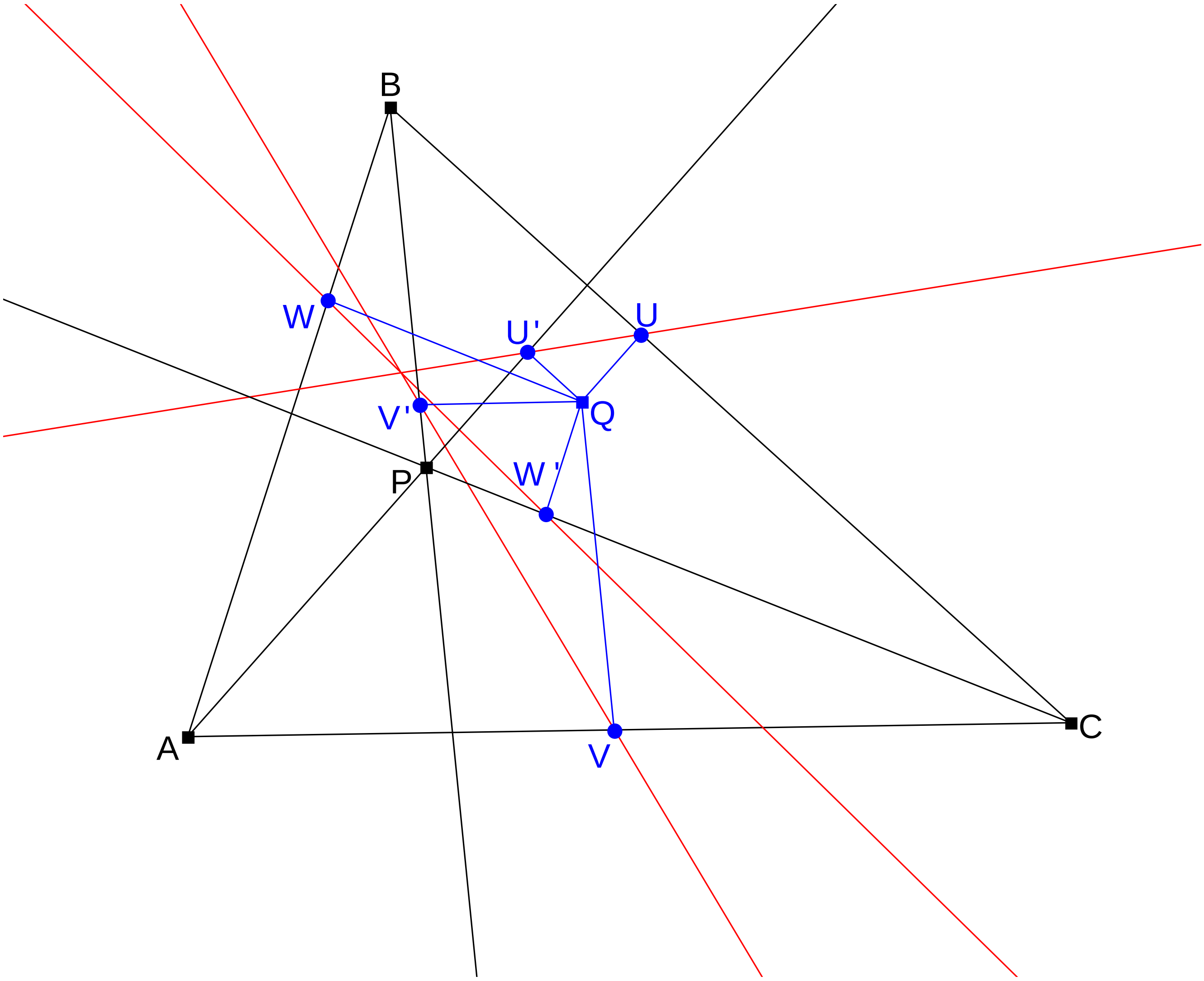

Fig. 1

The purpose of this note is to provide a synthetic proof of a theorem found by Eric Danneels and presented in Hyacinthos message #10135. The theorem goes at follows:

Theorem 1. Let P and Q be two points in the plane of a triangle ABC.

The parallels to the lines AP, BP, CP through the point Q intersect the lines BC, CA, AB at the points U, V, W.

The parallels to the lines BC, CA, AB through the point Q intersect the lines AP, BP, CP at the points U', V', W'.

Then, the lines UU', VV', WW' concur at one point.

Note. This point is called the **paracevian perspector** of the points P and Q with respect to the triangle ABC. (See Fig. 1.)

Now, Theorem 1 turns out to be by far not as easy to prove as it is formulated. Here is a *synthetic proof*:

Let A', B', C' be the points of intersection of the lines AP, BP, CP with the lines BC, CA, AB. In other words, we construct the cevian triangle $A'B'C'$ of the point P with respect to the triangle ABC.

Let the lines $A'Q$, $B'Q$, $C'Q$ intersect the lines $B'C'$, $C'A'$, $A'B'$ in the points X, Y, Z.

The parallels to the lines UU', VV', WW' through the points A', B', C' meet the lines $B'C'$, $C'A'$, $A'B'$ in the points X', Y', Z'.

(See Fig. 2.) At first we will prove that the lines $A'X'$, $B'Y'$, $C'Z'$ are concurrent.

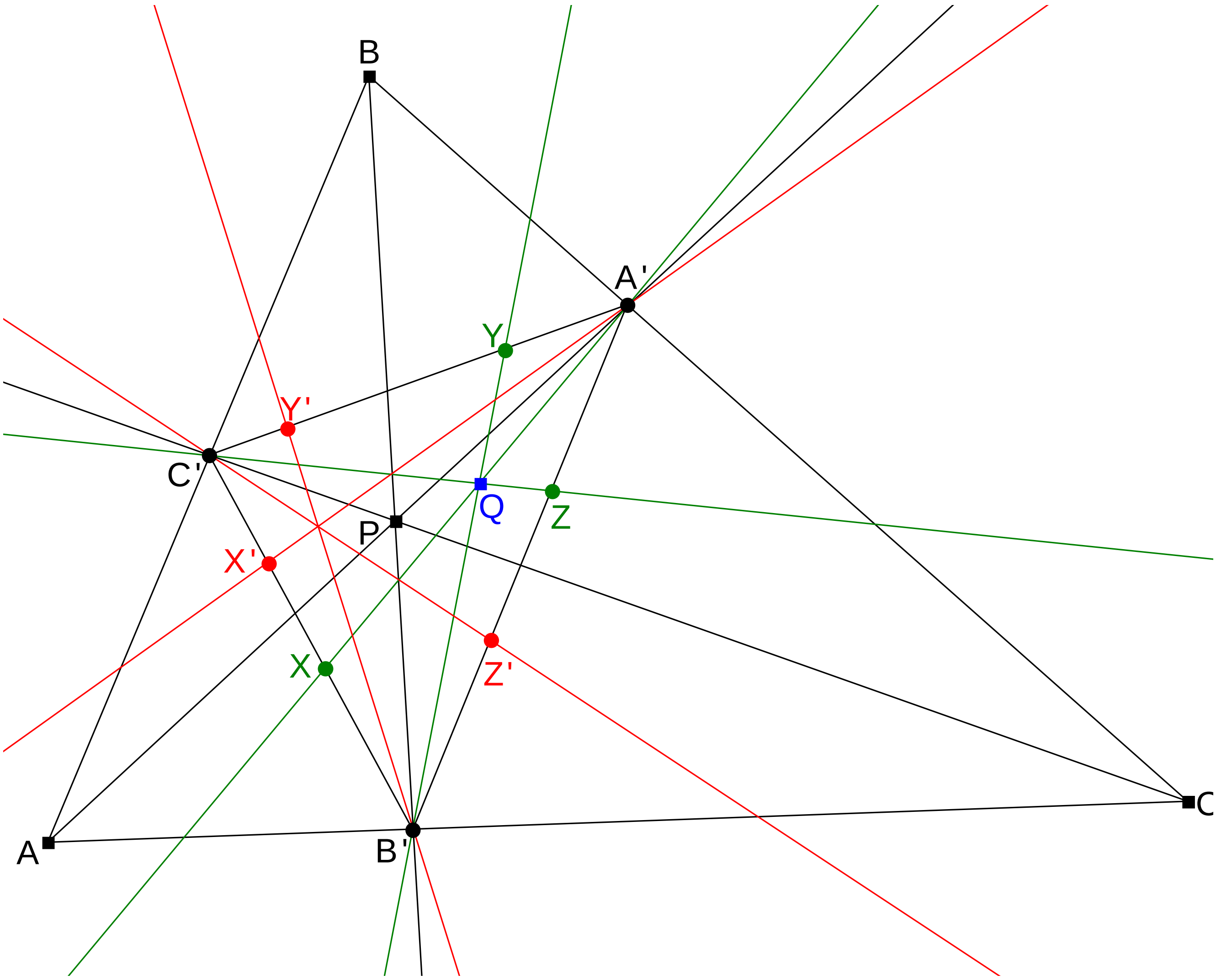

Fig. 2

Let the lines QV' and $B'Y'$ meet at F. The parallel to the line CA through the point B meets the lines $A'B'$, $B'C'$, $B'Y$ and $B'Y'$ at the points T, S, G and G', respectively. In the following, we will use directed segments, where the lines QV', CA, ST, being all parallel to each other, are assumed to have the same direction.

We have $QV \parallel V'B'$ and $QV' \parallel VB'$. Hence, the quadrilateral $QVB'V'$ is a parallelogram, and we get $QV' = VB'$. On the other hand, $B'F \parallel VV'$ and $FV' \parallel B'V$, so that the quadrilateral $V'VB'F$ is a parallelogram, and we get $V'F = VB'$.

Now, from $QV' = VB'$ and $V'F = VB'$, it follows that $QV' = V'F$. In other words, the point V' is the midpoint of the segment QF. But since the lines ST and QV' are parallel to each other (both of them being parallel to CA), we have $GB : BG' = QV' : V'F$. Now, from $QV' = V'F$, we have $QV' : V'F = 1$; thus, we conclude $GB : BG' = 1$, and consequently $GB = BG'$. Therefore, the point B is the midpoint of the segment GG'.

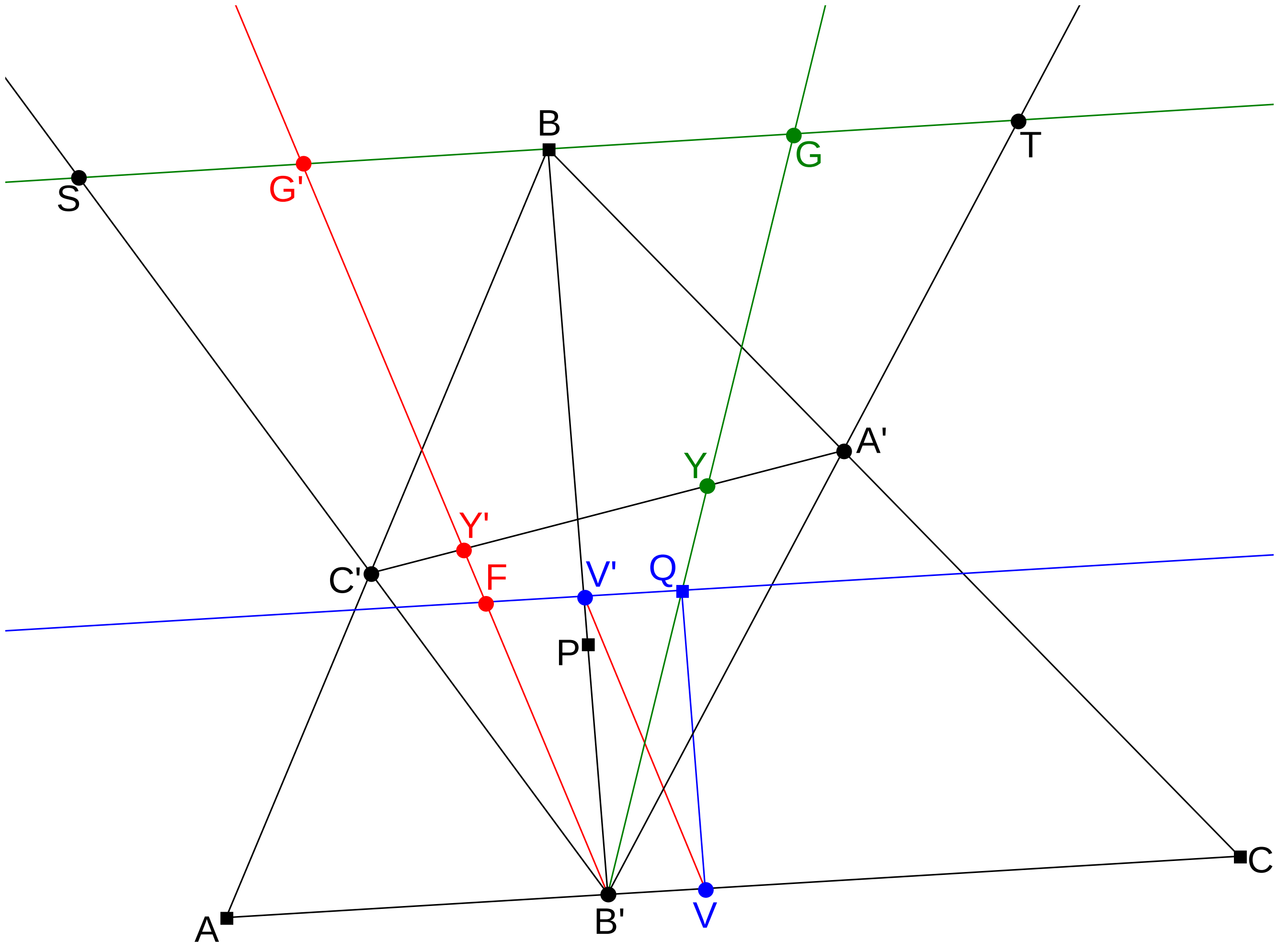

Fig. 3

On the other hand, $ST \parallel CA$ yields

$$\frac{SB}{B'A} = \frac{BC'}{AC'} \qquad \text{and} \qquad \frac{BT}{CB'} = \frac{BA'}{CA'}, \qquad \text{so that}$$

$$SB = B'A \bullet \frac{BC'}{AC'} \qquad \text{and} \qquad BT = CB' \bullet \frac{BA'}{CA'}.$$

Consequently,

$$\frac{SB}{BT} = \frac{B'A \bullet \frac{BC'}{AC'}}{CB' \bullet \frac{BA'}{CA'}} = \frac{B'A \bullet BC' \bullet CA'}{CB' \bullet BA' \bullet AC'} = \frac{CA'}{BA'} \bullet \frac{BC'}{AC'} \bullet \frac{B'A}{CB'}$$
$$= \frac{CA'}{-A'B} \bullet \frac{BC'}{-C'A} \bullet \frac{-AB'}{-B'C} = \frac{CA'}{A'B} \bullet \frac{BC'}{C'A} \bullet \frac{AB'}{B'C}.$$

But since the lines AA', BB', CC' concur (at the point P), the Ceva theorem shows that

$$\frac{CA'}{A'B} \bullet \frac{BC'}{C'A} \bullet \frac{AB'}{B'C} = 1. \tag{1}$$

Thus, we also have $\frac{SB}{BT} = 1$, so that $SB = BT$. In other words, B is the midpoint of the segment ST.

Now, $GB = BG'$ and $SB = BT$, so we have $SG' = SB + BG' = BT + GB = GT$ and $G'T = BT - BG' = SB - GB = SG$. Therefore,

$$\frac{SG'}{G'T} = \frac{GT}{SG}. \tag{2}$$

Moreover, since $ST \parallel CA$, we have

$$\frac{TB'}{B'A'} = \frac{BC}{CA'} \tag{3}$$

and $$\frac{C'B'}{B'S} = \frac{C'A}{AB}. \tag{4}$$

Now we will use an analogue of the Menelaos theorem, applying to quadrilaterals instead of triangles:

Menelaos theorem for quadrilaterals. If $A'_{1,2},\ A'_{2,3},\ A'_{3,4},\ A'_{4,1}$ are four collinear points on the sides $A_1A_2,\ A_2A_3,\ A_3A_4,\ A_4A_1$ of a quadrilateral $A_1A_2A_3A_4,$ then

$$\frac{A_1A'_{1,2}}{A'_{1,2}A_2} \bullet \frac{A_2A'_{2,3}}{A'_{2,3}A_3} \bullet \frac{A_3A'_{3,4}}{A'_{3,4}A_4} \bullet \frac{A_4A'_{4,1}}{A'_{4,1}A_1} = 1.$$

Note. The number on the right hand side of this equation is indeed 1 (not −1 in contrast to the classical Menelaos theorem for triangles).

Now, we apply the Menelaos theorem for quadrilaterals to the quadrilateral $A'C'ST,$ with the collinear points $Y',\ B',\ G',\ B'$ on its sides $A'C',\ C'S,\ ST,\ TA',$ respectively:

$$\frac{A'Y'}{Y'C'} \bullet \frac{C'B'}{B'S} \bullet \frac{SG'}{G'T} \bullet \frac{TB'}{B'A'} = 1.$$

Using (2), (3), (4), we can rewrite this as

$$\frac{A'Y'}{Y'C'} \bullet \frac{C'A}{AB} \bullet \frac{GT}{SG} \bullet \frac{BC}{CA'} = 1. \tag{5}$$

On the other hand, we can apply the Menelaos theorem for quadrilaterals to the quadrilateral $A'C'ST,$ with the collinear points $Y,\ B',\ G,\ B'$ on its sides $A'C',\ C'S,\ ST,\ TA',$ respectively, and obtain

$$\frac{A'Y}{YC'} \bullet \frac{C'B'}{B'S} \bullet \frac{SG}{GT} \bullet \frac{TB'}{B'A'} = 1.$$

Division by $\frac{SG}{GT}$ yields

$$\frac{A'Y}{YC'} \bullet \frac{C'B'}{B'S} \bullet \frac{TB'}{B'A'} = \frac{GT}{SG}.$$

After (3) and (4), we can rewrite this as

$$\frac{A'Y}{YC'} \bullet \frac{C'A}{AB} \bullet \frac{BC}{CA'} = \frac{GT}{SG}.$$

This can be considered as an expression for $\frac{GT}{SG},$ and substituting this expression in (5), we obtain

$$\frac{A'Y'}{Y'C'} \bullet \frac{C'A}{AB} \bullet \left(\frac{A'Y}{YC'} \bullet \frac{C'A}{AB} \bullet \frac{BC}{CA'} \right) \bullet \frac{BC}{CA'} = 1, \qquad \text{i. e.}$$

$$\frac{A'Y'}{Y'C'} \bullet \frac{A'Y}{YC'} \bullet \left(\frac{C'A}{AB} \right)^2 \bullet \left(\frac{BC}{CA'} \right)^2 = 1, \qquad \text{i. e.}$$

$$\frac{A'Y'}{Y'C'} \bullet \frac{A'Y}{YC'} \bullet \left(\frac{C'A}{CA'} \right)^2 \bullet \left(\frac{BC}{AB} \right)^2 = 1.$$

Similarly,

$$\frac{B'Z'}{Z'A'} \bullet \frac{B'Z}{ZA'} \bullet \left(\frac{A'B}{AB'} \right)^2 \bullet \left(\frac{CA}{BC} \right)^2 = 1;$$

$$\frac{C'X'}{X'B'} \bullet \frac{C'X}{XB'} \bullet \left(\frac{B'C}{BC'} \right)^2 \bullet \left(\frac{AB}{CA} \right)^2 = 1.$$

Multiplying these three equations, we obtain

$$\left(\frac{A'Y'}{Y'C'} \bullet \frac{A'Y}{YC'} \bullet \left(\frac{C'A}{CA'}\right)^2 \bullet \left(\frac{BC}{AB}\right)^2\right) \bullet \left(\frac{B'Z'}{Z'A'} \bullet \frac{B'Z}{ZA'} \bullet \left(\frac{A'B}{AB'}\right)^2 \bullet \left(\frac{CA}{BC}\right)^2\right)$$
$$\bullet \left(\frac{C'X'}{X'B'} \bullet \frac{C'X}{XB'} \bullet \left(\frac{B'C}{BC'}\right)^2 \bullet \left(\frac{AB}{CA}\right)^2\right) \quad = \quad 1 \bullet 1 \bullet 1 = 1.$$

After a rearrangement of the terms on the left hand side, this equation becomes

$$\left(\frac{C'X'}{X'B'} \bullet \frac{B'Z'}{Z'A'} \bullet \frac{A'Y'}{Y'C'}\right) \bullet \left(\frac{C'X}{XB'} \bullet \frac{B'Z}{ZA'} \bullet \frac{A'Y}{YC'}\right)$$
$$\bullet \left(\frac{C'A}{CA'} \bullet \frac{A'B}{AB'} \bullet \frac{B'C}{BC'}\right)^2 \bullet \left(\frac{BC}{AB} \bullet \frac{CA}{BC} \bullet \frac{AB}{CA}\right)^2 \quad = \quad 1. \tag{6}$$

Now,

$$\frac{C'X}{XB'} \bullet \frac{B'Z}{ZA'} \bullet \frac{A'Y}{YC'} = 1$$

after the Ceva theorem, since the lines $A'X,\ B'Y,\ C'Z$ concur (at Q). Furthermore,

$$\begin{aligned}\frac{C'A}{CA'} \bullet \frac{A'B}{AB'} \bullet \frac{B'C}{BC'} &= \frac{A'B}{CA'} \bullet \frac{C'A}{BC'} \bullet \frac{B'C}{AB'} = 1 : \left(\frac{CA'}{A'B} \bullet \frac{BC'}{C'A} \bullet \frac{AB'}{B'C}\right) \\ &= 1 : 1 \qquad \text{(since } \frac{CA'}{A'B} \bullet \frac{BC'}{C'A} \bullet \frac{AB'}{B'C} = 1 \text{ after (1))} \\ &= 1.\end{aligned}$$

And finally, obviously

$$\frac{BC}{AB} \bullet \frac{CA}{BC} \bullet \frac{AB}{CA} = 1.$$

Hence, the equation (6) takes the form

$$\left(\frac{C'X'}{X'B'} \bullet \frac{B'Z'}{Z'A'} \bullet \frac{A'Y'}{Y'C'}\right) \bullet 1 \bullet 1^2 \bullet 1^2 = 1,$$

so that

$$\frac{C'X'}{X'B'} \bullet \frac{B'Z'}{Z'A'} \bullet \frac{A'Y'}{Y'C'} = 1.$$

With the help of the Ceva theorem, this shows that the lines $A'X',\ B'Y',\ C'Z'$ concur at one point. Call this point R.

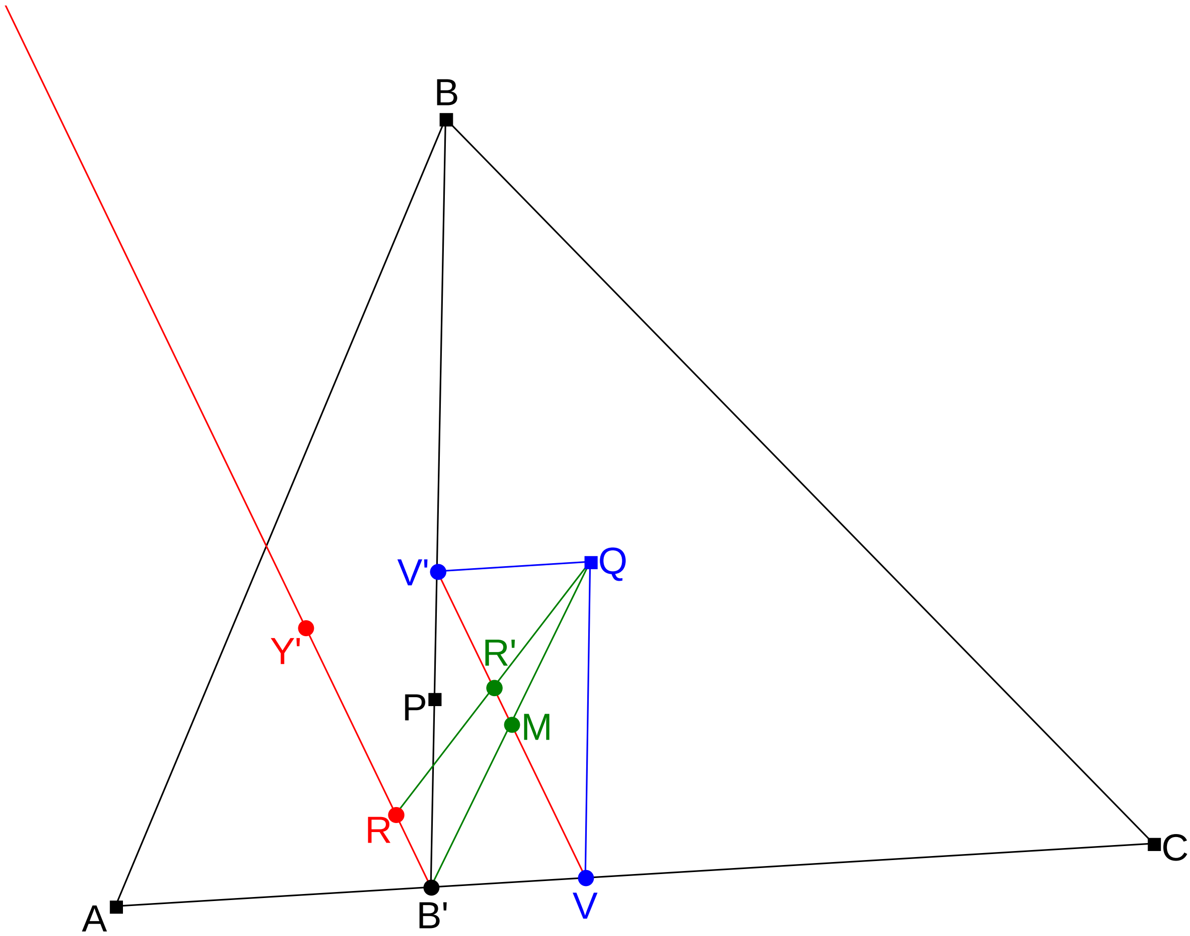

Fig. 4

Now, let R' be the midpoint of the segment $QR,$ and let M be the midpoint of the segment QB'. Since the quadrilateral $QVB'V'$ is a parallelogram, its diagonals QB' and VV' bisect each other. Hence, the midpoint M of the segment QB' is simultaneously the midpoint of the segment VV'. Thus, this point M lies on the line VV'. Now, as the line VV' is parallel to the line $B'Y'$, we can say that the line VV' is the parallel to the line $B'Y'$ through the point M.

Since R' and M are the midpoints of the sides QR and QB' of triangle RQB', we have $R'M \parallel RB'$, or, equivalently, $R'M \parallel B'Y'$. Thus, the point R' lies on the parallel to the line $B'Y'$ through the point M. But we already know that the parallel to the line $B'Y'$ through the point M is the line VV'. Hence, the point R' lies on the line VV'. Similarly, the same point R' lies on the lines WW' and UU', and it follows that the lines UU', VV', WW' concur at one point (the point R'). This completes the proof of Theorem 1. (See Fig. 5.)

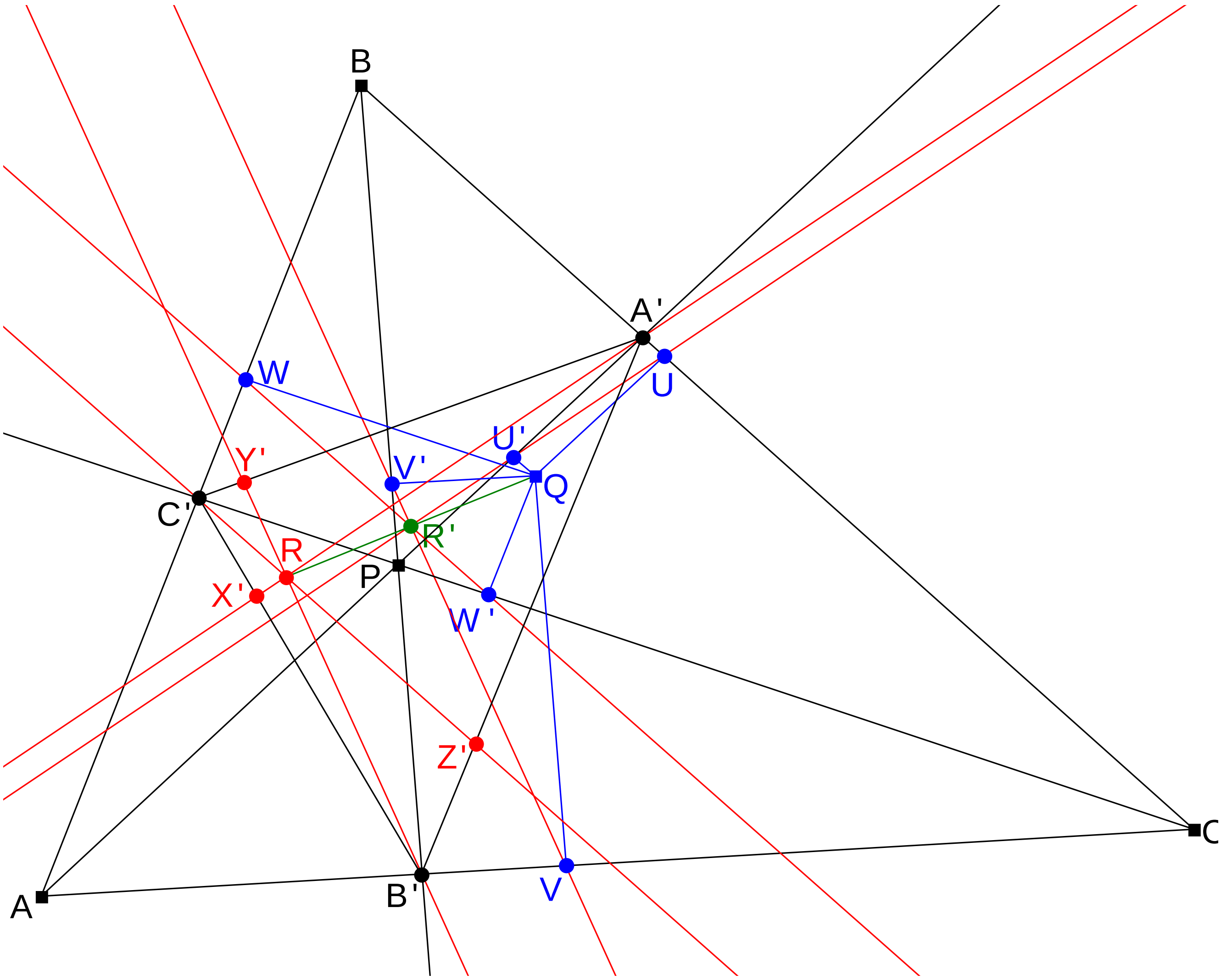

Fig. 5

Finally, we consider a remarkable special case of Theorem 1, namely the case when P is the orthocenter of triangle ABC. In this case, the lines $AP,\ BP,\ CP$ are the altitudes of triangle ABC. In other words, $AP \perp BC,\ BP \perp CA$ and $CP \perp AB$.

Since $QU \parallel AP$ and $AP \perp BC$, it follows that $QU \perp BC$; thus, U is the orthogonal projection of the point Q on the side BC of triangle ABC. Similarly, V and W are the orthogonal projections of the point Q on the sides CA and AB. Further, since $QU' \parallel BC$ and $AP \perp BC$, we have $QU' \perp AP$; thus, U' is the orthogonal projection of the point Q on the line AP, i. e. on the altitude of triangle ABC issuing from A. Similarly, V' and W' are the orthogonal projections of the point Q on the altitudes issuing from B and C, respectively. Altogether, this allows us to apply Theorem 1 and state the conclusion in the following way:

Theorem 2. Let Q be an arbitrary point in the plane of a triangle ABC.

Let $U,\ V,\ W$ be the orthogonal projections of the point Q on the sides $BC,\ CA,\ AB$ of triangle ABC.

Let $U',\ V',\ W'$ be the orthogonal projections of the point Q on the altitudes of triangle ABC issuing from the vertices $A,\ B,\ C$.

Then, the lines $UU',\ VV',\ WW'$ concur at one point. (See Fig. 6.)

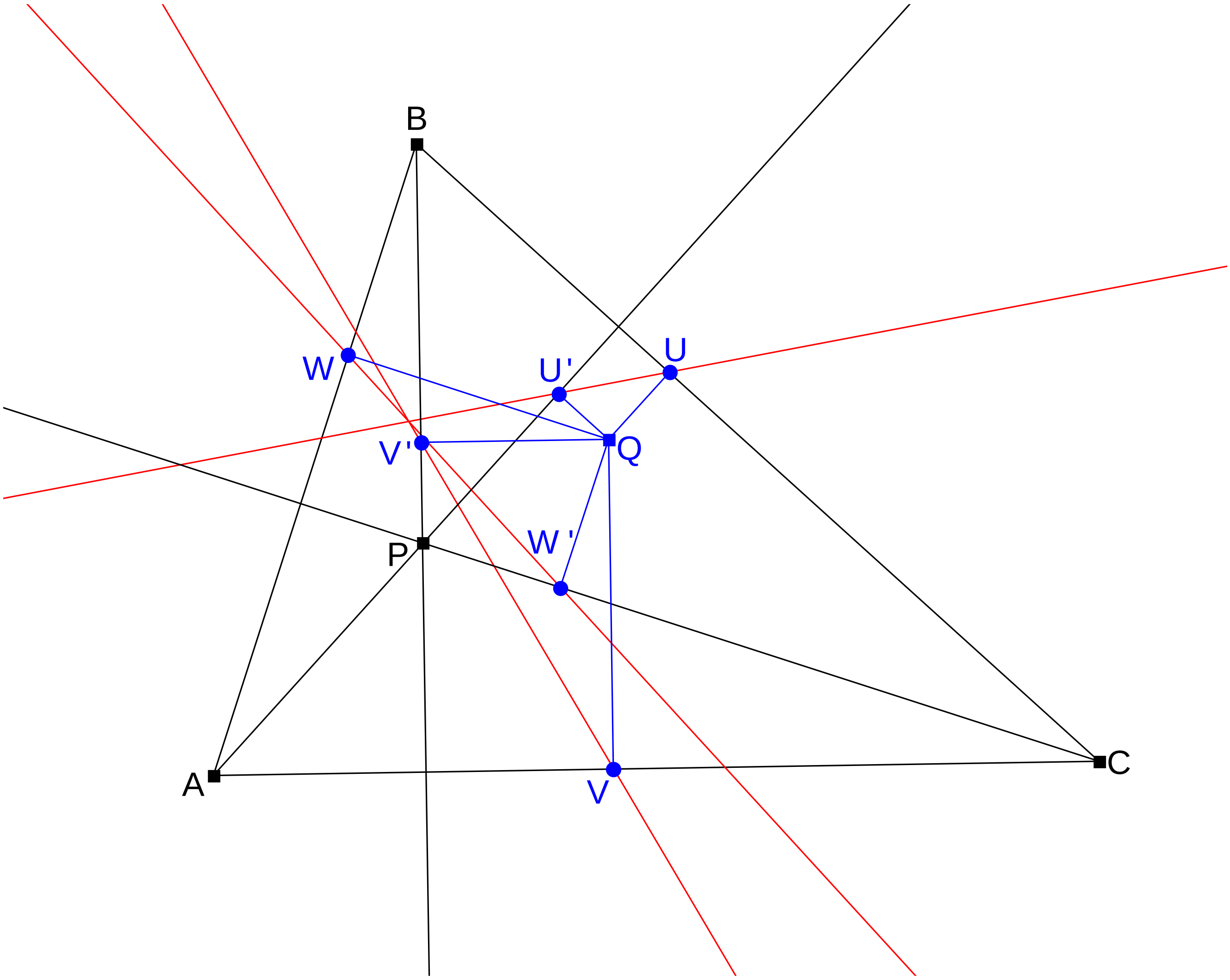

Fig. 6