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Problem 11453 by Richard Stanley, generalized

Let Δ be a finite set of finite sets. Assume that

$$\text{every set } F \in \Delta \text{ and every subset } G \text{ of } F \text{ satisfy } G \in \Delta. \tag{1}$$

Let k be a nonnegative integer. Assume that

$$\text{for every set } G \in \Delta \text{ satisfying } |G| \leq k, \text{ we have } 2^{k+1-|G|} \mid \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}. \tag{2}$$

Prove that $2^{k+1} \mid |\Delta|$.

Solution by Darij Grinberg.

Remark about notation: Let us denote by $\mathbb{N}$ the set $\{0, 1, 2, ...\}$ (and not the set $\{1, 2, 3, ...\}$, as some authors do).

We start with a simple lemma:

Lemma 1. Let G be a finite set. Then,

$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|}.$$

Proof of Lemma 1. For every $k \in \mathbb{N}$, we have

$$|\{F \in \mathcal{P}(G) \mid |F| = k\}| = \binom{|G|}{k}. \tag{3}$$

Now,

$$\sum_{F \in \mathcal{P}(G)} (-2)^{|F|} = \sum_{k \in \mathbb{N}} \sum_{\substack{F \in \mathcal{P}(G); \\ |F| = k}} (-2)^{|F|} = \sum_{k \in \mathbb{N}} \sum_{\substack{F \in \mathcal{P}(G); \\ |F| = k}} (-2)^{k} = \sum_{k \in \mathbb{N}} \underbrace{\sum_{\substack{F \in \mathcal{P}(G); \\ |F| = k}} 1}_{\substack{= |\{F \in \mathcal{P}(G) \mid |F| = k\}| \cdot 1 \\ = |\{F \in \mathcal{P}(G) \mid |F| = k\}| \underset{\text{(by (3))}}{=} \binom{|G|}{k}}} \cdot (-2)^{k}$$

$$= \sum_{k \in \mathbb{N}} \binom{|G|}{k} \cdot (-2)^{k} = \sum_{\substack{k \in \mathbb{N}; \\ k \leq |G|}} \binom{|G|}{k} \cdot (-2)^{k}$$

$$\left(\begin{array}{c} \text{here we replaced the } \sum\limits_{k \in \mathbb{N}} \text{ sign by an } \sum\limits_{\substack{k \in \mathbb{N}; \\ k \leq |G|}} \text{ sign, since the addends for} \\ k > |G| \text{ are zero (as } \binom{|G|}{k} = 0 \text{ for } k > |G| \text{)} \end{array} \right)$$

and

$$\left(\underbrace{-1}_{=(-2)+1}\right)^{|G|} = ((-2)+1)^{|G|} = \sum_{k=0}^{|G|} \binom{|G|}{k} \cdot (-2)^k \cdot \underbrace{1^{|G|-k}}_{=1} \qquad \text{(by the binomial formula)}$$

$$= \underbrace{\sum_{k=0}^{|G|}}_{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\leq|G|}}} \binom{|G|}{k} \cdot (-2)^k = \sum_{\substack{k\in\mathbb{N};\\ k\leq|G|}} \binom{|G|}{k} \cdot (-2)^k .$$

Thus,

$$\sum_{F\in\mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|} .$$

This completes the proof of Lemma 1.

Now let us solve the problem: Let $G \in \Delta$. Then, $\{F \in \Delta \mid G \supseteq F\} \subseteq \mathcal{P}(G)$ [1] and $\mathcal{P}(G) \subseteq \{F \in \Delta \mid G \supseteq F\}$ [2]. Hence, $\{F \in \Delta \mid G \supseteq F\} = \mathcal{P}(G)$.

Now,

$$\sum_{G\in\Delta} (-2)^{|G|} \sum_{\substack{F\in\Delta;\\ F\supseteq G}} (-1)^{|F|}$$

$$= \sum_{F\in\Delta} (-2)^{|F|} \sum_{\substack{G\in\Delta;\\ G\supseteq F}} (-1)^{|G|} \qquad \text{(here we renamed } G \text{ and } F \text{ as } F \text{ and } G \text{ in the sums)}$$

$$= \underbrace{\sum_{F\in\Delta} \sum_{\substack{G\in\Delta;\\ G\supseteq F}}}_{=\sum\limits_{G\in\Delta}\sum\limits_{\substack{F\in\Delta;\\ G\supseteq F}}} (-2)^{|F|} (-1)^{|G|} = \sum_{G\in\Delta} \sum_{\substack{F\in\Delta;\\ G\supseteq F}} (-2)^{|F|} (-1)^{|G|} = \sum_{G\in\Delta} (-1)^{|G|} \sum_{\substack{F\in\Delta;\\ G\supseteq F}} (-2)^{|F|}$$

$$= \sum_{G\in\Delta} (-1)^{|G|} \sum_{F\in\mathcal{P}(G)} (-2)^{|F|}$$

$$\left(\text{since } \{F \in \Delta \mid G \supseteq F\} = \mathcal{P}(G) \text{ yields } \sum_{\substack{F\in\Delta;\\ G\supseteq F}} (-2)^{|F|} = \sum_{F\in\mathcal{P}(G)} (-2)^{|F|}\right)$$

$$= \sum_{G\in\Delta} \underbrace{(-1)^{|G|} (-1)^{|G|}}_{=((-1)\cdot(-1))^{|G|}=1^{|G|}=1} \qquad \left(\text{since Lemma 1 yields } \sum_{F\in\mathcal{P}(G)} (-2)^{|F|} = (-1)^{|G|}\right)$$

$$= \sum_{G\in\Delta} 1 = |\Delta| \cdot 1 = |\Delta| . \tag{4}$$

[1] In fact, let $U \in \{F \in \Delta \mid G \supseteq F\}$. Then, $G \supseteq U$, thus $U \subseteq G$, which means that $U \in \mathcal{P}(G)$. Hence, we have proven that $U \in \mathcal{P}(G)$ for every $U \in \{F \in \Delta \mid G \supseteq F\}$. Thus, $\{F \in \Delta \mid G \supseteq F\} \subseteq \mathcal{P}(G)$.

[2] In fact, let $U \in \mathcal{P}(G)$. Then, $U \subseteq G$, thus $U \in \Delta$ (by (1), applied to G and U instead of F and G) and $G \supseteq U$, so that $U \in \{F \in \Delta \mid G \supseteq F\}$. Hence, we have proven that $U \in \{F \in \Delta \mid G \supseteq F\}$ for every $U \in \mathcal{P}(G)$. Thus, $\mathcal{P}(G) \subseteq \{F \in \Delta \mid G \supseteq F\}$.

But for every $G \in \Delta$, we have

$$2^{k+1} \mid (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|} \tag{5}$$

[3] Thus,

$$2^{k+1} \mid \sum_{G \in \Delta} (-2)^{|G|} \sum_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}.$$

This becomes $2^{k+1} \mid |\Delta|$ (due to (4)). Thus, the problem is solved.

[3] In fact, since $|G| \in \mathbb{N}$, we have either $|G| \leq k$ or $|G| \geq k+1$. In both of these cases, (5) holds, since:

- if $|G| \leq k$, then $2^{k+1-|G|} \mid \sum\limits_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$ (by (2)) and $2^{|G|} \mid (-2)^{|G|}$ (since $(-2)^{|G|} = (-1)^{|G|} 2^{|G|}$) lead to $2^{|G|} 2^{k+1-|G|} \mid (-2)^{|G|} \sum\limits_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$, what becomes $2^{k+1} \mid (-2)^{|G|} \sum\limits_{\substack{F \in \Delta; \\ F \supseteq G}} (-1)^{|F|}$ (since $2^{|G|} 2^{k+1-|G|} = 2^{|G|+(k+1-|G|)} = 2^{k+1}$), so that (5) holds;
- if $|G| \geq k+1$, then $2^{k+1} \mid 2^{|G|}$ and thus $2^{k+1} \mid (-1)^{|G|} 2^{|G|} = (-2)^{|G|}$, so that (5) holds.

Hence, (5) always holds. This completes the proof of (5).