A combinatorial generalization of the boson-fermion correspondence
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`lam - boson-fermion - 0507341v1.pdf`
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Errata and addenda by Darij Grinberg

I will refer to the results appearing in the article "A combinatorial generalization of the boson-fermion correspondence" by the numbers under which they appear in this article (specifically, in its version of 17 July 2005, posted on arXiv under the identifier arXiv:math/0507341v1).

11. Errata

- **Page 1:** Replace "composition" by "weak composition". (A *weak composition* is an infinite sequence $(r_1, r_2, r_3, \ldots)$ of nonnegative integers such that only finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $r_i \neq 0$. In contrast, a composition is a finite sequence of positive integers. The weight of a semistandard tableau T is a weak composition; it is not necessarily a composition, because it can have a 0 followed by a positive integer.)
- **Page 2:** In "the definition of a tableaux", replace "tableaux" by "tableau".
- **Page 2:** In "a semi-standard Young tableaux", replace "tableaux" by "tableau".
- **Page 3:** After "The ring Λ_K should be thought of as the ring of formal power series in countably many variables $x_1, x_2, \ldots$, of bounded degree", add ", which are invariant under permutations of $x_1, x_2, \ldots$". (Or else, you should replace "thought of as the ring" by "thought of as a subring".)
- **Page 4, §2:** Replace "a Young tableaux T" by "a Young tableau T".
- **Page 4, §2:** Replace "a Young tableaux of skew shape" by "a Young tableau of skew shape".
- **Page 4, §3:** Replace "$[B_k, B_l] = l \cdot a_l \cdot \delta_{k,-l}$" by "$[B_k, B_l] = k \cdot a_k \cdot \delta_{k,-l}$".
- **Page 4, §3:** Replace "$a_l = -a_{-l}$" (or else replace "$[B_k, B_l] = k \cdot a_k \cdot \delta_{k,-l}$" by "$[B_k, B_l] = a_k \cdot \delta_{k,-l}$"; but this option would require further changes in various other places).
- **Page 5, §3:** I think it would be useful to state the fact that the H-module Λ_K is faithful. This shows that any identity in H can be proven by verifying the corresponding identity for operators on Λ_K (which, to a combinatorialist, is far more familiar terrain).

The proof of the faithfulness of the H-module Λ_K is not difficult: One can first prove that the family

$$\left(\overrightarrow{\prod_{i \in \mathbb{Z} \setminus \{0\}}} (B_i)^{a_i} \right)_{\substack{(a_i)_{i \in \mathbb{Z} \setminus \{0\}} \text{ is a family} \\ \text{of nonnegative integers} \\ \text{such that only finitely many} \\ i \in \mathbb{Z} \setminus \{0\} \text{ satisfy } a_i \neq 0}}$$

(where the $\overrightarrow{\prod_{i \in \mathbb{Z} \setminus \{0\}}}$ symbol signifies a product taken in increasing order, i.e., we have $\overrightarrow{\prod_{i \in \mathbb{Z} \setminus \{0\}}} (B_i)^{a_i} = \cdots (B_{-2})^{a_{-2}} (B_{-1})^{a_{-1}} (B_1)^{a_1} (B_2)^{a_2} \cdots$) generates the K-module H (mainly because the relation $[B_k, B_l] = k \cdot a_k \cdot \delta_{k,-l}$ allows us to rearrange the terms in any monomial of the form $B_{p_1} B_{p_2} \cdots B_{p_s}$ into weakly increasing order, at the cost of creating smaller monomials). Moreover, the actions of the elements of this family on Λ_K are linearly independent (as one can easily see as well). Thus, the H-module Λ_K is faithful.

- **Page 5, Lemma 1:** Here the notation B_λ is being used; this notation is not defined until later (in §4).

- **Page 5, Lemma 1:** Replace "$B_{-k} B_\lambda = k a_k m_k(\lambda) B_\mu + B_\lambda B_{-k}$" by "$B_{-k} B_\lambda = -k a_k m_k(\lambda) B_\mu + B_\lambda B_{-k}$".

- **Page 5:** Replace "the parameters $a_l = 1$ for $l \geq 1$ and $a_l = -1$ for $l \leq -1$" by "the parameters $a_l = 1$".

- **Page 5:** Replace "$\{v_j : j \in Z\}$" by "$\{v_j : j \in \mathbb{Z}\}$".

- **Page 5, (5):** Replace "$v_{i_0} \wedge v_{i_{-1}} \wedge \cdots \wedge v_{i_{j-1}} \wedge v_{i_j - k} \wedge v_{i_{j+1}} \wedge \cdots$" by "$v_{i_0} \wedge v_{i_{-1}} \wedge \cdots \wedge v_{i_{j+1}} \wedge v_{i_j - k} \wedge v_{i_{j-1}} \wedge \cdots$". (The two subscripts "$i_{j-1}$" and "$i_{j+1}$" have been switched.)

- **Page 5, Theorem 3:** After "$\sigma(v_{i_0} \wedge v_{i_{-1}} \wedge \cdots) = s_\lambda$", add ", where $\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots)$".

- **Page 6, §3:** Replace "have been studied previously" by "has been studied previously".

- **Page 6, §4:** Remove the "and $U_k := \sum\limits_{\lambda \vdash k} z_\lambda^{-1} B_{-\lambda}$" part from "Let $D_k := \sum\limits_{\lambda \vdash k} z_\lambda^{-1} B_\lambda$ and $U_k := \sum\limits_{\lambda \vdash k} z_\lambda^{-1} B_{-\lambda}$". (It is entirely unnecessary, since the same definition of U_k appears again in the next paragraph.)

- **Page 6, §4:** In "$\langle v_s, v_s' \rangle = \delta_{ss'}$", replace "$v_s'$" by "$v_{s'}$".

- **Page 6, §4:** Replace "Define the generating functions

$$F^V_{s/t}(x_1, x_2, \ldots) = F_{s/t}(x_1, x_2, \ldots) := \sum_\alpha x^\alpha \langle U_{\alpha_l} U_{\alpha_{l-1}} \cdots U_{\alpha_1} \cdot t, s \rangle,$$

where the sum is over all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$. Similarly define

$$G^V_{s/t}(x_1, x_2, \ldots) = G_{s/t}(x_1, x_2, \ldots) = \sum_\alpha x^\alpha \langle D_{\alpha_l} D_{\alpha_{l-1}} \cdots D_{\alpha_1} \cdot s, t \rangle.$$

" by "Define the generating functions

$$F^V_{s/t}(x_1, x_2, \ldots) = F_{s/t}(x_1, x_2, \ldots) := \sum_\alpha x^\alpha \langle \cdots U_{\alpha_3} U_{\alpha_1} U_{\alpha_1} \cdot v_t, v_s \rangle,$$

where the sum is over all weak compositions $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$. Similarly define

$$G^V_{s/t}(x_1, x_2, \ldots) = G_{s/t}(x_1, x_2, \ldots) = \sum_\alpha x^\alpha \langle \cdots D_{\alpha_3} D_{\alpha_2} D_{\alpha_1} \cdot v_s, v_t \rangle.$$

"

I have made two changes here: First, the "t" and "s" on the right hand sides have been replaced by "v_t" and "v_s", respectively (since t and s themselves are not vectors, but just elements of the indexing set S). Second, instead of summing over all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$, I am summing over all weak compositions, since we need to allow 0's followed by positive integers here, and also since the "l" is confusing (if you sum over finite sequences, do you count $(2, 0, 3)$ and $(2, 0, 3, 0)$ as two distinct finite sequences?).

- **Page 6, §4:** For a combinatorialist reader, it would be helpful to explain how the various elements of H you have defined act on Λ_K. Namely, the map $H \to \operatorname{End}_K(\Lambda_K)$ given by the action of H on Λ_K sends

$$\begin{aligned} B_{-k} &\mapsto a_k p_k && \text{for every } k \geq 1; \\ B_k &\mapsto k \frac{\partial}{\partial p_k} = p_k^\perp && \text{for every } k \geq 1; \\ D_k &\mapsto h_k^\perp && \text{for every } k \geq 0; \\ D_\lambda &\mapsto h_\lambda^\perp && \text{for every partition } \lambda. \end{aligned}$$

It also sends

$$\begin{aligned} U_k &\mapsto h_k && \text{for every } k \geq 0; \\ U_\lambda &\mapsto h_\lambda && \text{for every partition } \lambda; \\ S_\lambda &\mapsto s_\lambda && \text{for every partition } \lambda \end{aligned}$$

when all a_l are 1.

- **Page 7, §4:** I do not understand what you mean by "The element $\Omega(H_-, X) \cdot v_b \in V \hat{\otimes} \Lambda_K(X)$ depends only on the choice of v_b.".

- **Page 7, §5:** In "By Proposition 2, there is a canonical map of H-modules $\phi : H \cdot b \to \Lambda_K$ sending $v_b \mapsto 1$.", replace "$H \cdot b$" by "$H \cdot v_b$". More importantly, I do not understand how this follows from Proposition 2. How do we know that $H \cdot v_b$ is irreducible?

  Here is the simplest proof I can find for the existence of a canonical H-module homomorphism $\phi : H \cdot v_b \to \Lambda_K$:

  We use the following fact:

  Proposition 2a. Let V be a representation of H. Let $v \in V$ be a highest weight vector. Then, there exists a unique H-module homomorphism $\phi : K[B_{-1}, B_{-2}, \ldots] \to V$ such that $\phi(1) = v$.

  Proposition 2a is relatively well-known (it appeared in Pavel Etingof's class on infinite-dimensional Lie algebras), and I am sure that your proof of Proposition 2 uses Proposition 2a as an intermediate step.

  Now, applying Proposition 2a to $H \cdot v_b$ and v_b instead of V and v, we conclude that there exists a unique H-module homomorphism $\phi : K[B_{-1}, B_{-2}, \ldots] \to H \cdot v_b$ such that $\phi(1) = v_b$. This homomorphism ϕ is injective (because it is nonzero, and because $K[B_{-1}, B_{-2}, \ldots]$ is an irreducible H-module) and surjective (because it satisfies $H \cdot \underbrace{v_b}_{=\phi(1)} = H \cdot \phi(1) = \phi(H \cdot 1) \subseteq \phi(K[B_{-1}, B_{-2}, \ldots])$), and thus an isomorphism. Hence, the inverse Φ of ϕ is an H-module isomorphism $\phi : H \cdot v_b \to \Lambda_K$.

- **Page 7, Theorem 6:** According to what you wrote in the mail, the assumptions of Theorem 6 need to be changed. Namely, instead of assuming that $v_b \in V$ is a highest weight vector of H, you want to assume that

  $$\langle B_l x, v_b \rangle = 0 \qquad \text{for every } x \in V \text{ and } l < 0. \tag{1}$$

  This assumption is used in deriving $\langle B_\lambda B_l \cdot v_s, v_b \rangle = k a_k m_k(\lambda) \langle B_\mu v_s, v_b \rangle$ from Lemma 1 in the proof of Theorem 6. (Namely, we have

  $$\left\langle B_\lambda \underbrace{B_l}_{\substack{=B_{-k} \\ (\text{since } l=-k)}} \cdot v_s, v_b \right\rangle = \left\langle \underbrace{B_\lambda B_{-k}}_{=B_{-k}B_\lambda + k a_k m_k(\lambda) B_\mu} \cdot v_s, v_b \right\rangle = \left\langle \left(B_{-k} B_\lambda + k a_k m_k(\lambda) B_\mu \right) \cdot v_s, v_b \right\rangle$$
  $$= \underbrace{\langle B_{-k} B_\lambda \cdot v_s, v_b \rangle}_{\substack{=0 \\ (\text{by (1), applied to } -k \text{ and } B_\lambda \cdot v_s \\ \text{instead of } l \text{ and } x)}} + k a_k m_k(\lambda) \langle B_\mu \cdot v_s, v_b \rangle = k a_k m_k(\lambda) \langle B_\mu \cdot v_s, v_b \rangle.$$

)

- **Page 8, proof of Theorem 6:** In "$\sum_c \langle B_l \cdot v_s, v_c \rangle \left(\sum_\lambda z_\lambda^{-1} \langle B_\lambda \cdot v_c, v_b \rangle \right)$", add a "$p_\lambda$" before the "$\langle B_\lambda \cdot v_c, v_b \rangle$".

- **Page 8, §5:** I find it worthwhile to mention that the map $\Phi : V \to \Lambda_K$ constructed in Theorem 6 is a $K[B_1, B_2, \ldots]$-modules even if we don't require (1) to hold. (Indeed, this follows from the second part of the proof of Theorem 6, that begins with "Now suppose $k > 0$"; this part does not use (1).)

- **Page 8, §5:** Shouldn't "a different action of H on Λ_K" be "a different action of H on V" ? Anyway, I fear I don't fully understand this paragraph, and it appears to me that it might use some additional assumptions.

- **Page 8, §6:** In "under the map $\kappa : \Lambda_K \to K$ given by $\kappa(p_k) = a_k$", replace "map" by "K-algebra homomorphism".

- **Page 9, Theorem 7:** In "$\langle U_k \cdot s, t \rangle$", replace "$s$" and "$t$" by "$v_s$" and "$v_t$", respectively. Do the analogous replacements in the other three equalities.

- **Page 9, Theorem 7:** I am not sure what assumptions this theorem needs. Obviously, (1) is required to apply Theorem 6, but we might need more for "the comments immediately after it".

- **Page 9, proof of Lemma 8:** The notation "$\lambda \cup \mu$" should be defined. (Here is a simple definition: If α and β are two partitions, then $\alpha \cup \beta$ will denote the partition obtained by sorting the sequence $\left(\alpha_1, \alpha_2, \ldots, \alpha_{l(\alpha)}, \beta_1, \beta_2, \ldots, \beta_{l(\beta)}\right)$ in weakly decreasing order.)

- **Page 9, proof of Lemma 8:** Replace both appearances of "θ" by "κ".

- **Page 9, Theorem 9:** In order to obtain the first equality of Theorem 9 from the second equality, you again use the assumption (1). (Indeed, this assumption is what guarantees that $F_{b/s} = \delta_{b,s}$ for all $s \in S$.)

- **Page 10, proof of Theorem 9:** The letter "i" is used in two different meanings here: Once as a bound variable in "$h_k \langle a_i \rangle$" (shorthand for "$h_k \langle a_1, a_2, \ldots \rangle$"), and another time as a bound variable in the products "$\prod_{i,j \geq 1}^{\infty}$".

- **Page 11:** Replace "Suppose further that B_k and B_l commute" by "Suppose further that B'_k and B'_l commute".

- **Page 11:** Add a whitespace in "Let$D'_k := \sum_{\lambda \vdash k} z_\lambda^{-1} B'_\lambda$".

- **Page 11:** Replace "$F'_{s/t}(x_1, x_2, \ldots) := \sum\limits_{\alpha} x^\alpha \langle U'_{\alpha_l} U'_{\alpha_{l-1}} \cdots U'_{\alpha_1} \cdot t, s \rangle$" by "$F'_{s/t}(x_1, x_2, \ldots) := \sum\limits_{\alpha} x^\alpha \langle \cdots U'_{\alpha_3} U'_{\alpha_2} U'_{\alpha_1} \cdot v_t, v_s \rangle$".
- **Page 12, §8.1:** Replace "$h_i \langle a_i \rangle$" by "$h_k \langle a_i \rangle$".
- **Page 12, §8.2:** Add a semicolon before "otherwise".
- **Page 12, §8.3:** Replace "$\tilde{B}_k \cdot v_1 \otimes v_2 = (B_k \cdot v_1) \otimes v_2 + v_1 \cdot (B_k \cdot v_2)$" by "$\tilde{B}_k \cdot (v_1 \otimes v_2) = (B_k \cdot v_1) \otimes v_2 + v_1 \otimes (B_k \cdot v_2)$" (notice that I've made two changes here).