American Mathematical Monthly Problem 11440 by Stefano Siboni, Trento, Italy.

[...]

Incomplete solution by Darij Grinberg.

Remark: Since $\dfrac{\mathbf{x}(t)}{||\mathbf{x}(t)||}$ is undefined for $\mathbf{x}(t) = 0$, we are going to consider only those functions $\mathbf{x}$ which never vanish. This renders the word "nonzero" in problem **(c)** useless.

We recall a fact from first-course differential equations:

Theorem 1. Let $V \subseteq \mathbb{R} \times \mathbb{R}^m$ be an open and connected subset of $\mathbb{R} \times \mathbb{R}^m$. We denote points in V by $(t, \mathbf{y})$, where t is the $\mathbb{R}$-coordinate and $\mathbf{y}$ is the $\mathbb{R}^m$-coordinate. Let $f : V \to \mathbb{R}^m$ be a continuous map which is locally Lipschitz with respect to $\mathbf{y}$.

(a) For every $(t_0, \mathbf{y}_0) \in V$, the initial value problem

$$\mathbf{y}' = f(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{2}$$

has one and only one maximal solution, which is a function $\lambda :]a, b[\to \mathbb{R}^m$ defined on some open interval $]a, b[$ (where a and b may be reals but may also be values such as $+\infty$ and $-\infty$) such that every solution of (2) on an interval is a restriction of λ and its domain is a subset of $]a, b[$.

(b) If $b < \infty$, then the set $\{\lambda(t) \mid t \in [t_0, b[\}$ is unbounded, or

$$\partial V \neq \varnothing \qquad \text{and} \qquad \lim_{t \nearrow b} \operatorname{dist}[(t, \lambda(t)), \partial V] = 0.$$

(c) If $a > -\infty$, then the set $\{\lambda(t) \mid t \in]a, t_0]\}$ is unbounded, or

$$\partial V \neq \varnothing \qquad \text{and} \qquad \lim_{t \searrow a} \operatorname{dist}[(t, \lambda(t)), \partial V] = 0.$$

[The word "or" means a logical "or" here (not an "or, equivalently").]

We are going to apply this theorem to our problem, but first let us prepare.
Let $m = 6$. Define a subset $V \subseteq \mathbb{R} \times (\mathbb{R}^3 \times \mathbb{R}^3)$ by

$$V = \mathbb{R} \times \left(\left(\mathbb{R}^3 \setminus 0\right) \times \mathbb{R}^3\right) = \left\{(t, (\mathbf{y}_1, \mathbf{y}_2)) \in \mathbb{R} \times \left(\mathbb{R}^3 \times \mathbb{R}^3\right) \mid \mathbf{y}_1 \neq 0\right\}.$$

Obviously, V is an open and connected subset of $\mathbb{R} \times (\mathbb{R}^3 \times \mathbb{R}^3)$.

For every vector function $\mathbf{x} \in \mathcal{C}^2(I, \mathbb{R}^3)$ (where $I \subseteq \mathbb{R}$ is some interval), we can define a vector function $\mathbf{y} \in \mathcal{C}^1(I, \mathbb{R}^3 \times \mathbb{R}^3)$ by $\mathbf{y}(t) = (\mathbf{x}(t), \mathbf{x}'(t))$ for every $t \in I$. Our vector differential equation (1) is equivalent to the vector differential equation

$$\mathbf{y}' = f(t, \mathbf{y})$$

for this function $\mathbf{y}$, where the map $f : V \to \mathbb{R}^3 \times \mathbb{R}^3$ is defined by

$$v\left(t,\left(\mathbf{y}_1,\mathbf{y}_2\right)\right)=\left(\mathbf{y}_2, p\left(t,\mathbf{y}_1,\mathbf{y}_2\right)\mathbf{y}_2\times\frac{\mathbf{y}_1}{\left|\left|\mathbf{y}_1\right|\right|}\right)\qquad\text{for all }\left(t,\left(\mathbf{y}_1,\mathbf{y}_2\right)\right)\in V.$$

This map $f : V \to \mathbb{R}^3 \times \mathbb{R}^3$ is continuous and locally Lipschitz with respect to $\mathbf{y}$ (in fact, the local Lipschitz property is clear because $\frac{\mathbf{y}_1}{\left|\left|\mathbf{y}_1\right|\right|}$, $\mathbf{y}_2$ and $p\left(t,\mathbf{y}_1,\mathbf{y}_2\right)$ are all locally Lipschitz with respect to $\mathbf{y}$, where the local Lipschitz property for the map $p\left(t,\mathbf{y}_1,\mathbf{y}_2\right)$ follows from the continuity of its partial derivatives).

We can now apply Theorem 1 **(a)** to our situation (identifying the space $\mathbb{R}^3 \times \mathbb{R}^3$ with $\mathbb{R}^6 = \mathbb{R}^m$). We conclude that for every $(t_0, \mathbf{y}_0) \in V$, the initial value problem (2) has one and only one maximal solution $\lambda :]a, b[\to \mathbb{R}^3 \times \mathbb{R}^3$ defined on some open interval $]a, b[$. So let us fix some $(t_0, \mathbf{y}_0) \in V$ and consider the maximal solution λ for this $(t_0, \mathbf{y}_0) \in V$. The function λ has the form $\lambda(t) = (\lambda_1(t), \lambda_2(t))$ for every $t \in]a, b[$, where $\lambda_1 :]a, b[\to \mathbb{R}^3$ and $\lambda_2 :]a, b[\to \mathbb{R}^3$ are two vector functions. Since λ satisfies (2), we must have $\lambda_2 = \lambda_1'$, so that $\lambda(t) = (\lambda_1(t), \lambda_1'(t))$ for every $t \in]a, b[$. Besides, the equation (2) yields

$$\lambda_2'(t)=p\left(t,\lambda_1(t),\lambda_2(t)\right)\lambda_2(t)\times\frac{\lambda_1(t)}{\left|\left|\lambda_1(t)\right|\right|},$$

which (in view of $\lambda_2 = \lambda_1'$) becomes

$$\lambda_1''(t)=p\left(t,\lambda_1(t),\lambda_1'(t)\right)\lambda_1'(t)\times\frac{\lambda_1(t)}{\left|\left|\lambda_1(t)\right|\right|}. \tag{3}$$

In other words, λ_1 is a solution of the equation (1) for every $(t_0, \mathbf{y}_0) \in V$. Conversely, every solution $\mathbf{x} : I \to \mathbb{R}^3$ (where I is some interval) of the equation (1) is the function λ_1 (restricted to I) for a suitable choice of the initial value $(t_0, \mathbf{y}_0) \in V$ [1]. Hence, in order to solve the problem, it is enough to prove the following three assertions for every $(t_0, \mathbf{y}_0) \in V$:

Assertion α. The solution λ is defined on all of $\mathbb{R}$; in other words, $a = -\infty$ and $b = \infty$ (since $]a, b[$ is the domain of λ).

Assertion β. We have $\lim\limits_{t\to\infty} \lambda_1(t) = \infty$, unless λ_1 is a constant function.

Assertion γ. The limit $\lim\limits_{t\to\infty} \frac{\lambda_1(t)}{\left|\left|\lambda_1(t)\right|\right|}$ exists.

Before we start proving these assertions, let us study λ.

The equation (3) yields $\langle \lambda_1''(t), \lambda_1(t)\rangle = 0$ and $\langle \lambda_1''(t), \lambda_1'(t)\rangle = 0$ for every $t \in]a, b[$ (where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of vectors in $\mathbb{R}^3$, defined by $\langle (a_1, a_2, a_3), (b_1, b_2, b_3)\rangle = a_1b_1 + a_2b_2 + a_3b_3$). In other words, $\langle \lambda_1'', \lambda_1\rangle = 0$ and $\langle \lambda_1'', \lambda_1'\rangle = 0$ (where $\langle \cdot, \cdot \rangle$ means the pointwise scalar product of functions).

We will often use the (obvious) fact that

$$\langle f, g\rangle' = \langle f', g\rangle + \langle f, g'\rangle \tag{4}$$

[1] In fact, take some $t_0 \in I$ and set $\mathbf{y}_0 = (\mathbf{x}(t_0), \mathbf{x}'(t_0))$, and define a function $\mathbf{y} : I \to \mathbb{R}^3 \times \mathbb{R}^3$ by $\mathbf{y}(t) = (\mathbf{x}(t), \mathbf{x}'(t))$ for every $t \in I$. Then, $\mathbf{y}$ solves the initial value problem (2), and thus is the restriction of its maximal solution λ to I, so that $\mathbf{x}$ is the restriction of λ_1 to I.

for any two vector functions $f \in \mathcal{C}^1\left(\left]a,b\right[\to \mathbb{R}^3\right)$ and $g \in \mathcal{C}^1\left(\left]a,b\right[\to \mathbb{R}^3\right)$. We have

$$\begin{aligned} \left\langle \lambda_1', \lambda_1' \right\rangle' &= \left\langle \lambda_1'', \lambda_1' \right\rangle + \underbrace{\left\langle \lambda_1', \lambda_1'' \right\rangle}_{=\left\langle \lambda_1'', \lambda_1' \right\rangle} \qquad \text{(by (4))} \\ &= 2 \cdot \left\langle \lambda_1'', \lambda_1' \right\rangle = 2 \cdot 0 = 0, \end{aligned}$$

so that $\left\langle \lambda_1', \lambda_1' \right\rangle$ is a constant function. Also,

$$\begin{aligned} \left\langle \lambda_1', \lambda_1 \right\rangle' &= \underbrace{\left\langle \lambda_1'', \lambda_1 \right\rangle}_{=0} + \left\langle \lambda_1', \lambda_1' \right\rangle \qquad \text{(by (4))} \\ &= \left\langle \lambda_1', \lambda_1' \right\rangle, \end{aligned}$$

and

$$\begin{aligned} \left\langle \lambda_1, \lambda_1 \right\rangle' &= \left\langle \lambda_1', \lambda_1 \right\rangle + \underbrace{\left\langle \lambda_1, \lambda_1' \right\rangle}_{=\left\langle \lambda_1', \lambda_1 \right\rangle} \qquad \text{(by (4))} \\ &= 2 \left\langle \lambda_1', \lambda_1 \right\rangle, \end{aligned}$$

so that

$$\left\langle \lambda_1, \lambda_1 \right\rangle'' = \left(\underbrace{\left\langle \lambda_1, \lambda_1 \right\rangle'}_{=2\left\langle \lambda_1', \lambda_1 \right\rangle} \right)' = 2 \underbrace{\left\langle \lambda_1', \lambda_1 \right\rangle'}_{=\left\langle \lambda_1', \lambda_1' \right\rangle} = 2 \underbrace{\left\langle \lambda_1', \lambda_1' \right\rangle}_{\text{a constant function}}$$

is a constant function. This means that $\left\langle \lambda_1, \lambda_1 \right\rangle$ is a quadratic polynomial in t. In other words, there exist reals u, v, w such that

$$\left\langle \lambda_1, \lambda_1 \right\rangle (t) = ut^2 + vt + w \qquad \text{for every } t \in \left]a,b\right[.$$

Then, $\left\langle \lambda_1, \lambda_1 \right\rangle' = 2 \left\langle \lambda_1', \lambda_1 \right\rangle$ yields that

$$\left\langle \lambda_1', \lambda_1 \right\rangle (t) = ut + \frac{v}{2} \qquad \text{for every } t \in \left]a,b\right[.$$

Using $\left\langle \lambda_1', \lambda_1 \right\rangle' = \left\langle \lambda_1', \lambda_1' \right\rangle$, this leads to

$$\left\langle \lambda_1', \lambda_1' \right\rangle (t) = u \qquad \text{for every } t \in \left]a,b\right[.$$

For every $t \in \left]a,b\right[$, we have

$$\begin{aligned} \underbrace{\left(ut^2 + vt + w\right)}_{\substack{=\left\langle \lambda_1, \lambda_1 \right\rangle (t) \\ =\left\langle \lambda_1(t), \lambda_1(t) \right\rangle}} \cdot \underbrace{u}_{\substack{=\left\langle \lambda_1', \lambda_1' \right\rangle (t) \\ =\left\langle \lambda_1'(t), \lambda_1'(t) \right\rangle}} &= \left\langle \lambda_1(t), \lambda_1(t) \right\rangle \cdot \left\langle \lambda_1'(t), \lambda_1'(t) \right\rangle \\ &\geq \underbrace{\left\langle \lambda_1'(t), \lambda_1(t) \right\rangle}_{=\left\langle \lambda_1', \lambda_1 \right\rangle (t) = ut + v/2}^{2} \qquad \text{(by the Cauchy-Schwarz inequality)} \\ &= \left(ut + \frac{v}{2}\right)^2, \end{aligned} \tag{5}$$

what rewrites as $4uw \geq v^2$.

We now distinguish between two cases:

Case 1. We have $4uw > v^2$.

Case 2. We have $4uw = v^2$.

In Case 2, we must have $(ut^2 + vt + w) \cdot u = \left(ut + \frac{v}{2}\right)^2$ for every $t \in \left]a, b\right[$, so that the chain of inequalities (5) is an equality, and therefore equality must hold in the Cauchy-Schwarz inequality $\langle \lambda_1(t), \lambda_1(t) \rangle \cdot \langle \lambda_1'(t), \lambda_1'(t) \rangle \geq \langle \lambda_1'(t), \lambda_1(t) \rangle^2$ for every $t \in \left]a, b\right[$, so that the vectors $\lambda_1(t)$ and $\lambda_1'(t)$ are collinear for every $t \in \left]a, b\right[$, so that $\lambda_1''(t) = 0$ for every $t \in \left]a, b\right[$ (by (3), since the cross product of two collinear vectors is always 0), so that λ_1 is a linear function and thus defined on the whole $\mathbb{R}$, and consequently $\lambda_2 = \lambda_1'$ is a constant function defined on the whole $\mathbb{R}$, so that $\lambda = (\lambda_1, \lambda_2)$ is defined on the whole $\mathbb{R}$. Thus, all three Assertions α, β and γ are obvious in this case (actually, Assertion β simply says nothing in this case).

So it remains to consider Case 1. In order to prove Assertion α, we have to verify $a = -\infty$ and $b = \infty$. Let us only show that $b = \infty$ (the proof of $a = -\infty$ is analogous).

In fact, assume (for the sake of contradiction) that $b = \infty$ is not the case. Then, $b < \infty$, so that Theorem 1 **(a)** yields that the set $\{\lambda(t) \mid t \in [t_0, b[\}$ is unbounded, or

$$\partial V \neq \varnothing \qquad \text{and} \qquad \lim_{t \nearrow b} \operatorname{dist}\left[(t, \lambda(t)), \partial V\right] = 0.$$

However, the set $\{\lambda(t) \mid t \in [t_0, b[\}$ cannot be unbounded (since

$$\begin{aligned} \|\lambda(t)\| &= \sqrt{\|\lambda_1(t)\|^2 + \Big\|\underbrace{\lambda_2}_{=\lambda_1'}(t)\Big\|^2} = \sqrt{\|\lambda_1(t)\|^2 + \|\lambda_1'(t)\|^2} \\ &= \sqrt{\underbrace{\langle \lambda_1(t), \lambda_1(t) \rangle}_{\substack{=\langle \lambda_1, \lambda_1 \rangle(t) \\ = ut^2+vt+w}} + \underbrace{\langle \lambda_1'(t), \lambda_1'(t) \rangle}_{\substack{=\langle \lambda_1', \lambda_1' \rangle(t) \\ =u}}} = \sqrt{ut^2 + vt + w + u} \end{aligned}$$

is a bounded function on any bounded interval). Hence, we must have

$$\partial V \neq \varnothing \qquad \text{and} \qquad \lim_{t \nearrow b} \operatorname{dist}\left[(t, \lambda(t)), \partial V\right] = 0.$$

But actually, $\partial V = \mathbb{R} \times (0 \times \mathbb{R}^3)$ (since $V = \mathbb{R} \times ((\mathbb{R}^3 \setminus 0) \times \mathbb{R}^3)$) and thus

$$\operatorname{dist}\left[(t, \lambda(t)), \partial V\right] = \|\lambda_1(t)\| = \sqrt{\langle \lambda_1(t), \lambda_1(t) \rangle} = \sqrt{\langle \lambda_1, \lambda_1 \rangle(t)} = \sqrt{ut^2 + vt + w}.$$

Thus, $\lim_{t \nearrow b} \operatorname{dist}\left[(t, \lambda(t)), \partial V\right] = 0$ becomes $\lim_{t \nearrow b} \sqrt{ut^2 + vt + w} = 0$. Thus, $\lim_{t \nearrow b} (ut^2 + vt + w) = 0$. But this is impossible, since the quadratic function $ut^2 + vt + w$ is bounded away from 0 (because its discriminant $4uw - v^2$ is positive, since we are in Case 1). Thus, we get our contradiction, and it is proved that $b = \infty$. Similarly, we can see that $a = -\infty$. This proves Assertion α.

In order to verify Assertion β, we notice that

$$\lim_{t \to \infty} \|\lambda_1(t)\| = \lim_{t \to \infty} \sqrt{\underbrace{\langle \lambda_1(t), \lambda_1(t) \rangle}_{\substack{=\langle \lambda_1, \lambda_1 \rangle(t) \\ =ut^2+vt+w}}} = \lim_{t \to \infty} \sqrt{ut^2 + vt + w} = \infty,$$

unless $u = 0$ which can only hold if the function λ_1 is a constant function (because $u = 0$ yields $\langle \lambda_1' (t), \lambda_1' (t) \rangle = \langle \lambda_1', \lambda_1' \rangle (t) = u = 0$ for every $t \in \mathbb{R}$, thus $\lambda_1' (t) = 0$ for every $t \in \mathbb{R}$, and thus $\lambda_1 = \text{const}$). This proves Assertion β.

I am yet unable to prove Assertion γ.