Combinatorics 1: The art of counting (vol. 1 of St Andrews Notes on Advanced Combinatorics)
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`https://cameroncounts.wordpress.com/lecture-notes/`
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Errata and addenda by Darij Grinberg (version of September 10, 2018)

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I will refer to the results appearing in "Combinatorics 1: The art of counting" by the numbers under which they appear in these notes (specifically, in their version of 28 March 2016, available from `https://cameroncounts.wordpress.com/lecture-notes/`).

1. Errata

- **page 6, §1.1, Definition:** Worth saying that you are assuming $n \geq 0$ and $k \geq 0$. This is not how $\binom{n}{k}$ is defined for negative n...
- **page 7, §1.2:** The long formula for $\binom{200}{2}$ goes beyond the page margins. Unless this is meant to make a point about its uselessness, I suggest scaling it down with one of the standard LaTeX techniques.
- **page 11, Proposition 1.9:** A closing parenthesis is missing at the end of the sentence.
- **page 11, §1.4:** "nonp-zero" $\to$ "non-zero".
- **page 12:** In the last displayed equation (or, to be pedantic, align environment) on this page, "$1 - (y/(1-y)\,x)$" should better be "$1 - (y/(1-y))\,x$". (Otherwise, it hinges on correctly interpreting an expression of the form a/bc as $(a/b)\,c$, which some will disagree with.)
- **page 13:** "by the same rule as for the usual binomial coefficients" is not what you mean (that "rule" would be "the number of k-subsets of an n-set", which is not a way to define binomial coefficients for n negative). You mean "by the rule in Proposition 1.2" instead.
- **page 13, Definition:** Replace both "k"s by "l" (or vice versa).
- **page 15:** "only if $k \leq j \leq i$" $\to$ "only if $j \leq k \leq i$".

- **page 15:** In the leftmost of the three 4×4-matrices, the $(4, 3)$-th entry should be 3, not 1.
- **page 16, Exercise 1.5:** "$0 \leq a_i, b_i \leq p-1$" is a dangerous notation: I've seen at least one paper where it meant "$0 \leq a_i$ and $b_i \leq p-1$". Worth explaining what it really means, seeing that this is an exercise.
- **page 16, Exercise 1.5:** Add a period before "Prove that".
- **page 17, Exercise 1.8:** The second sum should start at $k = 0$, not at $k = 1$.
- **page 20, proof of Theorem 2.1:** You talk of "points" and "edges" here, which have not been defined. (I know the picture, but not every reader might...)
- **page 23:** "which is treu" $\to$ "which is true".
- **page 24:** "contraru" $\to$ "contrary".
- **page 32, §3.1:** The condition "$0 \neq 1$" in the definition of a ring is useless and harmful. Often, one has to work with ideals in (e.g., polynomial) rings, and one wants to form the quotient, without knowing whether the quotient will be the trivial ring or not. I understand that requiring $0 \neq 1$ results in some notational simplifications (e.g., one can say that different monomials in the polynomial ring are actually distinct), but these are not worth the damage.
- **page 32, §3.1:** Add a closing parenthesis after "modulo a prime p, $\mathbb{F}_p$".
- **page 32, §3.1:** After "Note that 0 is not a unit", add "unless $0 = 1$".
- **page 33, Definition:** "a formal power series (a_0, a_1, a_2)" $\to$ "a formal power series $(a_0, a_1, a_2, \ldots)$" on the third line of this definition.
- **page 33:** Replace "is the formal power series" by "as the formal power series" twice (in both bullet points), or else change the structure of the sentence so it makes sense grammatically.

  (Also, is it just me being pedantic, or is it worth saying that $\sum_{n \geq 0} \frac{b_n x^n}{n!}$ really means $\sum_{n \geq 0} \frac{b_n}{n!} x^n$? With the way you proceed, infinite sums aren't really defined yet, and so far there is only the notation $\sum_{n \geq 0} a_n x^n$ defined as a shorthand for $(a_0, a_1, a_2, \ldots)$.)
- **page 36:** After "The coefficient a_n is the number of expressions for n as a sum of distinct positive integers", add "(up to order)".

- **pages 36–37, Substitution:** This paragraph appears to suggest that $f(g)$ is defined **only** when g has constant term 0. This is not so. Another important case when $f(g)$ is defined is when f is a polynomial. Yet another is when the constant term of g is nilpotent. These three cases still don't cover all possibilities, since it can also happen that the early coefficients of g only become nilpotent once multiplied with the appropriate coefficients of f; I must admit never having seen this case in nature, but it can occur.

- **page 37:** "descibe" $\to$ "describe".

- **page 37, proof of Proposition 3.2:** The "(Why?)" is misplaced here. You have already tacitly used the uniqueness of an inverse in the Example on page 35, when you wrote "$(1-ax)^{-1}$". Probably a better place for this "(Why?)" is just after the definition of a ring: You can mention that (unlike in a field) inverses don't always exist, but when they do, at least they are unique.

- **page 37, proof of Proposition 3.2:** Add a period after "(Why?)" (or in its place).

- **page 40, Example (Connected permutations):** "permutes the numbers $1, \ldots k$" needs a comma before the "k" (on line 5 of this example).

- **page 41, Substitution inverse:** Again, the word "require" suggests that constant term 0 is necessary, which is incorrect.

- **page 41, Substitution inverse:** "the last result" sounds so vague; why not "Proposition 3.2"?

- **page 42:** "the function $(a+x)^a$" $\to$ "the function $(1+x)^a$".

- **page 42:** In the very last display of this page, a closing parenthesis is missing after "$\log(1+(\exp(x)-1)$".

- **page 43, §3.5:** The word "only" in "we can only substitute" again misleads into thinking that constant term 0 is necessary for substitution.

- **page 43, §3.6:** "in to" $\to$ "into".

- **page 43:** "of any given prime power order" $\to$ "of order p^n for any prime p and any integer $n \geq 1$". (I tend to think that this is clearer for someone not jargon-savvy. Plus, 1 is a prime power depending on whom you ask...)

- **page 44–45:** You spend some time walking around in circles in this computation. You don't need Proposition 3.4 to arrive at the equality $(1-qx)^{-1} =$

$\prod\limits_{i\geq 1}\left(1-x^i\right)^{-a_i}$. Rather, this equality follows from

$$\begin{aligned}
&\prod_{i\geq 1}\left(1-x^i\right)^{-a_i} \\
&= \prod_{\substack{P\in F[x] \\ \text{monic irreducible}}} \left(1-x^{\deg P}\right)^{-1} \\
&= \prod_{\substack{P\in F[x] \\ \text{monic irreducible}}} \left(1+x^{\deg P}+x^{2\deg P}+x^{3\deg P}+\cdots\right) \\
&= \sum_{\substack{(m_P)_{P\in F[x] \text{ monic irreducible}} \text{ a family of} \\ \text{nonnegative integers with finite sum}}} x^{\sum\limits_P m_P \deg P} \qquad \text{(by expanding the product)} \\
&= \sum_{\substack{(m_P)_{P\in F[x] \text{ monic irreducible}} \text{ a family of} \\ \text{nonnegative integers with finite sum}}} x^{\deg\left(\prod\limits_P P^{m_P}\right)} \\
&= \sum_{\substack{Q\in F[x] \\ \text{monic}}} x^{\deg Q} \\
&\qquad \left(\begin{array}{c} \text{since each monic } Q\in F[x] \text{ can be uniquely written as } \prod\limits_P P^{m_P} \\ \text{for some family } (m_P)_{P\in F[x] \text{ monic irreducible}} \text{ of nonnegative integers} \end{array}\right) \\
&= \sum_{n\geq 0} q^n x^n = (1-qx)^{-1}.
\end{aligned}$$

All infinite products make sense for reasons like the ones you discussed.

On the other hand, your argument, I think, includes in it some pieces of a generating-function proof of Proposition 3.4 (if read backwards).

- **page 47, proof of Proposition 3.6:** After "choices of n non-negative integers", add "(order matters)".
- **page 47, proof of Proposition 3.6:** After "put "barriers" into $n-1$ of the boxes", add "(at most one barrier per box)".
- **page 48, Exercise 3.5:** Might be helpful to clarify that the "sequence of partial sums" is $(a_0, a_0+a_1, a_0+a_1+a_2, \ldots)$, rather than $(0, a_0, a_0+a_1, a_0+a_1+a_2, \ldots)$ (which also has a good claim to the title).
- **page 49, Exercise 3.10:** I take it that the list L is assumed to consist of distinct numbers?
- **page 50, §4.1:** After "The general set-up here is sequences", I'd add "$(a_0, a_1, a_2, \ldots)$" (to drive home the point that indexing starts at 0).

- **page 52, §4.1:** "Note that $\dbinom{-2}{n}$" → "Note that $(-1)^n \dbinom{-2}{n}$". Similarly, "Note that $\dbinom{-j}{n}$" → "Note that $(-1)^n \dbinom{-j}{n}$".
- **page 52, Theorem 4.3:** Replace "$h_i(x) = \sum\limits_{j=1}^{m_i} b_{ij} \dbinom{-j}{n}$" by "$h_i(n) = \sum\limits_{j=1}^{m_i} b_{ij} (-1)^n \dbinom{-j}{n}$". (You can also add that $h_i(x) = \sum\limits_{j=1}^{m_i} b_{ij} \dbinom{j+x-1}{j-1}$ so as to make it clear that h_i is a polynomial.)
- **page 54, proof of Theorem 4.4:** "the number of walks" → "the number of walks of length n" (on the second line of this proof).
- **page 55, Proposition 4.5:** Remove the "$\deg(p) < \deg(q)$" condition (here and in the proof of Proposition 4.6 and everywhere else). This condition shouldn't be there; it does not follow from Proposition 4.1 (since c_k may be 0, so the denominator can have degree $< k$), and it rules out various valid linear recurrent sequences (e.g., the sequence $(1, 0, 0, 0, \ldots)$ with $a_n = 1a_{n-1} + 0a_{n-2}$ with $a_0 = 1$ and $a_1 = 0$).
- **page 55, Proposition 4.6:** It is not clear what you mean by "differentiation": Are you taking the derivative of the generating function, or the first difference $(a_{n+1} - a_n)_{n \geq 0}$ of the sequence itself? You mean the former (judged by the proof), but this should be clarified. (Also, maybe mention shifting the sequence, so that the latter also follows.)
- **page 55, proof of Proposition 4.6:** In the last displayed equation on page 55, replace "$A_1(x) A_2(x)$" by "$A_1(x) + A_2(x)$".
- **page 56:** "We had on an earlier problem sheet" seems to be out of date: The exercise you are talking about is Exercise 4.2, which is not on an "earlier problem sheet".
- **page 56:** "this follows from the general result" is an overstatement: The general result shows that the sequence of the squares of the Fibonacci numbers is C-finite, but it does not show that the recurrence relation has 4 terms.
- **page 57, proof of the theorem:** "Proof of the theorem" → "Proof of Proposition 4.7".
- **page 57, proof of the theorem:** In "$b_n = d_a b_{n-1} + \cdots + d_l b_{n-l}$", replace "$d_a$" by "$d_1$".
- **page 57, proof of the theorem:** "the rows satisfy" → "each row satisfy". (Otherwise you are being ambiguous: "the rows satisfy" can also be read as "the sequence of all the rows satisfies".)

- **page 57, proof of the theorem:** "the columns satisfy" → "each column satisfy".
- **page 57, proof of the theorem:** "belongs to the vector space V" → "belongs to the vector space W".
- **page 57, proof of the theorem:** "by the characterisation of C-finite sequences" → "by Lemma 4.8" (as Proposition 4.5 is also a characterisation).
- **page 57, proof of the theorem:** "so it satisfies a recurrence of degree at most kl" is not really clear. Your proof of Lemma 4.8 shows that if all shifts of a lie in a d-dimensional vector space, then a satisfies a linear recurrence, but it does **not** yield a good bound on the degree of this recurrence. In particular, it does not show that the recurrence must have degree d (or less). A slight modification of the proof would do that: Instead of picking a maximal linearly independent set $a^{(l_1)}, \ldots, a^{(l_r)}$ of shifts, you should pick the smallest $m \in \{0, 1, \ldots, d\}$ (where d is the dimension of the subspace containing all shifts of a) such that $a^{(m)}$ is a linear combination of $a^{(0)}, a^{(1)}, \ldots, a^{(m-1)}$. Such an m must exist, since otherwise $a^{(0)}, a^{(1)}, \ldots, a^{(d)}$ would be linearly independent, which would cause the subspace containing the shifts of a to have dimension $> d$.
- **page 57:** After "if we guess a formula for the terms in a C-finite sequence", add "of known degree". Otherwise, we don't know how many values we have to check...
- **page 58, proof of Proposition 4.9:** "of length at most $k + l + 1$" → "of degree at most $k + l$" (you have not defined the "length" of a recurrence, I think).
- **page 58, Proposition 4.10:** What is a_n ?

  (Also, if you are giving so few details, maybe you should cite some paper telling the full story?)
- **page 62, §5.4:** What is the "length" of a Dyck path? (The geometric meaning of "length" is not what you mean, obviously.)=
- **page 62, §5.5:** I'd clarify here that the votes are anonymous, and thus two votes for the same candidates are indistinguishable.

  Same comment for Exercise 3.8 on page 48, by the way.
- **page 63, §5.6:** "maay" → "may".
- **page 64, §5.8:** "Figure 1" → "Figure 4".
- **page 64, §5.8:** "swap the first and by reflecting" → "swap the first and the fourth by reflecting".

- **page 64, §5.8:** Add a closing parenthesis after "the *Wedderburn–Etherington numbers*".
- **page 64, §5.8:** The recurrence relation requires $n > 1$.
- **page 67, §6.2:** "with the space $GF(q)^n$" → "with the space $\operatorname{GF}(q)^n$" (notice the \operatorname).
- **page 68, proof of Theorem 6.3:** At the beginning of this proof, V should be defined ("Let V be an n-dimensional vector space over $\operatorname{GF}(q)$").
- **page 68:** The definition of a "reduced echelon form" that I know also requires that the zero rows are all concentrated at the bottom of the matrix. This is not really relevant at this particular place in the notes, since your matrices don't have zero rows at all; but it becomes important in Exercise 6.4, where the matrices can have zero rows.
- **page 69, §6.2, Example:** After "in reduced echelon form", add "with no zero rows".
- **page 70, proof of Proposition 6.5:** Replace "$\begin{bmatrix} n \\ k-1 \end{bmatrix}_q$" by "$\begin{bmatrix} n-1 \\ k \end{bmatrix}_q$".
- **page 71, Theorem 6.8:** I'd replace "positive integer n" by "nonnegative integer n" here, because why not? Likewise, the induction in the proof should better start at $n = 0$.
- **page 73, proof of Theorem 6.9:** "$F(0,n) = F(m,0) = 0$" should be "$F(0,n) = F(m,0) = 1$".
- **page 73, proof of Theorem 6.9:** After "Now consider $F(m,n)$", add "for $m > 0$ and $n > 0$".
- **page 73, proof of Theorem 6.9:** "using Proposition 6.5" → "using Proposition 6.7".
- **page 73, Exercise 6.4:** "an integer greater than 2" → "any prime power".
- **page 73, Exercise 6.4:** Add "over $\operatorname{GF}(q)$" at the end.
- **page 75, proof of Theorem 7.1:** Replace "$\sum_{i=0}^{k} (-1)^i$" by "$\sum_{i=0}^{k} (-1)^i \binom{k}{i}$".
- **page 76, Theorem 7.2:** The sum should range over "$i \in \{0, \ldots, n\}$", not "$i \in \{1, \ldots, n\}$".
- **page 76, Theorem 7.3:** The sum should end at $i = n$, not at $i = n - 1$. (This is important for the case when $m = 0$.)

- **page 76, proof of Theorem 7.3:** After "to an n-set", add "(which we assume to be $\{1, 2, \ldots, n\}$)".
- **page 76, proof of Theorem 7.3:** The sum in the displayed equation should end at $i = n$, not at $i = n - 1$.
- **page 77:** "close to e" $\to$ "close to e^{-1}".
- **page 77, proof of Corollary 7.5:** "decreasing function of n" $\to$ "decreasing function of i".
- **page 78:** After the inductive proof of the $d_n = nd_{n-1} + (-1)^n$ formula, add a closing parenthesis.
- **page 78:** "we can deduce our formula for d_n" $\to$ "we can deduce Theorem 7.4". There are several formulas for d_n at this point.
- **page 79:** "let G be a given graph" $\to$ "Let G be a given graph".
- **page 79:** "all the connected components of G_I have the same colour" $\to$ "each connected component of G_I uses only one color".
- **page 79:** Somewhere here, the graph G should be required to be loopless.
- **page 80:** In the "antisymmetric" axiom, replace "$B \leq a$" by "$b \leq a$".
- **page 80:** "Every poset" $\to$ "Every finite poset".
- **page 80:** The definition of the incidence algebra only makes sense if the sums $\sum\limits_{a \leq c \leq b}$ are well-defined. This holds, for example, when each interval of P is finite.
- **page 81:** Your argument for the existence of the Möbius function doesn't actually show that it belongs to the incidence algebra. (Each element of the incidence algebra is represented by an upper triangular matrix, but not every upper triangular matrix represents an element of the incidence algebra, unless you define "upper-triangular" with respect to the poset order.)
- **page 81:** "any values of $f(a, b)$" $\to$ "any values $f(a, b)$".
- **page 83:** I'd add that f and g are assumed to be functions $P \to \mathbb{R}$ throughout this page (not $P \times P \to \mathbb{R}$ as on earlier pages).
- **page 83:** You have two different statements called (a), and two different statements called (b). May be worth renaming to clarify what you are referring to when you speak of (a) and (b).

- **page 83:** More details would certainly not hurt around the derivation of the PIE from theorem 7.7. First of all, how do you get the first (a) $\Longleftrightarrow$ (b) equivalence from Theorem 7.7? (You apply it to the functions $f' : P \times P \to \mathbb{R}$ and $g' : P \times P \to \mathbb{R}$ that send each $(a, b) \in P \times P$ to $g(b \setminus a)$ and $f(b \setminus a)$, respectively.) Second, how do you get the second (a) $\Longleftrightarrow$ (b) equivalence from the first (a) $\Longleftrightarrow$ (b) equivalence? (You apply it to the functions $\widetilde{f} : P \to \mathbb{R}$ and $\widetilde{g} : P \to \mathbb{R}$ that send each $a \in P$ to $f(\{1, 2, \ldots, n\} \setminus a)$ and $g(\{1, 2, \ldots, n\} \setminus a)$, respectively.)

- **page 83, The classical Möbius function:** "natural numbers" $\to$ "positive integers". (Having 0 allowed would break the finiteness of intervals.)

- **page 83, The classical Möbius function:** "the product of chains" $\to$ "a down-set of a product of infinite chains $\{0, 1, 2, \ldots\}$". (The whole product of chains is too large, since it would contain elements like $(1, 1, 1, \ldots)$, which would correspond to "supernatural numbers".)

  Perhaps it is better to consider not the whole poset of positive integers, but rather an interval of the form $[1, c]$. This one is a finite product of finitely many chains with nothing up its sleeves.

- **page 83, The classical Möbius function:** "one for each prime power" $\to$ "one for each prime".

- **page 84, The classical Möbius function:** "following functions" $\to$ "following statements for any two functions".

- **page 84, The classical Möbius function:** "Möbis" $\to$ "Möbius".

- **page 84, The classical Möbius function:** Replace "$\geq q^n - nq^{n/2}$" by "$\geq q^n - (n-1) q^{n/2} > 0$". This improved estimate is even more obvious than yours, and renders the special-casing of the case $q = n = 2$ unnecessary.

- **page 84, Subspaces of a vector space:** I'd explain how A becomes a poset.

- **page 85, Subspaces of a vector space:** After "the left-hand side becomes 0", add "whenever $n > 0$".

- **page 85, Exercise 7.1:** You need to assume that $g(0) = 1$ (as you yourself point out in the solution). Or, better, replace "$f(0) = 1$" by "$f(0) = g(0)$". (Of course, you will then have to modify the solution.)

- **page 85, Exercise 7.2:** Worth clarifying that you want the letters of a word to be distinct here (rather than just chosen from a set of n distinct letters).

- **page 85, Exercise 7.4:** "in other words" $\to$ "thus". ("In other words" suggests equivalence.)

- **page 87, Proposition 8.1:** On the right hand side, "k^k" should be "x^k".

- **page 87, proof of Proposition 8.1:** "Given any" $\to$ "Any".
- **page 87, proof of Proposition 8.1:** It should be said that "part of length k" is simply a picturesque way to say "part equal to k".
- **page 87:** "We know that obtaining a recurrence relation for a sequence is equivalent to computing its multiplicative inverse": This is a bit of an oversimplification. I guess it's true for linear recurrences with more-or-less constant coefficients.
- **page 87:** "$\prod_{k\geq 0}\left(1-x^k\right)$" $\to$ "$\prod_{k\geq 1}\left(1-x^k\right)$".
- **page 88, Proposition 8.2:** "$\prod_{k\geq 0}\left(1-x^k\right)$" $\to$ "$\prod_{k\geq 1}\left(1-x^k\right)$".
- **page 88, Proposition 8.2:** I'd clarify that "into an even number of distinct parts" means "into an even number of parts which are furthermore distinct", rather than the weaker "into some number of parts, of which an even number are distinct". (So $3+3+2$ does not count as a partition into an even number of distinct parts, even though it has only an even number of distinct parts.)
- **page 89:** "McMahon" $\to$ "MacMahon".
- **page 89, proof of Euler's Pentagonal Numbers Theorem:** It is worth mentioning the following (easy) facts:
  - If two integers k and l satisfy $k\left(3k-1\right)/2 = l\left(3l-1\right)/2$, then $k=l$. (This follows from noticing that $k\left(3k-1\right)/2 = l\left(3l-1\right)/2$ rewrites as $\left(k-l\right)\left(3k+3l-1\right)=0$, which yields $k-l=0$ because $3k+3l-1\equiv 1\not\equiv 0 \bmod 3$.)
  - If two nonnegative integers k and l satisfy $k\left(3k-1\right)/2 = l\left(3l+1\right)/2$, then $k=l=0$. (This follows by applying the preceding fact to $-l$ instead of l, thus obtaining $k=-l$.)

  These facts are used in the proof of the Theorem, e.g. when you claim at the very end that Class 3 contains only a single partition when n is pentagonal.
- **page 89, proof of Euler's Pentagonal Numbers Theorem:** You need to WLOG assume $n>0$; otherwise the base and the slope aren't well-defined.
- **page 90, proof of Euler's Pentagonal Numbers Theorem:** "removing the slope of λ'" $\to$ "removing the slope of λ".
- **page 91, proof of Euler's Pentagonal Numbers Theorem:** "with $|k|$ parts" $\to$ "with k parts".
- **page 92:** "previoiusly" $\to$ "previously".

- **page 92:** "the three infinite products on the left" → "the infinite product on the left". (The way you have written up this product, it's a single product, with three factors inside; there is no reason to split it now.)

- **page 92:** In the first paragraph on this page, you argue that the power series in Theorem 8.5 are well-defined. This is fine (except for the fact that you have never defined power series in two indeterminates), but perhaps insufficient to ensure that the substitution made in Exercise 8.3 is well-defined. For that purpose, I'd suggest a different argument: Theorem 8.5 can be regarded as an equality in the ring $\left(\mathbb{Z}\left[z^{-1}\right]\right)[[qz]]$ (the ring of formal power series in qz over the polynomial ring $\mathbb{Z}\left[z^{-1}\right]$). Indeed, it can be rewritten as the identity

  $$\prod_{n>0}\left(\left(1+u^{2n-1}v^{2n-2}\right)\left(1+u^{2n-1}v^{2n}\right)\left(1-u^{2n}v^{2n}\right)\right)=\sum_{l\in\mathbb{Z}}u^{l^2}v^{l(l-1)} \quad (1)$$

  in the ring $(\mathbb{Z}[v])[[u]]$, which makes sense because every $l \in \mathbb{Z}$ satisfies $l(l-1) \geq 0$. (The variables u and v correspond to your qz and z^{-1}.) The substitution made in Exercise 8.3 now turns into setting $u \mapsto -x$ and $v \mapsto -x^{1/2}$.

  (You may actually replace v by $v^{1/2}$ in (1) already, since v only appears in even powers in (1).)

- **page 92:** Maybe this is obvious, but I find it worthwhile to state this clearly before you use this language on page 93: An *electron* of a state S means a level in S, whereas a *hole* of a state S means a level that is not in S.

- **page 95, proof of Proposition 9.1:** "remaining k points" → "remaining $k-1$ points".

- **page 95, proof of Proposition 9.2:** The appearance of expressions such as "$\exp(-\exp(x))$" and "$\exp(\exp(x))$" here is unfortunate: They are defined as analytic functions, but not as formal power series. But it's not clear whether $F(x)$ is an analytic function (the convergence has not been studied), so comparing $F(x)$ with an analytic function is not obviously meaningful.

  In my opinion, the easiest way to fix this is to clarify what exactly it means to solve the ODE $\frac{d}{dx}F(x) = \exp(x)F(x)$ in formal power series. Namely, we are using the following fact: If $H(x)$ is a formal power series with constant term 0, and if $F(x)$ is a formal power series satisfying $\frac{d}{dx}F(x) = \left(\frac{d}{dx}H(x)\right)F(x)$, then $F(x) = A\exp(H(x))$ for some scalar A. You are applying this to $H(x) = \exp(x) - 1$ (since $\exp(x) - 1$ has constant term 0 but has the derivative $\exp(x)$), so you obtain $F(x) = A\exp(\exp(x)-1)$. Now, checking the value $F(0) = 1$ results in $A = 1$.

- **page 95, proof of Proposition 9.3:** "behviour" $\to$ "behavior".
- **page 96, proof of Proposition 9.3:** "the sum of" $\to$ "the sum".
- **page 98, first proof of Theorem 9.5:** On the second line of the displayed computation, "x_{l-1}" should be "$(x)_{l-1}$". Also, a linebreak wouldn't hurt.
- **page 99, second proof of Theorem 9.5:** "where $x \geq n$" $\to$ "where $x \in \mathbb{N}$". Similarly, in "for every positive integer $x \geq n$", remove the "$\geq n$". Nothing in your argument seems to require x to be $\geq n$.
- **page 101, proof of Proposition 9.6:** "$s(n,n)$ counts the cyclic permutations" $\to$ "$s(n,1)$ counts the cyclic permutations".
- **page 101, proof of Proposition 9.6:** "with $k-1$ classes" $\to$ "with $k-1$ cycles".
- **page 103, proof of Theorem 9.7:** "equivalence relation on X" $\to$ "equivalence relation on T".
- **page 103, proof of Theorem 9.7:** You argue by "replacing x by $-x$". But at this point, x is still assumed to be a positive integer, so you can't just replace it by its negative. Instead, the replacement should be done after the "both sides are polynomials, hence the equality is a polynomial identity" argument.
- **page 104, Theorem 9.8:** What do you mean by "a_i"?
- **page 105, Exercise 9.4:** "$(a_1.a_2,\ldots)$" $\to$ "$(a_1,a_2,\ldots)$".
- **page 106, Exercise 9.4:** "$(a_1.a_2,\ldots)$" $\to$ "$(a_1,a_2,\ldots)$".
- **page 109:** "betweem" $\to$ "between".
- **page 109:** "it is the largest of the $n+1$ binomial coefficients $\binom{n}{k}$": really? It doesn't even have the form $\binom{n}{k}$.
- **page 110, Exercise 10.2:** "x^m" should be "x^n" in the display.
- **page 112, solution to Exercise 1.3:** In the formula "$1 + i = \sqrt{2}\,\mathrm{e}^{\pi\mathrm{i}/4}$", the first "$i$" should be an "$\mathrm{i}$".

- **page 113, solution to Exercise 1.5:** "lies in a cycle" may mean two different things: either "is a subset of a cycle of σ on $\{1, 2, \ldots, a\}$", or "is an element of a cycle of σ on the set all b-element subsets of $\{1, 2, \ldots, a\}$". You mean the latter, but this is worth disambiguating.

- **page 113, solution to Exercise 1.6:** "a set of n elements" $\rightarrow$ "a set of n unlabelled elements".

- **page 117, solution to Exercise 3.1:** In the first displayed equation, replace "$(-4)^n$" by "$(-4)^n x^n$".

- **page 117, solution to Exercise 3.1:** Add a closing parenthesis after "$(-(2n-1)$".

- **page 118, solution to Exercise 3.6:** "by choosing x_{a_1}" $\rightarrow$ "by choosing x^{a_1}".

- **page 119, solution to Exercise 3.6:** "after the mid-term break" $\rightarrow$ "in §8.2".

- **page 120, solution to Exercise 4.1:** "$b/(1-x)$" $\rightarrow$ "$b/(1+x)$".

- **page 121, solution to Exercise 5.2:** "we have to prove that $f(n) = C_{n+1}$" $\rightarrow$ "we have to prove that $f(n) = C_{n-1}$".

- **page 121, solution to Exercise 5.2:** "by induction, that $f(n) = C_{n+1}$" $\rightarrow$ "by induction, that $f(n) = C_{n-1}$".

- **page 122, solution to Exercise 5.5:** "If k is even" $\rightarrow$ "If n is even".

- **page 123, solution to Exercise 6.2:** After "where A' is a $(k-1) \times (n-1)$ matrix in reduced echelon form", add "with no zero rows".

- **page 123, solution to Exercise 6.2:** "the last row is completely arbitrary" $\rightarrow$ "the last column is completely arbitrary".

- **page 123, solution to Exercise 6.2:** "reduced echelon with" $\rightarrow$ "reduced echelon form with".

- **page 123, solution to Exercise 6.2:** The last paragraph of this solution is somewhat confusing. The problem asks to prove the recurrence relation (Proposition 6.5) using the reduced echelon form interpretation (Proposition 6.4), not the other way round. It makes no sense to refer to "the preceding question" here, since the solution of "the preceding question" already uses Proposition 6.5. What you probably want to say is: If we define $\begin{bmatrix} n \\ k \end{bmatrix}_q$ recursively via Proposition 6.5, then the thus-defined $\begin{bmatrix} n \\ k \end{bmatrix}_q$ will be a polynomial in q; but if we define $\begin{bmatrix} n \\ k \end{bmatrix}_q$ via Proposition 6.4, then we also obtain a polynomial in q. Thus, the equivalence of these definitions holds for all q once it has been proven when q is a prime power.

- **page 123, solution to Exercise 6.3:** "As we did in the lectures" $\to$ "As we did in the Example after Proposition 6.4".
- **page 123, solution to Exercise 6.3:** "the reduced echelon matrices" $\to$ "the reduced echelon $k \times n$ matrices with no zero rows".
- **page 124, solution to Exercise 6.3:** In "not in the column with index in A", replace "the column" by "a column".
- **page 124, solution to Exercise 6.3:** "Recall that if" $\to$ "Recall that".
- **page 124, solution to Exercise 6.3:** Replace "$a(p)$" by "$A(p)$" (twice). (That's the notation you used in Theorem 6.9.)
- **page 124, solution to Exercise 6.3:** "108°" $\to$ "180°".
- **page 124, solution to Exercise 6.3:** Let me suggest another solution, easier than both of yours: Just substitute $1/q$ for q into the formula defining $\begin{bmatrix} n \\ k \end{bmatrix}_q$, and conclude (after some obvious simplifications) that $\begin{bmatrix} n \\ k \end{bmatrix}_{1/q} = \dfrac{1}{q^{k(n-k)}} \begin{bmatrix} n \\ k \end{bmatrix}_q$.
- **page 124, solution to Exercise 6.4:** In the second paragraph: "For each matrix in reduced echelon form" $\to$ "For each matrix in echelon form".
- **page 124, solution to Exercise 6.4:** Here is an easier solution:

  An $n \times n$-matrix M over $\mathbb{F}_q$ in echelon form can be constructed as follows:

  - Decide which of the n columns of M will contain leading ones. Say you have chosen the columns $i_1, i_2, \ldots, i_k$ to contain leading ones, with $i_1 < i_2 < \cdots < i_k$. The leading ones in these columns will necessarily lie in rows $1, 2, \ldots, k$, respectively (by the definition of echelon form).
  - Arbitrarily fill in the entries of M to the right of the leading ones. For each $h \in \{1, 2, \ldots, k\}$, there are exactly $n - i_h$ such entries in row h. Thus, the total number of such entries is $\sum\limits_{h=1}^{k} (n - i_h)$. Hence, there is a total of $q^{\sum\limits_{h=1}^{k}(n-i_h)} = \prod\limits_{h=1}^{k} q^{n-i_h}$ possibilities to fill in these entries.

  Thus, the total number of ways to construct such a matrix M is

  $$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \prod_{h=1}^{k} q^{n-i_h} = \prod_{i=1}^{n} \left(1 + q^{n-i}\right) = \prod_{i=1}^{n} \left(1 + q^{i-1}\right)$$

  (here, we have substituted $n + 1 - i$ for i in the product). The exercise is solved.

- **page 124, solution to Exercise 6.5:** Here is an alternative way to prove that $C(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$:

  A string consisting of $n-k$ letters x and k letters y can be identified with a lattice path from $(0,0)$ to $(n-k,k)$ using northerly and easterly steps only. (In fact, read the string from left to right, taking a step east for each letter x encountered and a step north for each letter y encountered.) The number J of jumps required to transform such a string into $x^{n-k}y^k$ is precisely the area $A(p)$ induced by the lattice path p corresponding to this string (because each jump presses the lattice path one box to the right, hence reducing the area by 1). Hence, $C(n,k)$ is the sum of $q^{A(p)}$ over all lattice paths from $(0,0)$ to $(n-k,k)$ using northerly and easterly steps only. But by Theorem 6.9 (applied to $n-k$ and k instead of m and n), this latter sum equals $\begin{bmatrix} n \\ n-k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q$. So we have proven that $C(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

- **page 126, solution to Exercise 7.2:** "If $F(n-1)$ words" $\to$ "We know that $F(n-1)$ words".

- **page 126, solution to Exercise 7.2:** "$F(n) - n!\,\mathrm{e}$" $\to$ "$|F(n) - n!\,\mathrm{e}|$" (twice).

- **page 126, solution to Exercise 7.2:** The "$\leq$" sign in the long computation should be replaced by "$<$", since otherwise you don't get the conclusion you want.

- **page 126, solution to Exercise 7.2:** "for $n \geq 1$" $\to$ "for $n > 1$".

- **page 126, solution to Exercise 7.3:** "for $d(n)/n!$" $\to$ "for $d_n/n!$".

- **page 126, solution to Exercise 7.3:** "clearlyu" $\to$ "clearly".

- **page 127, solution to Exercise 7.4:** In the first paragraph, it might be easier to argue not by removing edges from the full tree, but rather by adding edges to the empty forest. This way, you don't need to use the fact that a tree on n vertices has $n-1$ edges, but instead obtain this fact as a side-result.

- **page 127, solution to Exercise 7.4:** The right hand side of the displayed equality should be "$q(q-1)^{n-1}$", not "$q(q-1)^n$".

- **page 127, solution to Exercise 7.4:** The "(starting at a leaf)" part is unnecessary and complicates the proof. (It is true that you can pick any leaf as the first vertex of the list; but the only way I see to prove this is to show the stronger claim that you can pick any **vertex** as the first vertex of the list.)

- **page 127, solution to Exercise 7.4:** After "each vertex", add "(starting with the second one)".

- **page 127, solution to Exercise 7.5:** I don't think you have defined what you mean by "$P_{C_n}(q)$".
- **page 127, solution to Exercise 7.6:** "if $s \not\leq y$" $\to$ "if $x \not\leq y$".
- **page 127, solution to Exercise 7.6:** In the first displayed equation of this solution, add a closing parenthesis at the end.
- **page 128, solution to Exercise 7.7:** "This a number" $\to$ "This is a number".
- **page 128, solution to Exercise 7.8:** Replace "a_I" by "b_I" (twice), and replace "a'_I" by "b'_I" (twice).
- **page 128, solution to Exercise 7.9:** "so there are $(n-1)^{2k}$ such choices" $\to$ "so there are $(n-1)^{n-2k}$ such choices".
- **page 129, solution to Exercise 7.9:** The three appearances of the expressions "$2^k k!$" on this page should all be parenthesized, methinks.
- **page 129, solution to Exercise 7.9:** "in which at least a given collection of k pairs are looking at each other (and maybe more)" $\to$ "of a set of k pairs and of a configuration such that each of the k pairs (and maybe more) are looking at each other". There is certainly no "given collection" here, or else there would be no $(n)_{2k}$ factor!
- **page 129, solution to Exercise 7.9:** Replace "$n(n-1)^{n-2k}$" by "$(n-1)^{n-2k}$" twice (in the two displayed equations).
- **page 129, solution to Exercise 8.4:** Where is Exercise 8.4?
- **page 130, solution to Exercise 8.4:** Headless summation sign in the display.
- **page 132, solution to Exercise 10.1:** Overfull hbox.