Structure theory for the group algebra of the symmetric group, with applications to polynomial identities for the octonions
Murray R. Bremner, Sara Madariaga, Luiz A. Peresi
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Errata by Darij Grinberg

Marginalia

The following are my comments on specific places in the paper "Structure theory for the group algebra of the symmetric group, with applications to polynomial identities for the octonions" by Murray R. Bremner, Sara Madariaga, Luiz A. Peresi (Comment. Math. Univ. Carolin. 57,4 (2016), pp. 413–452). Very few of them are corrections (there is barely anything wrong in the paper); most of them are additional details and steps that have been omitted from the proofs.

- **page 414:** You write that the matrices obtained by restricting ϕ to S_n "have entries in $\{0, \pm 1\}$". If you are talking about the matrices $R^{\lambda}(p)$ from Definition 1.49, then I don't see why this is true (and I suspect it is not).

- **page 415, Definition 1.5:** "A **Young tableau** T^{λ}" should be "A **Young tableau** T". (The "λ" superscript is unnecessary and confusing; you just call it "T" afterwards.)

- **page 416, Definition 1.9:** It would be good to point out that this action of S_n on the set $\{\text{tableaux of shape } \lambda\}$ is free and transitive. (This is being used tacitly further below.)

- **page 417, Remark 1.14:** This is perhaps a bit out of place: You have yet to use the notation hvT at this point!

- **page 417, proof of Proposition 1.15:** In the second paragraph of this proof, replace "obtaining tableaux $T^{\lambda'} \succ T^{\mu'}$ where λ' and μ' are partitions of $n - n_1$" by "obtaining tableaux $T^{\lambda'}$ and $T^{\mu'}$ whose shapes λ' and μ' are partitions of $n - n_1$ satisfying $Y^{\lambda'} \succ Y^{\mu'}$ ".

- **page 419, proof of Lemma 1.19:** It would be helpful to point out that the third equality sign in the displayed equation relies on the facts that $\epsilon(v)^{-1} = \epsilon(v)$ (because $\epsilon(v) \in \{1, -1\}$) and that ϵ is a group homomorphism.

- **page 419, proof of Proposition 1.20:** Remove "Since $\epsilon(p) = \epsilon(p^{-1})$". (You don't use the fact that $\epsilon(p) = \epsilon(p^{-1})$ here.)

- **page 419, Definition 1.21:** Replace "in lex order" by "in some fixed order chosen in such a way that the standard tableaux of shape λ will be $T_1, T_2, \ldots, T_{d_\lambda}$ in lex order". Indeed, you don't need all the $n!$ tableaux $T_1, T_2, \ldots, T_{n!}$ to be in lex order; but you will later want the standard tableaux of shape λ to be $T_1, T_2, \ldots, T_{d_\lambda}$ in lex order. If you list all the $n!$ tableaux of shape λ in lex order, then (in general) the first d_λ tableaux in your list will not be the standard tableaux of shape λ.

- **page 420, Corollary 1.24:** After "be the standard tableaux", add "of shape λ".

- **page 421, proof of Proposition 1.25:** At the beginning of this proof, add "Again, set $H_i = H_{T_i}$ and $V_i = V_{T_i}$; thus, $D_i = H_i V_i$.".

- **page 421, proof of Proposition 1.25:** After "$hD_i^2 v = (hH_i) V_i H_i (V_i v) = \epsilon(v) H_i V_i H_i V_i = \epsilon(v) D_i^2$", add "$= \epsilon(v) \sum_{p \in S_n} x_p p$" (in order to make the step to the next equality clearer).

- **page 421, proof of Proposition 1.25:** Replace "On the left side of (7) take $p = \iota$, on the right side take $p = hv$, and compare coefficients" by "Comparing coefficients of hv on both sides of (7), we obtain".

- **page 421, proof of Proposition 1.25:** Replace "Setting $p = q$ on both sides, we obtain
$$x_q t q q^{-1} t q = \epsilon\left(q^{-1} t q\right) x_q q,$$
and this simplifies to $x_q q = -x_q q$" by "Comparing coefficients of q on both sides of this equation, we obtain $x_q = \epsilon\left(q^{-1} t q\right) x_q$ (since the only $p \in S_n$ satisfying $tpq^{-1} tq = q$ is q), and this simplifies to $x_q = -x_q$".

- **page 421, proof of Proposition 1.25:** After "Combining the results of the two cases,", add "we obtain that $D_i^2 = \sum_{h \in G_H(T_i)} \sum_{v \in G_V(T_i)} x_\iota \epsilon(v) hv$, since Lemma 1.12 shows that any permutation of the form hv can be written in this form in exactly one way. In view of $D_i = \sum_{h \in G_H(T_i)} \sum_{v \in G_V(T_i)} \epsilon(v) hv$, this rewrites as".

- **page 422, proof of Proposition 1.25:** The equality sign in "$\sum_{h,v} \epsilon(v) \operatorname{trace}(hv) = \operatorname{trace}(I_{\mathbb{F}S_n})$", again, relies on Lemma 1.12. (Indeed, Lemma 1.12 shows that the only pair (h, v) satisfying $hv = \iota$ is (ι, ι).)

- **page 422, Definition 1.26:** It is worth saying that E_i^λ will be denoted by E_i when λ is clear from the context.

- **page 422, §1.5:** After Corollary 1.27, I suggest adding another corollary (which is being tacitly used in the proof of Lemma 1.29):

  Corollary 1.27a. Let $\lambda \vdash n$. Let $i, j \in \{1, 2, \ldots, n!\}$. Then, $f_i = f_j$ and $c_i = c_j$ and $E_j = s_{ji} E_i s_{ij}$ and $E_i s_{ij} = s_{ij} E_j$.

  Proof of Corollary 1.27a. From (5), we obtain $D_j = s_{ji} D_i s_{ij}$, so that $\mathbb{F} S_n D_j = \mathbb{F} \underbrace{S_n s_{ji}}_{= S_n} D_i s_{ij} = \mathbb{F} S_n D_i s_{ij} \cong \mathbb{F} S_n D_i$ as vector spaces (since $s_{ij} \in \mathbb{F} S_n$ is invertible). Hence, $\dim \left(\mathbb{F} S_n D_j \right) = \dim \left(\mathbb{F} S_n D_i \right)$. In other words, $f_j = f_i$ (since the numbers f_i and f_j are defined to be $\dim \left(\mathbb{F} S_n D_i \right)$ and $\dim \left(\mathbb{F} S_n D_j \right)$, respectively). In other words, $f_i = f_j$. This yields $n!/f_i = n!/f_j$. In other words, $c_i = c_j$ (since the numbers c_i and c_j are defined to be $n!/f_i$ and $n!/f_j$, respectively). Finally, the definition of c_i yields $c_i = n!/f_i$, so that $\dfrac{f_i}{n!} = \dfrac{1}{c_i}$. But the definition of E_i yields $E_i = \underbrace{\dfrac{f_i}{n!}}_{= \dfrac{1}{c_i}} D_i = \dfrac{1}{c_i} D_i$. Likewise, $E_j = \dfrac{1}{c_j} D_j$. Hence,

  $$E_j = \underbrace{\frac{1}{c_j}}_{\substack{= \frac{1}{c_i} \\ \text{(since } c_i = c_j\text{)}}} \underbrace{D_j}_{= s_{ji} D_i s_{ij}} = \frac{1}{c_i} \cdot s_{ji} D_i s_{ij} = s_{ji} \underbrace{\left(\frac{1}{c_i} D_i \right)}_{= E_i} s_{ij} = s_{ji} E_i s_{ij}.$$

  Hence,

  $$\underbrace{s_{ij}}_{= \left(s_{ji} \right)^{-1}} \underbrace{E_j}_{= s_{ji} E_i s_{ij}} = \underbrace{\left(s_{ji} \right)^{-1} s_{ji}}_{= \iota} E_i s_{ij} = E_i s_{ij},$$

  so that $E_i s_{ij} = s_{ij} E_j$. This completes the proof of Corollary 1.27a. ■

- **page 422, §1.5:** I would also add another corollary (which is being tacitly used in the proof of Lemma 1.32):

  Corollary 1.27b. Let $\lambda \vdash n$. Let $i \in \{1, 2, \ldots, n!\}$. Then, $D_i \neq 0$ and $f_i \neq 0$ and $E_i \neq 0$.

  Proof of Corollary 1.27b. The definition of D_i yields

  $$\begin{aligned} D_i &= \underbrace{H_{T_i}}_{= \sum_{h \in G_H(T_i)} h} \underbrace{V_{T_i}}_{= \sum_{v \in G_V(T_i)} \epsilon(v) v} = \left(\sum_{h \in G_H(T_i)} h \right) \left(\sum_{v \in G_V(T_i)} \epsilon(v) v \right) \\ &= \sum_{h \in G_H(T_i)} \sum_{v \in G_V(T_i)} \epsilon(v) hv = \sum_{(h,v) \in G_H(T_i) \times G_V(T_i)} \epsilon(v) hv. \end{aligned}$$

The group elements hv on the right hand side of this equality are all distinct (by the second sentence of Lemma 1.12); thus, the sum $\sum\limits_{(h,v)\in G_H(T_i)\times G_V(T_i)} \epsilon(v)\, hv$ has no cancellations and therefore is nonzero. In other words, $D_i \neq 0$. Hence, the left ideal $\mathbb{F}S_n D_i$ is nonzero, and thus $\dim(\mathbb{F}S_n D_i) > 0$. The definition of f_i now yields $f_i = \dim(\mathbb{F}S_n D_i) > 0$. Hence, $f_i \neq 0$. Now, the definition of E_i yields $E_i = \dfrac{f_i}{n!} D_i \neq 0$ (since $f_i \neq 0$ and $D_i \neq 0$). This proves Corollary 1.27b. ■

- **page 422, proof of Lemma 1.29:** At the beginning of the proof, add the following sentence: "Again, set $H_i = H_{T_i}$ and $V_i = V_{T_i}$; thus, $D_i = H_i V_i$.".
- **page 422, proof of Lemma 1.29:** I would replace this proof with the following more detailed version:

  "Again, set $H_i = H_{T_i}$ and $V_i = V_{T_i}$; thus, $D_i = H_i V_i$. Define H_j and V_j likewise, so that $D_j = H_j V_j$.

  First, assume that $s_{ji} = vh$ for some $h \in G_H(T_i)$ and $v \in G_V(T_i)$. Then, Lemma 1.19 yields $hH_i = H_i$. Now, from $D_i = H_i V_i$, we obtain $hD_i = \underbrace{hH_i}_{=H_i} V_i = H_i V_i = D_i$. Multiplying this by $\dfrac{f_i}{n!}$, we obtain $hE_i = E_i$ (since $E_i = \dfrac{f_i}{n!} D_i$). Also, Lemma 1.19 yields $V_i v = \epsilon(v) V_i$. Now, from $D_i = H_i V_i$, we obtain $D_i v = H_i \underbrace{V_i v}_{=\epsilon(v) V_i} = \epsilon(v) \underbrace{H_i V_i}_{=D_i} = \epsilon(v) D_i$. Multiplying this by $\dfrac{f_i}{n!}$, we obtain $E_i v = \epsilon(v) E_i$ (since $E_i = \dfrac{f_i}{n!} D_i$).

  Now, Corollary 1.27a yields $E_j = s_{ji} E_i s_{ij}$, so that

$$E_i E_j = E_i \left(\underbrace{s_{ji}}_{=vh} E_i s_{ij} \right) = \underbrace{E_i v}_{=\epsilon(v) E_i} \underbrace{h E_i}_{=E_i} s_{ij} = \epsilon(v) \underbrace{E_i E_i}_{\substack{=(E_i)^2 = E_i \\ \text{(by Corollary 1.27)}}} s_{ij} = \epsilon(v) E_i s_{ij}$$
$$= \xi_{ij} E_i s_{ij} \qquad \left(\text{since } \xi_{ij} = \epsilon(v)\right).$$

  Second, assume that $s_{ji} \neq vh$ for any $h \in G_H(T_i)$ and $v \in G_V(T_i)$. Thus, $(s_{ji})^{-1} \neq (vh)^{-1}$ for any $h \in G_H(T_i)$ and $v \in G_V(T_i)$. In other words, $s_{ij} \neq h^{-1} v^{-1}$ for any $h \in G_H(T_i)$ and $v \in G_V(T_i)$ (since $(s_{ji})^{-1} = s_{ij}$ and $(vh)^{-1} = h^{-1} v^{-1}$). Equivalently, $s_{ij} \neq hv$ for any $h \in G_H(T_i)$ and $v \in G_V(T_i)$ (since $G_H(T_i)$ and $G_V(T_i)$ are subgroups of S_n and thus invariant under inversion). Hence, Lemma 1.16 (applied to $T = T_j$ and $p = s_{ij}$) shows that there exist two distinct numbers k, ℓ that lie in the same row of

T_j and in the same column of $s_{ij}T_j$. In view of $s_{ij}T_j = T_i$, this shows that k and ℓ lie in the same column of T_i; therefore, the transposition $t = (k, \ell)$ satisfies $t \in G_V(T_i)$. Hence, $V_i t = -V_i$. But k and ℓ lie in the same row of T_j; thus, $t \in G_H(T_j)$ and therefore $tH_j = H_j$. Now,

$$\underbrace{D_i}_{=H_iV_i}\underbrace{D_j}_{=H_jV_j} = H_iV_iH_jV_j = H_i\underbrace{V_it}_{=-V_i}\underbrace{tH_j}_{=H_j}V_j = -\underbrace{H_iV_i}_{=D_i}\underbrace{H_jV_j}_{=D_j} = -D_iD_j,$$

so that $D_iD_j = 0$. Since $E_i = \dfrac{f_i}{n!}D_i$ and $E_j = \dfrac{f_j}{n!}D_j$, this entails $E_iE_j = 0 = \xi_{ij}E_is_{ij}$ (since $\xi_{ij} = 0$). This completes the proof of Lemma 1.29."

- **page 423, proof of Lemma 1.32:** After "and so $\xi_{ij} = 0$", I would add "(since Corollary 1.27b yields $E_i \neq 0$, and thus $E_is_{ij} \neq 0$)".
- **page 423, proof of Lemma 1.32:** Replace "and so Lemma 1.29 gives $E_i = \xi_{ii}E_i$, hence $\xi_{ii} = 1$" by the simpler argument "and so the definition of ξ_{ii} yields $\xi_{ii} = \epsilon(\iota) = 1$".
- **page 423, proof of Proposition 1.33:** "Using Proposition 1.22" $\to$ "Using Corollary 1.27a (specifically, the $E_is_{ij} = s_{ij}E_j$ part)".
- **page 423, Corollary 1.36:** Please say that your definition of "subalgebra" does not require that the unity of the subalgebra equals the unity of the algebra! (This is far from standard.)
- **page 423:** After Corollary 1.36, I would add another corollary for later use:

  Corollary 1.36a. Let $\lambda, \mu \vdash n$ be distinct. Then, $N^\lambda N^\mu = 0$.

  Proof. It suffices to show that $E_i^\lambda s_{ij}^\lambda E_k^\mu s_{k\ell}^\mu = 0$ for any $i, j \in \{1, 2, \ldots, d_\lambda\}$ and $k, \ell \in \{1, 2, \ldots, d_\mu\}$. So let us consider such i, j and k, ℓ. Proposition 1.23 yields $D_j^\lambda D_k^\mu = 0$ (since $\lambda \neq \mu$). But the definitions of E_j^λ and E_k^μ yield $E_j^\lambda = \dfrac{f_j}{n!}D_j^\lambda$ and $E_k^\mu = \dfrac{f_k}{n!}D_k^\mu$. But Corollary 1.27a yields $E_is_{ij}^\lambda = s_{ij}^\lambda E_j^\lambda$ and therefore

  $$\underbrace{E_i^\lambda s_{ij}^\lambda}_{=s_{ij}^\lambda E_j^\lambda} E_k^\mu s_{k\ell}^\mu = s_{ij}^\lambda \underbrace{E_j^\lambda}_{=\frac{f_j}{n!}D_j^\lambda} \underbrace{E_k^\mu}_{=\frac{f_k}{n!}D_k^\mu} s_{k\ell}^\mu = \frac{f_j}{n!}\cdot\frac{f_k}{n!}s_{ij}^\lambda \underbrace{D_j^\lambda D_k^\mu}_{=0} s_{k\ell}^\mu = 0.$$

  This proves Corollary 1.36a. ■
- **page 423:** After Lemma 1.37, I would add another corollary for later use:

  Corollary 1.37a. For any partition $\lambda \vdash n$ and any two $d_\lambda \times d_\lambda$-matrices B and C, we have

  $$\alpha^\lambda(B)\,\alpha^\lambda(C) = \alpha^\lambda\left(B\mathcal{E}^\lambda C\right).$$

Proof. Let $\lambda \vdash n$, and let B and C be two $d_\lambda \times d_\lambda$-matrices B and C. Then, we can write B and C in the forms $B = \left(b_{ij}\right)$ and $C = \left(c_{ij}\right)$. Hence, $B = \left(b_{ij}\right) = \sum_{i,j} b_{ij} E_{ij}$ and $C = \left(c_{ij}\right) = \left(c_{k\ell}\right) = \sum_{k,\ell} c_{k\ell} E_{k\ell}$. Hence,

$$\begin{aligned}
&\alpha^\lambda \Bigg(\underbrace{B}_{=\sum_{i,j} b_{ij} E_{ij}} \Bigg) \alpha^\lambda \Bigg(\underbrace{C}_{=\sum_{k,\ell} c_{k\ell} E_{k\ell}} \Bigg) \\
&= \alpha^\lambda \left(\sum_{i,j} b_{ij} E_{ij} \right) \alpha^\lambda \left(\sum_{k,\ell} c_{k\ell} E_{k\ell} \right) = \sum_{i,j} b_{ij} \sum_{k,\ell} c_{k\ell} \underbrace{\alpha^\lambda \left(E_{ij}\right) \alpha^\lambda \left(E_{k\ell}\right)}_{\substack{=\alpha^\lambda\left(E_{ij}\mathcal{E}^\lambda E_{k\ell}\right) \\ \text{(by Lemma 1.37)}}} \\
&= \sum_{i,j} b_{ij} \sum_{k,\ell} c_{k\ell} \alpha^\lambda \left(E_{ij} \mathcal{E}^\lambda E_{k\ell}\right) = \alpha^\lambda \Bigg(\underbrace{\left(\sum_{i,j} b_{ij} E_{ij} \right)}_{=B} \mathcal{E}^\lambda \underbrace{\left(\sum_{k,\ell} c_{k\ell} E_{k\ell} \right)}_{=C} \Bigg) \\
&= \alpha^\lambda \left(B \mathcal{E}^\lambda C \right).
\end{aligned}$$

This proves Corollary 1.37a. ■

- **page 423, proof of Proposition 1.38:** After "$\sum_{\mu \vdash n} \alpha^\mu \left(A^\mu\right) = 0$", add "for some matrices $A^\mu = \left(a_{ij}^\mu\right)_{i,j=1,2,\ldots,d_\mu}$".

- **pages 423–424, proof of Proposition 1.38:** I think this whole proof would become clearer if rewritten as follows:

  "Assume that $\sum_{\mu \vdash n} \sum_{i,j=1}^{d_\mu} a_{ij}^\mu E_i^\mu s_{ij}^\mu = 0$ for some family of scalars $a_{ij}^\mu \in \mathbb{F}$. We shall show that $a_{ij}^\mu = 0$ for all μ and i, j.

  Fix a partition λ. Let A be the $d_\lambda \times d_\lambda$-matrix $\left(a_{ij}^\lambda\right)_{i,j=1,2,\ldots,d_\lambda}$. We shall show that $A = 0$.

Fix $u, v \in \{1, 2, \ldots, d_\lambda\}$. We have $E_u^\lambda \underbrace{\sum_{\mu \vdash n} \sum_{i,j=1}^{d_\mu} a_{ij}^\mu E_i^\mu s_{ij}^\mu}_{=0} = 0$, so that

$$0 = E_u^\lambda \sum_{\mu \vdash n} \sum_{i,j=1}^{d_\mu} a_{ij}^\mu E_i^\mu s_{ij}^\mu = \sum_{\mu \vdash n} \sum_{i,j=1}^{d_\mu} a_{ij}^\mu E_u^\lambda E_i^\mu s_{ij}^\mu = \sum_{i,j=1}^{d_\lambda} a_{ij}^\lambda E_u^\lambda E_i^\lambda s_{ij}^\lambda$$

$$\left(\begin{array}{c} \text{since Proposition 1.23 yields that } E_u^\lambda E_i^\mu = 0 \text{ whenever } \lambda \neq \mu, \\ \text{and this entails that all addends } a_{ij}^\mu E_u^\lambda E_i^\mu s_{ij}^\mu \text{ with } \lambda \neq \mu \text{ vanish} \end{array} \right)$$

$$= \sum_{i,j=1}^{d_\lambda} a_{ij} \underbrace{E_u E_i}_{\substack{= \zeta_{ui} E_u s_{ui} \\ \text{(by Lemma 1.29)}}} s_{ij}$$

(from now on, we are omitting the superscripts λ)

$$= \sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} E_u \underbrace{s_{ui} s_{ij}}_{= s_{uj}} = \sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} E_u s_{uj}.$$

Multiplying both sides of this equality with E_v on the right, we obtain

$$0 = \left(\sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} E_u s_{uj} \right) E_v = \sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} E_u s_{uj} \underbrace{E_v}_{\substack{= E_v s_{vv} \\ \text{(since } s_{vv} = \iota)}}$$

$$= \sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} \underbrace{(E_u s_{uj})(E_v s_{vv})}_{\substack{= \zeta_{jv} E_u s_{uv} \\ \text{(by Proposition 1.33)}}} = \sum_{i,j=1}^{d_\lambda} a_{ij} \zeta_{ui} \zeta_{jv} E_u s_{uv} = \left(\sum_{i,j=1}^{d_\lambda} \zeta_{ui} a_{ij} \zeta_{jv} \right) E_u s_{uv}.$$

Since $s_{uv} \in S_n$ is invertible, we can cancel s_{uv} from this equality and obtain

$$0 = \left(\sum_{i,j=1}^{d_\lambda} \zeta_{ui} a_{ij} \zeta_{jv} \right) E_u.$$

Since $E_u \neq 0$ (by Corollary 1.27b), we thus obtain

$$\sum_{i,j=1}^{d_\lambda} \zeta_{ui} a_{ij} \zeta_{jv} = 0$$

(since $\sum\limits_{i,j=1}^{d_\lambda} \zeta_{ui} a_{ij} \zeta_{jv}$ is a scalar).

But $A = \left(a_{ij}^\lambda \right)_{i,j=1,2,\ldots,d_\lambda} = \left(a_{ij} \right)_{i,j=1,2,\ldots,d_\lambda}$ (since we are omitting the superscript λ) and $\mathcal{E}^\lambda = \left(\zeta_{ij} \right)_{i,j=1,2,\ldots,d_\lambda}$ (by the definition of $\mathcal{E}^\lambda$). Hence,

$\sum\limits_{i,j=1}^{d_\lambda} \xi_{ui} a_{ij} \xi_{jv}$ is the (u,v)-th entry of the matrix $\mathcal{E}^\lambda A \mathcal{E}^\lambda$. Thus, we have showed that the (u,v)-th entry of the matrix $\mathcal{E}^\lambda A \mathcal{E}^\lambda$ is 0 (since we have showed that $\sum\limits_{i,j=1}^{d_\lambda} \xi_{ui} a_{ij} \xi_{jv} = 0$).

Forget that we fixed u, v. We thus have proved that the (u,v)-th entry of the matrix $\mathcal{E}^\lambda A \mathcal{E}^\lambda$ is 0 for each $u, v \in \{1, 2, \ldots, d_\lambda\}$. In other words, all entries of the matrix $\mathcal{E}^\lambda A \mathcal{E}^\lambda$ are 0. In other words, $\mathcal{E}^\lambda A \mathcal{E}^\lambda = 0$. Since $\mathcal{E}^\lambda$ is invertible (by Lemma 1.32), we thus obtain $A = 0$. Thus, all entries of A are 0. In other words, $a_{ij}^\lambda = 0$ for any $i, j \in \{1, 2, \ldots, d_\lambda\}$ (since the entries of A are a_{ij}^λ). Since we have proved this for any $\lambda \vdash n$, we thus conclude that all our scalars a_{ij}^μ are 0. This proves Proposition 1.38. ■"

- **page 424, proof of Corollary 1.40:** "by Proposition 1.23" → "by Proposition 1.38".
- **page 424, Remark at the end of §1.6:** You give the reference [35, §5.1.4, Theorem A]. Here are a few alternative references for proofs of the equality $\sum\limits_{\lambda} d_\lambda^2 = n!$:
  - Proposition 1.3.3 in Marc A. A. van Leeuwen, *The Robinson-Schensted and Schützenberger algorithms, an elementary approach*, version 25 Nov 2011.
    `http://www-math.univ-poitiers.fr/~maavl/`
  - Corollary 8.5 in Richard P. Stanley, *Algebraic Combinatorics: Walks, Trees, Tableaux, and More*, Undergraduate Texts in Mathematics, Springer 2013.
    `http://www-math.mit.edu/~rstan/algcomb/index.html`

    (This book also has a second edition; the equality still is Corollary 8.5 in it.)
  - Theorem 2.6.5 part 3. in Bruce E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd edition, Springer 2001.
- **page 424, §1.7:** "We prove that the map ψ in (1) is" → "We shall now construct the map ψ in (1), and prove that it is".
- **page 425, Proposition 1.43:** It is worth saying that the expression "$\delta_{\lambda\mu}\delta_{jk}U_{i\ell}^\lambda$" is understood to be 0 if $\lambda = \mu$ (even if $U_{i\ell}^\lambda$ is undefined in this case).
- **page 425, proof of Proposition 1.43:** After "If $\lambda = \mu$ then", add "the definitions of U_{ij} and $U_{k\ell}$ and Lemma 1.37a yield (with the notation $\mathcal{E}$ being used for $\mathcal{E}^\lambda$)".

- **page 425, proof of Proposition 1.43:** Replace "orthogonality of Proposition 1.23" by "orthogonality of Corollary 1.36a".

- **page 425, proof of Theorem 1.45:** Here is a more detailed version of this proof:

  "Proposition 1.43 shows that the map ψ is an $\mathbb{F}$-algebra homomorphism. It remains to prove that ψ is bijective. The Remark at the end of §1.6 shows that $\sum\limits_{\lambda} d_\lambda^2 = n!$; in other words, $\dim M = \dim(\mathbb{F}S_n)$. Hence, ψ is an $\mathbb{F}$-linear map between two $\mathbb{F}$-vector spaces of the same (finite) dimension. Thus, if ψ is injective, then ψ is bijective. Therefore, it will suffice to show that ψ is injective.

  Recall that $M = \bigoplus\limits_{i=1}^{r} M_{d_i}(\mathbb{F})$. Let $\gamma : M \to M$ be the $\mathbb{F}$-linear map that sends each $(A_1, A_2, \ldots, A_r) \in M$ to $\left(A_1 \left(\mathcal{E}^{\lambda_1}\right)^{-1}, A_2 \left(\mathcal{E}^{\lambda_2}\right)^{-1}, \ldots, A_r \left(\mathcal{E}^{\lambda_r}\right)^{-1}\right) \in M$. This map γ is well-defined (since the r matrices $\mathcal{E}^{\lambda_1}, \mathcal{E}^{\lambda_2}, \ldots, \mathcal{E}^{\lambda_r}$ are all invertible) and injective (since the r matrices $\left(\mathcal{E}^{\lambda_1}\right)^{-1}, \left(\mathcal{E}^{\lambda_2}\right)^{-1}, \ldots, \left(\mathcal{E}^{\lambda_r}\right)^{-1}$ are all invertible). Moreover, it is easy to see that $\psi = \alpha \circ \gamma$. (Indeed, by linearity, it suffices to show that $\psi\left(E_{ij}^\lambda\right) = (\alpha \circ \gamma)\left(E_{ij}^\lambda\right)$ for all $\lambda \vdash n$ and $i, j \in \{1, 2, \ldots, d_\lambda\}$. But this is easy to check, since the definition of ψ yields

  $$\psi\left(E_{ij}^\lambda\right) = U_{ij}^\lambda = \alpha^\lambda\left(E_{ij}^\lambda \left(\mathcal{E}^\lambda\right)^{-1}\right) = \alpha \underbrace{\left(0, 0, \ldots, 0, E_{ij}^\lambda \left(\mathcal{E}^\lambda\right)^{-1}, 0, 0, \ldots, 0\right)}_{=\gamma\left(0,0,\ldots,0,E_{ij}^\lambda,0,0,\ldots,0\right)=\gamma\left(E_{ij}^\lambda\right)}$$
  $$= \alpha\left(\gamma\left(E_{ij}^\lambda\right)\right) = (\alpha \circ \gamma)\left(E_{ij}^\lambda\right).$$

  Thus, $\psi = \alpha \circ \gamma$ is proven.)

  Now, the maps α and γ are both injective (indeed, the map α is injective by Corollary 1.40). Hence, their composition $\alpha \circ \gamma$ is injective. In other words, the map ψ is injective (since $\psi = \alpha \circ \gamma$). As we have seen above, this completes the proof of Theorem 1.45. ■

- **page 427:** After "Our next goal is to compute explicitly the algebra homomorphism ϕ", add "inverse to ψ".

- **page 427:** Replace "Proposition 1.22 and Lemma 1.29" by "Lemma 1.29 and Corollary 1.27a (specifically, the $E_i s_{ij} = s_{ij} E_j$ part of it)".

- **page 428:** I'd replace "The Wedderburn decomposition of $\mathbb{F}S_n$ shows that" by "The surjectivity of ψ in Theorem 1.45 shows that" (this is more concrete).

- **page 428, Definition 1.49:** This definition tacitly uses the fact that the $r_{ij}^{\lambda}(p)$ are uniquely determined by p, λ, i and j. This follows from the fact that the family $\left(U_{ij}^{\lambda}\right)$ (with λ ranging over all partitions of n and with i, j ranging over $\{1, 2, \ldots, d_\lambda\}$ each) is a basis of the $\mathbb{F}$-vector space $\mathbb{F}S_n$. (And this fact follows from Theorem 1.45, since the family $\left(E_{ij}^{\lambda}\right)$ forms a basis of M and is sent to the family $\left(U_{ij}^{\lambda}\right)$ by the map ψ.)

- **page 428, Lemma 1.50:** It is worth saying that Lemma 1.50 is a consequence of Proposition 1.43.

- **page 429, proof of Proposition 1.51:** Replace both "α"s in this proof by "α^λ". (It is dangerous to omit the superscript on an α, since α already has a different meaning given to it in Definition 1.39.)

- **page 429, proof of Proposition 1.51:** Remove the "write $\mathcal{E} = A_\iota^\lambda$ and" part of the first sentence of the proof. Instead, at the beginning of the proof, I'd add "The matrix A_ι^λ is the matrix $\mathcal{E}^\lambda$ from Definition 1.31, and thus is invertible (by Lemma 1.32). We shall omit the superscripts λ, so we write A_p for A_p^λ, and we write $\mathcal{E}$ for $\mathcal{E}^\lambda = A_\iota^\lambda = A_\iota$".

- **page 429, proof of Proposition 1.51:** Replace "We have" by "Thus, $E_{ii}\mathcal{E}^{-1}$ is the $d_\lambda \times d_\lambda$-matrix whose i-th row has entries $\eta_{i1}, \eta_{i2}, \ldots, \eta_{id_\lambda}$ while all other rows are 0. Therefore, the definition of α^λ yields

$$\alpha^\lambda \left(E_{ii}\mathcal{E}^{-1}\right) = \sum_{k=1}^{d_\lambda} \eta_{ik} E_i s_{ik}.$$

Similarly,

$$\alpha^\lambda \left(E_{jj}\mathcal{E}^{-1}\right) = \sum_{\ell=1}^{d_\lambda} \eta_{j\ell} E_j s_{j\ell}.$$

Hence,".

- **page 429, proof of Proposition 1.51:** The second-to-last equality sign in the long (displayed) computation relies on the equality

$$\sum_{\ell=1}^{d_\lambda} \eta_{j\ell} E_i s_{i\ell} = U_{ij},$$

which is not completely obvious. Here is how it can be proved: The matrix $\left(\mathcal{E}^\lambda\right)^{-1} = \mathcal{E}^{-1}$ has entries η_{ij}. Thus, $E_{ij}^{\lambda}\left(\mathcal{E}^\lambda\right)^{-1}$ is the $d_\lambda \times d_\lambda$-matrix whose i-th row has entries $\eta_{j1}, \eta_{j2}, \ldots, \eta_{jd_\lambda}$ while all other rows are 0. Therefore, the definition of α^λ yields

$$\alpha^\lambda\left(E_{ij}^\lambda\left(\mathcal{E}^\lambda\right)^{-1}\right)=\sum_{\ell=1}^{d_\lambda}\eta_{j\ell}E_i s_{i\ell}.$$

Now, the definition of U_{ij} yields

$$U_{ij}=\alpha^\lambda\left(E_{ij}^\lambda\left(\mathcal{E}^\lambda\right)^{-1}\right)=\sum_{\ell=1}^{d_\lambda}\eta_{j\ell}E_i s_{i\ell}.$$

Thus, $\sum\limits_{\ell=1}^{d_\lambda}\eta_{j\ell}E_i s_{i\ell}=U_{ij}$ is proven.

- **page 429, proof of Proposition 1.51:** The last equality sign in the long (displayed) computation relies on the equality

$$\sum_{k=1}^{d_\lambda}\eta_{ik}\xi_{kj}^p=\left(A_\iota^{-1}A_p\right)_{ij},$$

which is not completely obvious. Here is how it can be proved: We have $A_\iota=\mathcal{E}$, so that $A_\iota^{-1}=\mathcal{E}^{-1}$. Thus, the entries of the matrix A_ι^{-1} are the entries of the matrix $\mathcal{E}^{-1}$, which are the scalars η_{ij} (by the definition of η_{ij}). On the other hand, the entries of the matrix $A_p=A_p^\lambda$ are ξ_{ij}^p (by the definition of A_p^λ). Hence, the (i,j)-th entry of the matrix $A_\iota^{-1}A_p$ is $\sum\limits_{k=1}^{d_\lambda}\eta_{ik}\xi_{kj}^p$ (by the definition of the product of two matrices). In other words, $\left(A_\iota^{-1}A_p\right)_{ij}=\sum\limits_{k=1}^{d_\lambda}\eta_{ik}\xi_{kj}^p$. Thus, $\sum\limits_{k=1}^{d_\lambda}\eta_{ik}\xi_{kj}^p=\left(A_\iota^{-1}A_p\right)_{ij}$ is proven.

- **page 429, proof of Proposition 1.51:** "Therefore $r_{ij}^\lambda(p)$" $\to$ "Therefore, by Lemma 1.50 (and because $U_{ij}\neq 0$), we obtain $r_{ij}^\lambda(p)$".

- **page 429:** It is worth explaining why exactly Proposition 1.51 provides an explicit way of computing the homomorphism ϕ in (11). Indeed, each

$p \in S_n$ satisfies

$$\psi\left(R^{\lambda_1}(p), R^{\lambda_2}(p), \ldots, R^{\lambda_r}(p)\right)$$
$$= \sum_{\lambda \vdash n} \psi\left(\underbrace{R^{\lambda}(p)}_{\substack{=\sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\lambda} r_{ij}^{\lambda}(p) E_{ij}^{\lambda} \\ \text{(by the definition of } R^{\lambda}(p))}}\right) = \sum_{\lambda \vdash n} \underbrace{\psi\left(\sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\lambda} r_{ij}^{\lambda}(p) E_{ij}^{\lambda}\right)}_{\substack{=\sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\lambda} r_{ij}^{\lambda}(p) U_{ij}^{\lambda} \\ \text{(by the definition of } \psi)}}$$
$$= \sum_{\lambda \vdash n} \sum_{i=1}^{d_\lambda}\sum_{j=1}^{d_\lambda} r_{ij}^{\lambda}(p) U_{ij}^{\lambda} = p \qquad \text{(by (13))}$$

and therefore $\left(R^{\lambda_1}(p), R^{\lambda_2}(p), \ldots, R^{\lambda_r}(p)\right) = \psi^{-1}(p) = \phi(p)$ (since $\psi^{-1} = \phi$). Hence, by computing the matrices $R^{\lambda}(p)$ for all $\lambda \vdash n$, we can obtain an explicit formula for $\phi(p)$.