A note on bilinear forms

Darij Grinberg

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1. Introduction

In this note, we shall prove some elementary linear-algebraic properties of bilinear forms. These properties generalize some of the standard facts about non-degenerate bilinear forms on finite-dimensional vector spaces (e.g., the fact that \((V^\perp)^\perp = V\) for a vector subspace \(V\) of a vector space \(A\) equipped with a non-degenerate symmetric bilinear form) to bilinear forms which may be degenerate. They are unlikely to be new, but I have not found them explored anywhere, whence this note.

2. Bilinear maps and forms

We fix a field \(k\). This field will be fixed for the rest of this note. The notion of a “vector space” will always be understood to mean “\(k\)-vector space”. The
word “subspace” will always mean “k-vector subspace”. The word “linear” will always mean “k-linear”. If V and W are two k-vector spaces, then Hom (V, W) denotes the vector space of all linear maps from V to W. If S is a subset of a vector space V, then span S will mean the span of S (that is, the subspace of V spanned by S).

We recall the definition of a bilinear map:

**Definition 2.1.** Let V, W and U be three vector spaces. Let f : V × W → U be any map.
(a) We say that the map f is linear in its first argument if, for every w ∈ W, the map V → U, v ↦→ f (v, w) is linear.
(b) We say that the map f is linear in its second argument if, for every v ∈ V, the map W → U, w ↦→ f (v, w) is linear.
(c) We say that the map f is bilinear if it is both linear in its first argument and linear in its second argument.

More explicitly, this definition can be rewritten as follows: A map f : V × W → U is linear in its first argument if and only if every λ₁, λ₂ ∈ k and v₁, v₂ ∈ V and w ∈ W satisfy

\[ f (λ₁v₁ + λ₂v₂, w) = λ₁f (v₁, w) + λ₂f (v₂, w). \]  

(1)

A map f : V × W → U is linear in its second argument if and only if every λ₁, λ₂ ∈ k and v ∈ V and w₁, w₂ ∈ W satisfy

\[ f (v, λ₁w₁ + λ₂w₂) = λ₁f (v, w₁) + λ₂f (v, w₂). \]  

(2)

A map f : V × W → U is bilinear if and only if it satisfies both (1) and (2).

Bilinear maps are also known as “k-bilinear maps” (actually, this latter notion is used when k is not clear from the context).

Here are some examples:

**Example 2.2.** Let V, W and U be three vector spaces. Let 0 : V × W → U be the map which sends every p ∈ V × W to 0. Then, the map 0 is bilinear.

**Example 2.3.** Consider kⁿ as the vector space of all row vectors of size n (with entries in k).

- The map kⁿ × kⁿ → k sending every ((x₁, x₂, …, xₙ), (y₁, y₂, …, yₙ)) ∈ kⁿ × kⁿ to x₁y₁ + x₂y₂ + ⋯ + xₙyₙ ∈ k is a bilinear map. (This map is called the dot product map, and the image of a pair (v, w) ∈ kⁿ × kⁿ under this map is called the dot product of v and w.)

- The map kⁿ × kⁿ → kⁿ sending every ((x₁, x₂, …, xₙ), (y₁, y₂, …, yₙ)) ∈ kⁿ × kⁿ to (x₁y₁, x₂y₂, …, xₙyₙ) ∈ kⁿ is a bilinear map.
• The map $k^n \times k^n \to k^n \times k^n$ (where $k^n \times k^n$ denotes the ring of $n \times n$-matrices over $k$) sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in k^n \times k^n$ to the matrix
\[
\begin{pmatrix}
x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\
x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\
\vdots & \vdots & \ddots & \vdots \\
x_n y_1 & x_n y_2 & \cdots & x_n y_n \\
\end{pmatrix}
\in k^{n \times n}
\]
is a bilinear map.

• The map $k^n \times k^n \to k$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in k^n \times k^n$ to $x_1 + y_1 + x_2 + y_2 + \cdots + x_n + y_n \in k$ is not bilinear (unless $n = 0$). It is neither linear in its first argument nor linear in its second argument.

• The map $k^n \times k^n \to k$ sending every $((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in k^n \times k^n$ to $(x_1 + x_2 + \cdots + x_n) y_1 y_2 \cdots y_n \in k$ is linear in its first argument, but not bilinear (unless $n \leq 1$).

**Example 2.4.** Let $V$ and $W$ be two vector spaces. The map $\text{Hom} (V, W) \times V \to W$, $(f, v) \mapsto f (v)$ is bilinear.

Let us see two ways to construct new bilinear maps out of old:

**Definition 2.5.** Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be any map. Then, $f^{\text{op}}$ will be denote the map
\[
W \times V \to U, \quad (w, v) \mapsto f (v, w).
\]

Thus, $f^{\text{op}} (w, v) = f (v, w)$ for every $(w, v) \in W \times V$.

The following proposition is obvious:

**Proposition 2.6.** Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be any map.

(a) We have $(f^{\text{op}})^{\text{op}} = f$.

(b) Assume that the sets $V$, $W$ and $U$ are vector spaces. Assume that the map $f$ is bilinear. Then, the map $f^{\text{op}} : W \times V \to U$ is bilinear.

The following proposition is equally trivial:

**Proposition 2.7.** Let $V$, $W$ and $U$ be three vector spaces. Let $A$ be a subspace of $V$, and let $B$ be a subspace of $W$. Let $f : V \times W \to U$ be a bilinear map. Then, the restriction $f |_{A \times B}$ is a bilinear map $A \times B \to U$.

Let us introduce some more terminology, for convenience:
Definition 2.8. Let $V$, $W$ and $U$ be three sets. Let $f : V \times W \to U$ be a bilinear map.

The map $f : V \times W \to U$ is bilinear. In other words, the map $f$ is both linear in its first argument and linear in its second argument (by the definition of “bilinear”).

(a) The map $f$ is linear in its first argument. In other words, for every $w \in W$, the map $V \to U, v \mapsto f(v, w)$ is linear (by the definition of “linear in its first argument”). This latter map will be denoted by $f(-, w)$. This map $f(-, w)$ is thus a linear map $V \to U$; in other words, $f(-, w) \in \text{Hom}(V, U)$.

(b) The map $f$ is linear in its second argument. In other words, for every $v \in W$, the map $W \to U, w \mapsto f(v, w)$ is linear (by the definition of “linear in its second argument”). This latter map will be denoted by $f(v, -)$. This map $f(v, -)$ is thus a linear map $W \to U$; in other words, $f(v, -) \in \text{Hom}(W, U)$.

The notations $f(v, -)$ and $f(-, w)$ introduced in Definition 2.8 should not be confused with the notation $f(v, w)$ for the image of a pair $(v, w) \in V \times W$ under $f$. (Fortunately, they can easily be distinguished from the notation $f(v, w)$, because a dash cannot be mistaken for an element of $V$ or an element of $W$.) Of course, the dash in the notations $f(v, -)$ and $f(-, w)$ indicates “insert element here”.

“Bilinear form” is just an alternative terminology for a bilinear map whose target is $k$:

Definition 2.9. Let $V$ and $W$ be two vector spaces. A bilinear form means a bilinear map $V \times W \to k$.

Using the notion of a bilinear form, we can restate Proposition 2.6 (b) and Proposition 2.7 in the particular case of $U = k$ as follows:

Proposition 2.10. Let $V$ and $W$ be two vector spaces. Let $f : V \times W \to k$ be a bilinear form.

(a) The map $f^\text{op} : W \times V \to k$ is a bilinear form.

(b) Let $A$ be a subspace of $V$, and let $B$ be a subspace of $W$. Then, the restriction $f|_{A \times B}$ is a bilinear form $A \times B \to k$.

Proof of Proposition 2.10 Proposition 2.10 (a) follows immediately from Proposition 2.6 (b) (applied to $U = k$). Proposition 2.10 (b) follows immediately from Proposition 2.7 (applied to $U = k$).

3. Left and right orthogonal spaces

We shall now introduce the notions of left and right orthogonal spaces with respect to a bilinear map:
Definition 3.1. Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

(a) If $B$ is a subset of $W$, then $\mathcal{L}_f (B)$ denotes the subset $\{ v \in V \mid f (v, b) = 0 \text{ for all } b \in B \}$ of $V$. We call $\mathcal{L}_f (B)$ the left orthogonal space of $B$.

(b) If $A$ is a subset of $V$, then $\mathcal{R}_f (A)$ denotes the subset $\{ w \in W \mid f (a, w) = 0 \text{ for all } a \in A \}$ of $W$. We call $\mathcal{R}_f (A)$ the left orthogonal space of $B$.

We will mostly use the notations $\mathcal{L}_f (B)$ and $\mathcal{R}_f (A)$ in the case when $f$ is a bilinear map. However, we have chosen to introduce them in full generality, since others occasionally use them in more general settings (e.g., for sesquilinear maps).

We shall first prove some simple facts:

Proposition 3.2. Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

(a) If $B$ is a subset of $W$, then $\mathcal{L}_f (B) = \mathcal{R}_{f^\text{op}} (B)$.

(b) If $A$ is a subset of $V$, then $\mathcal{R}_f (A) = \mathcal{L}_{f^\text{op}} (A)$.

Notice that $A$ and $B$ are not required to be subspaces, only subsets, in Proposition 3.2; furthermore, the map $f$ is not required to be bilinear.

Proof of Proposition 3.2. (a) Let $A$ be a subset of $V$. Then, the definition of $\mathcal{L}_{f^\text{op}} (A)$ yields

$\mathcal{L}_{f^\text{op}} (A) = \left\{ v \in W \mid f^\text{op} (v, b) = 0 \text{ for all } b \in A \right\}$

(by the definition of $f^\text{op}$)

$= \{ v \in W \mid f (b, v) = 0 \text{ for all } b \in A \}$

(here, we have renamed the index $b$ as $a$)

$= \{ w \in W \mid f (a, w) = 0 \text{ for all } a \in A \}$

(here, we have renamed the index $v$ as $w$)

This proves Proposition 3.2 (a).

(c) Let $B$ be a subset of $W$. Proposition 2.6 (a) yields $(f^\text{op})^\text{op} = f$. But Proposition 3.2 (b) (applied to $W, V, f^\text{op}$ and $B$ instead of $V, W, f$ and $A$) shows that $\mathcal{R}_{f^\text{op}} (B) = \mathcal{L}_{\left(f^\text{op}\right)^\text{op}} (B)$. Since $(f^\text{op})^\text{op} = f$, this rewrites as $\mathcal{R}_{f^\text{op}} (B) = \mathcal{L}_f (B)$. This proves Proposition 3.2 (a).
Proposition 3.3. Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a bilinear map.

(a) If $B$ is a subset of $W$, then $\mathcal{L}_f (B)$ is a subspace of $V$.

(b) If $A$ is a subset of $V$, then $\mathcal{R}_f (A)$ is a subspace of $W$.

Notice that $A$ and $B$ are not required to be subspaces, only subsets, in Proposition 3.3.

Proof of Proposition 3.3. (a) Let $B$ be a subset of $W$.

Recall that a linear map $f (\cdot , w) \in \text{Hom} (V, U)$ is defined for every $w \in W$ (according to Definition 2.8 (a)). Thus, $\text{Ker} (f (\cdot , w))$ is a subspace of $V$ for every $w \in W$ (because the kernel of a linear map is always a subspace of its domain). Consequently, $\bigcap_{w \in B} \text{Ker} (f (\cdot , w))$ is an intersection of subspaces of $V$, and therefore itself a subspace of $V$.

We claim that

$$\mathcal{L}_f (B) = \bigcap_{w \in B} \text{Ker} (f (\cdot , w)) \quad (3)$$

Proof of (3): Let $x \in \mathcal{L}_f (B)$. Thus,

$$x \in \mathcal{L}_f (B) = \{ v \in V \mid f (v, b) = 0 \text{ for all } b \in B \}$$

(by the definition of $\mathcal{L}_f (B)$). In other words, $x$ is an element of $V$ and satisfies

$$f (x, b) = 0 \quad \text{for all } b \in B. \quad (4)$$

Now, we have $x \in \text{Ker} (f (\cdot , w))$ for every $w \in B$. In other words, $x \in \bigcap_{w \in B} \text{Ker} (f (\cdot , w))$.

Let us now forget that we fixed $x$. We thus have shown that $x \in \bigcap_{w \in B} \text{Ker} (f (\cdot , w))$ for every $x \in \mathcal{L}_f (B)$. In other words,

$$\text{Ker} (f (\cdot , w)) \subseteq \bigcap_{w \in B} \text{Ker} (f (\cdot , w)). \quad (5)$$

On the other hand, let $y \in \bigcap_{w \in B} \text{Ker} (f (\cdot , w))$. Thus, $y \in \bigcap_{w \in B} \text{Ker} (f (\cdot , w)) \subseteq V$. Moreover, we have $f (y, b) = 0$ for all $b \in B$. Thus, $y$ is an element of $V$ and satisfies $f (y, b) = 0$ for all $b \in B$. In other words, $y \in$}

1. Here, the intersection $\bigcap_{w \in B} \text{Ker} (f (\cdot , w))$ is taken in the ambient set $V$. Thus, if $B = \emptyset$, then $\bigcap_{w \in B} \text{Ker} (f (\cdot , w))$ is understood to be $V$.

2. Proof. Let $w \in B$. Applying (4) to $b = w$, we obtain $f (x, w) = 0$.

But the definition of $f (\cdot , w)$ yields $(f (\cdot , w))(x) = f (x, w) = 0$. In other words, $x \in \text{Ker} (f (\cdot , w))$, qed.

3. Proof. Let $b \in B$. We have $y \in \bigcap_{w \in B} \text{Ker} (f (\cdot , w)) \subseteq \text{Ker} (f (\cdot , b))$ (since $b \in B$). Thus, $(f (\cdot , b))(y) = 0$. But the definition of $f (\cdot , b)$ yields $(f (\cdot , b))(y) = f (y, b)$. Hence, $f (y, b) = (f (\cdot , b))(y) = 0$, qed.
{v \in V \mid f(v, b) = 0 \text{ for all } b \in B}. Since \( \mathcal{L}_f(B) = \{v \in V \mid f(v, b) = 0 \text{ for all } b \in B\} \), this rewrites as \( y \in \mathcal{L}_f(B) \).

Let us now forget that we fixed \( y \). We thus have shown that \( y \in \mathcal{L}_f(B) \) for every \( y \in \bigcap_{w \in B} \text{Ker } (f(-, w)) \). In other words,

\[
\mathcal{L}_f(B) \subseteq \bigcap_{w \in B} \text{Ker } (f(-, w)).
\]

Combining this with (5), we obtain \( \mathcal{L}_f(B) = \bigcap_{w \in B} \text{Ker } (f(-, w)) \). This proves (3).

Now, recall that the subset \( \bigcap_{w \in B} \text{Ker } (f(-, w)) \) of \( V \) is a subspace of \( V \). In view of (3), this rewrites as follows: The subset \( \mathcal{L}_f(B) \) of \( V \) is a subspace of \( V \). This proves Proposition 3.3 (a).

(b) Let \( A \) be a subset of \( V \). Proposition 3.2 (b) yields \( \mathcal{R}_f(A) = \mathcal{L}_{\text{op}}(A) \). But the map \( f_{\text{op}} \) is bilinear (by Proposition 2.6 (b)). Hence, Proposition 3.3 (a) (applied to \( W, V, f_{\text{op}} \) and \( A \) instead of \( V, W, f \) and \( B \)) shows that \( \mathcal{L}_{\text{op}}(A) \) is a subspace of \( W \). Since \( \mathcal{R}_f(A) = \mathcal{L}_{\text{op}}(A) \), this yields that \( \mathcal{R}_f(A) \) is a subspace of \( W \). This proves Proposition 3.3 (b).

Of course, Proposition 3.3 (b) is an analogue of Proposition 3.3 (a), and could be proven in the same way as we proved Proposition 3.3 (a). However, we have chosen to derive it from Proposition 3.3 (a) instead (using Proposition 3.2 (b)), since this way is shorter.

**Proposition 3.4.** Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a map.

(a) If \( B \) and \( B' \) are two subsets of \( W \) satisfying \( B \subseteq B' \), then \( \mathcal{L}_f(B) \supseteq \mathcal{L}_f(B') \).

(b) If \( A \) and \( A' \) are two subsets of \( V \) satisfying \( A \subseteq A' \), then \( \mathcal{R}_f(A) \supseteq \mathcal{R}_f(A') \).

**Proof of Proposition 3.4** (a) Let \( B \) and \( B' \) be two subsets of \( W \) satisfying \( B \subseteq B' \). The definition of \( \mathcal{L}_f(B) \) yields

\[
\mathcal{L}_f(B) = \{v \in V \mid f(v, b) = 0 \text{ for all } b \in B\}. \tag{6}
\]

The definition of \( \mathcal{L}_f(B') \) yields

\[
\mathcal{L}_f(B') = \{v \in V \mid f(v, b) = 0 \text{ for all } b \in B'\}. \tag{7}
\]

Now, let \( x \in \mathcal{L}_f(B') \). Thus, \( x \in \mathcal{L}_f(B') = \{v \in V \mid f(v, b) = 0 \text{ for all } b \in B'\} \). In other words, \( x \) is an element \( v \) of \( V \) satisfying \( f(v, b) = 0 \text{ for all } b \in B' \). In other words, \( x \) is an element of \( V \) and satisfies

\[
f(x, b) = 0 \quad \text{for all } b \in B'. \tag{7}
\]

Therefore, \( f(x, b) = 0 \text{ for all } b \in B \) \( ^4 \). Hence, \( x \) is an element of \( V \) and satisfies \( f(x, b) = 0 \text{ for all } b \in B \). In other words, \( x \) is an element \( v \) of \( V \) satisfying

\[
^4\text{Proof. Let } b \in B. \text{ Then, } b \in B \subseteq B'. \text{ Hence, } f(x, b) = 0 \text{ (by (7)). Qed.}
\]
In other words, \( x \in \{ v \in V \mid f(v, b) = 0 \text{ for all } b \in B \} \). In light of (6), this rewrites as \( x \in \mathcal{L}_f(B) \).

Now, let us forget that we fixed \( x \). We thus have proven that \( x \in \mathcal{L}_f(B) \) for every \( x \in \mathcal{L}_f(B') \). In other words, \( \mathcal{L}_f(B) \supseteq \mathcal{L}_f(B') \). This proves Proposition 3.4 (a).

(b) Let \( A \) and \( A' \) be two subsets of \( V \) satisfying \( A \subseteq A' \). Proposition 3.2 (b) yields \( \mathcal{R}_f(A) = \mathcal{L}_{f^\text{op}}(A) \). Proposition 3.2 (b) (applied to \( A' \) instead of \( A \)) yields \( \mathcal{R}_f(A') = \mathcal{L}_{f^\text{op}}(A') \). But Proposition 3.4 (a) (applied to \( W, V, f^\text{op}, A \) and \( A' \) instead of \( V, W, f, B \) and \( B' \)) shows that \( \mathcal{L}_{f^\text{op}}(A) \supseteq \mathcal{L}_{f^\text{op}}(A') \). Thus, \( \mathcal{R}_f(A) = \mathcal{L}_{f^\text{op}}(A) \supseteq \mathcal{L}_{f^\text{op}}(A') = \mathcal{R}_f(A') \). This proves Proposition 3.4 (b).

Proposition 3.5. Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a map. Let \( A \) be a subset of \( V \). Let \( B \) be a subset of \( W \). Then, \( A \subseteq \mathcal{L}_f(B) \) holds if and only if \( B \subseteq \mathcal{R}_f(A) \).

Proof of Proposition 3.5. We have the following chain of logical equivalences:

\[
(A \subseteq \mathcal{L}_f(B))
\iff \left( \begin{array}{c}
\text{every } x \in A \text{ satisfies } x \in \mathcal{L}_f(B) \cr
\{v \in V \mid f(v, b) = 0 \text{ for all } b \in B\} \cr
\text{(by the definition of } \mathcal{L}_f(B))
\end{array} \right)
\iff (\text{every } x \in A \text{ satisfies } x \in \{v \in V \mid f(v, b) = 0 \text{ for all } b \in B\})
\iff (\text{every } x \in A \text{ satisfies } f(x, b) = 0 \text{ for all } b \in B)
\quad (\text{since every } x \in A \text{ satisfies } x \in V \text{ already (because } A \subseteq V))
\iff (\text{every } x \in A \text{ and } b \in B \text{ satisfy } f(x, b) = 0)
\iff (\text{every } a \in A \text{ and } b \in B \text{ satisfy } f(a, b) = 0)
\quad (8)
\quad (\text{here, we renamed the index } x \text{ as } a).
Also, we have the following chain of logical equivalences:

\[
(B \subseteq \mathcal{R}_f(A))
\]

\[\iff \left( \text{every } y \in B \text{ satisfies } y \in \mathcal{R}_f(A) \right) \right. \]

\[\iff \left( \text{every } y \in B \text{ satisfies } y \in \{ w \in W \mid f(a, w) = 0 \text{ for all } a \in A \} \right) \right. \]

\[\iff \left( \text{every } y \in B \text{ satisfies } \forall a \in A, f(a, y) = 0 \right) \right. \]

\[\iff \left( \text{every } a \in A \text{ and } b \in B \text{ satisfy } f(a, b) = 0 \right) \right. \]

Comparing this with (8), we obtain \((A \subseteq \mathcal{L}_f(B)) \iff (B \subseteq \mathcal{R}_f(A)).\) This proves Proposition 3.5.

**Corollary 3.6.** Let \(V, W\) and \(U\) be three vector spaces. Let \(f : V \times W \to U\) be a map.

(a) If \(B\) is a subset of \(W\), then \(\mathcal{L}_f(B) = \{ v \in V \mid B \subseteq \mathcal{R}_f(\{v\}) \}.\)

(b) If \(A\) is a subset of \(V\), then \(\mathcal{R}_f(A) = \{ w \in W \mid A \subseteq \mathcal{L}_f(\{w\}) \}.\)

**Proof of Corollary 3.6** (a) Let \(B\) be a subset of \(W\). For every \(v \in V\), we have the following logical equivalence:

\[(B \subseteq \mathcal{R}_f(\{v\})) \iff (v \in \mathcal{L}_f(B))\] \hfill (9)

Now,

\[
\begin{align*}
\{ v \in V \mid B & \subseteq \mathcal{R}_f(\{v\}) \} = \mathcal{L}_f(B)
\end{align*}
\]

(by \(\text{(9)}\))

\[
\text{this is equivalent to } \{ v \in \mathcal{L}_f(B) \} = \mathcal{L}_f(B) \quad \text{(since } \mathcal{L}_f(B) \subseteq V).\]

**Proof of (9):** Let \(v \in V\). Proposition 3.5 (applied to \(A = \{v\}\)) yields that \(\{v\} \subseteq \mathcal{L}_f(B)\) holds if and only if \(B \subseteq \mathcal{R}_f(\{v\})\). In other words, we have the following logical equivalence \(\{v\} \subseteq \mathcal{L}_f(B) \iff (B \subseteq \mathcal{R}_f(\{v\}))\). But clearly, we have the logical equivalence \(\{v\} \subseteq \mathcal{L}_f(B) \iff (v \in \mathcal{L}_f(B))\). Thus, we have the chain of logical equivalences

\[(v \in \mathcal{L}_f(B)) \iff (\{v\} \subseteq \mathcal{L}_f(B)) \iff (B \subseteq \mathcal{R}_f(\{v\})).\]

This proves (9).
This proves Corollary \[3.6\] (a).

(b) Let \( A \) be a subset of \( V \). Then, Corollary \[3.6\] (a) (applied to \( W, V, \text{op} \) and \( A \) instead of \( V, W, f \) and \( B \)) shows that
\[
\mathcal{L}_{\text{op}}(A) = \{ v \in W \mid A \subseteq \mathcal{R}_{\text{op}}\{v\}\} = \{ w \in W \mid A \subseteq \mathcal{R}_{\text{op}}\{w\}\} \quad (10)
\]
(here, we renamed the index \( v \) as \( w \)).

But Proposition \[3.2\] (b) shows that \( \mathcal{R}_f(A) = \mathcal{L}_{\text{op}}(A) \). Furthermore, every \( w \in W \) satisfies \( \mathcal{L}_f(\{w\}) = \mathcal{R}_{\text{op}}\{w\} \) (by Proposition \[3.2\] (a), applied to \( B = \{w\} \)). Hence,
\[
\left\{ w \in W \mid A \subseteq \mathcal{L}_f(\{w\}) \right\} = \left\{ w \in W \mid A \subseteq \mathcal{R}_{\text{op}}\{w\} \right\} = \mathcal{L}_{\text{op}}(A) \quad \text{(by (10))}
\]
\[
= \mathcal{R}_f(A).
\]

This proves Corollary \[3.6\] (b). \( \square \)

**Corollary 3.7.** Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a map.

(a) If \( B \) is a subset of \( W \), then \( B \subseteq \mathcal{R}_f(\mathcal{L}_f(B)) \).

(b) If \( A \) is a subset of \( V \), then \( A \subseteq \mathcal{L}_f(\mathcal{R}_f(A)) \).

**Proof of Corollary 3.7** (a) Let \( B \) be a subset of \( W \). Proposition \[3.5\] (applied to \( A = \mathcal{L}_f(B) \)) shows that \( \mathcal{L}_f(B) \subseteq \mathcal{L}_f(B) \) holds if and only if \( B \subseteq \mathcal{R}_f(\mathcal{L}_f(B)) \). Hence, \( B \subseteq \mathcal{R}_f(\mathcal{L}_f(B)) \) (since \( \mathcal{L}_f(B) \subseteq \mathcal{L}_f(B) \)). This proves Corollary 3.7 (a).

(b) Let \( A \) be a subset of \( V \). Proposition \[3.5\] (applied to \( B = \mathcal{R}_f(A) \)) shows that \( A \subseteq \mathcal{L}_f(\mathcal{R}_f(A)) \) holds if and only if \( \mathcal{R}_f(A) \subseteq \mathcal{R}_f(A) \). Hence, \( A \subseteq \mathcal{L}_f(\mathcal{R}_f(A)) \) (since \( \mathcal{R}_f(A) \subseteq \mathcal{R}_f(A) \)). This proves Corollary 3.7 (b). \( \square \)

We have so far not focussed on computing the subspaces \( \mathcal{L}_f(B) \) and \( \mathcal{R}_f(A) \) in Definition 3.1. This is straightforward to do when \( B \) (resp., \( A \)) is a finite set and \( V, W \) and \( U \) are finite-dimensional vector spaces (because in this case, the statement that \( f(v,b) = 0 \) for all \( b \in B \)) is a conjunction of finitely many linear equations, and thus easy to solve using linear algebra). However, when \( B \) (resp., \( A \)) is infinite, this becomes harder (as one would need to solve an infinite system of linear equations). However, when \( B \) (resp., \( A \)) is a finite-dimensional subspace of \( W \) (resp. \( V \)), then it is still easy, due to the following fact:

**Proposition 3.8.** Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a bilinear map.

(a) If \( B \) is a subset of \( W \), then \( \mathcal{L}_f(B) = \mathcal{L}_f(\text{span} B) \).

(b) If \( A \) is a subset of \( V \), then \( \mathcal{R}_f(A) = \mathcal{R}_f(\text{span} A) \).
Using Proposition 3.8(a), it is easy to compute $\mathcal{L}_f(B)$ whenever $V, W$ and $U$ are three finite-dimensional vector spaces and $B$ is a subspace of $W$. Similarly, using Proposition 3.8(b), it is easy to compute $\mathcal{R}_f(A)$ whenever $V, W$ and $U$ are three finite-dimensional vector spaces and $A$ is a subspace of $V$.

**Proof of Proposition 3.8** (a) Let $B$ be a subset of $W$. Recall that span $B$ is the smallest subspace of $W$ containing $B$ as a subset. Thus, if $G$ is any subspace of $W$ containing $B$ as a subset, then

$$\text{span } B \subseteq G.$$  \hspace{1cm} (11)

Clearly, $B \subseteq \text{span } B$. Thus, Proposition 3.4(a) (applied to $B' = \text{span } B$) shows that $\mathcal{L}_f(B) \supseteq \mathcal{L}_f(\text{span } B)$.

Corollary 3.6(a) yields $\mathcal{L}_f(B) = \{v \in V \mid B \subseteq \mathcal{R}_f(\{v\})\}$. Also, Corollary 3.6(a) (applied to span $B$ instead of $B$) yields

$$\mathcal{L}_f(\text{span } B) = \{v \in V \mid \text{span } B \subseteq \mathcal{R}_f(\{v\})\}.$$ \hspace{1cm} (12)

But for any $v \in V$, we have the following logical equivalence:

$$(B \subseteq \mathcal{R}_f(\{v\})) \iff (\text{span } B \subseteq \mathcal{R}_f(\{v\}))$$ \hspace{1cm} (13)

6**Proof.** Namely, let us assume that $V, W$ and $U$ are three finite-dimensional vector spaces, and that $B$ is a subspace of $W$. The vector space $B$ is a subspace of the finite-dimensional vector space $W$, and thus itself finite-dimensional. Hence, $B$ has a finite basis $B'$. Consider this $B'$.

Proposition 3.8(a) (applied to $B'$ instead of $B$) shows that $\mathcal{L}_f(B') =

$$\mathcal{L}_f\left(\text{span } (B')\right) = \mathcal{L}_f(B).$$

But since $B'$ is finite, it is easy to compute $\mathcal{L}_f(B')$ using linear algebra. In other words, it is easy to compute $\mathcal{L}_f(B)$ using linear algebra (since $\mathcal{L}_f(B') = \mathcal{L}_f(B)$). Qed.
Now,

$$\mathcal{L}_f (B) = \left\{ v \in V \mid \underbrace{B \subseteq \mathcal{R}_f (\{v\})}_{\text{this is equivalent to } \left( \text{span } B \subseteq \mathcal{R}_f (\{v\}) \right)} \right\}$$

This proves Proposition 3.8 (a).

(b) Let $A$ be a subset of $W$. Then, Proposition 3.2 (b) yields $\mathcal{R}_f (A) = \mathcal{L}_{f^{\mathsf{op}}} (A)$. Also, Proposition 3.2 (b) (applied to span $A$ instead of $A$) yields $\mathcal{R}_f (\text{span } A) = \mathcal{L}_{f^{\mathsf{op}}} (\text{span } A)$.

But the map $f^{\mathsf{op}}$ is bilinear (by Proposition 2.6 (b)). Hence, Proposition 3.8 (a) (applied to $W$, $V$, $f^{\mathsf{op}}$ and $A$ instead of $V$, $W$, $f$ and $B$) shows that $\mathcal{L}_{f^{\mathsf{op}}} (A) = \mathcal{L}_{f^{\mathsf{op}}} (\text{span } A)$. Thus, $\mathcal{R}_f (A) = \mathcal{L}_{f^{\mathsf{op}}} (A) = \mathcal{L}_{f^{\mathsf{op}}} (\text{span } A) = \mathcal{R}_f (\text{span } A)$. This proves Proposition 3.8 (b).

Next, we explore the orthogonal spaces of unions:

**Proposition 3.9.** Let $V$, $W$ and $U$ be three vector spaces. Let $f : V \times W \to U$ be a map.

(a) If $B_1$ and $B_2$ are two subsets of $W$, then $\mathcal{L}_f (B_1 \cup B_2) = \mathcal{L}_f (B_1) \cap \mathcal{L}_f (B_2)$.

(b) If $A_1$ and $A_2$ are two subsets of $V$, then $\mathcal{R}_f (A_1 \cup A_2) = \mathcal{R}_f (A_1) \cap \mathcal{R}_f (A_2)$.

Of course, Proposition 3.9 does not hold (in general) when the $\cup$ and $\cap$ signs are switched.

**Proof of Proposition 3.9** (a) Let $B_1$ and $B_2$ be two subsets of $W$. The definition of...
This proves Proposition 3.9 (a).

(b) Proposition 3.9 (b) can be derived from Proposition 3.9 (a) using the same tactic that we used (for example) to derive Proposition 3.4 (b) from Proposition 3.4 (a). (Alternatively, Proposition 3.9 (b) can be proven analogously to Proposition 3.9 (a).)

A similar result holds for orthogonal spaces of sums of subspaces when \( f \) is bilinear:

**Proposition 3.10.** Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a bilinear map.

(a) If \( B_1 \) and \( B_2 \) are two subspaces of \( W \), then \( \mathcal{L}_f (B_1 + B_2) = \mathcal{L}_f (B_1) \cap \mathcal{L}_f (B_2) \).

(b) If \( A_1 \) and \( A_2 \) are two subspaces of \( V \), then \( \mathcal{R}_f (A_1 + A_2) = \mathcal{R}_f (A_1) \cap \mathcal{R}_f (A_2) \).

**Proof of Proposition 3.10**

(a) Let \( B_1 \) and \( B_2 \) be two subspaces of \( W \). Then, \( \text{span} (B_1 \cup B_2) = B_1 + B_2 \). Now, Proposition 3.8 (a) (applied to \( B = B_1 \cup B_2 \)) shows that

\[
\text{span} B \subseteq G. \tag{14}
\]

Now, \( B_1 \) and \( B_2 \) are two subspaces of \( W \). Thus, \( B_1 + B_2 \) is again a subspace of \( W \). Also, \( B = B_1 \cup B_2 \subseteq B_1 + B_2 \) (since \( B_1 \subseteq B_1 + B_2 \) and \( B_2 \subseteq B_1 + B_2 \)). Thus, \( B_1 + B_2 \) is a subspace of \( W \) containing \( B_1 \cup B_2 \) as a subset. Therefore, (14) (applied to \( G = B_1 + B_2 \)) shows that \( \text{span} B \subseteq B_1 + B_2 \).

On the other hand, combining \( B_1 \subseteq B_1 \cup B_2 = B \subseteq \text{span} B \) and \( B_2 \subseteq B_1 \cup B_2 = B \subseteq \text{span} B \), we obtain \( B_1 + B_2 \subseteq \text{span} B \). Since \( \text{span} B \) is a vector subspace of \( W \). Combined with \( \text{span} B \subseteq B_1 + B_2 \), this yields \( B_1 + B_2 = \text{span} B \). Since \( B = B_1 \cup B_2 \), this rewrites as \( B_1 + B_2 = \text{span} (B_1 \cup B_2) \). Qed.
\[ \mathcal{L}_f (B_1 \cup B_2) = \mathcal{L}_f \left( \text{span} \left( B_1 \cup B_2 \right) \right) = \mathcal{L}_f (B_1 + B_2). \]  

Hence,

\[ \mathcal{L}_f (B_1 + B_2) = \mathcal{L}_f (B_1 \cup B_2) = \mathcal{L}_f (B_1) \cap \mathcal{L}_f (B_2) \]

(by Proposition 3.9 (a)). This proves Proposition 3.10 (a).

(b) The map \( f^{\text{op}} \) is bilinear (by Proposition 2.6 (b)). Hence, Proposition 3.10 (b) can be derived from Proposition 3.10 (a) using the same tactic that we used (for example) to derive Proposition 3.4 (b) from Proposition 3.4 (a). (Alternatively, Proposition 3.10 (b) can be proven analogously to Proposition 3.10 (a).) \( \Box \)

4. Interlude: Symmetric and antisymmetric bilinear forms

We make a digression to define the notions of symmetric and antisymmetric bilinear forms and maps:

**Definition 4.1.** Let \( V \) and \( U \) be two vector spaces. Let \( f : V \times V \to U \) be a map.

(a) The map \( f \) is said to be symmetric if and only if it satisfies

\[ (f (v, w) = f (w, v) \quad \text{for all } (v, w) \in V \times W). \]

(b) The map \( f \) is said to be antisymmetric if and only if it satisfies

\[ (f (v, w) = -f (w, v) \quad \text{for all } (v, w) \in V \times W). \]

Antisymmetric maps are also called skew-symmetric maps.

**Proposition 4.2.** Let \( V \) and \( U \) be two vector spaces. Let \( f : V \times V \to U \) be a map.

(a) If \( f \) is symmetric, then \( \mathcal{L}_f (A) = \mathcal{R}_f (A) \).

(b) If \( f \) is antisymmetric, then \( \mathcal{L}_f (A) = -\mathcal{R}_f (A) \).

Proposition 4.2 allows for the following definition (which is actually standard):

**Definition 4.3.** Let \( V \) and \( U \) be two vector spaces. Let \( f : V \times V \to U \) be a map. Let \( A \) be a subset of \( V \). Assume that \( f \) is symmetric or antisymmetric. Then, Proposition 4.2 shows that \( \mathcal{L}_f (A) = \mathcal{R}_f (A) \). The subset \( \mathcal{L}_f (A) = \mathcal{R}_f (A) \) of \( V \) is denoted by \( A^\perp \), at least when \( f \) is clear from the context.

**Proof of Proposition 4.2.** Straightforward and left to the reader. \( \Box \)
5. The morphism of quotients I

We shall now construct a certain morphism between quotient spaces induced by any bilinear map. First, we recall the universal property of quotient spaces:

Proposition 5.1. Let \( V \) be a vector space. Let \( A \) be a subspace of \( V \). Let \( \pi_{V,A} \) be the canonical projection \( V \to V/A \).

Let \( W \) be a further vector space. Let \( g : V \to W \) be a linear map such that \( g \left( A \right) = 0 \). Then, there exists a unique linear map \( g' : V/A \to W \) such that \( g = g' \circ \pi_{V,A} \).

Let us fix a notation for projections onto quotient spaces:

Definition 5.2. Let \( V \) be a vector space. Let \( A \) be a subspace of \( V \). Let \( v \in V \). Then, the residue class of \( v \) modulo \( A \) (that is, the image of \( v \) under the canonical projection \( V \to V/A \)) will be denoted by \( [v]_A \). (Other widespread notations for this residue class are \( v \mod A \), \( v + A \) and \( \overline{v}_A \).)

We now state the main theorem of this section:

Theorem 5.3. Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a bilinear map.

(a) Then, there exists a unique linear map \( \alpha : V/\mathcal{L}_f(W) \to \text{Hom} \left( W/\mathcal{R}_f(V), U \right) \) satisfying

\[
\left( \left( \alpha \left( [v]_{\mathcal{L}_f(W)} \right) \right) \left( [w]_{\mathcal{R}_f(V)} \right) \right) = f \left( v, w \right) \quad \text{for all } (v, w) \in V \times W . \quad (15)
\]

(b) This map \( \alpha \) is injective.

Proof of Theorem 5.3 First, we notice that \( \mathcal{L}_f(W) \) is a subspace of \( V \) (by Proposition 3.3 (a), applied to \( B = W \)). Hence, the quotient space \( V/\mathcal{L}_f(W) \) is well-defined. Furthermore, \( \mathcal{R}_f(V) \) is a subspace of \( W \) (by Proposition 3.3 (b), applied to \( A = V \)). Hence, the quotient space \( W/\mathcal{R}_f(V) \) is well-defined.

(a) It is easy to see that there exists at most one linear map \( \alpha : V/\mathcal{L}_f(W) \to \text{Hom} \left( W/\mathcal{R}_f(V), U \right) \) satisfying \(...\)

Proof. Let \( a_1 \) and \( a_2 \) be two linear maps \( \alpha : V/\mathcal{L}_f(W) \to \text{Hom} \left( W/\mathcal{R}_f(V), U \right) \) satisfying \(...\). We shall show that \( a_1 = a_2 \).

We know that \( \alpha \) is a linear map \( \alpha : V/\mathcal{L}_f(W) \to \text{Hom} \left( W/\mathcal{R}_f(V), U \right) \) satisfying \(...\). In other words, \( \alpha \) is a linear map \( V/\mathcal{L}_f(W) \to \text{Hom} \left( W/\mathcal{R}_f(V), U \right) \) and satisfies

\[
\left( \left( \alpha \left( [v]_{\mathcal{L}_f(W)} \right) \right) \left( [w]_{\mathcal{R}_f(V)} \right) \right) = f \left( v, w \right) \quad \text{for all } (v, w) \in V \times W . \quad (16)
\]

Now, let \( x \in V/\mathcal{L}_f(W) \), Let \( y \in W/\mathcal{R}_f(V) \).

We have \( x \in V/\mathcal{L}_f(W) \). Hence, we can write \( x \) in the form \( [v]_{\mathcal{L}_f(W)} \) for some \( v \in V \).
We shall now construct such a map.

Let \( \pi_{V,\mathcal{L}_f(W)} \) be the canonical projection \( V \rightarrow V/\mathcal{L}_f(W) \). Thus,

\[
\pi_{V,\mathcal{L}_f(W)}(v) = [v]_{\mathcal{L}_f(W)} \quad \text{for every } v \in V. \tag{17}
\]

Let \( \pi_{W,\mathcal{R}_f(V)} \) be the canonical projection \( W \rightarrow W/\mathcal{R}_f(V) \). Thus,

\[
\pi_{W,\mathcal{R}_f(V)}(w) = [w]_{\mathcal{R}_f(V)} \quad \text{for every } w \in V. \tag{18}
\]

Let \( v \in V \). The map \( f \) is bilinear.

Recall that a linear map \( f(v, -) \in \text{Hom}(W, U) \) is defined (according to Definition \ref{def:bilinear_map} (b)). We have \( f(v, -)(\mathcal{R}_f(V)) = 0 \). Hence, Proposition \ref{prop:linear_map} (applied to \( W, \mathcal{R}_f(V), \pi_{W,\mathcal{R}_f(V)}, U \) and \( f(v, -) \)) shows that there exists a unique linear map \( g' : W/\mathcal{R}_f(V) \rightarrow U \) such that \( f(v, -) = g' \circ \pi_{W,\mathcal{R}_f(V)} \). Let us denote this \( g' \) by \( g_v \). Thus, \( g_v \) is a linear map \( W/\mathcal{R}_f(V) \rightarrow U \) and satisfies \( f(v, -) = g_v \circ \pi_{W,\mathcal{R}_f(V)} \).

We have

\[
g_v([w]_{\mathcal{R}_f(V)}) = f(v, w) \quad \text{for all } w \in W \tag{19}
\]

Consider this \( v \). Thus, \( x = [v]_{\mathcal{L}_f(W)} \).

We have \( y \in W/\mathcal{R}_f(V) \). Hence, we can write \( y \) in the form \([w]_{\mathcal{R}_f(V)}\) for some \( w \in W \).

Consider this \( w \). Thus, \( y = [w]_{\mathcal{R}_f(V)} \).

Now,

\[
\begin{pmatrix}
\alpha_1 \\
\begin{pmatrix}
x \\
= [v]_{\mathcal{L}_f(W)}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
y \\
= [w]_{\mathcal{R}_f(V)}
\end{pmatrix}
= \begin{pmatrix}
\alpha_1 \\
\begin{pmatrix}
x \\
= [v]_{\mathcal{L}_f(W)}
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
[w]_{\mathcal{R}_f(V)}
\end{pmatrix}
= f(v, w)
\]

(by (16)). The same reasoning (but applied to \( \alpha_2 \) instead of \( \alpha_1 \)) shows that \( (\alpha_2(x))(y) = f(v, w) \). Thus, \( (\alpha_1(x))(y) = f(v, w) \).

Let us now forget that we fixed \( x \). We thus have shown that \( (\alpha_1(x))(y) = f(v, w) \) for every \( y \in W/\mathcal{R}_f(V) \). In other words, \( \alpha_1(x) = \alpha_2(x) \).

Let us now forget that we fixed \( y \). We thus have shown that \( \alpha_1(x) = \alpha_2(x) \) for every \( x \in V/\mathcal{L}_f(W) \). In other words, \( \alpha_1 = \alpha_2 \).

Let us now forget that we fixed \( \alpha_1 \) and \( \alpha_2 \). We thus have shown that \( \alpha_1 = \alpha_2 \) whenever \( \alpha_1 \) and \( \alpha_2 \) are two linear maps \( \alpha : V/\mathcal{L}_f(W) \rightarrow \text{Hom}(W/\mathcal{R}_f(V), U) \) satisfying (15). In other words, there exists at most one linear map \( \alpha : V/\mathcal{L}_f(W) \rightarrow \text{Hom}(W/\mathcal{R}_f(V), U) \) satisfying (15). Qed.

\[10\text{ Proof.}\] Let \( y \in \mathcal{R}_f(V) \). We shall prove that \( f(v, -)(y) = 0 \).

We have \( y \in \mathcal{R}_f(V) = \{w \in W \mid f(a, w) = 0 \text{ for all } a \in V\} \) (by the definition of \( \mathcal{R}_f(V) \)).

In other words, \( y \) is an element \( w \) of \( W \) satisfying \( f(a, w) = 0 \) for all \( a \in V \). In other words, \( y \) is an element of \( W \) and satisfies \( f(a, y) = 0 \) for all \( a \in V \).

We know that \( f(a, y) = 0 \) for all \( a \in V \). Applying this to \( a = v \), we obtain \( f(v, y) = 0 \). But the definition of \( f(v, -) \) yields \( f(v, -)(y) = f(v, y) = 0 \).

Now, let us forget that we fixed \( y \). We thus have shown that \( f(v, -)(y) = 0 \) for every \( y \in \mathcal{R}_f(V) \). In other words, \( f(v, -) \left(\mathcal{R}_f(V)\right) = 0 \). Qed.
Now, let us forget that we fixed \( v \). We thus have constructed a linear map \( g_v : W/\mathcal{R}_f (V) \to U \) for every \( v \in V \). We have furthermore shown that this map satisfies (19) for every \( v \in V \). For every \( v \in V \), the map \( g_v \) is a linear map \( W/\mathcal{R}_f (V) \to U \), and thus belongs to \( \text{Hom} \left( W/\mathcal{R}_f (V), U \right) \).

Now, we define a map \( g : V \to \text{Hom} \left( W/\mathcal{R}_f (V), U \right) \) by

\[
(g \left( v \right) = g_v \quad \text{for every } v \in V).
\]

(This is well-defined, because for every \( v \in V \), the map \( g_v \) belongs to \( \text{Hom} \left( W/\mathcal{R}_f (V), U \right) \).) Thus, for every \( v \in V \) and \( w \in W \), we have

\[
\left( g \left( v \right) \right) \left( \left[ w \right]_{\mathcal{R}_f (V)} \right) = g_v \left( \left[ w \right]_{\mathcal{R}_f (V)} \right) = f \left( v, w \right) \quad \text{(by (19)).}
\]

**Proof of (19):** Let \( w \in W \). From (18), we obtain \( \left[ w \right]_{\mathcal{R}_f (V)} = \pi_{W,\mathcal{R}_f (V)} (w) \). Applying the map \( g_v \) to both sides of this equality, we obtain

\[
\begin{align*}
g_v \left( \left[ w \right]_{\mathcal{R}_f (V)} \right) &= g_v \left( \pi_{W,\mathcal{R}_f (V)} (w) \right) = \left( g_v \circ \pi_{W,\mathcal{R}_f (V)} \right) (w) \\
&= f(v, -)(w) = f \left( v, w \right) \quad \text{(by the definition of } f \left( v, - \right) \text{).}
\end{align*}
\]

This proves (19).
The map $g$ is linear. Moreover, $g \left( \mathcal{L}_f (W) \right) = 0$. Hence, Proposition 5.1 (applied to $\mathcal{L}_f (W)$, $\pi_{V, \mathcal{L}_f (W)}$ and $\text{Hom} (W/\mathcal{R}_f (V), U)$ instead of $A$, $\pi_{V,A}$ and $g$) shows that there exists a unique linear map $g' : V/\mathcal{L}_f (W) \rightarrow W$. 

**Proof.** Let $v_1$ and $v_2$ be two elements of $V$. Let $\lambda_1$ and $\lambda_2$ be two elements of $k$. We shall show that $g (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g (v_1) + \lambda_2 g (v_2)$.

Let $y \in W/\mathcal{R}_f (V)$. Thus, $y$ can be written in the form $[w]_{\mathcal{R}_f (V)}$ for some $w \in W$. Consider this $w$. Thus, $y = [w]_{\mathcal{R}_f (V)}$. Hence, for every $v \in V$, we have

\[
(g (v)) \left( \begin{array}{c} y \\ = [w]_{\mathcal{R}_f (V)} \end{array} \right) = g (v) \left( [w]_{\mathcal{R}_f (V)} \right) = f (v, w) \quad \text{by (21)}.
\]

Recall that a linear map $f (-, w) \in \text{Hom} (V, U)$ is defined (according to Definition 2.8 (a)). This map is linear, and thus we have $f (-, w) \left( \lambda_1 v_1 + \lambda_2 v_2 \right) = \lambda_1 f (-, w) (v_1) + \lambda_2 f (-, w) (v_2)$. Comparing this with $(f (-, w) \left( \lambda_1 v_1 + \lambda_2 v_2 \right) = f (\lambda_1 v_1 + \lambda_2 v_2, w)$ (by the definition of $f (-, w)$), we obtain

\[
f (\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \left( f (-, w) (v_1) \right) + \lambda_2 \left( f (-, w) (v_2) \right).
\]

Now, (21) (applied to $v = v_1$) yields $g (v_1) (y) = f (v_1, w)$. Also, (21) (applied to $v = v_2$) yields $g (v_2) (y) = f (v_2, w)$. But (21) (applied to $v = \lambda_1 v_1 + \lambda_2 v_2$) yields

\[
g (\lambda_1 v_1 + \lambda_2 v_2) (y) = f (\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f (v_1, w) + \lambda_2 f (v_2, w).
\]

Let us now forget that we fixed $y$. We thus have shown that $g (\lambda_1 v_1 + \lambda_2 v_2) (y) = (\lambda_1 g (v_1) + \lambda_2 g (v_2)) (y)$ for every $y \in W/\mathcal{R}_f (V)$. In other words, $g (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g (v_1) + \lambda_2 g (v_2)$.

Let us now forget that we fixed $v_1$, $v_2$, $\lambda_1$ and $\lambda_2$. Thus, we have proven that $g (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 g (v_1) + \lambda_2 g (v_2)$ for every $v_1 \in V$, $v_2 \in V$, $\lambda_1 \in k$ and $\lambda_2 \in k$. In other words, the map $g$ is linear, qed.

---

13Proof. Let $p \in \mathcal{L}_f (W)$. Thus, $p \in \mathcal{L}_f (W) = \{ v \in V \mid f (v, b) = 0 \text{ for all } b \in W \}$ (by the definition of $\mathcal{L}_f (W)$). In other words, $p$ is an element $v$ of $V$ satisfying $f (v, b) = 0$ for all $b \in W$. In other words, $p$ is an element of $V$ and satisfies $f (p, b) = 0$ for all $b \in W$.

Let $y \in W/\mathcal{R}_f (V)$. Thus, $y$ can be written in the form $[w]_{\mathcal{R}_f (V)}$ for some $w \in W$. Consider this $w$. Thus, $y = [w]_{\mathcal{R}_f (V)}$.

But we have $f (p, b) = 0$ for all $b \in W$. Applying this to $b = w$, we obtain $f (p, w) = 0$. But (20) (applied to $v = p$) yields $g (p) \left( [w]_{\mathcal{R}_f (V)} \right) = f (p, w) = 0$. Hence,

\[
(g (p)) \left( \begin{array}{c} y \\ = [w]_{\mathcal{R}_f (V)} \end{array} \right) = g (p) \left( [w]_{\mathcal{R}_f (V)} \right) = 0.
\]

Now, let us forget that we fixed $y$. We thus have shown that $g (p) (y) = 0$ for every
Hom\((W/R_f(V), U)\) such that \(g = g' \circ \pi_{V,L_f(W)}\). Let us denote this \(g'\) by \(\beta\). Thus, \(\beta\) is a linear map \(V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) and satisfies \(g = \beta \circ \pi_{V,L_f(W)}\).

Now, for all \((v,w) \in V \times W\), we have

\[
\begin{pmatrix}
\beta \left( \left[ v \right]_{L_f(W)} \right)
\end{pmatrix}
\begin{pmatrix}
\left[ w \right]_{R_f(V)}
\end{pmatrix}
= \begin{pmatrix}
\beta \left( \pi_{V,L_f(W)}(v) \right)
\end{pmatrix}
\begin{pmatrix}
\left[ w \right]_{R_f(V)}
\end{pmatrix}
= \begin{pmatrix}
\left( \beta \circ \pi_{V,L_f(W)} \right)(v)
\end{pmatrix}
\begin{pmatrix}
\left[ w \right]_{R_f(V)}
\end{pmatrix}
= \left( g(v) \right) \left[ w \right]_{R_f(V)} = f(v,w) \quad \text{(by (20)).}
\]

In other words, the linear map \(\beta\) satisfies

\[
\left( \beta \left( \left[ v \right]_{L_f(W)} \right) \right) \left[ w \right]_{R_f(V)} = f(v,w) \quad \text{for all } (v,w) \in V \times W.
\]

Hence, the map \(\beta\) is a linear map \(\alpha : V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) satisfying (15). Therefore, there exists at least one linear map \(\alpha : V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) satisfying (15) (namely, \(\beta\)). Consequently, there exists a unique linear map \(\alpha : V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) satisfying (15) (since we already know that there exists at most one linear map \(\alpha : V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) satisfying (15)). This proves Theorem 5.3 (a).

(b) Consider the unique linear map \(\alpha : V/L_f(W) \to \text{Hom}(W/R_f(V), U)\) satisfying (15).

We have

\[
\begin{array}{ll}
L_f(W) = \{ v \in V \mid f(v,b) = 0 \text{ for all } b \in W \} & \text{(by the definition of } L_f(W)\text{)} \\
= \{ p \in V \mid f(p,b) = 0 \text{ for all } b \in W \} & \text{(here, we have renamed the index } v \text{ as } p \} \\
= \{ p \in V \mid f(p,w) = 0 \text{ for all } w \in W \} & \text{(here, we have renamed the index } b \text{ as } w \}.
\end{array}
\]

\(y \in W/R_f(V)\). In other words, \(g(p) = 0\).

Now, let us forget that we fixed \(p\). We thus have shown that \(g(p) = 0\) for every \(p \in L_f(W)\).

In other words, \(g(L_f(W)) = 0\), qed.
Let $x \in \text{Ker} \alpha$. We have $x \in \text{Ker} \alpha \subseteq V/\mathcal{L}_f(W)$. Hence, we can write $x$ in the form $[v]_{\mathcal{L}_f(W)}$ for some $v \in V$. Consider this $v$. Thus, $x = [v]_{\mathcal{L}_f(W)}$.

We have $x \in \text{Ker} \alpha$, so that $\alpha(x) = 0$. Now, let $w \in W$. Then, $(v,w) \in V \times W$ (since $v \in V$ and $w \in W$). Thus, (15) shows that $\left(\alpha\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = f(v,w)$ (since we know that $\alpha$ satisfies (15)). Hence,

$$f(v,w) = \left(\alpha\left([v]_{\mathcal{L}_f(W)}\right)\right)\left([w]_{\mathcal{R}_f(V)}\right) = 0\left([w]_{\mathcal{R}_f(V)}\right) = 0.$$ 

Let us now forget that we fixed $w$. We thus have shown that $f(v,w) = 0$ for all $w \in W$. Thus, $v$ is an element of $V$ and satisfies $f(v,w) = 0$ for all $w \in W$. In other words, $v \in \{p \in V \mid f(p,w) = 0 \text{ for all } w \in W\}$. In view of (22), this rewrites as $v \in \mathcal{L}_f(W)$. Hence, $[v]_{\mathcal{L}_f(W)} = 0$. Thus, $x = [v]_{\mathcal{L}_f(W)} = 0$.

Let us now forget that we fixed $x$. We thus have shown that $x = 0$ for every $x \in \text{Ker} \alpha$. In other words, $\text{Ker} \alpha = 0$. Thus, the linear map $\alpha$ is injective. This proves Theorem 5.3 (b).

6. The morphism of quotients II

So far, we have not used the assumption that $k$ is a field; we could have just as well let $k$ be any commutative ring. (Of course, we would have to talk about $k$-modules instead of vector spaces, and similarly; but apart from this, the results and proofs would have been the same.) We shall now prove some slightly deeper result where we will actually need this assumption (that $k$ is a field). We will also make some assumptions on finite-dimensionality.

Definition 6.1. Let $V$ be a vector space. Then, $V^*$ will denote the vector space $\text{Hom}(V,k)$. This space $V^*$ is called the dual space of $V$. It is well-known that $\dim(V^*) = \dim V$ when $V$ is finite-dimensional. Moreover, there is a canonical injective linear map $V \to V^{**}$ (which sends every $v \in V$ to the linear map $\tilde{v} \in V^{**}$ which sends every $f \in V^*$ to $f(v) \in k$). When $V$ is finite-dimensional, this linear map $V \to V^{**}$ is a vector space isomorphism.

We shall first state some well-known properties of finite-dimensional vector spaces. The first of these properties is the fact that an injective linear map between two finite-dimensional vector spaces of equal dimension must be an isomorphism:
**Proposition 6.2.** Let $A$ and $B$ be two finite-dimensional vector spaces. Let $i : A \rightarrow B$ be an injective linear map. Assume that $\dim A = \dim B$. Then, $i$ is a vector space isomorphism.

The next property is the particular case of Proposition 6.2 when the map $i$ is an inclusion:

**Proposition 6.3.** Let $B$ be a finite-dimensional vector space. Let $A$ be a subspace of $B$ such that $\dim A = \dim B$. Then, $A = B$.

Furthermore, the domain of an injective linear map always has at most the same dimension as its target:

**Proposition 6.4.** Let $A$ and $B$ be two finite-dimensional vector spaces. Let $i : A \rightarrow B$ be an injective linear map. Then, $\dim A \leq \dim B$.

Now, we state the next crucial theorem:

**Theorem 6.5.** Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \rightarrow k$ be a bilinear form.

(a) Then, there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \rightarrow (W/\mathcal{R}_f(V))^*$ satisfying (15). This map $\alpha$ is a vector space isomorphism.

(b) We have $\dim (V/\mathcal{L}_f(W)) = \dim (W/\mathcal{R}_f(V))$.

**Proof of Theorem 6.5.** The map $f : V \times W \rightarrow k$ is a bilinear form. In other words, the map $f : V \times W \rightarrow k$ is a bilinear map (according to the definition of a “bilinear form”).

(a) Theorem 5.3 (applied to $U = k$) yields that there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \rightarrow \text{Hom}(W/\mathcal{R}_f(V), k)$ satisfying (15). In other words, there exists a unique linear map $\alpha : V/\mathcal{L}_f(W) \rightarrow (W/\mathcal{R}_f(V))^*$ satisfying (15) (since $(W/\mathcal{R}_f(V))^* = \text{Hom}(W/\mathcal{R}_f(V), k)$). This proves Theorem 6.5 (a).

Theorem 5.3 (applied to $U = k$) yields that the unique linear map $\alpha : V/\mathcal{L}_f(W) \rightarrow \text{Hom}(W/\mathcal{R}_f(V), k)$ satisfying (15) is injective. In other words, the unique linear map $\alpha : V/\mathcal{L}_f(W) \rightarrow (W/\mathcal{R}_f(V))^*$ satisfying (15) is injective (since $(W/\mathcal{R}_f(V))^* = \text{Hom}(W/\mathcal{R}_f(V), k)$). Consider this map $\alpha$.

The vector spaces $V/\mathcal{L}_f(W)$ and $W/\mathcal{R}_f(V)$ are finite-dimensional (since $V$ and $W$ are finite-dimensional). The vector space $(W/\mathcal{R}_f(V))^*$ is finite-dimensional (since $W/\mathcal{R}_f(V)$ is finite-dimensional). We know that the map $\alpha$ is injective. Thus, Proposition 6.4 (applied to $A = V/\mathcal{L}_f(W)$, $B = (W/\mathcal{R}_f(V))^*$ and $i = \alpha$) shows that $\dim (V/\mathcal{L}_f(W)) \leq \dim (W/\mathcal{R}_f(V))^*$.

But every finite-dimensional vector space $G$ satisfies $\dim (G^*) = \dim G$. Applying this to $G = W/\mathcal{R}_f(V)$, we obtain $\dim (W/\mathcal{R}_f(V))^* = \dim (W/\mathcal{R}_f(V))$. 

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Hence,
\[
\dim \left( \frac{V}{L_f(W)} \right) \leq \dim \left( \left( \frac{W}{R_f(V)} \right)^* \right) = \dim \left( \frac{W}{R_f(V)} \right). \tag{23}
\]

We thus have proven the inequality \( (23) \) for every two finite-dimensional vector spaces \( V \) and \( W \) and every bilinear form \( f : V \times W \to k \).

But \( f^{\text{op}} : W \times V \to k \) is a bilinear form (by Proposition 2.10(a)). Hence, we can apply \( (23) \) to \( W, V \) and \( f^{\text{op}} \) instead of \( V, W \) and \( f \). As a result, we obtain
\[
\dim \left( \frac{W}{L_{f^{\text{op}}}(V)} \right) \leq \dim \left( \frac{W}{R_{f^{\text{op}}}(V)} \right).
\]

But Proposition 3.2(a) (applied to \( U = k \) and \( B = W \)) yields \( L_f(W) = R_{f^{\text{op}}}(W) \). Also, Proposition 3.2(b) (applied to \( U = k \) and \( A = V \)) yields \( R_f(V) = L_{f^{\text{op}}}(V) \). Now,
\[
\dim \left( \frac{W}{R_f(V)} \right) = \dim \left( \frac{W}{L_{f^{\text{op}}}(V)} \right) \leq \dim \left( \frac{V}{R_{f^{\text{op}}}(W)} \right).
\]

Combining this inequality with \( (23) \), we obtain \( \dim \left( \frac{V}{L_f(W)} \right) = \dim \left( \frac{W}{R_f(V)} \right) \).

This proves Theorem 6.5(c).

It remains to prove Theorem 6.5(b). But this is now easy: We have
\[
\dim \left( \frac{V}{L_f(W)} \right) = \dim \left( \frac{W}{R_f(V)} \right) = \dim \left( \left( \frac{W}{R_f(V)} \right)^* \right).
\]

Hence, Proposition 6.2 (applied to \( A = V/L_f(W), B = \left( \frac{W}{R_f(V)} \right)^* \) and \( i = \alpha \)) shows that \( \alpha \) is a vector space isomorphism (since \( \alpha \) is injective). This proves Theorem 6.5(b).

Theorem 6.5(c) leads to a particularly useful corollary:

**Corollary 6.6.** Let \( V \) and \( W \) be two finite-dimensional vector spaces. Let \( f : V \times W \to k \) be a bilinear form. Let \( A \) be a subspace of \( V \). Let \( B \) be a subspace of \( W \). Then,
\[
\dim \left( \frac{A}{(A \cap L_f(B))} \right) = \dim \left( \frac{B}{(B \cap R_f(A))} \right).
\]

We shall derive this corollary from Theorem 6.5(c) using the following simple fact:

**Proposition 6.7.** Let \( V, W \) and \( U \) be three vector spaces. Let \( f : V \times W \to U \) be a map. Let \( A \) be a subspace of \( V \). Let \( B \) be a subspace of \( W \).

(a) We have \( A \cap L_f(B) = L_{f|A \times B}(B) \).

(b) We have \( B \cap R_f(A) = R_{f|A \times B}(A) \).
Proof of Proposition \[6.7\] (a) The definition of $\mathcal{L}_f (B)$ yields $\mathcal{L}_f (B) = \{ v \in V \mid f(v, b) = 0 \text{ for all } b \in B \}$.

The map $f \mid_{A \times B}$ is a map $A \times B \to U$. Hence, the definition of $\mathcal{L}_{f \mid_{A \times B}} (B)$ yields

$$\mathcal{L}_{f \mid_{A \times B}} (B) = \left\{ v \in A \mid \left( f \mid_{A \times B} \right)(v, b) = 0 \text{ for all } b \in B \right\}$$

$$= \{ v \in A \mid f(v, b) = 0 \text{ for all } b \in B \}$$

(since $A \subseteq V$)

$$= \{ v \in V \mid v \in A \text{ and } f(v, b) = 0 \text{ for all } b \in B \}$$

(since $A \subseteq V$)

$$= A \cap \mathcal{L}_f (B).$$

This proves Proposition \[6.7\] (a).

(b) This proof is similar to that of Proposition \[6.7\] (a).

\[\square\]

Proof of Corollary \[6.6\] The vector spaces $V$ and $W$ are finite-dimensional. Hence, their subspaces $A$ and $B$ also are finite-dimensional. Proposition \[2.10\] (b) shows that the restriction $f \mid_{A \times B}$ is a bilinear form $A \times B \to k$. Hence, Theorem \[6.5\] (c) (applied to $A$, $B$ and $f \mid_{A \times B}$ instead of $V$, $W$ and $f$) shows that $\dim \left( A / \mathcal{L}_{f \mid_{A \times B}} (B) \right) = \dim \left( B / \mathcal{R}_{f \mid_{A \times B}} (A) \right)$.

But Proposition \[6.7\] (a) (applied to $U = k$) shows that $A \cap \mathcal{L}_f (B) = \mathcal{L}_{f \mid_{A \times B}} (B)$. Also, Proposition \[6.7\] (b) (applied to $U = k$) shows that $B \cap \mathcal{R}_f (A) = \mathcal{R}_{f \mid_{A \times B}} (A)$. Hence,

$$\dim \left( A / (A \cap \mathcal{L}_f (B)) \right) = \dim \left( A / \mathcal{L}_{f \mid_{A \times B}} (B) \right) = \dim \left( B / \mathcal{R}_{f \mid_{A \times B}} (A) \right) = \dim \left( B / (B \cap \mathcal{R}_f (A)) \right).$$

This proves Corollary \[6.6\].

\[\square\]

7. More on orthogonal spaces

We are now in the position to prove less obvious results about orthogonal spaces. First of all, using Corollary \[6.6\] we can derive a formula for the dimension of of an orthogonal space:
Corollary 7.1. Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to k$ be a bilinear form.

(a) If $A$ is a subspace of $V$, then
\[
\dim \left( \mathcal{R}_f(A) \right) = \dim W - \dim A + \dim \left( A \cap \mathcal{L}_f(W) \right).
\]

(b) If $B$ is a subspace of $W$, then
\[
\dim \left( \mathcal{L}_f(B) \right) = \dim V - \dim B + \dim \left( B \cap \mathcal{R}_f(V) \right).
\]

Proof of Corollary 7.1. (a) Let $A$ be a subspace of $V$. Then, Corollary 6.6 (applied to $B = W$) yields
\[
\dim \left( \frac{A}{A \cap \mathcal{L}_f(W)} \right) = \dim \left( \frac{W}{W \cap \mathcal{R}_f(A)} \right) = \dim W - \dim \left( \mathcal{R}_f(A) \right),
\]
so that
\[
\dim W - \dim \left( \mathcal{R}_f(A) \right) = \dim \left( \frac{A}{A \cap \mathcal{L}_f(W)} \right) = \dim A - \dim \left( A \cap \mathcal{L}_f(W) \right).
\]
Solving this equation for $\dim \left( \mathcal{R}_f(A) \right)$, we obtain
\[
\dim \left( \mathcal{R}_f(A) \right) = \dim W - (\dim A - \dim \left( A \cap \mathcal{L}_f(W) \right)) = \dim W - \dim A + \dim \left( A \cap \mathcal{L}_f(W) \right).
\]
This proves Corollary 7.1 (a).

(b) The proof of Corollary 7.1 (b) is analogous to that of Corollary 7.1 (a) (but now we need to apply Corollary 6.6 to $A = V$).

Let us now state another fact from linear algebra:

Proposition 7.2. Let $V$ be a vector space. Let $A$ and $B$ be two subspaces of $V$.

(a) We have $(A + B) / B \cong A / (A \cap B)$ as vector spaces.

(b) Assume that $V$ is finite-dimensional. Then, $\dim (A + B) = \dim A + \dim B - \dim (A \cap B)$.

Proof of Proposition 7.2. (a) Let $\iota$ be the canonical inclusion $A \to A + B$. (This is well-defined, since $A \subseteq A + B$.) Let $\pi_{A,A\cap B}$ be the canonical projection $A \to A / (A \cap B)$. Let $\pi_{A+B,B}$ be the canonical projection $A + B \to (A + B) / B$.
The map \( \pi_{A+B,B} \circ \iota : A \rightarrow (A + B) / B \) is linear (since it is the composition of the two linear maps \( \pi_{A+B,B} \) and \( \iota \)). It furthermore satisfies \( (\pi_{A+B,B} \circ \iota) (A \cap B) = 0 \) \[14\] Proposition 5.1 (applied to \( A, A \cap B, \pi_{A,A\cap B}, (A + B) / B \) and \( \pi_{A+B,B} \circ \iota \) instead of \( V, A, \pi_{V,A}, W \) and \( g \)) thus shows that there exists a unique linear map \( g' : (A / (A \cap B)) \rightarrow (A + B) / B \) such that \( \pi_{A+B,B} \circ \iota = g' \circ \pi_{A,A\cap B} \). Consider this \( g' \).

We have

\[
g'([a]_{A\cap B}) = [a]_B \quad \text{for every } a \in A \quad \text{(24)}
\]

Now, the map \( g' \) is surjective\[15\] and injective\[16\]. Hence, the map \( g' \) is bijective, and thus a vector space isomorphism (since \( g' \) is linear). Thus, there exists

\[
\pi_{A+B,B} \circ \iota \quad \text{such that} \quad \pi_{A+B,B} (x) = [x]_B \quad \text{for every } x \in A \cap B.
\]

Now, let us forget that we fixed \( a \). We thus have shown that \( (\pi_{A+B,B} \circ \iota) (a) = 0 \) for every \( a \in A \cap B \). In other words, \( (\pi_{A+B,B} \circ \iota) (A \cap B) = 0 \), qed.

\[15\] Proof. Let \( a \in A \cap B \). Thus, \( a \in A \cap B \subseteq A \); therefore, \( \iota (a) \) is well-defined. Now, \( \iota \) is an inclusion map; therefore, \( \iota (a) = a \). But \( \pi_{A+B,B} \) is the canonical projection \( A + B \rightarrow (A + B) / B \). Hence, \( \pi_{A+B,B} (a) = [a]_B = 0 \) (since \( a \in A \cap B \subseteq B \)). Therefore, \( (\pi_{A+B,B} \circ \iota) (a) = \pi_{A+B,B} (\iota (a)) = \pi_{A+B,B} (a) = 0 \).

\[16\] Proof of (24). Let \( a \in A \). Recall that \( \pi_{A,A\cap B} \) is the canonical projection \( A \rightarrow A / (A \cap B) \). Hence, \( \pi_{A,A\cap B} (a) = [a]_{A\cap B} \). Hence, \( [a]_{A\cap B} = \pi_{A,A\cap B} (a) \). Hence, applying the map \( g' \) to both sides of this equality, we obtain

\[
g' ([a]_{A\cap B}) = g' (\pi_{A,A\cap B} (a)) = (g' \circ \pi_{A,A\cap B}) (a) = (\pi_{A+B,B} \circ \iota) (a) = \pi_{A+B,B} \left( \iota (a) \right) = \pi_{A+B,B} (a) = [a]_B
\]

(since \( \pi_{A+B,B} \) is the canonical projection \( A + B \rightarrow (A + B) / B \)). This proves (24).

\[17\] Proof of (24). Let \( x \in (A + B) / B \). We shall show that \( x \in g' (A / (A \cap B)) \).

We have \( x \in (A + B) / B \). Thus, \( x \) can be written in the form \( [u]_B \) for some \( u \in A + B \). Consider this \( u \). Thus, \( x = [u]_B \).

We have \( u \in A + B \). Thus, \( u \) can be written in the form \( a + b \) for some \( a \in A \) and \( b \in B \). Consider these \( a \) and \( b \). Thus, \( u = a + b \). Now,

\[
x = \left[ \begin{array}{c} u \\ a+b \\ \hline \end{array} \right]_B = [a+b]_B = [a]_B + [b]_B = [a]_B = g' \left[ \begin{array}{c} a \\ \hline \end{array} \right]_{A\cap B} = g' [a]_{A\cap B} \quad \text{(by (24))}
\]

\[
\in g' (A / (A \cap B)).
\]

Now, let us forget that we fixed \( x \). We thus have shown that every \( x \in (A + B) / B \) satisfies \( x \in g' (A / (A \cap B)) \). In other words, \( (A + B) / B \subseteq g' (A / (A \cap B)) \). In other words, the map \( g' \) is surjective, qed.

\[17\] Proof. Let \( x \in \ker (g') \). We shall show that \( x = 0 \).

We have \( x \in \ker (g') \subseteq A / (A \cap B) \). Thus, \( x \) can be written in the form \( [a]_{A\cap B} \) for some \( a \in A \). Consider this \( a \). Thus, \( x = [a]_{A\cap B} \).
a vector space isomorphism \( A / (A \cap B) \to (A + B) / B \) (namely, \( g' \)). In other words, \((A + B) / B \cong A / (A \cap B)\) as vector spaces. Proposition 7.2 (a) is now proven.

**Proposition 7.2** (a) yields \((A + B) / B \cong A / (A \cap B)\). This yields \(\dim((A + B) / B) = \dim(A / (A \cap B))\) (since isomorphic vector spaces have equal dimensions). Thus, \(\dim((A + B) / B) = \dim(A / (A \cap B)) = \dim A - \dim(A \cap B)\). Comparing this with \(\dim((A + B) / B) = \dim(A + B) - \dim B = \dim A - \dim(A \cap B)\), we obtain \(\dim(A + B) = \dim A + \dim B - \dim(A \cap B)\).

This proves Proposition 7.2 (b).

We notice that Proposition 7.2 (a) holds even if \(k\) is not a field but just a commutative ring (and vector spaces are replaced by \(k\)-modules).

We shall now prove the following fact:

**Proposition 7.3.** Let \(V\) and \(W\) be two finite-dimensional vector spaces. Let \(f : V \times W \to k\) be a bilinear form.

(a) If \(A\) is a subspace of \(V\), then \(\mathcal{L}_f(\mathcal{R}_f(A)) = A + \mathcal{L}_f(W)\).

(b) If \(B\) is a subspace of \(W\), then \(\mathcal{R}_f(\mathcal{L}_f(B)) = B + \mathcal{R}_f(V)\).

**Proof of Proposition 7.3.** The map \(f : V \times W \to k\) is a bilinear form. In other words, the map \(f : V \times W \to k\) is a bilinear map (according to the definition of a “bilinear form”).

(a) Let \(A\) be a subspace of \(V\).

Straightforward applications of Proposition 3.3 (a) show that \(\mathcal{L}_f(W)\) and \(\mathcal{L}_f(\mathcal{R}_f(A))\) are subspaces of \(V\), and thus finite-dimensional.

Corollary 5.7 (b) yields \(A \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\). Also, \(\mathcal{R}_f(A) \subseteq W\), and thus \(\mathcal{L}_f(\mathcal{R}_f(A)) \subseteq \mathcal{L}_f(W)\) (by Proposition 3.4 (a), applied to \(U = k\), \(B = \mathcal{R}_f(A)\) and \(B' = W\)). In other words, \(\mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\). Combined with \(A \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\), this yields \(A + \mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\).

We have \(x \in \text{Ker}(g')\) and thus \(g'(x) = 0\). Hence, \(0 = g'\left(\frac{x}{[a]_{A \cap B}}\right) = g'( [a]_{A \cap B} ) = [a]_B \) (by (24)). Hence, \([a]_B = 0\). In other words, \(a \in B\). Combining this with \(a \in A\), we obtain \(a \in A \cap B\). Hence, \([a]_{A \cap B} = 0\). Thus, \(x = [a]_{A \cap B} = 0\).

Let us now forget that we fixed \(x\). We thus have proven that every \(x \in \text{Ker}(g')\) satisfies \(x = 0\). In other words, \(\text{Ker}(g') = 0\). This shows that \(g'\) is injective (since \(g'\) is a linear map).

**Proof.** Recall the following simple fact from linear algebra: If \(X\), \(Y\) and \(Z\) are three subspaces of \(V\) satisfying \(X \subseteq Y \subseteq Z\), then \(X + Y \subseteq Z\). This fact (applied to \(X = A\), \(Y = \mathcal{L}_f(W)\) and \(Z = \mathcal{L}_f(\mathcal{R}_f(A))\)) shows that \(A + \mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\) (since \(A \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\) and \(\mathcal{L}_f(W) \subseteq \mathcal{L}_f(\mathcal{R}_f(A))\)). Qed.
We have \( A \subseteq V \). Thus, Proposition 3.4 (applied to \( A' = V \) and \( U = k \)) yields \( \mathcal{R}_f (A) \supseteq \mathcal{R}_f (V) \). Therefore, \( \mathcal{R}_f (A) \cap \mathcal{R}_f (V) = \mathcal{R}_f (V) \).

Theorem 6.5 (c) shows that \( \dim (V/\mathcal{L}_f (W)) = \dim (W/\mathcal{R}_f (V)) = \dim W - \dim (\mathcal{R}_f (V)) \). Hence,

\[
\dim W - \dim (\mathcal{R}_f (V)) = \dim (V/\mathcal{L}_f (W)) = \dim V - \dim (\mathcal{L}_f (W)). \tag{25}
\]

Proposition 7.2 (applied to \( B = \mathcal{L}_f (W) \)) shows that

\[
\dim (A + \mathcal{L}_f (W)) = \dim A + \dim (\mathcal{L}_f (W)) - \dim (A \cap \mathcal{L}_f (W)). \tag{26}
\]

But \( \mathcal{R}_f (A) \) is a subspace of \( W \) (by Proposition 3.3 (a), applied to \( U = k \)). Hence, Corollary 7.1 (b) (applied to \( B = \mathcal{R}_f (A) \)) yields

\[
\dim (\mathcal{L}_f (\mathcal{R}_f (A))) = \dim V - \dim (\mathcal{R}_f (A)) + \dim (\mathcal{R}_f (A) \cap \mathcal{R}_f (V)) = \dim V - \dim A + \dim (A \cap \mathcal{L}_f (W)) + \dim (\mathcal{R}_f (V)) = \dim V - \dim W + \dim A - \dim (A \cap \mathcal{L}_f (W)) + \dim (\mathcal{R}_f (V)) = \dim V - \dim (\mathcal{L}_f (W)) + \dim A - \dim (A \cap \mathcal{L}_f (W)) \]

(by 25).

Thus, we know that \( \mathcal{L}_f (\mathcal{R}_f (A)) \) is a finite-dimensional vector space; we know that \( A + \mathcal{L}_f (W) \) is a subspace of \( \mathcal{L}_f (\mathcal{R}_f (A)) \) (since \( A + \mathcal{L}_f (W) \subseteq \mathcal{L}_f (\mathcal{R}_f (A)) \)); we furthermore know that \( \dim (A + \mathcal{L}_f (W)) = \dim (\mathcal{L}_f (\mathcal{R}_f (A))) \) (by 27).

Thus, Proposition 6.3 (applied to \( \mathcal{L}_f (\mathcal{R}_f (A)) \) and \( A + \mathcal{L}_f (W) \) instead of \( B \) and \( A \)) shows that \( A + \mathcal{L}_f (W) = \mathcal{L}_f (\mathcal{R}_f (A)) \). This proves Proposition 7.3 (a).

(b) Let \( B \) be a subspace of \( V \). Proposition 2.10 (a) shows that the map \( f^\text{op} : W \times V \to k \) is a bilinear form. Hence, we can apply Proposition 7.3 (a) to \( W, V, f^\text{op} \) and \( B \) instead of \( V, W, f \) and \( A \). As a result, we obtain \( \mathcal{L}_f (\mathcal{R}_f (B)) = B + \mathcal{L}_f (V) \).

But using straightforward (by now) applications of Proposition 3.2 we can rewrite this as \( \mathcal{R}_f (\mathcal{L}_f (B)) = B + \mathcal{R}_f (V) \). This proves Proposition 7.3 (b).
Corollary 7.4. Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to k$ be a bilinear form.

(a) If $A$ is a subspace of $V$ satisfying $\mathcal{L}_f (W) \subseteq A$, then $\mathcal{L}_f (\mathcal{R}_f (A)) = A$.

(b) If $B$ is a subspace of $W$ satisfying $\mathcal{R}_f (V) \subseteq B$, then $\mathcal{R}_f (\mathcal{L}_f (B)) = B$.

Proof of Corollary 7.4 (a) Let $A$ be a subspace of $V$ satisfying $\mathcal{L}_f (W) \subseteq A$. Proposition 7.3 (a) shows that $\mathcal{L}_f (\mathcal{R}_f (A)) = A + \mathcal{L}_f (W) = A$ (since $\mathcal{L}_f (W) \subseteq A$). Corollary 7.4 (a) is proven.

(b) Let $B$ be a subspace of $W$ satisfying $\mathcal{R}_f (V) \subseteq B$, then $\mathcal{R}_f (\mathcal{L}_f (B)) = B$. Proposition 7.3 (b) shows that $\mathcal{R}_f (\mathcal{L}_f (B)) = B + \mathcal{R}_f (V) = B$ (since $\mathcal{R}_f (V) \subseteq B$). Corollary 7.4 (b) is proven.

Our next claim is an analogue of Proposition 3.10 with the roles of $\cap$ and $+$ interchanged (but also with stricter assumptions, since in general it would not hold):

Proposition 7.5. Let $V$ and $W$ be two finite-dimensional vector spaces. Let $f : V \times W \to k$ be a bilinear form.

(a) If $B_1$ and $B_2$ are two subspaces of $W$ satisfying $\mathcal{R}_f (V) \subseteq B_1$ and $\mathcal{R}_f (V) \subseteq B_2$, then $\mathcal{L}_f (B_1 \cap B_2) = \mathcal{L}_f (B_1) + \mathcal{L}_f (B_2)$.

(b) If $A_1$ and $A_2$ are two subspaces of $V$ satisfying $\mathcal{L}_f (W) \subseteq A_1$ and $\mathcal{L}_f (W) \subseteq A_2$, then $\mathcal{R}_f (A_1 \cap A_2) = \mathcal{R}_f (A_1) + \mathcal{R}_f (A_2)$.

Proof of Proposition 7.5 (a) Let $B_1$ and $B_2$ be two subspaces of $W$ satisfying $\mathcal{R}_f (V) \subseteq B_1$ and $\mathcal{R}_f (V) \subseteq B_2$.

Straightforward applications of Proposition 3.3 (a) show that $\mathcal{L}_f (B_1)$ and $\mathcal{L}_f (B_2)$ are subspaces of $V$.

From $B_1 \subseteq W$, we obtain $\mathcal{L}_f (B_1) \supseteq \mathcal{L}_f (W)$ (by Proposition 3.4 (a), applied to $U = k$, $B = B_1$ and $B' = W$). In other words, $\mathcal{L}_f (W) \subseteq \mathcal{L}_f (B_1)$.

Corollary 7.4 (b) (applied to $B = B_1$) yields $\mathcal{R}_f (\mathcal{L}_f (B_1)) = B_1$ (since $\mathcal{R}_f (V) \subseteq B_1$). Corollary 7.4 (b) (applied to $B = B_2$) yields $\mathcal{R}_f (\mathcal{L}_f (B_2)) = B_2$ (since $\mathcal{R}_f (V) \subseteq B_2$).

Now, $\mathcal{L}_f (W) \subseteq \mathcal{L}_f (B_1) \subseteq \mathcal{L}_f (B_1) + \mathcal{L}_f (B_2)$. Hence, Corollary 7.4 (a) (applied to $A = \mathcal{L}_f (B_1) + \mathcal{L}_f (B_2)$) yields

$$\mathcal{L}_f (\mathcal{R}_f (\mathcal{L}_f (B_1) + \mathcal{L}_f (B_2))) = \mathcal{L}_f (B_1) + \mathcal{L}_f (B_2).$$
Hence,

\[
\mathcal{L}_f(B_1) + \mathcal{L}_f(B_2) = \mathcal{L}_f \left( \frac{\mathcal{R}_f (\mathcal{L}_f (B_1) + \mathcal{L}_f (B_2))}{= \mathcal{R}_f (\mathcal{L}_f (B_1)) \cap \mathcal{R}_f (\mathcal{L}_f (B_2))} \right) \\
= \mathcal{L}_f \left( \frac{\mathcal{R}_f (\mathcal{L}_f (B_1)) \cap \mathcal{R}_f (\mathcal{L}_f (B_2))}{= B_1 \cap = B_2} \right) = \mathcal{L}_f (B_1 \cap B_2).
\]

This proves Proposition 7.5 \((a)\).

\((b)\) The map \(f^{\text{op}}\) is a bilinear form (by Proposition 2.10 \((a)\)). Hence, Proposition 7.5 \((b)\) can be derived from Proposition 7.5 \((a)\) using the same tactic that we used (for example) to derive Proposition 3.4 \((b)\) from Proposition 3.4 \((a)\).