The Redei–Berge symmetric function of a directed graph

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joint work with Richard P. Stanley

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/badboll2023.pdf

paper (draft): https://arxiv.org/abs/2307.05569

- **Definition.** A **digraph** ($=$ directed graph) means a pair (V, A) of a finite set V and a subset $A \subseteq V \times V$.
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  The elements $(u, v) \in A$ are called **arcs** of this digraph, and are drawn accordingly.
  We allow loops ($(u, u) \in A$) and antiparallel arcs ($(u, v) \in A$ and $(v, u) \in A$) but not parallel arcs (A is not a multiset).

The complement of a digraph

- **Definition.** Let $D = (V, A)$ be a digraph. Then, $\overline{D}$ denotes the **complement** of D; this is the digraph $(V, \ (V \times V) \setminus A)$. Its arcs are the **non-arcs** of D.

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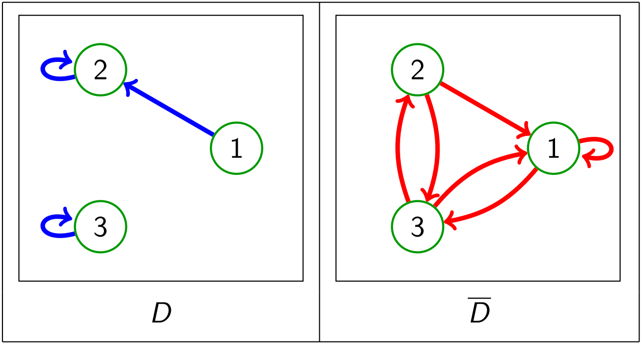
- **Example.**



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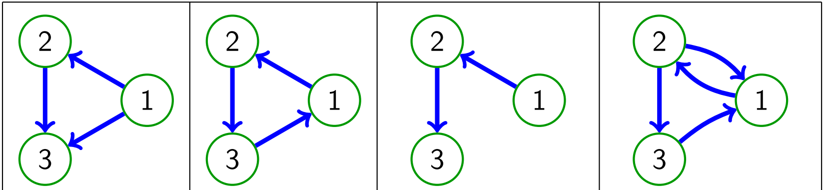
- **Examples.**



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- **Definition.** Let $D = (V, A)$ be a digraph. A **Hamiltonian path** (short: **hamp**) of D means a V-listing $(v_1, v_2, \ldots, v_n)$ such that

  $$(v_i, v_{i+1}) \in A \qquad \text{for each } i \in \{1, 2, \ldots, n-1\}.$$

  In other words (for $V \neq \varnothing$), it means a path of D that contains each vertex.

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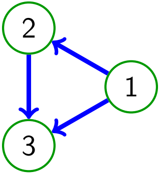
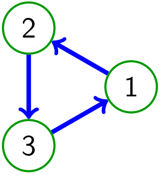
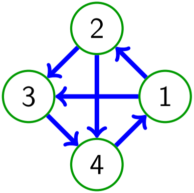
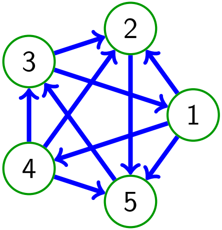
- **Example.** Here are some tournaments:

			
has 1 hamp: $(1, 2, 3)$	has 3 hamps: e.g., $(2, 3, 1)$	has 5 hamps: e.g., $(1, 2, 3, 4)$	has 9 hamps: e.g., $(1, 4, 2, 5, 3)$

- Recall **Redei's Theorem:** Let D be a tournament. Then,

$$(\# \text{ of hamps of } D) \text{ is odd.}$$

- Rédei's proof is complicated and intransparent (see Moon, *Topics on Tournaments* for an English version).

Berge's theorem

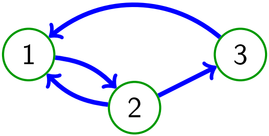
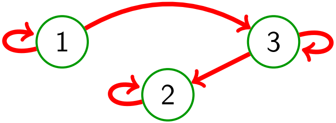
- Recall **Redei's Theorem:** Let D be a tournament. Then,

$$(\# \text{ of hamps of } D) \text{ is odd.}$$

- Rédei's proof is complicated and intransparent (see Moon, *Topics on Tournaments* for an English version).
  To give a more conceptual proof, Berge discovered the following:
- **Theorem (Berge 1976):** Let D be a digraph. Then,

$$(\# \text{ of hamps of } \overline{D}) \equiv (\# \text{ of hamps of } D) \bmod 2.$$

- **Example.**

 D	 $\overline{D}$
has 3 hamps	has 1 hamp

Berge's theorem: questions and further directions

- Berge proves his theorem (in his *Graphs* textbook) using an elegant inclusion-exclusion argument.
  Then he uses his theorem to prove Rédei's theorem via induction on the number of "inversions" (arcs directed the "wrong way").
  This proof is much cleaner than Rédei's, but still far from simple.
  For a detailed exposition, see https://www.cip.ifi.lmu.de/~grinberg/t/17s/5707lec7.pdf .

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- **Remark.** Can we improve on Rédei's theorem even further? MathOverflow question #232751 asks for the possible values of (# of hamps of D) for a tournament D.
  Among the numbers between 1 and 80555, the answer is "all odd numbers except for 7 and 21" (proved by bof and Gordon Royle).
  Question: Are these the only exceptions?

- Independently, Chow (*The Path-Cycle Symmetric Function of a Digraph*, 1996) introduced a symmetric function assigned to each digraph D.
  (This was inspired by Chung/Graham's cover polynomial in rook theory.)
- We only discuss a coarsening of his construction (Chow has two families of variables, and we set the second family to 0).
  Question: Which of the results below can be generalized to the full version?

- **Definition.** Let $n \in \mathbb{N}$, and let I be a subset of $\{1, 2, \ldots, n-1\}$. Then, we define the power series

$$L_{I,n} := \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_p < i_{p+1} \text{ for each } p \in I}} x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$$

(where the indices $i_1, i_2, \ldots, i_n$ range over $\{1, 2, 3, \ldots\}$).
Remark: This is a **(Gessel's) fundamental quasisymmetric function**.

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  Remark: This is a **(Gessel's) fundamental quasisymmetric function**.
- **Definition.** Let $n \in \mathbb{N}$. Let $D = (V, A)$ be a digraph with n vertices. We define the **Redei–Berge symmetric function**

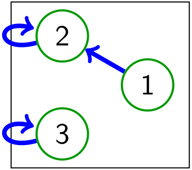
$$U_D := \sum_{w \text{ is a } V\text{-listing}} L_{\mathrm{Des}(w,D),\ n} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]],$$

  where

$$\mathrm{Des}(w, D) := \{i \in \{1, 2, \ldots, n-1\} \mid (w_i, w_{i+1}) \in A\}$$
$$\text{for each } V\text{-listing } w = (w_1, w_2, \ldots, w_n).$$

The Rédei–Berge symmetric function: example

- **Example:** Let

$D =$ 

.

Then,

$$U_D = \sum_{w \text{ is a } V\text{-listing}} L_{\text{Des}(w,D),\ 3}$$

$$= L_{\text{Des}((1,2,3),D),\ 3} + L_{\text{Des}((1,3,2),D),\ 3} + L_{\text{Des}((2,1,3),D),\ 3} + L_{\text{Des}((2,3,1),D),\ 3} + L_{\text{Des}((3,1,2),D),\ 3} + L_{\text{Des}((3,2,1),D),\ 3}$$

$$= L_{\{1\},\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\varnothing,\ 3} + L_{\{2\},\ 3} + L_{\varnothing,\ 3}$$

$$= 4 \cdot L_{\varnothing,\ 3} + L_{\{1\},\ 3} + L_{\{2\},\ 3}$$

$$= 4 \cdot \sum_{i_1 \le i_2 \le i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 < i_2 \le i_3} x_{i_1} x_{i_2} x_{i_3} + \sum_{i_1 \le i_2 < i_3} x_{i_1} x_{i_2} x_{i_3}.$$

- We can restate the definition of U_D directly as follows:
- **Proposition.** Let $D = (V, A)$ be a digraph. Then,

$$U_D = \sum_{f:V\to\{1,2,3,\ldots\}} a_{D,f} \prod_{v\in V} x_{f(v)},$$

where $a_{D,f}$ is the # of all V-listings $w = (w_1, w_2, \ldots, w_n)$ such that

  - we have $f(w_1) \leq f(w_2) \leq \cdots \leq f(w_n)$;
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The Rédei–Berge symmetric function: restatement

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- This is similar (though not directly related) to P-partition enumerators and chromatic symmetric functions.

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- **Remark.** We can restate the definition of $a_{D,f}$ in nicer terms. Namely, fix a digraph $D = (V, A)$ and a map $f : V \to \{1, 2, 3, \ldots\}$. For any $j \in f(V)$, let $\overline{D_j}$ denote the induced subdigraph of the complement $\overline{D}$ on the vertex set $f^{-1}(j) = \{v \in V \mid f(v) = j\}$. Then,

$$a_{D,f} = \prod_{j\in f(V)} \left(\text{\# of hamps of } \overline{D_j}\right).$$

The Rédei–Berge symmetric function: context

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- Thus, any results about U_D might give us information about the # of hamps!
- Formulas for U_D in some specific cases (D acyclic, D poset, D path) can be found in Additional Problem 120 to Chapter 7 of Stanley's EC2.

- I called U_D the "Rédei–Berge symmetric function", but is it actually symmetric? Yes, and in fact something better holds:
- **Definition.** For each $k \geq 1$, let

$$p_k := x_1^k + x_2^k + x_3^k + \cdots$$

be the k**-th power-sum symmetric function**.

- **Theorem.** For any digraph D, we have

$$U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots].$$

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  That is, U_D can be written as a polynomial in $p_1, p_2, p_3, \ldots$ over $\mathbb{Z}$.
- Which polynomial, though?

- **Definition.** Fix a digraph $D = (V, A)$.
  Let $\mathfrak{S}_V$ be the symmetric group on the set V.
  For any $\sigma \in \mathfrak{S}_V$, we let

$$p_{\text{type}\,\sigma} := \prod_{\gamma \text{ is a cycle of } \sigma} p_{\text{length of } \gamma}.$$

  In other words, if σ has cycles of lengths $a, b, \ldots, k$ (including 1-cycles), then $p_{\text{type}\,\sigma} = p_a p_b \cdots p_k$.

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- **Main Theorem I.** Let $D = (V, A)$ be a digraph. Set

  $$\varphi(\sigma) := \sum_{\substack{\gamma \text{ is a cycle of } \sigma; \\ \gamma \text{ is a } D\text{-cycle}}} ((\text{length of } \gamma) - 1) \qquad \text{for each } \sigma \in \mathfrak{S}_V.$$

  Then,

  $$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} (-1)^{\varphi(\sigma)} p_{\text{type}\,\sigma}.$$

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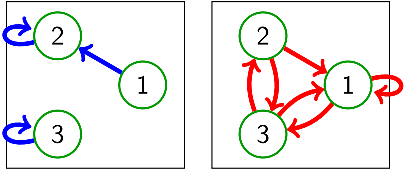
- This yields the $U_D \in \mathbb{Z}[p_1, p_2, p_3, \ldots]$ theorem, of course.

p-expansions: Main Theorem I, example 1

- **Example.** Recall our favorite example:



D $\qquad\qquad$ $\overline{D}$

The cycles of D are $(2)_\sim$ and $(3)_\sim$, whereas the cycles of $\overline{D}$ are $(1)_\sim$, $(2,3)_\sim$, $(3,1)_\sim$ and $(1,3,2)_\sim$.
Thus, the $\sum\limits_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has four addends, corresponding to (σ written in one-line notation)

$\sigma =$	$[1,2,3]$	$[3,1,2]$	$[1,3,2]$	$[3,2,1]$
$(-1)^{\varphi(\sigma)} =$	1	1	1	1
$p_{\operatorname{type}\sigma} =$	p_1^3	p_3	$p_2 p_1$	$p_2 p_1$

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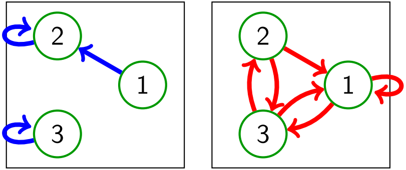

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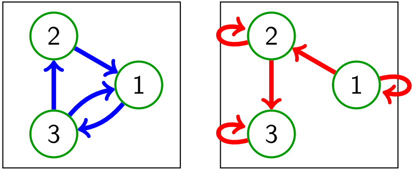
Hence, Main Theorem I yields

$$U_D = p_1^3 + p_3 + p_2 p_1 + p_2 p_1 = p_1^3 + 2 p_1 p_2 + p_3.$$

- **Example.** Let



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Thus, the $\displaystyle\sum_{\substack{\sigma\in\mathfrak{S}_V;\\ \text{each cycle of } \sigma \text{ is}\\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}}$ sum in Main Theorem I has three addends, with

$\sigma =$	$[1,2,3]$	$[3,1,2]$	$[3,2,1]$
$(-1)^{\varphi(\sigma)} =$	1	1	-1
$p_{\text{type}\,\sigma} =$	p_1^3	p_3	$p_2 p_1$

Hence, Main Theorem I yields $U_D = p_1^3 + p_3 - p_2 p_1$.

- Recall **Main Theorem I:** Let $D = (V, A)$ be a digraph. Set

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- Main Theorem I yields Berge's theorem, since the sum for D and the sum for $\overline{D}$ range over the same σ's, and the addends only differ in sign.
- **Corollary.** Let $D = (V, A)$ be a digraph. Assume that every D-cycle has odd length. Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle}}} p_{\text{type}\,\sigma} \in \mathbb{N}[p_1, p_2, p_3, \ldots].$$

- **Main Theorem II.** Let $D = (V, A)$ be a tournament. For each $\sigma \in \mathfrak{S}_V$, let $\psi(\sigma)$ denote the number of nontrivial cycles of σ. (A cycle is called **nontrivial** if it has length > 1.) Then,

$$U_D = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is a } D\text{-cycle}; \\ \text{all cycles of } \sigma \text{ have odd length}}} 2^{\psi(\sigma)} p_{\operatorname{type} \sigma}$$

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- Main Theorem II easily yields Rédei's theorem, as the only addend with $2^{\psi(\sigma)}$ odd is the $\sigma = \operatorname{id}$ addend.

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- **Remark.** Not all p-positive U_D's are explained by this theorem.

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$$\sum_{\substack{f:V\to\{1,2,3,\ldots\};\\ f\circ\sigma=f}} \prod_{v\in V} x_{f(v)} = p_{\operatorname{type}\sigma}.$$

Proof. Easy exercise.

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- Using this lemma (and the above formula for $a_{D,f}$), we can easily reduce Main Theorem I to the following lemma:
- **Main combinatorial lemma.** Let $D = (V, A)$ be a digraph with n vertices. Let $f : V \to \{1, 2, 3, \ldots\}$ be any map. Then,

$$\prod_{j \in f(V)} \left(\# \text{ of hamps of } \overline{D_j}\right) = \sum_{\substack{\sigma \in \mathfrak{S}_V; \\ \text{each cycle of } \sigma \text{ is} \\ \text{a } D\text{-cycle or a } \overline{D}\text{-cycle;} \\ f \circ \sigma = f}} (-1)^{\varphi(\sigma)},$$

where $\overline{D_j}$ is the induced subdigraph of $\overline{D}$ on the vertex set $f^{-1}(j)$.

- So we need to prove the **Main combinatorial lemma:** Let $D = (V, A)$ be a digraph with n vertices. Let $f : V \to \{1, 2, 3, \ldots\}$ be any map. Then,

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- Work on each "level set" $f^{-1}(j)$ separately:
  Main combinatorial lemma (simplified). Let $D = (V, A)$ be a digraph with n vertices. Then,

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- This can be proved using a nontrivial exclusion-inclusion.

- Main Theorems II and III follow from Main Theorem I by combining σ's into equivalence classes by reversing certain cycles.

p-expansions: Proof ideas, 3

- Main Theorems II and III follow from Main Theorem I by combining σ's into equivalence classes by reversing certain cycles.
- The proof of Main Theorem I is detailed in the preprint (https://arxiv.org/abs/2307.05569); the proofs of II and III are outlined.
  These would make a good project for formalization (Coq, Lean, etc.): only elementary combinatorics but some tricky reasoning with cycles and sums.

A surprise

- Rédei's theorem determines the # of hamps of a tournament D modulo 2. What about mod 4?

A surprise

- Rédei's theorem determines the # of hamps of a tournament D modulo 2. What about mod 4?
- **Theorem.** Let D be a tournament. Then,

$$(\#\text{ of hamps of } D) \equiv 1 + 2\,(\#\text{ of nontrivial odd-length } D\text{-cycles}) \bmod 4.$$

  Here, "nontrivial" means "having length > 1".
- We can prove this using Main Theorem II. We have not seen this anywhere in the literature.

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- **Hsin Chieh Liao, Anna Pun, Bruce Sagan, Mike Zabrocki** for helpful comments.
- **the organizers** for the invitation.
- **you** for your patience.