Math Time problem proposal #1
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Problem. Let x_1, x_2, ..., x_n be real numbers such that $x_1+x_2+...+x_n=1$ and such that $x_i<1$ for every $i\in\{1,2,...,n\}$. Prove that

$$\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}\geq\frac{n}{2(n-1)}.$$

Solution. First, for the sake of brevity[1], we introduce a notation: If i_1, i_2, ..., i_k are some free variables, if f is a function of k variables, and if $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_l$ are some assertions containing some of the variables i_1, i_2, ..., i_k, then

$$\sum_{\mathcal{A}_1,\ \mathcal{A}_2,\ ...,\ \mathcal{A}_l}^{i_1,\ i_2,\ ...,\ i_k} f(i_1,i_2,...,i_k)$$

will mean the sum of the values $f(i_1,i_2,...,i_k)$ over all "good" k-tuples $(i_1,\ i_2,\ ...,\ i_k)$; hereby, a k-tuple $(i_1,\ i_2,\ ...,\ i_k)$ is called *"good"* if it satisfies $i_u\in\{1,2,...,n\}$ for every $u\in\{1,2,...,k\}$ and also satisfies all the assertions $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_l$.

Using this notation, we can write many typical sums in a simpler form: For instance, $\sum\limits_{1\leq i\leq n}f(i)=\sum\limits^{i}f(i)$, and $\sum\limits_{1\leq i<j\leq n}f(i,j)=\sum\limits_{i<j}^{i,\ j}f(i,j)$.

Note that we won't use the notation $\sum\limits_{k=u}^{v}f(k)$ for $f(u)+f(u+1)+...+f(v)$ in this solution, since it (theoretically) could be misunderstood because $\sum\limits_{k=u}^{v}f(k)$ has a different meaning in our above notation.

With the notation defined above, we have

$$\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}=\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}.$$

Hence, the inequality in question,

$$\sum_{1\leq i<j\leq n}\frac{x_ix_j}{(1-x_i)(1-x_j)}\geq\frac{n}{2(n-1)},$$

rewrites as

$$\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}\geq\frac{n}{2(n-1)}.$$

Multiplication by $2(n-1)^2$ transforms this inequality into

$$(n-1)^2\cdot 2\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}\geq n(n-1).$$

[1] In as far as brevity is possible for this solution...

Since

$$2\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}=\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}+\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}$$

$$=\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}+\sum_{j<i}^{i,\ j}\frac{x_jx_i}{(1-x_j)(1-x_i)}$$

(at this step we have interchanged i with j in the second sum)

$$=\sum_{i<j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}+\sum_{j<i}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}$$

$$=\sum_{i\neq j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)},$$

this inequality rewrites as

$$(n-1)^2\cdot\sum_{i\neq j}^{i,\ j}\frac{x_ix_j}{(1-x_i)(1-x_j)}\geq n(n-1). \tag{1}$$

Thus, in order to solve the problem, it remains to prove this inequality (1).

Set $t_i=1-x_i$ for every $i\in\{1,2,...,n\}$. Then, $t_i>0$ for every $i\in\{1,2,...,n\}$ (since $x_i<1$ and thus $1-x_i>0$, so that $t_i=1-x_i>0$). Besides, $t_i=1-x_i$ yields $x_i=1-t_i$. Finally,

$$\sum^k t_k=\sum^k(1-x_k)=n-\sum^k x_k=n-(x_1+x_2+...+x_n)=n-1. \tag{2}$$

Consequently,

$$(n-1)^2 \cdot \sum_{i \neq j}^{i,\ j} \frac{x_i x_j}{(1-x_i)(1-x_j)} = \sum_{i \neq j}^{i,\ j} \frac{(n-1)\, x_i \cdot (n-1)\, x_j}{(1-x_i)(1-x_j)}$$

$$= \sum_{i \neq j}^{i,\ j} \frac{(n-1)(1-t_i) \cdot (n-1)(1-t_j)}{t_i t_j} = \sum_{i \neq j}^{i,\ j} \frac{((n-1)-(n-1)\, t_i) \cdot ((n-1)-(n-1)\, t_j)}{t_i t_j}$$

$$= \sum_{i \neq j}^{i,\ j} \frac{\left(\left(t_j + \sum\limits_{k \neq j}^{k} t_k\right) - (n-1)\, t_i\right) \cdot \left(\left(t_i + \sum\limits_{m \neq i}^{m} t_m\right) - (n-1)\, t_j\right)}{t_i t_j}$$

$$\left(\text{since (2) yields } n-1 = \sum^{k} t_k = t_j + \sum_{k \neq j}^{k} t_k \text{ and } n-1 = \sum^{k} t_k = \sum^{m} t_m = t_i + \sum_{m \neq i}^{m} t_m\right)$$

$$= \sum_{i \neq j}^{i,\ j} \frac{\left(\left(t_j + \sum\limits_{k \neq j}^{k} t_k\right) - \sum\limits_{k \neq j}^{k} t_i\right) \cdot \left(\left(t_i + \sum\limits_{m \neq i}^{m} t_m\right) - \sum\limits_{m \neq i}^{m} t_j\right)}{t_i t_j}$$

$$\left(\text{since } (n-1)\, t_i = \sum_{k \neq j}^{k} t_i \text{ and } (n-1)\, t_j = \sum_{m \neq i}^{m} t_j\right)$$

$$= \sum_{i \neq j}^{i,\ j} \frac{\left(t_j + \sum\limits_{k \neq j}^{k} (t_k - t_i)\right) \cdot \left(t_i + \sum\limits_{m \neq i}^{m} (t_m - t_j)\right)}{t_i t_j}$$

$$= \sum_{i \neq j}^{i,\ j} \frac{t_i t_j + \sum\limits_{k \neq j}^{k} (t_k - t_i) \cdot t_i + \sum\limits_{m \neq i}^{m} (t_m - t_j) \cdot t_j + \sum\limits_{k \neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m \neq i}^{m} (t_m - t_j)}{t_i t_j}$$

$$= \sum_{i \neq j}^{i,\ j} \left(1 + \frac{\sum\limits_{k \neq j}^{k} (t_k - t_i)}{t_j} + \frac{\sum\limits_{m \neq i}^{m} (t_m - t_j)}{t_i} + \frac{\sum\limits_{k \neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m \neq i}^{m} (t_m - t_j)}{t_i t_j}\right)$$

$$= \sum_{i \neq j}^{i,\ j} 1 + \sum_{i \neq j}^{i,\ j} \frac{\sum\limits_{k \neq j}^{k} (t_k - t_i)}{t_j} + \sum_{i \neq j}^{i,\ j} \frac{\sum\limits_{m \neq i}^{m} (t_m - t_j)}{t_i} + \sum_{i \neq j}^{i,\ j} \frac{\sum\limits_{k \neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m \neq i}^{m} (t_m - t_j)}{t_i t_j}. \qquad (3)$$

Now,

$$\sum_{i \neq j}^{i,\ j} 1 = n\,(n-1)$$

(because $n\,(n-1)$ is the number of all pairs $(i,\ j)$ with i and j being elements of the set $\{1, 2, ..., n\}$ and satisfying $i \neq j$). Besides,

$$\sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i)}{t_j} + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i} = \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i} + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i}$$

(here we have interchanged i with j and renamed k by m in the first sum)

$$= 2\sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i} = 2\sum^{i} \frac{1}{t_i} \sum_{i\neq j}^{j} \sum_{m\neq i}^{m} (t_m - t_j) = 2\sum^{i} \frac{1}{t_i} \left(\sum_{i\neq j}^{j} \sum_{m\neq i}^{m} t_m - \sum_{i\neq j}^{j} \sum_{m\neq i}^{m} t_j \right)$$

$$= 2\sum^{i} \frac{1}{t_i} \left(\sum_{m\neq i}^{m} \sum_{i\neq j}^{j} t_m - \sum_{i\neq j}^{j} \sum_{m\neq i}^{m} t_j \right) = 2\sum^{i} \frac{1}{t_i} \left(\sum_{m\neq i}^{m} (n-1)\, t_m - \sum_{i\neq j}^{j} (n-1)\, t_j \right)$$

$$= 2\sum^{i} \frac{1}{t_i} \left(\sum_{j\neq i}^{j} (n-1)\, t_j - \sum_{i\neq j}^{j} (n-1)\, t_j \right) \qquad \text{(here we have renamed } m \text{ by } j \text{ in the second sum)}$$

$$= 2\sum^{i} \frac{1}{t_i} \left(\sum_{i\neq j}^{j} (n-1)\, t_j - \sum_{i\neq j}^{j} (n-1)\, t_j \right) = 2\sum^{i} \frac{1}{t_i} \cdot 0 = 0.$$

Hence, (3) becomes

$$(n-1)^2 \cdot \sum_{i\neq j}^{i,\ j} \frac{x_i x_j}{(1-x_i)(1-x_j)}$$

$$= \underbrace{\sum_{i\neq j}^{i,\ j} 1}_{=n(n-1)} + \underbrace{\sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i)}{t_j} + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i}}_{=0} + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i t_j}$$

$$= n(n-1) + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i t_j}.$$

Therefore, the inequality that we have to prove, namely (1), rewrites as

$$n(n-1) + \sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i t_j} \geq n(n-1).$$

This is obviously equivalent to

$$\sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i t_j} \geq 0. \qquad (4)$$

Thus, we only have to prove (4) in order to complete the solution of the problem (because (4) is equivalent to (1), and proving (1) solves the problem).

Now,

$$\sum_{i\neq j}^{i,\ j} \frac{\sum\limits_{k\neq j}^{k} (t_k - t_i) \cdot \sum\limits_{m\neq i}^{m} (t_m - t_j)}{t_i t_j} = \sum_{i\neq j,\ k\neq j,\ m\neq i}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j}$$

$$= \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{i\neq j,\ k\neq j,\ m\neq i,\ (k=i\ \vee\ m=j)}^{i,\ j,\ k,\ m} \underbrace{\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j}}_{\substack{=0,\text{ since we have } k=i \text{ or } m=j \\ \text{and thus } t_k - t_i = 0 \text{ or } t_m - t_j = 0}}$$

$$\left(\begin{array}{c}\text{because every quadruple } (i,j,k,m) \text{ satisfies exactly one of the two assertions} \\ (k\neq i\ \wedge\ m\neq j) \text{ and } (k=i\ \vee\ m=j)\end{array}\right)$$

$$= \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + 0 = \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j}$$

$$= \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j,\ k\neq m}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j,\ k=m}^{i,\ j,\ k,\ m} \underbrace{\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j}}_{\substack{=\frac{(t_k - t_i)(t_k - t_j)}{t_i t_j}, \\ \text{since } k=m}}$$

$$= \sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j,\ k\neq m}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{i\neq j,\ k\neq j,\ k\neq i}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j}.$$

Thus, in order to prove (4), it is enough to prove the two inequalities

$$\sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j,\ k\neq m}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \geq 0; \qquad (5)$$

$$\sum_{i\neq j,\ k\neq j,\ k\neq i}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} \geq 0. \qquad (6)$$

Proving (5) is rather easy: First, the six conditions $i \neq j,\ k \neq j,\ m \neq i,\ k \neq i,\ m \neq j,\ k \neq m$ altogether (connected by a logical "and") are equivalent to the condition that the four numbers i, j, k, m are pairwisely distinct, i. e. that the set $\{i, j, k, m\}$

has exactly 4 elements, i. e. that $\left|\{i,j,k,m\}\right| = 4$. Hence,

$$\sum_{i\neq j,\ k\neq j,\ m\neq i,\ k\neq i,\ m\neq j,\ k\neq m}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} = \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j}$$

$$= \frac{1}{4} \cdot \left(\sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \right.$$
$$\left. + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} \right)$$

$$= \frac{1}{4} \cdot \left(\sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_i - t_k)(t_m - t_j)}{t_k t_j} \right.$$
$$\left. + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)(t_j - t_m)}{t_i t_m} + \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_i - t_k)(t_j - t_m)}{t_k t_m} \right)$$

(at this step, we have renamed some variables: in the second sum, we interchanged i with k; in the third sum, we interchanged j with m; in the fourth sum, we interchanged both i with k and j with m)

$$= \frac{1}{4} \cdot \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \left(\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} + \frac{(t_i - t_k)(t_m - t_j)}{t_k t_j} + \frac{(t_k - t_i)(t_j - t_m)}{t_i t_m} + \frac{(t_i - t_k)(t_j - t_m)}{t_k t_m} \right)$$

$$= \frac{1}{4} \cdot \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \left(\frac{(t_k - t_i)(t_m - t_j)}{t_i t_j} - \frac{(t_k - t_i)(t_m - t_j)}{t_k t_j} - \frac{(t_k - t_i)(t_m - t_j)}{t_i t_m} + \frac{(t_k - t_i)(t_m - t_j)}{t_k t_m} \right)$$

$$= \frac{1}{4} \cdot \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} (t_k - t_i)(t_m - t_j) \cdot \left(\frac{1}{t_i t_j} - \frac{1}{t_k t_j} - \frac{1}{t_i t_m} + \frac{1}{t_k t_m} \right)$$

$$= \frac{1}{4} \cdot \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} (t_k - t_i)(t_m - t_j) \cdot \frac{(t_k - t_i)(t_m - t_j)}{t_i t_j t_k t_m} = \frac{1}{4} \cdot \sum_{|\{i,j,k,m\}|=4}^{i,\ j,\ k,\ m} \frac{(t_k - t_i)^2 (t_m - t_j)^2}{t_i t_j t_k t_m} \geq 0$$

because squares are nonnegative; thus, (5) is proven.

In order to prove (6), we proceed similarly: First, the three conditions $i \neq j,\ k \neq j,\ k \neq i$ altogether (connected by a logical "and") are equivalent to the condition that the three numbers i, j, k are pairwise distinct, i. e. that the set $\{i,j,k\}$ has exactly

3 elements, i. e. that $|\{i, j, k\}| = 3$. Hence,

$$\sum_{i\neq j,\ k\neq j,\ k\neq i}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} = \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j}$$

$$= \frac{1}{3} \cdot \left(\sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} \right)$$

$$= \frac{1}{3} \cdot \left(\sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_i - t_j)(t_i - t_k)}{t_j t_k} + \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{(t_j - t_k)(t_j - t_i)}{t_k t_i} \right)$$

$$\left(\begin{array}{c} \text{at this step, we have renamed some variables:} \\ \text{in the second sum, we renamed } i,\ j,\ k \text{ by } j,\ k,\ i, \text{ respectively;} \\ \text{in the third sum, we renamed } i,\ j,\ k \text{ by } k,\ i,\ j, \text{ respectively} \end{array} \right)$$

$$= \frac{1}{3} \cdot \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \left(\frac{(t_k - t_i)(t_k - t_j)}{t_i t_j} + \frac{(t_i - t_j)(t_i - t_k)}{t_j t_k} + \frac{(t_j - t_k)(t_j - t_i)}{t_k t_i} \right)$$

$$= \frac{1}{3} \cdot \sum_{|\{i,j,k\}|=3}^{i,\ j,\ k} \frac{t_k (t_k - t_i)(t_k - t_j) + t_i (t_i - t_j)(t_i - t_k) + t_j (t_j - t_k)(t_j - t_i)}{t_i t_j t_k} \geq 0,$$

because the Schur inequality yields $t_k (t_k - t_i)(t_k - t_j) + t_i (t_i - t_j)(t_i - t_k) + t_j (t_j - t_k)(t_j - t_i) \geq 0$ for any triple $(i,\ j,\ k)$. Thus, the inequality (6) is proven.

As both inequalities (5) and (6) are verified now, the inequality (4) follows, and thus the problem is solved.

Remark. The above problem is a generalization of the Schur inequality (which states that $a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0$ for any three nonnegative reals a, b, c) to n variables (in fact, if you set $n = 3$ in the problem, you get the Schur inequality for the numbers $1 - x_1$, $1 - x_2$, $1 - x_3$).

In the particular case when the reals x_1, x_2, ..., x_n are nonnegative, the problem can be solved much more easily - a solution for this case was given in [1] (proof of Theorem 4.1).

References

[1] Darij Grinberg, *An inequality involving* $2n$ *numbers.*