Classical Invariant Theory - A Primer
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Errata and questions (by Darij Grinberg)[1]

Section 1

- **Page 2:** Between the definition and Exercise 4, you write: "and the *stabilizer* of w it the subgroup $G_w := \{g \in G \mid gw = w\}$". The word "it" should be "is" here.

- **Page 4, Example 3:** In the formula

$$\sigma_2 := x_1x_2 + x_1x_3 + \cdots x_{n-1}x_n,$$

a plus sign is missing before $x_{n-1}x_n$.

- **Page 5, first line:** The verb form of "proof" is "prove", not "proof". This mistake is repeated several times through the text, and if you wish to correct it, the fastest way would be to search for "proof" in your tex file.

- **Page 6, Exercise 9:** The sentence "The V_n are the classical *binary forms* of degree n." is a bit confusing. I think "The V_n are the classical *spaces of binary forms* of degree n." would be better.

- **Page 6, Exercise 11:** I think it would be better to introduce $\gamma_0, ..., \gamma_n$ before introducing f and h, since the point is to find $\gamma_0, ..., \gamma_n$ which do not depend on f and h.

- **Page 7:** Replace "is a restriction an invariant" by "is a restriction of an invariant".

- **Page 9, Exercise 21:** I am surprised that you never come back to this nice exercise! It nicely generalizes to SL_n for arbitrary $n \in \mathbb{N}$ whenever K is a field of characteristic 0. The coordinate ring of SL_n is $K[\mathrm{SL}_n] = K[\mathrm{M}_n] \diagup (\det - 1)$, and the invariant ring $K[\mathrm{SL}_n]^{U_n}$ (where U_n acts on SL_n by left multiplication) is generated by the $k \times k$ minors extracted from the last k rows of the matrix for $k = 1, 2, ..., n-1$ (or $k = 1, 2, ..., n$, which doesn't change anything). In order to prove this, we can proceed as follows[2]:

  - Fix $n \in \mathbb{N}$. It is easy to see that $K[\mathrm{SL}_n] = K[\mathrm{M}_n] \diagup (\det - 1)$.
  - Recall the fact that a surjective G-linear homomorphism $f : A \to B$ between two completely reducible representations A and B of a group G always restricts to a surjective homomorphism $A^G \to B^G$. This can be generalized as follows: If H is a subgroup of some group G, and if A and B

[1] updated 13 December 2016
[2] The following outline of a proof uses the results of Sections 5–7.

are two representations of G such that A is completely reducible (as a G-module), and if $f : A \to B$ is a surjective G-linear homomorphism, then f restricts to a surjective homomorphism $A^H \to B^H$ of vector spaces[3]. Let us call this generalized fact the "extended Schmid lemma" (since my use of this lemma is similar to the trick used by Barbara Schmid in her proof of your Exercise 33).

– Now, let V be the K-vector space K^n, and let $p \in \mathbb{N}$ be arbitrary. Then, we can identify the K-vector space V^p with the K-vector space $K^{n\times p}$ of $n \times p$-matrices (by equating every p-tuple $(v_1, v_2, \ldots, v_p) \in V^p$ with the $n \times p$-matrix whose columns are $v_1, v_2, \ldots, v_p$). The group GL_p thus acts on V^p via right multiplication.[4] Let us denote this action by $\rightharpoonup$ (that is, we write $A \rightharpoonup B$ for the image of a $B \in V^p$ under the action of an $A \in \mathrm{GL}_p$ with respect to this action).

– We let U_p^- denote the group of the lower triangular unipotent matrices in GL_n. The group U_p^- is a subgroup of GL_p, and thus also acts on V^p (by restricting the GL_p-action $\rightharpoonup$ on V^p). We denote this latter action by $\rightharpoonup$ as well.

– We define a group homomorphism $\zeta : U_p \to U_p^-$ by setting $\zeta(A) = \left(A^T\right)^{-1}$ for every $A \in U_p$. This ζ allows to transform the action $\rightharpoonup$ of U_p^- on V^p into an action of U_p on V^p (by restriction); let us denote this latter action by $\rightharpoondown$ (that is, we write $A \rightharpoondown B$ for the image of a $B \in V^p$

[3] *Sketch of a proof.* Let H be a subgroup of some group G. Let A and B be two representations of G such that A is completely reducible (as a G-module). Let $f : A \to B$ be a surjective G-linear homomorphism. We need to prove that f restricts to a surjective homomorphism $A^H \to B^H$ of vector spaces. In other words, we need to prove that $f\left(A^H\right) = B^H$ (since it is clear that f restricts to a homomorphism $A^H \to B^H$ of vector spaces (since $f\left(A^H\right) \subseteq \left(\underbrace{f(A)}_{\subseteq B}\right)^H \subseteq B^H$)). Clearly, $\operatorname{Ker} f$ is a G-submodule of A (since f is a G-linear homomorphism). But the G-module A is completely reducible. In other words, every G-submodule of A is a direct summand of A. Thus, $\operatorname{Ker} f$ is a direct summand of A (since $\operatorname{Ker} f$ is a G-submodule of A). In other words, there exists a G-submodule A' of A such that $A = A' \oplus \operatorname{Ker} f$. Consider this A'. We have $A = A' \oplus \operatorname{Ker} f = A' + \operatorname{Ker} f$ and $A' \cap \operatorname{Ker} f = 0$ (since $A' \oplus \operatorname{Ker} f$ is an internal direct sum). Since f is surjective, we have $B = f\left(\underbrace{A}_{=A'+\operatorname{Ker} f}\right) = f\left(A' + \operatorname{Ker} f\right) = f\left(A'\right) + \underbrace{f\left(\operatorname{Ker} f\right)}_{=0} = f\left(A'\right) = \left(f\mid_{A'}\right)\left(A'\right)$, and thus the map $f\mid_{A'} : A' \to B$ is surjective. The map $f\mid_{A'}$ is also injective (since $\operatorname{Ker}\left(f\mid_{A'}\right) = A' \cap \operatorname{Ker} f = 0$) and G-linear. Thus, $f\mid_{A'} : A' \to B$ is an isomorphism of G-modules. Hence, $\left(f\mid_{A'}\right)\left(\left(A'\right)^H\right) = B^H$. Thus, $B^H = \left(f\mid_{A'}\right)\left(\left(A'\right)^H\right) = f\left(\underbrace{\left(A'\right)^H}_{\subseteq A^H}\right) \subseteq f\left(A^H\right)$. Combined with $f\left(A^H\right) \subseteq \left(\underbrace{f(A)}_{\subseteq B}\right)^H \subseteq B^H$, this yields $f\left(A^H\right) = B^H$, qed.

[4] This is the same as saying that the group GL_p acts on V^p via the identification $V^p = \operatorname{Hom}\left(K^p, V\right)$.

under the action of an $A \in U_p$ with respect to this latter action). Then,

$$A \rightharpoondown B = \underbrace{\zeta(A)}_{=(A^T)^{-1}} \rightharpoonup B = \left(A^T\right)^{-1} \rightharpoonup B$$

for any $A \in U_p$ and $B \in V^p$. If we identify V^p with $K^{n\times p}$ as explained above, then this simplifies to

$$\begin{aligned} A \rightharpoondown B &= \left(A^T\right)^{-1} \rightharpoonup B = B\left(\left(A^T\right)^{-1}\right)^{-1} \\ &\qquad \text{(since the action } \rightharpoonup \text{ is given by right multiplication)} \\ &= BA^T \qquad (1) \end{aligned}$$

for any $A \in U_p$ and $B \in V^p = K^{n\times p}$.

– Notice that ζ is a group isomorphism. Hence, the action $\rightharpoonup$ of U_p^- on V^p and the action $\rightharpoondown$ of U_p on V^p are "the same action modulo renaming the group elements". In particular, this shows that $K[V^p]^{U_p^-} = K[V^p]^{U_p}$ (where the action $\rightharpoonup$ is used in defining $K[V^p]^{U_p^-}$, and the action $\rightharpoondown$ is used in defining $K[V^p]^{U_p}$).

– In 5.7 Corollary 1, we have shown that if W is a GL_n-module, then W is simple if and only if $\dim W^{U_n} = 1$. A similar argument shows that if W is a GL_n-module, then W is simple if and only if $\dim W^{U_n^-} = 1$ [5]. Using this fact, we can see that

$$\dim L_\lambda(p)^{U_p^-} = 1 \qquad (2)$$

whenever λ is a dominant weight of height $\leq p$. (The proof of (2) is analogous to the proof of $\dim L_\lambda(p)^{U_p} = 1$.)

– Proposition 7.8 shows that the invariant ring $K[V^p]^{U_p}$, where the action of U_p on V^p is given by restricting the action $\rightharpoonup$ of GL_p on V^p, is generated by the $k\times k$-minors extracted from the first k columns of the matrix X for $k = 1, 2, ..., n$. Similarly we can show that the invariant ring $K[V^p]^{U_p^-}$ is generated by the $k \times k$-minors extracted from the **last** k columns of the matrix X for $k = 1, 2, ..., n$. (The proof is analogous to the proof of Proposition 7.8, but now we need to use (2) instead of $\dim L_\lambda(p)^{U_p} = 1$.)

– Now, set $p = n$, so that $V^p = K^{n\times p} = K^{n\times n}$. Let $\mathbf{i} : \mathrm{M}_n \to K^{n\times n}$ be the K-vector space isomorphism which sends every matrix $A \in \mathrm{M}_n$ to its transpose $A^T \in K^{n\times n}$. Of course, M_n and $K^{n\times n}$ are identical as sets, but we regard them as endowed with two different U_n-module structures: Namely, the U_n-module structure on M_n is given by left multiplication, whereas the U_n-module structure on $K^{n\times n}$ is the U_p-module structure $\rightharpoondown$ on V^p defined above (this makes sense since $n = p$ and $K^{n\times n} = V^p$). Recall that this latter structure satisfies (1) for every $A \in U_n$ and $B \in K^{n\times n}$.

[5] To prove this, it is enough to make some straightforward changes to the proofs of Proposition 5.7 and 5.7 Corollary 1 (that is, replace "U_n", "U_n^-", "$i < j$", "$\prec$", "$\succ$", "maximal" and "lower triangular" by "U_n^-", "U_n", "$i > j$", "$\succ$", "$\prec$", "minimal" and "upper triangular", respectively).

- It is straightforward to show that any $A \in U_n$ and $B \in \mathrm{M}_n$ satisfy $\mathbf{i}(AB) = A \rightharpoondown \mathbf{i}(B)$. In other words, $\mathbf{i} : \mathrm{M}_n \to K^{n\times n}$ is a U_n-module homomorphism with respect to the U_n-module structures on M_n and on $K^{n\times n}$ that we have just described. Thus, $\mathbf{i} : \mathrm{M}_n \to K^{n\times n}$ is a U_n-module isomorphism with respect to these structures (since $\mathbf{i}$ is a K-vector space isomorphism). Thus, it induces a U_n-module isomorphism $K[\mathbf{i}] : K[K^{n\times n}] \to K[\mathrm{M}_n]$ (which sends every polynomial map $p \in K[K^{n\times n}]$ to the composition $p \circ \mathbf{i}$). We have $K[\mathrm{M}_n]^{U_n} = (K[\mathbf{i}])\left(K[K^{n\times n}]^{U_n}\right)$ (since $K[\mathbf{i}]$ is a U_n-module isomorphism). But since $K^{n\times n} = V^p$ and $n = p$, we have
$$K\left[K^{n\times n}\right]^{U_n} = K[V^p]^{U_p} = K[V^p]^{U_p^-}$$
(where the action $\rightharpoonup$ is used in defining $K[V^p]^{U_p^-}$, and the action $\rightharpoondown$ is used in defining $K[V^p]^{U_p}$). Thus,
$$K[\mathrm{M}_n]^{U_n} = (K[\mathbf{i}])\left(\underbrace{K\left[K^{n\times n}\right]^{U_n}}_{=K[V^p]^{U_p^-}}\right) = (K[\mathbf{i}])\left(K[V^p]^{U_p^-}\right).$$
Thus, the ring $K[\mathrm{M}_n]^{U_n}$ is generated by the $k\times k$-minors extracted from the last k **rows** of the matrix X for $k = 1, 2, ..., n$ (because the ring $K[V^p]^{U_p^-}$ is generated by the $k\times k$-minors extracted from the last k **columns** of the matrix X for $k = 1, 2, ..., n$, and because $\mathbf{i}$ is the map which sends every matrix to its transpose).
- We have $K[\mathrm{SL}_n] = K[\mathrm{M}_n] \diagup (\det -1)$. That is, we have a canonical surjection $K[\mathrm{M}_n] \to K[\mathrm{SL}_n]$. This surjection is a ring homomorphism and is SL_n-linear (where SL_n acts on both M_n and SL_n by left multiplication). Let us denote this surjection by f. Applying the extended Schmid lemma to $G = \mathrm{SL}_n$, $H = U_n$, $A = K[\mathrm{M}_n]$ and $B = K[\mathrm{SL}_n]$, we thus conclude that f restricts to a surjective homomorphism $K[\mathrm{M}_n]^{U_n} \to K[\mathrm{SL}_n]^{U_n}$ (since we know that $K[\mathrm{M}_n]$ is completely reducible as a SL_n-module (according to 5.4 Proposition 2)). Thus, $K[\mathrm{SL}_n]^{U_n} = f\left(K[\mathrm{M}_n]^{U_n}\right)$. Hence, the ring $K[\mathrm{SL}_n]^{U_n}$ is generated by the $k\times k$-minors extracted from the last k rows of the matrix X for $k = 1, 2, ..., n$. The $k\times k$-minors for $k = n$ in this generating set are redundant (because there is only one of them – namely, $\det X$ –, and it is identically to 1 (since we are in $K[\mathrm{SL}_n]$)); thus, we conclude that the ring $K[\mathrm{SL}_n]^{U_n}$ is generated by the $k\times k$-minors extracted from the last k rows of the matrix X for $k = 1, 2, ..., n-1$.

Thus, our proof is complete.

- **Page 9, second line from the bottom:** You write: "and let $p : V_1 \otimes V_2 \to U$ be a linear projection [...]". The "linear" means "G-linear" here, not "K-linear" as I first thought. This may be worth pointing out.
- **Page 10, Example 3:** There are several GL_n-module structures on M_n. Here you apparently mean the adjoint structure; better to state this explicitly?

- **Page 13, Exercise 30:** Nothing wrong here, but I believe you can strengthen the "canonical" to "unique" here.
  Besides, the K-algebra A needs not be commutative. It can even be any arbitrary K-vector space with a K-bilinear "multiplication", such as a Lie algebra.

- **Page 14, two lines below Exercise 31:** "characterstic" should be "characteristic".

- **Page 14, proof of Theorem 2:** You write: "Clearly, we have $p_j = \sum_{|\rho|=j} j_\rho \cdot z_1^{\rho_1} z_2^{\rho_2} \cdots z_n^{\rho_n}$." This should be $p_j = \sum_{|\rho|=j} \binom{j}{\rho_1, \rho_2, ..., \rho_n} j_\rho \cdot z_1^{\rho_1} z_2^{\rho_2} \cdots z_n^{\rho_n}$, where $\binom{j}{\rho_1, \rho_2, ..., \rho_n}$ denotes a multinomial coefficient. Fortunately, this multinomial coefficient is positive, so it doesn't create any troubles in the proof (neither in the $\operatorname{char} K = 0$ nor in the $\operatorname{char} K > |G|$ case).

- **Page 15, first line:** Typo: "symmertic" should be "symmetric".

Section 2

- **Page 18, second line from the bottom:** "bases" is the plural form of "basis". The right singular form is "basis", not "bases". This mistake is repeated some more times in your text.

- **Page 20, Exercise 4:** I am not sure about this one, but I *believe* that you mean "geometrically diagonalizable matrices" (i. e., matrices diagonalizable over the algebraic closure of K) when you say "diagonalizable matrices" here. Otherwise I really have no idea how to solve the exercise with your hint. Fortunately, the power of this exercise does not dwindle from restricting it to geometrically diagonalizable matrices.

- **Page 21, Example:** You write: "Since f is a polynomial function on $\mathrm{M}_2' \times \mathrm{M}_2'$ and the given invariants are algebraically independent, it follows that f must be a polynomial function in these invariants." I don't understand this step - as far as I understand (from `http://mathoverflow.net/questions/32427`) the problem of finding a generating set for the quotient field of the invariant ring is much easier than the problem of finding a generating set for the invariant ring itself, and algebraic independence of the generating set isn't enough either.

- **Page 22, Remark:** You refer to "Chapter II" – is this some sequel that is being planned for the text?

- **Page 22, Remark:** Maybe it would be better to remind the reader that n means $\dim V$ here.

Section 3

- **Page 24, Proof of Lemma:** Replace "an" by "and" in "[...] belong to the same orbit under $\mathcal{S}_m$ if an only if [...]".

- **Page 27, Decomposition Theorem, part (c):** In $\mathrm{M}_\lambda \otimes L_\lambda$, the M should be an italicized M.

- **Page 27, proof of the Decomposition Theorem:** You write: "For the last statement it remains to show that the endomorphism ring of every simple $\mathcal{S}_m$-modules M_λ [...]". There is a typo here ("$\mathcal{S}_m$-modules" should be "$\mathcal{S}_m$-module").

- **Page 28, Remark:** You write that "$M_\lambda = M_\lambda^\circ \otimes_\mathbb{Q} K$ and $L_\lambda = L_\lambda^\circ \otimes_\mathbb{Q} K$, where M_λ° is a simple $\mathbb{Q}[\mathcal{S}_m]$-module L_λ° a simple $\mathrm{GL}(\mathbb{Q})$-module." First, there is an "and" missing in this sentence, but there is some more substantial problem: What does $\mathrm{GL}(\mathbb{Q})$ mean? Probably you want to say $\mathrm{GL}(V')$ where V' is a $\mathbb{Q}$-vector space such that $V \cong V' \otimes_\mathbb{Q} K$. It seems to me that there is a better way to formulate this: If V is a $\mathbb{Q}$-vector space, then $L_\lambda(V \otimes_\mathbb{Q} K) = L_\lambda(V) \otimes_\mathbb{Q} K$.

Section 4

- **Page 32, the end of §4.2:** The fifth line of a 5-lines long computation says:

  $$= f_{\sigma^{-1}}(v_1 \otimes \cdots \otimes v_m \otimes \varphi_1 \otimes \cdots \otimes \varphi_m).$$

  The σ^{-1} should be a σ here, unless I am mistaken.

- **Page 32, the end of §4.2:** You write: "Thus, $\alpha\langle\mathcal{S}_m\rangle = \langle f_\sigma \mid \sigma \in \mathcal{S}_p\rangle$ and the claim follows." I guess the p here should be an m.

- **Page 33, proof of the Claim:** You write: "Hence, the dual map $\widetilde{\beta}^*$ identifies the multilinear invariants of $\mathrm{End}(V)^m$ with those of $V^m \otimes V^{*m}$." Isn't the $\otimes$ symbol supposed to be a $\oplus$ symbol?

- **Page 33, proof of the Claim:** The second line of a 5-lines long computation says:

  $$= \mathrm{Tr}_\sigma(\beta(v_1 \otimes \varphi_1)\,\beta(v_2 \otimes \varphi_2)\cdots).$$

  I think there should be commata between the β's here:

  $$= \mathrm{Tr}_\sigma(\beta(v_1 \otimes \varphi_1), \beta(v_2 \otimes \varphi_2), \cdots).$$

- **Page 33, proof of the Claim:** On the right hand side of the formula

  $$\prod_{i=1}^{m} \varphi_i\left(v_{\sigma(i)}\right) = f_\sigma(v_1 \otimes \cdots \otimes v_m \otimes \varphi_1 \otimes \cdots \otimes \varphi_m),$$

  the σ should be a σ^{-1} this time.

- **Page 35, Lemma:** In the formula

$$\mathcal{P} : K\left[V_1 \oplus \cdots \oplus V_r\right]_{(d_1,...,v_r)} \to K\left[V_1^{d_1} \oplus \cdots \oplus V_r^{d_r}\right]_{\text{multlin}},$$

  the index $(d_1, ..., v_r)$ should be $(d_1, ..., d_r)$.

- **Page 36, §4.7:** You write: "The restitution of the invariant Tr_σ is a product of functions of the form $\operatorname{Tr}(i_1, ..., i_k)$." What you call $\operatorname{Tr}(i_1, ..., i_k)$ here has originally been denoted by $\operatorname{Tr}_{i_1 \ldots i_k}$ in §2.5.

- **Page 37:** You write: "Now it follows from the FFT for $\operatorname{GL}(V)$ (2.1) that $rd = s$ and that H is a scalar multiple of the invariant $(1 \mid 1)^d (2 \mid 1)^d \cdots (r \mid 1)^d$." I think $(1 \mid 1)^d (2 \mid 1)^d \cdots (r \mid 1)^d$ should be $(1 \mid 1)^d (1 \mid 2)^d \cdots (1 \mid r)^d$ here.

- **Page 37:** You write: "On the other hand, starting with $h = \varepsilon$ [...]". I think that starting with $h = \varepsilon$ does not help, as the polarization of a polynomial of degree r (such as h) has nothing to do with the polarization of a polynomial of degree 1 (such as ε). It would rather make sense to start with $h = \varepsilon^r$. Am I missing something?

Section 5

- **Page 38, Exercise 1:** A word "be" is missing in "Let $\rho : \operatorname{GL}(V) \to \operatorname{GL}_N(K)$ an irreducible [...]".

- **Page 40, Exercise 5:** I would rather write

$$\lambda_1 \wedge \cdots \wedge \lambda_j \mapsto \left(v_1 \wedge \cdots \wedge v_j \mapsto \sum_{\sigma \in \mathcal{S}_j} \operatorname{sgn} \sigma \ \lambda_1\left(v_{\sigma(1)}\right) \cdots \lambda_j\left(v_{\sigma(j)}\right)\right)$$

  instead of

$$\lambda_1 \wedge \cdots \wedge \lambda_j : v_1 \wedge \cdots \wedge v_j \mapsto \sum_{\sigma \in \mathcal{S}_j} \operatorname{sgn} \sigma \ \lambda_1\left(v_{\sigma(1)}\right) \cdots \lambda_j\left(v_{\sigma(j)}\right)$$

  here.

- **Page 40, Exercise 6:** There are some mistakes here:

  - The definition of μ has two typos: A $\wedge$ sign is missing in the $e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots e_n$ term, and (more importantly) there is a factor of $(-1)^i$ (or $(-1)^{i-1}$, depending on your preferences) missing before this term.
  - In Assertion (a), the equality "$\mu(g\omega) = \det g \cdot \mu(\omega)$" should be "$\mu(g\omega) = \det g \cdot g\mu(\omega)$" instead.

- **Page 42, proof of Proposition:** You write: "Let $\rho : \operatorname{GL}(V) \to \operatorname{GL}(W)$ be a polynomial representation and $\widetilde{\rho} : \operatorname{End}(V) \to \operatorname{End}(W)$ its extension (Lemma 6.2(b))." I think the Lemma you are referring to is 5.2 (b), not 6.2(b).

- **Page 42, proof of Proposition:** At the end of the proof, you construct an embedding of $\mathrm{S}^m V$ into $V^{\otimes m}$. This is an embedding only if $\operatorname{char} K = 0$ (or at least $\operatorname{char} K > m$). Is it possible that you assume $\operatorname{char} K = 0$ in the Proposition? I am a bit confused here because you explicitely require $\operatorname{char} K = 0$ in Corollary 1 but you don't mention $\operatorname{char} K$ in the Proposition.

- **Page 42, Remark:** You write: "Then we show that every finite dimensional subrepresentation of $\operatorname{End}(V)$ is contained in a direct sum $\oplus_i V^{\otimes n_i}$." You mean $K[\operatorname{End}(V)]$ here, not $\operatorname{End}(V)$.

- **Page 43:** You write: "Clearly, the two coincide if and only if $G \subset \mathrm{SL}(V)$." I think that "clearly" the opposite is the case: take $G = \left\{ s \in \mathrm{GL}(V) \mid (\det s)^2 = 1 \right\}$. Or is it me who doesn't understand something here? I know that my counterexample is perverse from an algebraic-geometric viewpoint (it is not even connected), and I am wondering whether a simple additional condition rescues the assertion. Otherwise it would be probably wiser to explicitly list the groups G for which you claim the assertion to hold.

- **Page 45, first line:** There is a typo here: "representaiton" should be "representation".

- **Page 45:** In the last sentence before Exercise 13, you write: "Hence, every irreducible representation of $\mathrm{SL}(V)$ occurs in some $V^{\otimes m}$ [...]". To be more precise, "irreducible" should be "irreducible polynomial" here.

- **§5.5:** There is no mistake on your part here, but honestly I would find it better if you would explain once again that $K[G]$ means the ring of polynomial functions on G, while KG means the group ring of G (the ring of formal linear combinations of elements of G). Unfortunately, several people (one of them being myself) have the habit of reading *both* $K[G]$ and KG as the group ring of G, which conflicts with your notation here.

- **Page 46, Exercise 14:** What you call Map here was called Mor one page above.

- **Page 47:** You write: "In other words, $\chi = r_1\varepsilon_1 + r_2\varepsilon_2 + \cdots r_n\varepsilon_n \in \mathcal{X}(T_n)$ [...]". There is a plus sign missing (before $r_n\varepsilon_n$).

- **Page 47:** You write: "the eigenspaces W_λ are the corresponding *weight space*,". This should be a plural: "weight space**s**".

- **Page 50, between Corollary 1 and Definition 2:** You refer to "3.3 Corollary 1". I think you mean "5.3 Corollary 1".

- **Page 51, Example (2):** In the equation $p_n\varepsilon_1 + p_{n-1}\varepsilon_2 + \cdots p_1\varepsilon_n = \sigma_0\lambda$, there is a plus sign missing in front of the $p_1\varepsilon_n$ term. This is the third time I am seeing this in your text - maybe it has a meaning I don't understand?

- **Page 51, Exercise 22:** You write: "(Cf. 3.3. Exercise 4.)" Actually Exercise 4 is in §3.2.

- **Page 52:** You write: "Therefore, we get an action of $\mathcal{S}_n$ on the character group $\mathcal{X}(T_n)$ defined by $\sigma(\chi(t)) := \chi(\sigma^{-1}t\sigma)$ [...]". The $\sigma(\chi(t))$ term should be $(\sigma(\chi))(t)$, apparently.

- **Page 52, proof of Proposition 1:** There is a wrong reference in "where $\omega_j := \varepsilon_1 + \cdots + \varepsilon_j$ is the highest weight of $\wedge^j K^n$ (5.7 Example (2))." You want Example (1), not (2).

- **Page 52, proof of Proposition 1:** You claim that the element w "is fixed under U_n and has weight $\lambda' := \sum_{i=1}^{n-1} p_i\omega_i$". It seems to me that the weight should rather be $\lambda' := \sum_{i=1}^{n-1} m_i\omega_i$.

- **Page 52, proof of Proposition 1:** Another incorrect reference: "It follows from Proposition 6.6" should be "It follows from Proposition 5.7".

- **Page 53, definition of "height":** Unless I have overlooked it, there is no definition of $\operatorname{ht}\lambda$ in your text. It would be enough to say that $\operatorname{ht}\lambda$ is an abbreviation for the height of λ.

- **Page 53, Exercise 25:** Are you sure that $n = k \cdot |\lambda|$ and not $|\lambda| = nk$?

- **Page 53, §5.9:** Two wrong references in the first absatz here (before Proposition 1): "Theorem 3.5" should be "Theorem 3.3", and "In §7" should be "In §6".

- **Page 54, Remark 1:** You write: "Moreover, embedding $\mathrm{GL}_n \subset \mathrm{GL}_{n+1}$ we get a canonical inclusion

  $$L_\lambda(n) \subset L_\lambda(n+1) = \langle L_\lambda(n) \rangle_{\mathrm{GL}_{n+1}},$$

  [...]". How exactly do you get this inclusion? (My preferred way to get an inclusion $L_\lambda(n) \subset L_\lambda(n+1)$ is to use the functoriality of L_λ, but you don't introduce this until Remark 2.)

- **Page 54, Proof of Lemma:** You write: "Clearly, this is the regular representation, i.e., $(V^{\otimes n})_{\det} \simeq \bigoplus_\lambda M_\lambda \otimes M_\lambda$." I think you are silently using $M_\lambda^* \simeq M_\lambda$ here; maybe it would be better if you are more explicit about it.

- **Page 55:** In the formula

  $$V_n(\lambda)_{\det} \simeq M_\lambda \otimes L_\lambda(n)_{\det} \simeq M_\lambda \otimes M_\lambda,$$

  the $V_n(\lambda)$ should be $V_\lambda(n)$.

- **Page 57, Exercise 2:** "representation" should be "rational representation" here, I think.

Section 6

- **Page 59, Exercise 4:** I believe this is wrong (as, e.g., the example of $n = 2$, $\lambda = (2)$ and $r = 1$ shows, in which your definition yields $\lambda^\vee = (0) = \varnothing$). There might be several ways to fix it. The one that I know is the following: If λ is a partition of height $\leq n$, and if m is an integer such that $m \geq \lambda_1$, then

$$s_{\lambda^\vee}(x_1, x_2, \ldots, x_n) = (x_1 x_2 \cdots x_n)^m \cdot s_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\right),$$

  where $\lambda^\vee$ is the partition $(m - \lambda_n, m - \lambda_{n-1}, \ldots, m - \lambda_1)$.

- **Page 60, Cauchy's formula:** Replace y_n by y_m in "where both sides are considered as elements in the ring $\mathbb{Z}[[x_1, ..., x_n, y_1, ..., y_n]]$".

- **Page 60, Proof of Cauchy's formula:** In the second line of this proof, replace $x_{m+1} = ... = x_n = 0$ by $y_{m+1} = ... = y_n = 0$.

- **Page 61:** In the middle of the page, the determinant

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \dfrac{y_1}{1 - x_2 y_1} & \dfrac{y_2}{1 - x_2 y_2} & \cdots & \dfrac{y_n}{1 - x_2 y_n} \\ \vdots & & & \vdots \end{pmatrix}$$

  should either be

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \dfrac{y_1}{1 - x_2 y_1} & \dfrac{y_2}{1 - x_2 y_2} & \cdots & \dfrac{y_n}{1 - x_2 y_n} \\ \vdots & & & \vdots \end{pmatrix}$$

  (the 0's have been replaced by 1's) or be

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \dfrac{y_1}{1 - x_2 y_1} & \dfrac{y_2}{1 - x_2 y_2} - \dfrac{y_1}{1 - x_2 y_1} & \cdots & \dfrac{y_n}{1 - x_2 y_n} - \dfrac{y_n}{1 - x_2 y_n} \\ \vdots & & & \vdots \end{pmatrix}.$$

- **Page 62, §6.3:** In the uppermost formula on page 62, the term $\sum\limits_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \cdot y_{\tau\sigma(1)}^{\nu_1} \cdots y_{\tau\sigma(n)}^{\nu_n}$ should be $\sum\limits_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \cdot y_{\sigma\tau(1)}^{\nu_1} \cdots y_{\sigma\tau(n)}^{\nu_n}$, as I think. (Of course, $\sum\limits_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \cdot y_{\tau\sigma(1)}^{\nu_1} \cdots y_{\tau\sigma(n)}^{\nu_n}$ is correct as well, but $\sum\limits_{\sigma \in \mathcal{S}_n} \operatorname{sgn} \sigma \cdot y_{\sigma\tau(1)}^{\nu_1} \cdots y_{\sigma\tau(n)}^{\nu_n}$ is the term you get by replacing the summation by a double summation as described in your text.)

- **Page 62, §6.4:** When you define power sums, it might be useful to state your policy regarding n_0: is it undefined, is it defined as 1, is it defined as n ? Unless you define it as 1, the definition

$$n_\mu(x_1, ..., x_n) := \prod_{i \geq 1} n_{\mu_i}(x_1, ..., x_n)$$

should be

$$n_\mu(x_1, ..., x_n) := \prod_{i=1}^{k} n_{\mu_i}(x_1, ..., x_n)$$

where $k = \max\{i \in \mathbb{N} \mid \mu_i \neq 0\}$.

- **Page 63, Proof of** $\operatorname{Tr}\varphi = n_\mu(x_1, ..., x_n)$**:** Here you write "In fact, the lines $K(e_{i_1} \otimes \cdots \otimes e_{i_m}) \subset V^{\otimes m}$ are stable under φ." It seems to me that "stable" is not the right word here; they are permuted (i. e. mapped to each other) by φ.

- **Page 63, proof of Lemma 1:** Three times on this page, you write $\sum\limits_{\nu \geq 0}$ while you actually mean $\sum\limits_{\nu \geq 1}$.

- **Page 63, proof of Lemma 1:** Here you write "we can calculate the term of degree m [...]". Actually you mean the term of degree $2m$, at least as far as the total degree in all variables together is concerned. Of course, you can also describe it as the term of degree m in the variables x_1, x_2, ..., x_n.

- **Page 64:** In the long calculation of R_m (in the middle of page 64), there is a minor typo: $\sum\limits_{\mu \in \mathcal{P}_{\Updownarrow}}$ should be $\sum\limits_{\mu \in \mathcal{P}_m}$.

- **Page 65, Lemma 2 (c):** You might want to add that you consider b_λ as a class function on $\mathcal{S}_m$ here (and not just as a function on partitions of m).

- **Page 65, Lemma 2 (c):** You write $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_r}$. Probably the r means n here.

- **Page 65, Lemma 2:** It seems that you are using a normal italic letter S here for the symmetric groups, whereas you use a calligraphic $\mathcal{S}$ in the rest of the text.

- **Page 65, Proof of Lemma 2 (c):** You write: "This shows that $b_\lambda(\mu)$ is the number of possibilities to decompose the set $M = \{\mu_1, \mu_2, ..., \mu_m\}$ into m disjoint subsets $M = M_1 \cup M_2 \cup ... \cup M_m$ such that the sum of the μ_j's in M_i is equal to λ_i." There are three mistakes here: First, we don't want m disjoint subsets $M = M_1 \cup M_2 \cup ... \cup M_m$, but we want n disjoint subsets $M = M_1 \cup M_2 \cup ... \cup M_n$. Secondly, $\{\mu_1, \mu_2, ..., \mu_m\}$ should be $\{\mu_1, \mu_2, ..., \mu_\phi\}$, where ϕ is the greatest integer satisfying $\mu_\phi \neq 0$ (in fact, we do need this, because the product $(x_1^{\mu_1} + x_2^{\mu_1} + \cdots)(x_1^{\mu_2} + x_2^{\mu_2} + \cdots)\cdots$ is supposed to end with this $\left(x_1^{\mu_\phi} + x_2^{\mu_\phi} + \cdots\right)$ and not to go on infinitely). Finally, we are not decomposing the set $M = \{\mu_1, \mu_2, ..., \mu_\phi\}$, but rather the set $M = \{1, 2, ..., \phi\}$ (in such a way that $\sum\limits_{j \in M_i} \mu_j = \lambda_i$). The difference is that some μ_j's may be equal while the corresponding j's are not. Accordingly, "the set $M = \{\mu_1, ..., \mu_m\}$ of the cycle lengths of σ" should be "the set $M = \{1, 2, ..., \phi\}$ labeling the cycles of σ".

- **Page 66, Proof of Theorem:** You write: "Using Lemma 2 (b) (and again (c)) we see that the sign must be +1 since [...]". This is slightly incomplete - in fact, you need to know that $a_\lambda \neq -a_\mu$ for $\lambda \neq \mu$ here, because otherwise it could "cancel" against some a_μ with $\mu > \lambda$.

- **Page 66, Exercise 11:** Replace "where $\ell_i = \lambda_i + m - i$" by "where $\ell_i = \lambda_i + r - i$".

- **Page 66, Exercise 11:** Replace $\Delta\left(x_1 + \cdots + x_r\right)^r$ by $\Delta\left(x_1 + \cdots + x_r\right)^m$ in the Hint.

- **Page 66, Exercise 12:** After "we associate a *hook* consisting of all boxes below or to the right hand side of B", you might include "(including B itself)".

- **Page 67, Example (2):** At the end of this example, $K^n \diagup K\left(1,1,...,1\right)$ should be $K^m \diagup K\left(1,1,...,1\right)$.

- **Page 67, Proof of Theorem:** You write: "Since the a_λ form a $\mathbb{Z}$-basis of the class functions [...]". In fact they don't. They form a $\mathbb{Q}$-basis only (but this is enough for the proof). Directly after that, $\widetilde{s}_\lambda \in \mathbb{Z}\left[x_1,...,x_n\right]$ should be replaced by $\widetilde{s}_\lambda \in \mathbb{Q}\left[x_1,...,x_n\right]$.

- **Page 67, Proof of Theorem:** In the formula

$$\chi\left(\sigma, x_1, ..., x_n\right) = n_\mu\left(x_1, ..., x_n\right),$$

there should be a semicolon instead of a comma after the σ (at least, this is how you introduced the χ notation).

- **Page 69, proof of Corollary 2:** In the formula

$$\prod_{i=1...n,j=1...,m} \frac{1}{1 - x_i y_j},$$

the commata below the product sign are inconsistent.

- **Page 69, proof of Corollary 2:** What do you mean by "Now we can argue as in the proof of the Theorem above"? You only need to say that a representation is uniquely determined by its character, or is there some other trick that you are using here?

- **Page 69, Corollary 3:** You have misspelt "is" as "if" twice (in the context "where the sum if over all partitions [...]"; this appears one time after the S^m formula and once again after the $\bigwedge^m$ formula).

- **Page 69, Exercise 13:** "Show that $\operatorname{ht}\lambda$ is the smallest integer [...]" - are you sure about it? I think the smallest such integer is λ_1. The statement that "$\det^{\operatorname{ht}\lambda} L_\lambda\left(n\right)^* \simeq L_{\lambda^c}\left(n\right)$" should be replaced by "$\det^m L_\lambda\left(n\right)^* \cong L_{\lambda^\vee}\left(n\right)$ for any integer $m \geq \lambda_1$, where $\lambda^\vee$ denotes the partition $\left(m - \lambda_n, m - \lambda_{n-1}, \ldots, m - \lambda_1\right)$".

- **Page 70:** You write: "We will first show that there is an interesting relation between such multiplicities for the general linear group and those for the symmetric group." This is a bit misleading; the multiplicities are of different types for the general linear group and for the symmetric group. For the general linear group, you take the interior tensor product between two representations of *one and the same* GL_n. For the symmetric group, you take the tensor product of a representation of S_a with a representation of S_b, generally for different a and b. The question to decompose the interior tensor product of two representations of the *same* symmetric group S_n is harder.

- **Page 70:** "PIERI's formula" is misspelt "PERIE's formula" here.

Section 7

- **Page 74, Exercise 3:** Could not $N > 0$ be weakened to $N \geq 0$ here? This way the result would also encompass groups which don't have any nonconstant invariant.

- **Page 74, §7.2:** Before Corollary 1, you write: "Part (a) of the following corollary is WEYL's Theorem A of the previous section 7.1." I don't see how the $K[V^p]^G = \left\langle K[V^n]^G \right\rangle_{\mathrm{GL}_p}$ part of Weyl's Theorem A should directly follow from Corollary 1 (a). While Corollary 1 (a) clearly yields that $K[V^p]^G$ is *generated* by $\left\langle K[V^n]^G \right\rangle_{\mathrm{GL}_p}$, I don't see a direct reason why it *equals* $\left\langle K[V^n]^G \right\rangle_{\mathrm{GL}_p}$, i. e., why $\left\langle K[V^n]^G \right\rangle_{\mathrm{GL}_p}$ is a K-algebra (i. e., why it is closed under multiplication).

- **Page 74, §7.2:** You use the notion of a "multihomogeneous subspace" without defining it (you only defined a multihomogeneous component some time before). I guess you mean a subspace which is the (direct) sum of its multihomogeneous components?

- **Page 74, Lemma.** You might add that the lemma also holds more generally for every T_p-stable subspace $F \subset K[V^p]$.

- **Page 75, but also many more times throughout the text:** You use the $\subset$ and $\subseteq$ signs as synonyms (both meaning "subset", not "proper subset"). Find/Replace should do the job here.

- **Page 76, Lemma:** I think the right hand side only makes sense for $i \neq j$ (unless we are talking about a topological field such as $\mathbb{R}$ or $\mathbb{C}$).

- **Page 77, Proof of Lemma.** In the very first formula of page 77, the left hand side should be $f_\nu(v_1, ..., v_p)$ rather than $f_\nu(x_1, ..., x_p)$.

- **Page 77, Example (c):** You write: "assuming that $f = f(v_0)$ is a function depending only on the first copy of V^{d+1} ". This would be less ambiguous if worded "[...] on the first copy of V in V^{d+1} ".

- **Page 78, Proof of the Proposition:** After (3), you write: "Clearly the sum is finite [...]". It is only finite for $i \neq j$, and you need an additional argument (actually, a reference to the lemma that a T_p-stable subspace is always multihomogeneous) to handle the case $i = j$.

- **Page 79, §7.5:** The notion "unimodular" has never been defined in the text. Is a linear group said to be unimodular if it is included in $\mathrm{SL}\, V$? In this case, what is the relation to the standard definition of "unimodular" in Lie group theory?

- **Page 79, §7.5:** Nitpicking: You write: "the determinant of the $n \times n$ matrix consisting of the column vectors $v_1, ... v_n$". A comma is missing before v_n here.

- **Page 79, Exercise 5:** More nitpicking: "or again a determinant" should be a "or again $\pm$ a determinant", since you are only counting the $[i_1, ..., i_n]$ with $i_1 < i_2 < ... < i_n$ as determinants, while the polarization may change the order of the vectors.

- **Page 81, proof of Theorem:** "Therefore, it suffices to show that $\sum_i x_i^\alpha y_i^\beta \cdots z_i^\gamma \in A$ for all $\alpha, \beta, ..., \gamma \geq 0$." Why does this suffice?

- **Page 81, Definition:** The definition begins with "A G-algebra is called *multiplicity free* if A is a direct sum [...]". I would replace "G-algebra" by "G-algebra A" in order to have the label A introduced.

- **Page 82, directly above the Lemma:** It might be useful to say that A_λ means the λ-isotypic component, *not* the weight space (which used to be denoted by W_λ on page 47).

- **Page 82, directly above the Lemma:** You write

$$\Omega_A := \{\lambda \in \wedge_G \mid A_\lambda \neq 0\}.$$

  Replace the "$\wedge_G$" here by a "Λ_G". There is a subtle (but visible) difference between this sign $\wedge_G$ that you are using here and the Λ_G that you used above for the monoid of highest weights.

- **Page 82, Lemma:** What is a weight of G ? You have only introduced weights of $\mathrm{GL}(V)$. I guess that a weight of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times ... \times \mathrm{GL}(V_k)$ is a k-tuple of weights of the corresponding $\mathrm{GL}(V_i)$, but I don't understand what a weight of $\mathrm{SL}(V)$ would mean. As I see from the Proof of the Lemma, it was originally intended only for $G = \mathrm{GL}_n$.

- **Page 82, §7.8:** You write: "For $k \leq m := \min(p, \dim V)$ we have a $\mathrm{GL}(V)$-equivariant multilinear map [...]". The map is not multilinear, but polynomial of degree $\left(\underbrace{1, 1, ..., 1}_{k \text{ ones}}, 0, 0, ..., 0\right)$. (And, most importantly, it is surjective.)

- **Page 83, Proof:** You write: "and the claim follows by Lemma 1 above". This "Lemma 1" is actually Lemma 7.7 (b).

- **Page 83, Proposition:** Let me add one more remark: Another proof of this Proposition can be found in Theorem 3.3 0) of the paper

  C. DeConcini, David Eisenbud, C. Procesi,
  Young Diagrams and Determinantal Varieties,
  Inventiones Math. 56 (1980), pp. 129–165.

  The proof given there actually works in a more general setting, where K is merely assumed to be an infinite field, not necessarily of characteristic 0.

- **Page 84, before Exercise 10:** "identify" is misspelt "indentify".

- **Page 84:** Exercise 10 seems a bit too easy to me.

- **Page 84, proof of Lemma:** In the second line of this proof, "homomorphism" is misspelt "homomorphim".

- **Page 85, Example (a):** In the equation $-\varepsilon_n = \varepsilon_1 + \cdots \varepsilon_{n-1}$, there is a plus sign missing in front of the ε_{n-1} term.

- **Page 85, Example (b):** At the end of this example, "Similarly on finds" should be "Similarly one finds".

- **Page 85, Example (c):** In the equation $u := e_1 \otimes f_1 + \cdots e_m \otimes f_m$, there is a plus sign missing in front of the $e_m \otimes f_m$ summand.

- **Page 85, Example (d):** "symmetric 2×2 matrices" should probably be "symmetric $n \times n$ matrices".

- **Page 85, Example (d):** "The monoid of even dominant weights of height $\leq n = \dim V$" is not generated by $2\omega_1, ..., 2\omega_n$. You mean the monoid of even *positive* dominant weights.

- **Page 86, Example (d):** In the last formula, shouldn't $\bigoplus_{\substack{\lambda \text{ even} \\ |\lambda|=m,\ \operatorname{ht}\lambda\leq n}}$ be $\bigoplus_{\substack{\lambda \text{ even} \\ |\lambda|=2m,\ \operatorname{ht}\lambda\leq n}}$?

Section 8

- **Page 88, Proof of Theorem B:** Add "for all $f \in K\left[V^p\right]^G$" after the formula

$$A_i B_i f \in K\left[\left\langle K\left[V^{n-1}\right]^G\right\rangle_{\mathrm{GL}_p}, [i_1, ..., i_n]\right].$$

- **Page 89, directly above Theorem A:** In the formula

$$\mathrm{GL}_{p_1'} \times \cdots \mathrm{GL}_{p_r'} \hookrightarrow \mathrm{GL}_{p_1} \times \cdots \times \mathrm{GL}_{p_r},$$

  there is a $\times$ sign missing in front of the $\mathrm{GL}_{p_r'}$ factor.

- **Page 89, Theorem A:** "subset" should be "subspace" here.
- **Page 89, Theorem B:** Replace $V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}$ by $V_1^{n_1-1} \oplus \cdots \oplus V_r^{n_r-1}$.
- **Page 89, between Theorem 3 and the proof:** In the relation $\Delta_{ij}^{(\nu)} \in \operatorname{End}(K[V_\nu)^{p_\nu}])$, the brackets have gone berserk.
- **Page 89, between Theorem 3 and the proof:** Replace "rows" by "columns" in "all $k \times k$-minors extracted from the first k rows [...]".
- **Page 89, Proof of Theorem 3:** You write: "Clearly the operators $A_i{}^\nu$, $B_i{}^\nu$ commute with the operators $A_i{}^{\nu'}$, $B_i{}^{\nu'}$ for $\nu' \neq \nu$." The ν and ν' exponents on the operators should be bracketed: $A_i{}^{(\nu)}$, $B_i{}^{(\nu)}$, resp. $A_i{}^{(\nu')}$, $B_i{}^{(\nu')}$.
- **Page 90, Corollary 1:** In the relation $S \subset K\left[v_1^{n_1} \oplus \cdots \oplus V_r^{n_r}\right]^G$, the v_1 should be an uppercase V_1.
- **Page 90, Corollary 1:** Add "and restricting in case $p_i < n_i$ for some i" at the end of this corollary.
- **Page 90, Corollary 3:** "determinants" is misspelt "determinats" at the very end of the statement of this corollary.
- **Page 95, Remark:** I think that the s_ν in the formula

  $$f =^\lambda f = \sum_\nu \alpha_\nu s_\nu d_\nu d_\nu^* \lambda^{n(m_\nu - n_\nu)}$$

  should be a p_ν.
- **Sections 8.4 and 8.5:** You use the notation $\langle i \mid j \rangle$ for something that you called $(i \mid j)$ in Section 2. Maybe it would be useful to assimilate these notations.
- **Page 92, Example:** "If the group G is simple" should be "If the group G is simple but not cyclic".

Section 9

- **Page 95, Proof of Theorem:** After (4), you write: "Now we use the fact that $\Delta_{y,x}$ commutes with $[x, y]$ [...]". First, $\Delta_{y,x}$ should be Δ_{yx} (no comma). More importantly, you use not only that Δ_{yx} commutes with $[x, y]$, but also that Ω commutes with Δ_{xy}.
- **Page 96, before the Proposition:** Replace "(see Lemma 9.1)" by "(see Exercise 1)".
- **Page 96, before the Proposition:** Replace "and that $[x, y]$ commutes with $\mathrm{SL}(V)$" by "and that $[x, y]$ and Ω commute with $\mathrm{SL}(V)$".

- **Page 96, proof of Proposition:** You write: "It therefore suffices to show that the two spaces have the save dimension [...]". Trivial typo: "save" should be "same".

- **Page 96, proof of Proposition:** "same" is misspelt "save" in "the two spaces have the save dimension".

- **Page 97, §9.3:** You write: "CAPELLI was able to generalize the formula (1) of 9.1 [...]". There is no formula (1) in 9.1.

- **Page 98, Proof of Capelli's identity:** In part (c), you write: $C_{\xi,x} = \sum_{i_1,...,i_p} |\xi|_{i_1,...,i_p} \cdot \left|\frac{\partial}{\partial x}\right|_{i_1,...,i_p}$ for $n > p$. If I were you, I would replace the $\sum_{i_1,...,i_p}$ sign by a $\sum_{i_1<...<i_p}$ sign so that it is clear that every minor is to be counted only once.

- **Page 98, Proof of Capelli's identity:** On the fourth line from the bottom of page 98, you write: "where $|\xi|_{i_1,...,i_p}$ and $\left|\frac{\partial}{\partial x}\right|_{i_1,...,i_p}$ are the minors of the matrices $(\xi_1, ..., \xi_n)$ [...]". This $(\xi_1, ..., \xi_n)$ should be $(\xi_1, ..., \xi_p)$ instead.

- **Page 99, Proof of Capelli's identity:** The commutation relations (a), (b), (c), (d) are not sufficient. For example, they don't help in transforming $\Delta_{aa}\Delta_{bc}$ to $\Delta_{bc}\Delta_{aa}$. Instead I would propose the following formulation:
  Let $a, b, c, d \in \{x_i, \xi_i\}$ be arbitrary (I don't require them to be distinct). Then,
  (a) Δ_{ab} commutes with Δ_{cd} if $\{a, b\} \cap \{c, d\} = \varnothing$;
  (b) $\Delta_{ab}\Delta_{bc} = \Delta_{bc}\Delta_{ab} + \Delta_{ac}$ if $a \neq c$;
  (c) $\Delta_{ab}\Delta_{ca} = \Delta_{ca}\Delta_{ab} - \Delta_{cb}$ if $b \neq c$;
  (d) $\Delta_{ab}\Delta_{ba} = \Delta_{ba}\Delta_{ab} + \Delta_{aa} - \Delta_{bb}$.

- **Page 99, Proof of Capelli's identity:** On the fourth line from the bottom of page 98, you write: "i. e., C_k it is the determinant of the matrix [...]". This doesn't make much sense grammatically; I would write "this is the determinant of the matrix [...]".

- **Page 100, Proof of Capelli's identity:** A minor typo again: $f(x_1, .., x_p)$ should have three (and not just two) dots between x_1 and x_p. This typo is in the second absatz of the text on page 100.

- **Page 101, Proof of Capelli's identity:** In the first determinant on this page, the entry $\Delta_{x_k\xi_k}\Delta_{x_{k-1}x_k}$ should be $\Delta_{x_k\xi_k}\Delta_{x_{k-1}x_2}$.

- **Page 105, Proof of Capelli's identity:** In the second determinant on this page, the line

$$0 \quad \Delta_{x_i x_{k+1}} \quad \cdots \quad \Delta_{x_i x_p}$$

  should begin with zeroes:

$$0 \quad \cdots \quad 0 \quad 0 \quad \Delta_{x_i x_{k+1}} \quad \cdots \quad \Delta_{x_i x_p} \quad .$$

- **Page 106, Proof of Capelli's identity:** In the big determinant on this page, the line

$$0 \quad \Delta_{\xi_j x_{k+1}} \quad \cdots \quad \Delta_{\xi_j x_p}$$

should begin with zeroes:

$$0 \quad \cdots \quad 0 \quad 0 \quad \Delta_{\xi_j x_{k+1}} \quad \cdots \quad \Delta_{\xi_j x_p} \quad .$$

- **Page 106, Proof of Capelli's identity:** You write: "Now we see that these matrices are obtained from C'' by replacing the first k entries in the i-th row of W_i', $i = 1, ..., k-1$ and in the j-th row of W_j'', $j = k+1, ..., p$ by zero." This is slightly ambiguous - it sounds like you take W_i' and replace the first k entries in its i-row by zero, but in fact you do this to the matrix C'' and not to W_i'. To clear up this ambiguity, I would replace "of W_i'" by "for W_i'", and similarly for W_j''.

- **Page 106, Proof of Capelli's identity:** The equalities

$$\begin{aligned} W_i' &= \det D_i \qquad & i &= 1, ..., k-1 \\ W_j'' &= \det D_{j-1} \qquad & j &= k+1, ..., p \end{aligned}$$

should be

$$\begin{aligned} W_i' &= -\det D_i \qquad & i &= 1, ..., k-1 \\ W_j'' &= -\det D_{j-1} \qquad & j &= k+1, ..., p. \end{aligned}$$

- **Page 107, Lemma:** You write: "and $C_{ij} \in \mathcal{U}(p)$, the algebra generated by the Δ_{ij}". This is a little bit misleading, since it sounds like C_{ij} lies in the algebra generated by Δ_{ij} only (and no other Δ's), i. e. like C_{ij} is a polynomial in the single argument Δ_{ij}. It would be better to write "and $C_{ij} \in \mathcal{U}(p)$, the algebra generated by all polarization operators".

- **Page 107, proof of Lemma:** In the formula

$$C_p = \sum_{\sigma \in \S_p} \operatorname{sgn} \sigma \widetilde{\Delta}_{\sigma(1)1} \widetilde{\Delta}_{\sigma(2)2} \cdots \widetilde{\Delta}_{\sigma(p)p},$$

the $\S_p$ should be $\mathcal{S}_p$.

- **Page 108, Proof of Application 1:** In the last formula of this proof, the second $\sum\limits_{i<j}$ sign should be a double summation $\sum\limits_{i<j} \sum\limits_{\ell}$.

- **Page 109, Proof of Application 2:** Replace "(cf. Exercise 1)" by "(cf. Exercise 2)".

- **Page 110:** On the first line of page 110, $(x_1, ..., x_n)$ should be $(x_1, ..., x_p)$.

- **Page 110:** On the third line of page 110, $\left(\dfrac{\partial}{\partial x_{ij}}\right)_{i,j=1,...,n}$ should be $\left(\dfrac{\partial}{\partial x_{ij}}\right)_{\substack{i=1,...,n \\ j=1,...,p}}$.

- **Page 110, Proof of Proposition:** In the equation (1), the $\sum_{i_1,...,i_p}$ sign should be a $\sum_{i_1<...<i_p}$ sign.

- **Page 110, Proof of Proposition:** In the very last equation on page 110, replace "with B_l^{ij} of the required form" by "with $B_l^{ij} f$ of the required form".

- **Page 111, Proof of Proposition:** In the first formula on page 111, there should be an f at the end of the right-hand side.

- **Page 111, Proof of Proposition:** Directly after the first formula on page 111, you write: "where $A_l, B_l \in \mathcal{U}(p)$ and $B_l \left|\frac{\partial}{\partial x}\right|_{i_1...i_p} f$ are of the required form". Replace $\left|\frac{\partial}{\partial x}\right|_{i_1...i_p}$ by $\left|\frac{\partial}{\partial x}\right|_{i_1,...,i_p}$ (commata!) here.

- **Page 111, Proof of Proposition:** At the end of this proof, the $\sum_{i_1,...,i_p}$ sign occurs three times (two times in the last formula on page 111, and one time in the last line of page 111). Each time it should be $\sum_{i_1<...<i_p}$ instead.

Section 10

- **Page 112, Exercise 1:** "isomorphism" is misspelt "isomorpism" in the hint.

- The text in Section 10 seems to have holes:
  After Exercise 2, some text is missing (probably "Let $n = 2m$, let $V = K^n$, and let").
  After Exercise 7, it seems that a part of the next sentence has been lost.
  After Theorem 10.2 (b), some text has been eaten as well.
  Some text directly following the Proposition on page 117 also seems to have disappeared into oblivion.

- **Page 113, Exercise 4:** Replace "$2m \times 2m$-matrix S" by "invertible $2m \times 2m$-matrix S".

- **Page 113, Exercise 5:** I don't understand why you work with $\mathbb{C}$ rather than with the algebraic closure $\overline{K}$ of K. It can all be done easily with the help of the rational equivalence

$$(\text{skew-symmetric matrices})_n \to \mathrm{SO}_n, \qquad A \mapsto (E+A)^{-1}(E-A).$$

- **Page 114, between 10.2 and 10.3:** Probably it would be better to precise "invariant" to "Sp_{2m}-invariant".

- **Page 114, proof of the FFT for O_n and SO_n:** In the very last formula of page 114,

$$[i_1,...,i_n]\,[j_1,...,j_n] = \det\left(\left(i_k \mid i_l\right)_{k,l=1}^n\right),$$

  the i_l should be j_l.

- **Page 115, third absatz:** In "It follows that the restriction homomorphism $K\left[V^{n-1}\right] \mapsto K\left[V'^{n-1}\right]$ induces", replace the $\mapsto$ arrow by a $\to$ arrow.

- **Page 115, proof of the FFT for O_n and SO_n:** In the last absatz of this proof, you write "Given $v = (v_1, ..., v_n) \in Z$, the subspace $W(v)$ spanned by $v_1, ..., v_n$ has dimension $n-1$ and [...]". Both times $v_1, ..., v_n$ should be $v_1, ..., v_{n-1}$ here.

- **Page 116, proof of the FFT for Sp_{2m}:** You define V' and V'' by

$$V' := \{v = (x_1, ..., x_{2m-2}, 0, ..., 0)\} \qquad \text{and}$$
$$V'' := \{v = (0, ..., 0, x_{2m-1}, ..., x_{2m})\}.$$

Shouldn't these rather be

$$V' := \{v = (x_1, ..., x_{2m-2}, 0, 0)\} \qquad \text{and}$$
$$V'' := \{v = (0, ..., 0, x_{2m-1}, x_{2m})\}$$

?

- **Page 116, proof of the FFT for Sp_{2m}:** There is no need to assume K algebraically closed in this proof, because we can always map one vector to another by some $g \in \mathrm{Sp}_{2m}$ and thus we can also map the orthogonal complement of $W(v)$ to that of V'. (Actually, *I think* it can be avoided in the proof of the FFT for O_n and SO_n as well, at the price of of replacing

$$Z := \left\{v = (v_1, ..., v_{n-1}) \in V^{n-1} \mid \det\left((v_i \mid v_j)\right) \neq 0\right\}$$

by

$$Z := \left\{v = (v_1, ..., v_{n-1}) \in V^{n-1} \mid \det\left((v_i \mid v_j)\right) \text{ is a nonzero square}\right\};$$

however, this makes the proof that Z is Zariski-dense a bit harder and I haven't checked my proof.)

- **Page 116, proof of the FFT for Sp_{2m}:** In one of the formulas on this page, you write

$$\mathrm{Sp}_{2m} \cdot \left(V'^{\ 2m-1} \oplus V\right) = [...]$$

There is a typo here: $V'^{\ 2m-1}$ should be $V'^{\ 2m-2}$.

- **Page 117, before the Proposition:** You write: "In fact, we have $(A_i, A_j) = \frac{1}{2}\operatorname{Tr} A_i A_j$ and $[A_i A_j A_k] = \frac{1}{2}\operatorname{Tr} A_i A_j A_K$." Two minor typos here: (A_i, A_j) should be $(A_i \mid A_j)$, and A_K should be A_k.

- **Page 117:** You write: "As a consequence of the proposition above, every trace function $\mathrm{Tr}_{i_1,...,i_k}$ for $k > 2$ can be expressed as a polynomial in the traces Tr_i and Tr_{ij}." But what about Tr_{ijk} ?

References

- **[Der91]:** Typo: “fomres”.
- **[DiC70]:** “Als Buch bei” is German.
- **[For87]:** This is not vol. 1287 but vol. 1278.
- **[How95]:** This appears two times in the list of references.