1st QEDMO (QED Mathematical Olympiad) Gunzenhausen

1. Prove that every integer can be written as sum of 5 third powers of integers.
   (http://www.artofproblemsolving.com/Forum/viewtopic.php?t=33776)

2. Let ABC be a triangle. Let C' and A' be the reflections of its vertices C and A, respectively, in the altitude of triangle ABC issuing from B. The perpendicular to the line BA' through the point C' intersects the line BC at U; the perpendicular to the line BC' through the point A' intersects the line BA at V. Prove that $UV \parallel CA$.
   (Darij Grinberg)

3. At a tournament between n persons, each person plays against each other person exactly one time, and every game has a winner and a loser. Prove that after the tournament, one can arrange the n participants of the tournament in a chain $P_1 \to P_2 \to \ldots \to P_n$ such that, for every integer i with $1 \leq i < n$, the player P_i won against the player P_{i+1}.
   (Rédei's Theorem[1])

4. Solve the equation $x^3 + 2y^3 + 5z^3 = 0$ in integers.
   (Daniel Harrer)

5. Let ABC be a triangle, and let C' and A' be the feet of its altitudes issuing from the vertices C and A, respectively. Denote by P the midpoint of the segment $C'A'$. The circumcircles of triangles $AC'P$ and $CA'P$ have a common point apart from P. Denote this common point by Q. Prove that:
   a) The point Q lies on the circumcircle of triangle ABC.
   b) The line PQ passes through the point B.
   c) We have $\dfrac{AQ}{CQ} = \dfrac{AB}{CB}$.
   (Darij Grinberg or classical[2])

6. Prove that for any four real numbers a, b, c, d, the inequality $(a-b)(b-c)(c-d)(d-a) + (a-c)^2(b-d)^2 \geq 0$ holds.
   (Mihai Onucu Drimbe, "Inegalitati, idei si metode", Zalau: Gil, 2003, inequality (350) by Vlad Bazon)

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[1] In graph-theoretical terms, this problem requires proving that every tournament has a Hamiltonian path. This is well-known enough to have a proof on the Wikipedia: `http://en.wikipedia.org/wiki/Tournament_%28graph_theory%29#Paths_and_cycles`

[2] The problem is mostly a combination of well-known facts on antiparallels and harmonic quadrilaterals, so it does not make much sense to speak of authorship.

7. Prove that, for any positive integer n, there exists a subset S of the set $\{1, 2, \ldots, n\}$ such that this subset S has at most $2\lfloor\sqrt{n}\rfloor + 1$ elements, and such that the set $\{|x - y| \mid x \in S;\ y \in S\}$ equals the set $\{1, 2, \ldots, n - 1\}$.
   (*Remark.* For every real x, we denote by $\lfloor x \rfloor$ the greatest integer which is $\leq x$.)
   (Romania TST 1998, 3rd round, problem 1)

8. Prove that if an integer n can be written as $n = a^2 + ab + b^2$ with a and b being integers, then $7n$ can also be written this way.
   (Daniel Harrer[3])

9. Let ABC be a triangle with $AB \neq CB$. Let C' be a point on the ray $[AB$ such that $AC' = CB$. Let A' be a point on the ray $[CB$ such that $CA' = AB$. Let the circumcircles of triangles ABA' and CBC' intersect at a point Q (apart from B). Prove that the line BQ bisects the segment CA.
   (Darij Grinberg)

10. Let $n \geq 3$ be an integer. Also, let P_1, P_2, ..., P_n be n distinct two-element subsets of $M = \{1, 2, ..., n\}$, such that, for any two distinct numbers i and j from M, if the sets P_i and P_j have a common element, then there exists a $k \in M$ such that $P_k = \{i, j\}$.
    Prove that every element of M occurs in exactly two of the subsets P_1, P_2, ..., P_n.
    (Daniel Harrer's extension of an IMO Shortlist 1985 problem)

11. Let a, b, c be positive integers such that $a^2 + b^2 + c^2$ is divisble by $a + b + c$. Prove that at least two of the numbers a^3, b^3, c^3 leave the same remainder upon division through $a + b + c$.
    (Kolmogorov contest (a Russian mathematical tournament) 2002, Second Round, 5 Dec 02, First League)

12. For any three positive real numbers a, b, c, prove the inequality
    $$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \geq 6.$$
    (Peter Scholze and Darij Grinberg)

13. Let n be a positive integer. Find the number of all sequences $(a_1,\ a_2,\ ...,\ a_k)$ of k distinct numbers from the set $\{1, 2, 3, ..., n\}$ with the following property:
    For every member a of this sequence (except of the first one), there exists a member b that precedes a in this sequence and satisfies $|a - b| = 1$.
    (known exercise)

[3]This exercise "comes from" basic algebraic number theory (the multiplicativity of the norm on $\mathbb{Z}\left[\sqrt{-3}\right]$).

14. In the following, the abbreviation $g \cap h$ will mean the point of intersection of two lines g and h.

Let $ABCDE$ be a convex pentagon. Let $A' = BD \cap CE$, $B' = CE \cap DA$, $C' = DA \cap EB$, $D' = EB \cap AC$ and $E' = AC \cap BD$. Furthermore, let $A'' = AA' \cap EB$, $B'' = BB' \cap AC$, $C'' = CC' \cap BD$, $D'' = DD' \cap CE$ and $E'' = EE' \cap DA$. Prove that:

$$\frac{EA''}{A''B} \cdot \frac{AB''}{B''C} \cdot \frac{BC''}{C''D} \cdot \frac{CD''}{D''E} \cdot \frac{DE''}{E''A} = 1.$$

(Darij Grinberg)