

5th QEDMO (QED Mathematical Olympiad)

1. Let a , b and k be three positive integers.

We define two sequences (a_n) and (b_n) by the starting values $a_1 = a$ and $b_1 = b$ and the recurrent equations $a_{n+1} = ka_n + b_n$ and $b_{n+1} = kb_n + a_n$ for each positive integer n . Prove that if $a_1 \perp b_1$, $a_2 \perp b_2$ and $a_3 \perp b_3$ hold, then $a_n \perp b_n$ holds for every positive integer n .

Hereby, the abbreviation $x \perp y$ stands for "the numbers x and y are coprime".

(Darij Grinberg)

2. Let $ABCD$ be a (not self-intersecting) quadrilateral satisfying $\angle DAB = \angle BCD \neq 90^\circ$. Let X and Y be the orthogonal projections of the point D on the lines AB and BC , and let Z and W be the orthogonal projections of the point B on the lines CD and DA .

Establish the following facts:

a) The quadrilateral $XYZW$ is an isosceles trapezoid such that $XY \parallel ZW$.

b) Let M be the midpoint of the segment AC . Then, the lines XZ and YW pass through the point M .

c) Let N be the midpoint of the segment BD , and let X' , Y' , Z' , W' be the midpoints of the segments AB , BC , CD , DA . Then, the point M lies on the circumcircles of the triangles $W'X'N$ and $Y'Z'N$.

(Darij Grinberg)

3. Let a , b , c , d be four positive reals such that $d = a + b + c + 2\sqrt{ab + bc + ca}$. Prove that $a = b + c + d - 2\sqrt{bc + cd + db}$.

(Darij Grinberg, inspired by the Descartes-Soddy formula about the radii of four circles which touch each other in pairs)

4. Let n be a positive integer, and let (a_1, a_2, \dots, a_n) , (b_1, b_2, \dots, b_n) and (c_1, c_2, \dots, c_n) be three sequences of integers such that for any two distinct numbers i and j from the set $\{1, 2, \dots, n\}$, none of the seven integers

$$\begin{array}{ll} a_i - a_j ; & (b_i + c_i) - (b_j + c_j) ; \\ b_i - b_j ; & (c_i + a_i) - (c_j + a_j) ; \\ c_i - c_j ; & (a_i + b_i) - (a_j + b_j) ; \\ (a_i + b_i + c_i) - (a_j + b_j + c_j) \end{array}$$

is divisible by n .

Prove that:

- a) The number n is odd.

b) The number n is not divisible by 3.

(generalization of one direction of Theorem 2.1 in: Dean Alvis, Michael Kinyon, **Birkhoff's Theorem for Panstochastic Matrices**, American Mathematical Monthly, 1/2001 (Vol. 108), pp. 28-37; the original Theorem 2.1 is obtained if you require $b_i = i$ and $c_i = -i$ for all i , and add in a converse stating that such sequences (a_1, a_2, \dots, a_n) , (b_1, b_2, \dots, b_n) and (c_1, c_2, \dots, c_n) indeed exist if n is odd and not divisible by 3)

5. Let a, b, c be three integers. Prove that there exist six integers x, y, z, x', y', z' such that

$$a = yz' - zy'; \quad b = zx' - xz'; \quad c = xy' - yx'.$$

(Darij Grinberg)

6. Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that the equality $f((f(x))^2 + f(y)) = x \cdot f(x) + y$ holds for any two rationals x and y .

(<http://www.mathlinks.ro/Forum/viewtopic.php?t=16667>)

7. Given a positive integer n . One night, somewhere deep in the voids of the Internet, $2n$ SPAMmers SPAMmed each other. More precisely, each SPAMmer has sent SPAMs to at least n other SPAMmers. Prove that there were two SPAMmers each of whom has sent a SPAM to the other one.

(Norway MO 1994, final round, problem 4 (**a**)¹)

8. Let $A, B, C, A', B', C', X, Y, Z, X', Y', Z'$ and P be pairwise distinct points in space such that

$$\begin{aligned} A' &\in BC; B' \in CA; C' \in AB; X' \in YZ; Y' \in ZX; Z' \in XY; \\ P &\in AX; P \in BY; P \in CZ; P \in A'X'; P \in B'Y'; P \in C'Z'. \end{aligned}$$

Prove that

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = \frac{YX'}{X'Z} \cdot \frac{ZY'}{Y'X} \cdot \frac{XZ'}{Z'Y}.$$

(Darij Grinberg)

¹The original problem in Norway MO was the particular case $n = 10$, but it was obviously meant to be generalized.