} \otimes u_{i\ell,\pi} \otimes e_i,
\end{aligned}$$

and in general these two terms are not equal (unless G is abelian), so that (28) does not hold.

One possible way to correct this is to replace "$\Pi : V \to \mathcal{O}(G) \otimes V$" by "$\Pi : V \to V \otimes \mathcal{O}(G)$", replace (27) by

$$\Pi\left(e_j\right) = \sum_{i=1}^{d(\pi)} e_i \otimes u_{ij,\pi},$$

replace (28) by

$$\left(1_V \otimes \Delta\right) \circ \Pi = \left(\Pi \otimes 1_{\mathcal{O}(G)}\right) \circ \Pi,$$

and replace (29) by

$$\pi(g) = \left(1_V \otimes \delta_g\right) \circ \Pi.$$

- **Page 24, §3.3:** The footnote [29] (which explains that you use bra-ket notation) should be made much earlier: You already use bra-ket notation in (33) (the $\langle v_1|$ and $\langle v_3|$ are bras; the $v_2\rangle$ and $v_4\rangle$ are kets).

- **Page 26, §3.3, part (C):** Here it is written that:

  "Indeed, for $h \in H$, $h \neq 1$ we can write $h = \exp x$, with $x \in U_1$, $x \neq 0$, hence $h^2 = \exp 2x$ belongs to V but not to V_1, hence not to H."

I don't understand why h^2 does not belong to V_1. But the argument can be salvaged as follows:

For every $h \in H$ satisfying $h \neq 1$, we can write $h = \exp x$ with $x \in U_1$, $x \neq 0$, and we can find some $n \in \mathbb{N}$ such that $nx \in U \diagdown U_1$; for this n, we then have $h^n \in V$ but $h^n = \exp(nx) \notin V_1$, so that $h^n \notin H$, which is absurd.

- **Page 27, §3.3, proof of Lemma 3.3.1:** Replace "we find a real polynomial" by "we find a real polynomial P".

- **Page 27, §3.3:** Here, the notations $GL(m, R)$ and $GL(m; R)$ are used for one and the same thing.

- **Page 36, proof of Theorem 3.7.1:** In "by power series $\varphi^j(\mathbf{x}, \mathbf{y}) = \varphi^j\left(x^1, \ldots, x^N; y_1, \ldots, y^N\right)$", replace "$y_1$" by "$y^1$".

- **Page 47, Theorem 3.8.3:** In footnote [48], replace "$\bigotimes\limits_{p+q=n}$" by "$\bigoplus\limits_{p+q=n}$".

- **Page 47, Theorem 3.8.3:** In footnote [48], replace "$S(A_n) = A_n$" by "$S(A_n) \subset A_n$".

- **Page 47, proof of Theorem 3.8.3:** Is it really obvious that "An inverse map Λ_p to Θ_p can be defined as the composition of the iterated coproduct $\overline{\Delta}_p$ which maps $\pi_p(A)$ to $\pi_1(A)^{\otimes p}$ with the natural projection of $\pi_1(A)^{\otimes p}$ to $\operatorname{Sym}^p(\pi_1(A))$"? I don't see a simple reason for this.

- **Page 55, (126):** Add "where $n = p + q$" after this equality.

- **Page 61, (159):** This equality is not literally true for $m = 0$. Indeed, for $m = 0$, the two addends $1 \otimes [\gamma_1 \mid \ldots \mid \gamma_m]$ and $[\gamma_1 \mid \ldots \mid \gamma_m] \otimes 1$ should be regarded as only one addend. It would be better to replace the right hand side of (159) by $\sum\limits_{i=0}^{m} [\gamma_1 \mid \ldots \mid \gamma_i] \otimes [\gamma_{i+1} \mid \ldots \mid \gamma_m]$; this works for all m, including $m = 0$.

- **Page 61, (160):** Replace "n^r" by "n_r".

- **Page 62, §4.1:** Replace "$z\left(\underbrace{1, \ldots, 1}_{r}\right)$" by "$Z\left(\underbrace{1, \ldots, 1}_{r}\right)$" (on the last line of §4.1).

- **Page 66:** Replace "$z(k_1, \ldots, k_r)$" by "$Z(k_1, \ldots, k_r)$".

- **Page 63:** I have a hunch that "where Δ_k is the simplex $\{0 < t_1 < t_2 < \cdots < t_k\}$" should be "where Δ_k is the simplex $\{0 < t_1 < t_2 < \cdots < t_k < 1\}$".

- **Page 73:** Replace "and replaces" by "and replace".