American Mathematical Monthly Problem 11435 by Panagiote Ligouras, Noci, Italy.

Given a triangle ABC with sidelengths a, b, c, inradius r and circumradius R. Prove the inequality

$$\frac{a^2bc}{(a+b)(a+c)}+\frac{b^2ca}{(b+c)(b+a)}+\frac{c^2ab}{(c+a)(c+b)}\leq\frac{9}{2}rR.$$

Solution by Darij Grinberg.

Let Δ be the area of triangle ABC, and $s=\frac{1}{2}(a+b+c)$ its semiperimeter. It is known that $\Delta=sr$, so that $r=\frac{\Delta}{s}=\frac{\Delta}{\frac{1}{2}(a+b+c)}=\frac{2\Delta}{a+b+c}$, and it is also known that $R=\frac{abc}{4\Delta}$. Thus, $\frac{9}{2}rR=\frac{9}{2}\cdot\frac{2\Delta}{a+b+c}\cdot\frac{abc}{4\Delta}=\frac{9abc}{4(a+b+c)}$. Thus, the inequality that we have to prove,

$$\frac{a^2bc}{(a+b)(a+c)}+\frac{b^2ca}{(b+c)(b+a)}+\frac{c^2ab}{(c+a)(c+b)}\leq\frac{9}{2}rR,$$

rewrites as

$$\frac{a^2bc}{(a+b)(a+c)}+\frac{b^2ca}{(b+c)(b+a)}+\frac{c^2ab}{(c+a)(c+b)}\leq\frac{9abc}{4(a+b+c)}.$$

Upon multiplication by $\frac{a+b+c}{abc}$, this becomes

$$\frac{a(a+b+c)}{(a+b)(a+c)}+\frac{b(a+b+c)}{(b+c)(b+a)}+\frac{c(a+b+c)}{(c+a)(c+b)}\leq\frac{9}{4}. \tag{1}$$

Now, consider a new triangle with sidelengths $a'=b+c$, $b'=c+a$, $c'=a+b$ (such a triangle actually exists because $b+c$, $c+a$, $a+b$ satisfy the triangle inequalities: for instance, $(c+a)+(a+b)>(b+c)$). Let A', B', C' be the three angles of this triangle, and let $s'=\frac{1}{2}(a'+b'+c')$ be the semiperimeter of this triangle. Then,

$$s'=\frac{1}{2}(a'+b'+c')=\frac{1}{2}((b+c)+(c+a)+(a+b))=a+b+c,$$

and a standard formula yields

$$\cos^2\frac{A'}{2}=\frac{s'(s'-a')}{b'c'}=\frac{(a+b+c)((a+b+c)-(b+c))}{(c+a)(a+b)}=\frac{a(a+b+c)}{(a+b)(a+c)},$$

and similarly $\cos^2\frac{B'}{2}=\frac{b(a+b+c)}{(b+c)(b+a)}$ and $\cos^2\frac{C'}{2}=\frac{c(a+b+c)}{(c+a)(c+b)}$. Hence, the inequality in question, (1), is equivalent to

$$\cos^2\frac{A'}{2}+\cos^2\frac{B'}{2}+\cos^2\frac{C'}{2}\leq\frac{9}{4}.$$

But this is easy using Jensen's inequality: The function

$$[0,\pi] \to \mathbb{R}, \qquad \varphi \mapsto \cos^2 \frac{\varphi}{2},$$

is concave (since $\cos^2 \frac{\varphi}{2} = \frac{\cos\varphi + 1}{2}$, and the function

$$[0,\pi] \to \mathbb{R}, \qquad \varphi \mapsto \cos\varphi$$

is concave), and thus the Jensen inequality yields

$$\begin{aligned}
\cos^2 \frac{A'}{2} + \cos^2 \frac{B'}{2} + \cos^2 \frac{C'}{2} &\leq 3\cos^2 \frac{\left(\frac{A'+B'+C'}{3}\right)}{2} = 3\cos^2 \frac{A'+B'+C'}{6} \\
&= 3\cos^2 \frac{\pi}{6} \qquad \text{(since } A', B', C' \text{ are angles of a triangle)} \\
&= 3 \cdot \frac{3}{4} = \frac{9}{4},
\end{aligned}$$

qed.