3rd QEDMO (QED Mathematical Olympiad) Regensburg (9. - 12. April 2006)

1. With the abbreviation $|XYZ|$ for the area of an arbitrary triangle XYZ, prove that any convex pentagon $ABCDE$ satisfies the equality

$$|EAC| \cdot |EBD| = |EAB| \cdot |ECD| + |EBC| \cdot |EDA| .$$

   (problem P151 in the German periodical Praxis der Mathematik, proposed by Engel)

2. Let a, b, c and n be positive integers such that a^n is divisible by b, such that b^n is divisible by c, and such that c^n is divisible by a. Prove that $(a+b+c)^{n^2+n+1}$ is divisible by abc.
   (generalization of Norway MO 1998-99 finals, problem 2 ***b)****)*

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation $xf(x) - yf(y) = (x-y)\,f(x+y)$ for any two reals x and y.
   (Irish MO 1995 problem 5)

4. An order fanatic colors n of the numbers $1, 2, \ldots, 2n$ red, and the n others green. Now he orders the red numbers increasingly and thus obtains the sequence $a_1 < a_2 < \ldots < a_n$. Then he orders the green numbers decreasingly and thus obtains the sequence $b_1 > b_2 > \ldots > b_n$. He is surprised when he notices that the sum $|a_1 - b_1| + |a_2 - b_2| + \ldots + |a_n - b_n|$ equals n^2. Just a random coincidence?
   (Proizvolov identity, classical)

5. Find all positive integers n such that there are infinitely many lines of Pascal's triangle whose entries are all coprime to n. In other words: Find all positive integers n such that there are infinitely many positive integers k with the property that the numbers $\binom{k}{0}$, $\binom{k}{1}$, $\binom{k}{2}$, ..., $\binom{k}{k}$ are all coprime to n.
   (Daniel Harrer)

6. The incircle of a triangle ABC touches its sides BC, CA, AB at the points X, Y, Z, respectively. Let X', Y' and Z' be the reflections of these points X, Y, Z in the external angle bisectors of the angles CAB, ABC, BCA, respectively. Show that $Y'Z' \parallel BC$, $Z'X' \parallel CA$ and $X'Y' \parallel AB$.
   (Darij Grinberg, but may be an older result[1])

[1] If you replace "external" by "internal", then this problem becomes easier (and known). Be warned that it is not analogous to the problem with "exterior"!

7. A programmer creates a $2^n \times n$ table (i. e., a table with 2^n lines and n columns), writing down each of the 2^n possible sequences which have length n and consist of the numbers -1 and 1 only. Then, a h4xx0r replaces some of the numbers in the table by 0. Prove that: No matter how hard the h4xx0r tries, the programmer will always be able to find a (non-empty) set of lines such that the sum of these lines is the 0-vector (which means that for every $k \in \{1, 2, ..., n\}$, the sum of numbers in these lines on position k is 0).
(Peru TST 2006, problem 3, also http://www.mathlinks.ro/Forum/viewtopic.php?t=6196)

8. Let the sequence $(a_n)_{n\in\mathbb{N}}$ be defined recursively by $a_1 = 1$ and $a_1 + a_2 + \ldots + a_n = n^2 \cdot a_n$ for any natural n. Determine a_{2006}.
(Norway MO 1994-95 finals, problem 1 ***a)****; Britain 1996)*

9. Let ABC be a triangle, and C' and A' be the midpoints of its sides AB and BC. Consider two lines g and g' which both pass through the vertex A and are symmetric to each other with respect to the angle bisector of the angle CAB. Further, let Y and Y' be the orthogonal projections of the point B on these lines g and g'. Show that the points Y and Y' are symmetric to each other with respect to the line $C'A'$.
(Darij Grinberg)

10. Define a sequence $(a_n)_{n\in\mathbb{N}}$ by $a_1 = a_2 = a_3 = 1$ and $a_{n+1} = \dfrac{a_n^2 + a_{n-1}^2}{a_{n-2}}$ for every integer $n \geq 3$. Show that all elements a_i of this sequence are integers.
(L. J. Mordell and apparently Dana Scott, see also http://oeis.org/A064098)

11. Let a, b, c be three positive reals. Prove the inequality

$$\frac{a^2 + 2b^2}{b + c} + \frac{b^2 + 2c^2}{c + a} + \frac{c^2 + 2a^2}{a + b} \geq \frac{3}{2}(a + b + c).$$

(Darij Grinberg)

12. Albatross and Frankinfueter still have n cards each, but these time these cards are blank on both sides. Since they do not want to throw away the cards, each of the two writes the integers from 1 to $2n$ onto his n cards in an arbitrary order such that two numbers are written on each card - one on each of the two sides of the card -, and each of the two players uses every number from 1 to $2n$ exactly one time. After doing so they compare their $2n$ cards. Prove that they can lay out their $2n$ cards on a table in such a way that every integer from 1 to $2n$ is visible (i. e. it appears on the visible side of some card).
(Norway MO 1995-96 finals problem 3)