When is the Algebra of Multisymmetric Polynomials Generated by the Elementary Multisymmetric Polynomials?

Emmanuel Briand

https://www.emis.de/journals/BAG/vol.45/no.2/b45h2bri.pdf
Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry), Volume **45** (2004), No. 2, pp. 353–368.

Errata and addenda by Darij Grinberg

1. Errata

- **page 356, Definition 2:** In the last sentence of Definition 2, "for α running in the parts of $\mathfrak{p}$" might better be "for α running over the distinct parts of $\mathfrak{p}$".

- **page 358:** After you define polarization, you could add the formula

$$\Delta_{\alpha}^{r}(fg) = \sum \Delta_{\beta}^{|\beta|} f \cdot \Delta_{\gamma}^{|\gamma|} g$$

for any homogeneous polynomials f and g with $\deg f + \deg g = r$. Here, the sum is over all $(\beta, \gamma) \in (\mathbb{N}^r)^2$ such that $|\beta| = \deg f$, $|\gamma| = \deg g$ and $\beta + \gamma = \alpha$. This formula is easy and known, but since you are defining polarization, you might as well mention this formula, as you are using it several times (for example, you silently use it whenever you make an argument of the form "some polynomials P_i generate a polynomial $Q \Longrightarrow$ the polarizations of P_i generate the polarization of Q").

- **page 359, proof of Theorem 3:** You say "To determine the multiplicative coefficient, note that $m_{\mathfrak{p}}$ is the sum of $\dfrac{k!}{\mu_{\mathfrak{p}}!}$ monomials, while $e_{1,1,\ldots,1}$ is the sum of $k!$ monomials. So the multiplicative coefficient is $\mu_{\mathfrak{p}}!$.". I think that both numbers $\dfrac{k!}{\mu_{\mathfrak{p}}!}$ and $k!$ should be multiplied with $n!/(n-k)!$ here.

- **page 363, Lemma 12:** The dot at the end of the formula ($m_{\mathfrak{p}} \cdot m_{\mathfrak{q}} = \ldots$) should be a comma.

- **page 363, proof of the Lemma 12:** Some of the letters a, b, ..., z in the formulas should be boldface.

- **page 364, proof of Lemma 13:** This proof is incorrect. Let me explain where it goes wrong.

  First of all, what you call "partial ordering $\precsim_i$" is not actually a partial ordering, but just a pre-order: Indeed, two vector partitions $\mathfrak{p}$ and $\mathfrak{q}$ may satisfy $\lambda(j; \mathfrak{p}) = \lambda(j; \mathfrak{q})$ for ALL j (including $j = i$) but still not be equal.

Also, I suspect you want to add "and $\lambda(j;\mathfrak{p}) = \lambda(j;\mathfrak{q})$ for all $j \neq i$" after "$p \precsim_i q$ if and only if $\lambda(i;\mathfrak{p})$ is smaller than $\lambda(i;\mathfrak{q})$ in lexicographic order" (because otherwise, in your reduction algorithm it would be possible that some steps destroy what previous steps have achieved, and the algorithm goes around in circles). Besides, either you want to replace $\precsim_i$ by $\prec_i$, or "smaller" by "smaller or equal".

So let me assume that you want to define $\prec_i$ by: $p \prec_i q$ if and only if $\lambda(i;\mathfrak{p})$ is smaller than $\lambda(i;\mathfrak{q})$ in lexicographic order and $\lambda(j;\mathfrak{p}) = \lambda(j;\mathfrak{q})$ for all $j \neq i$.

Now, how do you make sure that, in the first of three cases, you have $q \prec_i p$? This is the case $\lambda(i;\mathfrak{p}) = (t_1, t_2, ..., t_s, k, ..., k, 0, ..., 0)$. Everything is okay when $t_s > k+1$, but when $t_s = k+1$, hell may break loose. For example, say $\mathfrak{p} = ((3,x),(2,y),(1,z))$ for some distinct positive integers x, y, z which I don't want to specify. Let $i = 1$ (so we are reducing the first coordinate). Then your reduction yields $m_{\mathfrak{p}} = m_{\mathfrak{r}} e_{2\xi_1} - \sum m_{\mathfrak{q}}$. The problem is now, one of the $\mathfrak{q}$'s is $((3,x),(2,z),(1,y))$. And this is in no way "smaller" than $\mathfrak{p}$, and if we try to reduce it further, we get $\mathfrak{p}$ back again as some of the $\mathfrak{q}$'s.

As far as I have understood, what your argument does show is that the multisymmetric polynomials with multidegree dominated by $(N, N, ..., N)$ generate the multisymmetric polynomials as a module over the elementary multisymmetric ones, where $N = n\,(n-1)\,/2$ (the proof seems to be similar to the one given by Göbel for what is nowadays called Göbel's bound). But this argument does not show $n-1$ is enough...

[**Update:** Emmanuel Briand has confirmed the above mistake. For a correct proof of Lemma 13, see Fleischmann's paper [5].]

- **page 365, §4.3:** You refer to "Proposition 13". It should be "Lemma 13".